CHAPTER X

DYNAMICS OF HOLONOMIC SYSTEMS

§ 1. Holonomic systems. In this chapter we shall consider the dynamics of certain constrained systems. We shall derive for them equations of motion in which the reactions will not appear.

Let a system of n material points be given. Let the constraints of the system be such that only certain positions of the system are possible at each moment. We do not assume (as in chap. IX) that the same positions are possible at each moment: the set of all possible positions of the system can change together with time.

An example is a point which can remain on a moving plane or a moving curve, e. g. a bead strung on a wire which moves or alters its shape.

If the constraints are bilateral (p. 419) at the moment t, then the coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, of the points of the system must satisfy certain equations (p. 421) at the time t:

$$F_1(x_1, \ldots, z_n, t) = 0, \ldots, F_m(x_1, \ldots, z_n, t) = 0; \tag{1}$$

we write them briefly as:

$$F_j(x_1, \ldots, z_n, t) = 0 \qquad (j = 1, 2, \ldots, m). \tag{I}$$

If the constraints are unilateral at the time t, then, in addition to (I), the relations ((9), p. 420)

$$\Phi_r(x_1, \ldots, z_n, t) \leqq 0 \qquad (r = 1, 2, \ldots, s) \tag{II}$$

hold.

A system whose constraints can be represented by means of relations of the form (I) and (II) is called *holonomic*.

If the functions F_j and Φ_r do not depend on the time t, we say that the *constraints* are *independent of the time* and the system is called *scleronomic*.

In chap. IX we investigated the conditions of equilibrium of holono-mo-scleronomic systems. The constraints of such systems can be defined by relations of the form (I), p. 421, and (9), p. 420:

$$F_j(x_1, y_1, z_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m), \tag{I'}$$

$$\Phi_r(x_1, y_1, z_1, \ldots, z_n) \leqq 0 \qquad (r = 1, 2, \ldots, s). \tag{II'}$$

If at least one of the functions in (I) or (II) depends on the time t, we say that the constraints *depend on the time* and the system is called *rheonomic*.

It is easy to see that a scleronomic system is a particular example of a rheonomic system; in other words, equations (I′) and (II′) are a particular case of equations (I) and (II).

In general, the functions F_j, Φ_r, are assumed to be continuous and to have continuous partial derivatives in a certain region of the variables

$$x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t.$$

The equations (I), (I′), (II), (II′), are said to represent the *constraints in a finite form.*

Just as in scleronomic systems (p. 421), we assume that the functions (I) are independent of each other and that $m < 3n$. The number $k = 3n - m$ is called the *number of degrees of freedom* of the given system.

Example. Let a material point $A(x, y, z)$ be constrained to remain on the surface of a certain sphere which moves with a uniform advancing motion.

Let us denote by r the radius of the sphere, by ξ_0, η_0, ζ_0, the coordinates of the centre of the sphere at the time $t_0 = 0$, by ξ, η, ζ, the coordinates at the time t, and by a, b, c, the projections of the velocity of the advancing motion on the coordinate axes. At the time t we have $\xi = \xi_0 + at$, $\eta = \eta_0 + bt$, and $\zeta = \zeta_0 + ct$. The sphere therefore has the equation $(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 - r^2 = 0$ at the time t; hence

$$(x - \xi_0 - at)^2 + (y - \eta_0 - bt)^2 + (z - \zeta_0 - ct)^2 - r^2 = 0. \tag{2}$$

Hence the coordinates of the point A must satisfy equation (2) at each moment; as it is of the form $F(x, y, z, t) = 0$, the constraints are bilateral, dependent on the time, and therefore the system is holonomo-rheonomic.

If we assume that the point A has to remain within the sphere or on its surface, then the constraints are expressed by the inequality

$$(x - \xi_0 - at)^2 + (y - \eta_0 - bt)^2 + (z - \zeta_0 - ct)^2 - r^2 \leqq 0, \tag{3}$$

and hence they will be unilateral in this case.

§ 2. Non-holonomic systems. Not always can the bilateral constraints of a system be represented in the finite form (I) or (I′).

Let us suppose, for example, that to each point A of space there corresponds a vector $\boldsymbol{H}$ whose projections depend on the coordinates x, y, z, of the point A. Consequently:

$$H_x = \alpha(x, y, z),\ H_y = \beta(x, y, z),\ H_z = \gamma(x, y, z), \tag{1}$$

where α, β, γ, are given functions.

Let us assume that a material point can move only in such a way that its velocity in every position is perpendicular to $\boldsymbol{H}$. Let us denote by $x^{\cdot}, y^{\cdot}, z^{\cdot}$ the projections of the velocity $\mathbf{v}$ of the material point. Therefore at each moment the relation $\boldsymbol{H}\mathbf{v} = 0$ must hold, whence

$$\alpha(x, y, z)\, x^{\cdot} + \beta(x, y, z)\, y^{\cdot} + \gamma(x, y, z)\, z^{\cdot} = 0. \tag{2}$$

If there exists a function $F(x, y, z)$ such that its partial derivatives are equal to the corresponding functions α, β, γ, then equation (2) can be written in the form $\mathrm{d}F / \mathrm{d}t = 0$, whence $F = \mathrm{const} = c$, i. e.

$$F(x, y, z) - c = 0. \tag{3}$$

Conversely, if equation (3) holds, then differentiating it, we obtain (2). Equations (2) and (3) are therefore equivalent in this case, and consequently the constraints are holonomic, since they can be represented in the finite form (3).

However, if the functions α, β, γ, are not the partial derivatives of a function, then equation (2) may be not equivalent to any equation of the form (3). In this case, therefore, the constraints cannot be represented in a finite form and the system is said to be *non-holonomic*.

Equation (2) is usually written in the form

$$\alpha(x, y, z)\, \mathrm{d}x + \beta(x, y, z)\, \mathrm{d}y + \gamma(x, y, z)\, \mathrm{d}z = 0. \tag{4}$$

An equation of a more general form is

$$\alpha(x, y, z, t)\, \mathrm{d}x + \beta(x, y, z, t)\, \mathrm{d}y + \gamma(x, y, z, t)\, \mathrm{d}z + \varepsilon(x, y, z, t)\, \mathrm{d}t = 0. \tag{5}$$

Equation (5) is equivalent to the equation

$$\alpha x^{\cdot} + \beta y^{\cdot} + \gamma z^{\cdot} + \varepsilon = 0, \tag{6}$$

where $\alpha, \beta, \gamma, \varepsilon$, are given functions of the variables x, y, z, and t. It constitutes the necessary condition which the velocities of the points of the system must satisfy. We shall not examine non-holonomic systems more closely.

§ 3. Virtual displacements. On p. 426 we defined the virtual displacement of holonomo-scleronomic systems. We shall now consider rheonomic systems.

Point on a surface. Let the point $A(x, y, z)$ be constrained to remain on a moving surface S whose equation at the moment t is

$$F(x, y, z, t) = 0. \tag{1}$$

The coordinates of the point A therefore satisfy equation (1) at the moment t.

A *virtual displacement* is said to be every displacement $\overline{\delta s}$ of the point A with projections $\delta x, \delta y, \delta z$, satisfying the equation

$$\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z = 0. \tag{2}$$

We see from this that a virtual displacement is such as if the surface S were fixed and had the position it occupies at the moment t. Consequently a virtual displacement is an arbitrary vector tangent at the moment t to the surface S at the point A (p. 423).

Let us give the point A an arbitrary motion compatible with the constraints. The coordinates of the point will therefore satisfy equation (1). Forming the derivative with respect to the time t, we obtain from (1)

$$\frac{\partial F}{\partial x}x^{\cdot} + \frac{\partial F}{\partial y}y^{\cdot} + \frac{\partial F}{\partial z}z^{\cdot} + \frac{\partial F}{\partial t} = 0. \tag{3}$$

Denoting by $\mathbf{v}$ the velocity of the point A, we obtain from (3)

$$\frac{\partial F}{\partial x}v_x + \frac{\partial F}{\partial y}v_y + \frac{\partial F}{\partial z}v_z + \frac{\partial F}{\partial t} = 0. \tag{4}$$

Comparing (2) and (4), we see that we cannot take $\delta x = v_x$, $\delta y = v_y$, and $\delta z = v_z$, i. e. $\overline{\delta s} = \mathbf{v}$, unless $\partial F / \partial t = 0$.

Therefore, in rheonomic systems the virtual displacements in general are not proportional to possible velocities (as in scleronomic systems), i. e. they are expressed by vectors other than possible velocities.

In particular, the displacement $\overline{\delta s} = 0$ is by (2) a virtual displacement (i. e. $\delta x = 0$, $\delta y = 0$, $\delta z = 0$), and from (4) it follows that if $\partial F / \partial t \neq 0$, then $\mathbf{v} = 0$ is not a possible velocity.

Remark. The total differential of function (1) is

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z + \frac{\partial F}{\partial t}\mathrm{d}t.$$

If we take the differential under the assumption that $t = \mathrm{const}$, then $\mathrm{d}t = 0$; consequently

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z.$$

Hence equation (2) is obtained by forming the differential of both sides of (1) under the assumption that the time $t =$ const, and then writing $\delta x, \delta y, \delta z$, for dx, dy, dz, respectively.

In the example on p. 467 the virtual displacement satisfies the equation which one obtains by differentiating equation (2), p. 467, under the assumption that $t =$ const. We get:

$$(x - \xi_0 - at)\,\delta x + (y - \eta_0 - bt)\,\delta y + (z - \zeta_0 - ct)\,\delta z = 0.$$

Choosing $\delta y, \delta z$, arbitrarily, we can determine δx from this equation.

Point on a curve. Let a material point A be constrained to remain on the moving curve C whose equations at the time t are:

$$F_1(x, y, z, t) = 0, \quad F_2(x, y, z, t) = 0. \tag{7}$$

A *virtual displacement* of the point A at the moment t is said to be a displacement $\overline{\delta s}$ (having the projections $\delta x, \delta y, \delta z$) which satisfies the equations:

$$\frac{\partial F_1}{\partial x}\delta x + \frac{\partial F_1}{\partial y}\delta y + \frac{\partial F_1}{\partial z}\delta z = 0, \quad \frac{\partial F_2}{\partial x}\delta x + \frac{\partial F_2}{\partial y}\delta y + \frac{\partial F_2}{\partial z}\delta z = 0. \tag{8}$$

Consequently the virtual displacement is such as if the curve C were fixed and had that position which it occupies at the time t. The virtual displacement is therefore an arbitrary vector tangent at the time t to the curve C at the point A (p. 424).

In this case also the virtual displacement is generally not proportional to a possible velocity. For by (7) the possible velocity $\mathbf{v}$ satisfies the equations (which are obtained by forming the derivatives of equations (7)):

$$\begin{aligned} \frac{\partial F_1}{\partial x}v_x + \frac{\partial F_1}{\partial y}v_y + \frac{\partial F_1}{\partial z}v_z + \frac{\partial F_1}{\partial t} &= 0, \\ \frac{\partial F_2}{\partial x}v_x + \frac{\partial F_2}{\partial y}v_y + \frac{\partial F_2}{\partial z}v_z + \frac{\partial F_2}{\partial t} &= 0. \end{aligned} \tag{9}$$

If $\partial F_1 / \partial t \neq 0$ or $\partial F_2 / \partial t \neq 0$, then by (8) and (9) we cannot take $\delta x = v_x$, $\delta y = v_y$, $\delta z = v_z$, i. e. $\overline{\delta s} = \mathbf{v}$.

Let us still note that equations (8) are obtained by forming the differentials of equations (7), under the assumption that $t =$ const, and writing $\delta x, \delta y, \delta z$, instead of dx, dy, dz.

Example I. A material point A is constrained to remain on a parabola rotating about the z-axis with a constant angular velocity ω (positive, if the rotation takes place from right to left). At $t_0 = 0$ the parabola lies in the xz-plane and has the equation

$$z = x^2. \tag{10}$$

The parabola generates a paraboloid of revolution $z = x^2 + y^2$. The position of the parabola at the time t is obtained as the intersection of the paraboloid with the plane $x \sin\omega t + y \cos\omega t = 0$. The coordinates of the point A consequently satisfy the equations:

$$x^2 + y^2 - z = 0, \quad x \sin\omega t + y \cos\omega t = 0. \tag{11}$$

The virtual displacement $\delta x, \delta y, \delta z$, satisfies the equations obtained by differentiating (11) under the assumption that $t =$ const. Therefore:

$$2x\,\delta x + 2y\,\delta y - \delta z = 0, \quad \delta x \sin\omega t + \delta y \cos\omega t = 0.$$

If $\omega t \neq \frac{1}{2}\pi$ and $\omega t \neq \frac{3}{2}\pi$, then:

$$\delta y = -\delta x \tan\omega t, \quad \delta z = 2(x - y\tan\omega t)\,\delta x,$$

where δx is arbitrary.

Systems of points. Let a holonomic system whose constraints are defined by the equations

$$F_j(x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t) = 0 \qquad (j = 1, 2, \ldots, m) \tag{12}$$

be given.

A *virtual displacement* of a system at the moment t in the position $(x_1, \ldots, z_n)$ compatible with the constraints is defined to be every displacement $\delta x_1, \ldots, \delta z_n$, satisfying the equations:

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\delta x_i + \frac{\partial F_j}{\partial y_i}\delta y_i + \frac{\partial F_j}{\partial z_i}\delta z_i\right) = 0 \qquad (j = 1, 2, \ldots, m). \tag{I}$$

Equations (I) are assumed to be linearly independent at each moment t; in other words, we assume that we can choose from among the unknowns $\delta x_i, \delta y_i, \delta z_i$ $k = 3n - m$ unknowns arbitrarily and determine the remaining m unknowns from equations (I).

Equations (I) have a form similar to those for a scleronomic system (cf. (I), p. 426). The virtual displacements of a system at the moment t are therefore such as if the constraints did not depend on the time and were constantly such as at the time t.

Equations (I) are obtained by forming the differentials of equations (12), under the assumption that $t =$ const, and then writing $\delta x_1, \ldots, \delta z_n$, instead of $dx_1, \ldots, dz_n$, respectively.

In the case of rheonomic systems we cannot say that the virtual displacements are proportional to the possible velocities. For let us give the system an arbitrary motion compatible with the constraints. Differentiating (12), we obtain

$$\frac{\partial F_j}{\partial x_1}\dot{x}_1 + \ldots + \frac{\partial F_j}{\partial z_n}\dot{z}_n + \frac{\partial F_j}{\partial t} = 0 \qquad (j = 1, 2, \ldots, m). \tag{13}$$

Denoting by $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the velocities of the points, we can write (13) in the form

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i} v_{i_x} + \frac{\partial F_j}{\partial y_i} v_{i_y} + \frac{\partial F_j}{\partial z_i} v_{i_z}\right) + \frac{\partial F_j}{\partial t} = 0 \qquad (j = 1, 2, \ldots, m). \quad (14)$$

Comparing (14) with (I) we see that we cannot in general take $\delta x_1 = v_{i_x}, \ldots, \delta z_n = v_{n_z}$, which can be done in the case of scleronomic systems.

If the relations ((II), p. 466)

$$\Phi_r(x_1, \ldots, z_n, t) \leqq 0 \qquad (r = 1, 2, \ldots, s) \quad (15)$$

hold in addition to equations (12), then, besides (I), the virtual displacement must satisfy those of the relations

$$\sum_{i=1}^{n}\left(\frac{\partial \Phi_r}{\partial x_i}\,\delta x_i + \frac{\partial \Phi_r}{\partial y_i}\,\delta y_i + \frac{\partial \Phi_r}{\partial z_i}\,\delta z_i\right) \leqq 0 \qquad (r = 1, 2, \ldots, s), \quad \text{(II)}$$

for which the equalities $\Phi_r = 0$ (cf. (II), p. 432) hold in a given position of the system at the moment t.

Generalized coordinates. Let the position of a holonomic system be defined by means of the parameters $q_1, \ldots, q_k$ (p. 451).

If the system is rheonomic, then the functions which define the natural coordinates $x_1, \ldots, z_n$, corresponding to the parameters $q_1, \ldots, q_k$, depend on the time. Consequently ((I), p. 452):

$$x_i = f_i(q_1, \ldots, q_k, t),\; y_i = \varphi_i(q_1, \ldots, q_k, t),\; z_i = \psi_i(q_1, \ldots, q_k, t), \quad (i = 1, 2, \ldots, n). \quad (16)$$

If the parameters are independent, then to every set of the variables $q_1, \ldots, q_k$, in a certain region of these variables (the region can depend on the time t) there corresponds a position of the system compatible with the constraints. If the parameters are dependent, then in the case of bilateral constraints certain equations (2), p. 453:

$$\Phi_r(q_1, \ldots, q_k, t) = 0 \qquad (r = 1, 2, \ldots, s) \quad (17)$$

must be satisfied, and in the case of unilateral constraints the parameters must satisfy the inequalities (3), p. 453:

$$\Psi_r(q_1, \ldots, q_k, t) \leqq 0 \qquad (r = 1, 2, \ldots, \varrho). \quad (18)$$

In particular, when the functions (16)—(18) do not depend on the time t, the system is scleronomic.

If a system is moving, then the parameters $q_1, \ldots, q_k$, depend on the

time t. The motion of the system will therefore be determined by giving the functions:

$$q_1 = q_1(t), \;\ldots,\; q_k = q_k(t) \quad (19)$$

defining the values of the parameters of the system at each moment t. The natural coordinates are obtained by substituting functions (19) in functions (16). If the parameters are dependent and satisfy equations (17) and possibly inequalities (18) too, then functions (19) must likewise satisfy these relations.

Let the positions of a holonomic system be defined parametrically by means of functions (16). The virtual displacement of the system at the moment t in a certain position compatible with the constraints is obtained by assuming that the constraints are independent of the time and such as they were at the moment t. Hence in virtue of (III), p. 454, we get:

$$\delta x_i = \sum_{j=1}^{k}\frac{\partial x_i}{\partial q_j}\,\delta q_j,\quad \delta y_i = \sum_{j=1}^{k}\frac{\partial y_i}{\partial q_j}\,\delta q_j,\quad \delta z_i = \sum_{j=1}^{k}\frac{\partial z_i}{\partial q_j}\,\delta q_j\; (i = 1, 2, \ldots, n). \quad \text{(III)}$$

Formulae (III) are obtained by forming the differentials of (16), under the assumption that $t = \text{const}$, and then writing $\delta x_i, \delta y_i, \delta z_i, \delta q_j$, instead of $\mathrm{d}x_i, \mathrm{d}y_i, \mathrm{d}z_i, \mathrm{d}q_j$.

If the parameters are independent, then δq_j in (III) are arbitrary. If the parameters defining the position of the system compatible with the constraints satisfy relations (17), then δq_j in (III) are not arbitrary: they must satisfy the equations ((IV), p. 455)

$$\sum_{j=1}^{k}\frac{\partial \Phi_r}{\partial q_j}\,\delta q_j = 0 \qquad (r = 1, 2, \ldots, s). \quad \text{(IV)}$$

Finally, if the parameters must satisfy certain inequalities (18) in addition to equations (17), then δq_j must satisfy, besides (IV), those of the relations

$$\sum_{j=1}^{k}\frac{\partial \Psi_r}{\partial q_j}\,q_j \leqq 0 \qquad (r = 1, 2, \ldots, \varrho), \quad \text{(V)}$$

for which the equations $\Psi_r = 0$ ((V), p. 455) hold in a given position of the system at the moment t.

Example 2. A material point is constrained to a line l lying in the xy-plane and passing through the origin O of the system. The line l rotates about O with a constant angular velocity ω.

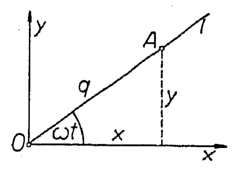

Fig. 317.

Let us take the point O as the origin of the coordinate system and give the line l an arbitrary sense (Fig. 317). Let us denote by q the coordinate of the point A with respect to the axis l; finally, let us assume that the axis l coincided with the axis of x at the moment t. For the coordinates x, y, of the point A we then obtain the formulae:

$$x = q\cos\omega t, \quad y = q\sin\omega t. \tag{20}$$

The variable q defines the position of the point at the moment t; it is therefore a parameter. Differentiating equations (20), under the assumption that $t = \text{const}$, we get:

$$\delta x = \delta q\cos\omega t, \quad \delta y = \delta q\sin\omega t. \tag{21}$$

§ 4. D'Alembert's principle. Equilibrium of forces. So far we have defined the concept of the equilibrium of acting forces for scleronomic systems. According to the definition given (p. 435), the acting forces are in equilibrium if the system of points can remain at rest despite the action of these forces.

This definition of equilibrium, however, is not suitable for rheonomic systems.

For example, if a system of material points is constrained to remain constantly in a horizontal plane moving vertically upwards with a uniform motion, then obviously the system can at no time remain at rest. According to the preceding definition, therefore, we could not say that any system of forces is in equilibrium.

The principle of virtual work (p. 436) gives the necessary and sufficient condition of the equilibrium of forces for scleronomic systems (if there is no friction). Now, for rheonomic systems (when there is no friction) we take the principle of virtual work as the definition of the equilibrium of the acting forces: we therefore say that *the forces acting on a holonomo-rheonomic system* (*in which there is no friction*) *are in equilibrium at a certain time t, if for every virtual displacement at the time t the virtual work of the forces is zero or a negative number.*

According to this definition the principle of virtual work applies to holonomic systems whether they are scleronomic or rheonomic.

D'Alembert's principle. Let the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, act on a holonomic system consisting of n material points $A(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$. Let us denote by $m_1, \ldots, m_n$, the masses, and by $\mathbf{p}_1, \ldots, \mathbf{p}_n$, the accelerations of these points.

The vectors $-m_1\mathbf{p}_1, \ldots, -m_n\mathbf{p}_n$ were called *forces of inertia* (p. 73). If the system is free, then according to d'Alembert's principle (p. 188) the acting forces balance the forces of inertia. Now, experience shows that d'Alembert's principle is also true for constrained holonomic systems, in which there is no friction. Therefore we can state it as follows:

The forces acting on the points of a holonomic system (in which there is no friction) *balance the forces of inertia at each moment.*

Hence the forces $\mathbf{P}_i - m_i\mathbf{p}_i$ (where $i = 1, 2, \ldots, n$) are in equilibrium. Denoting by $\overline{\delta s_i}$ the virtual displacements, we obtain ((I), p. 434 and (II), p. 437)

$$\sum_{i=1}^{n}(\mathbf{P}_i - m_i\mathbf{p}_i)\,\overline{\delta s_i} \leqq 0. \tag{I}$$

In the case of bilateral constraints (or reversible displacements) we have

$$\sum_{i=1}^{n}(\mathbf{P}_i - m_i\mathbf{p}_i)\,\overline{\delta s_i} = 0. \tag{I'}$$

Denoting by $x_i^{\cdot\cdot}, y_i^{\cdot\cdot}, z_i^{\cdot\cdot}$, the projections of the acceleration $\mathbf{p}_i$, and by $\delta x_i, \delta y_i, \delta z_i$, the projections of the displacements $\overline{\delta s_i}$, we can write formulae (I) and (I') in the form ((II), p. 437)

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i] \leqq 0, \tag{II}$$

and in the case of bilateral constraints we have ((III), p. 437)

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i] = 0. \tag{II'}$$

Therefore d'Alembert's principle reduces the problems of dynamics to problems of statics. This principle can be proved in many instances. In the cases when friction is defined, we accept it as a law verified by experience. In the general case we say that there is no friction if d'Alembert's principle applies to a given system.

Remark. Let us assume that a system is free. Consequently $\delta x_i, \delta y_i, \delta z_i$, are arbitrary numbers. Since equation (II') has to hold for every set of numbers $\delta x_i, \delta y_i, \delta z_i$, the coefficients of these numbers must be zero. Consequently:

$$P_{i_x} - m_i x_i^{\cdot\cdot} = 0, \quad P_{i_y} - m_i y_i^{\cdot\cdot} = 0, \quad P_{i_z} - m_i z_i^{\cdot\cdot} = 0,$$

whence

$$m_i x_i^{\cdot\cdot} = P_{i_x}, \quad m_i y_i^{\cdot\cdot} = P_{i_y}, \quad m_i z_i^{\cdot\cdot} = P_{i_z} \qquad (i = 1, 2, \ldots, n).$$

The above equations are obviously Newton's equations of motion ((II), p. 186).

Example 1. A heavy material point A of mass m falls (without friction) along an inclined plane making an angle α with the horizontal. Determine the motion of the point.

Denoting by $\boldsymbol{p}$ the acceleration, by $\mathbf{Q}$ the weight of the point A, and by $\overline{\delta s}$ the virtual displacement (Fig. 318), we obtain from d'Alembert's principle

$$(\mathbf{Q} - m\boldsymbol{p})\,\overline{\delta s} \leqq 0. \tag{1}$$

Let us take as the z-axis the line of the greatest fall on the inclined, plane giving it a downward sense. Let $\overline{\delta s}$ have the direction of the z-axis. Denoting the projections of $\overline{\delta s}$ and $\boldsymbol{p}$ on the z-axis by δs and p, as well as noting that $\overline{\delta s}$ is a reversible displacement, we obtain

$$(\mathbf{Q} - m\boldsymbol{p})\,\overline{\delta s} = 0, \text{ i.e. } \mathbf{Q}\,\overline{\delta s} - m\boldsymbol{p}\,\overline{\delta s} = 0,$$

from which $mg\,\delta s \sin\alpha - mp\,\delta s = 0$, and therefore

$$m(g\sin\alpha - p)\,\delta s = 0. \tag{2}$$

Since equation (2) holds for every δs, we have $g\sin\alpha - p = 0$, whence

$$p = g\sin\alpha. \tag{3}$$

The equation (3) determines the acceleration of the point. It is easy to show that at this acceleration formula (1) holds for every $\overline{\delta s}$ (lying in the plane or not).

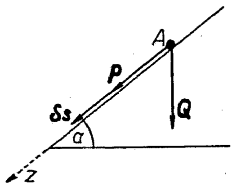

Fig. 318.

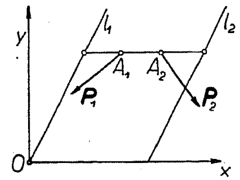

Fig. 319.

Example 2. Two material points A_1, A_2, of masses m_1, m_2, are strung on a massless rigid wire whose ends are constrained to remain on two parallel lines l_1, l_2. The forces $\mathbf{P}_1, \mathbf{P}_2$, lying in the plane of the lines l_1, l_2, act on the points (Fig. 319). Determine the motion of the points, assuming that there is no friction.

It is easy to see that the wire will have a constant direction. Let us choose the axes x and y in the plane of the lines l_1, l_2, giving the x axis the direction of the wire, and let us denote the coordinates of the points by x_1, y_1, and x_2, y_2. The constraints will therefore be defined by the equation

$$y_1 - y_2 = 0. \tag{4}$$

In virtue of d'Alembert's principle, ((II') p. 475), we obtain

$$(P_{1_x} - m_1\ddot{x}_1)\,\delta x_1 + (P_{1_y} - m_1\ddot{y}_1)\,\delta y_1 + (P_{2_x} - m_2\ddot{x}_2)\,\delta x_2 + \\ + (P_{2_y} - m_2\ddot{y}_2)\,\delta y_2 = 0. \tag{5}$$

From equation (4) we have $\delta y_1 - \delta y_2 = 0$, i.e. $\delta y_1 = \delta y_2$. Substituting this value in (5), we get

$$(P_{1_x} - m_1\ddot{x}_1)\,\delta x_1 + (P_{2_x} - m_2\ddot{x}_2)\,\delta x_2 + (P_{1_y} - m_1\ddot{y}_1 + \\ + P_{2_y} - m_2\ddot{y}_2)\,\delta y_1 = 0. \tag{6}$$

Since $\delta x_1, \delta x_2, \delta y_1$, are arbitrary numbers, their coefficients in equations (6) must be zero. Consequently:

$$m_1\ddot{x}_1 = P_{1_x}, \quad m_2\ddot{x}_2 = P_{2_x}, \tag{7}$$

$$m_1\ddot{y}_1 + m_2\ddot{y}_2 = P_{1_y} + P_{2_y}. \tag{8}$$

From (4) we have $\ddot{y}_1 - \ddot{y}_2 = 0$, i.e. $\ddot{y}_1 = \ddot{y}_2$. Equation (8) can therefore be written in the form

$$(m_1 + m_2)\,\ddot{y}_1 = P_{1_y} + P_{2_y}. \tag{9}$$

Equations (7), (9), and (4), determine the motion of the points.

Example 3. A vertical plane Π passing through the z axis directed vertically upwards, rotates about z with a constant angular velocity ω. A heavy point A of the mass m is constrained to the plane Π. (Fig. 320). Determine the motion of this point, assuming that there is no friction.

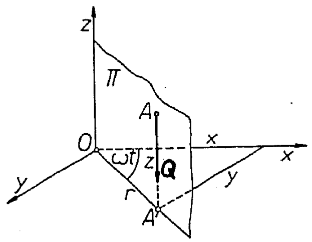

Fig. 320.

Let us assume that the plane Π had the position of the xz-plane at $t = 0$. The equation of the plane Π at the time t will hence be

$$y\cos\omega t - x\sin\omega t = 0. \tag{10}$$

The coordinates x, y, of the point A must therefore satisfy equation (10). Since the force of gravity has the projections, $0, 0, -mg$, on the coordinate axes, from d'Alembert's principle it follows that

$$-m\ddot{x}\,\delta x - m\ddot{y}\,\delta y + (-mg - m\ddot{z})\,\delta z = 0. \tag{11}$$

By (10) the virtual displacement $\delta x, \delta y, \delta z$, satisfies the equation

$$\delta y\cos\omega t - \delta x\sin\omega t = 0. \tag{12}$$

Consequently δz is arbitrary and $\delta x, \delta y$, satisfy equation (12).

Assuming $\delta x = 0$, $\delta y = 0$, in (11), we obtain $(-mg - m\ddot{z})\,\delta z = 0$. Since δz is arbitrary, $-mg - m\ddot{z} = 0$, i.e.

$$\ddot{z} = -g, \tag{13}$$

from which $z = -\frac{1}{2}gt^2 + ct + c'$, where c and c' are certain constants.

From (11) and (13) we obtain $x^{\cdot\cdot}\,\delta x + y^{\cdot\cdot}\,\delta y = 0$; hence

$$x^{\cdot\cdot}\,\delta x \cos\omega t + y^{\cdot\cdot}\,\delta y \cos\omega t = 0, \tag{14}$$

whence by (12) $(x^{\cdot\cdot}\cos\omega t + y^{\cdot\cdot}\sin\omega t)\,\delta x = 0$, and, since δx is arbitrary,

$$x^{\cdot\cdot}\cos\omega t + y^{\cdot\cdot}\sin\omega t = 0. \tag{15}$$

Let us put $r = OA'$, where A' denotes the projections of A on the xy-plane. Consequently

$$x = r\cos\omega t, \quad y = r\sin\omega t, \tag{16}$$

from which

$$x^{\cdot\cdot} = r^{\cdot\cdot}\cos\omega t - 2r^{\cdot}\omega\sin\omega t - r\omega^2\cos\omega t,$$
$$y^{\cdot\cdot} = r^{\cdot\cdot}\sin\omega t + 2r^{\cdot}\omega\cos\omega t - r\omega^2\sin\omega t$$

and by substituting in (15) $r^{\cdot\cdot} - r\omega^2 = 0$; therefore (cf. example 4, p. 139) $r = c_1e^{\omega t} + c_2e^{-\omega t}$, whence in virtue of (16):

$$x = (c_1e^{\omega t} + c_2e^{-\omega t})\cos\omega t, \quad y = (c_1e^{\omega t} + c_2e^{-\omega t})\sin\omega t.$$

The constants c, c', c_1, c_2, are determined from the initial conditions.

§ 5. Work and kinetic energy in scleronomic systems. Let a holonomo-scleronomic system composed of n material points of masses $m_1, \ldots, m_n$, and having coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, be subjected to the action of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

For the moment, let us assume that the constraints are bilateral.

From d'Alembert's principle we have for every virtual displacement $\delta x_i, \delta y_i, \delta z_i$

$$\sum_{i=1}^{n}[(P_{i_x} - m_ix_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_iy_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_iz_i^{\cdot\cdot})\,\delta z_i] = 0. \tag{1}$$

Since the system is scleronomic, the velocities of the points can be considered as virtual displacements (p. 425). Therefore, putting $x_i^{\cdot} = \delta x_i$, $y_i^{\cdot} = \delta y_i$, $z_i^{\cdot} = \delta z_i$, we obtain from (1)

$$\sum_{i=1}^{n}[(P_{i_x} - m_ix_i^{\cdot\cdot})\,x_i^{\cdot} + (P_{i_y} - m_iy_i^{\cdot\cdot})\,y_i^{\cdot} + (P_{i_z} - m_iz_i^{\cdot\cdot})\,z_i^{\cdot}] = 0, \tag{2}$$

i. e.

$$\sum_{i=1}^{n}(P_{i_x}x_i^{\cdot} + P_{i_y}y_i^{\cdot} + P_{i_z}z_i^{\cdot}) - \sum_{i=1}^{n}m_i(x_i^{\cdot\cdot}x_i^{\cdot} + y_i^{\cdot\cdot}y_i^{\cdot} + z_i^{\cdot\cdot}z_i^{\cdot}) = 0. \tag{3}$$

The kinetic energy is expressed by the formula

$$E = \sum_{i=1}^{n}\tfrac{1}{2}m_i(x_i^{\cdot 2} + y_i^{\cdot 2} + z_i^{\cdot 2}),$$

whence $E^{\cdot} = \sum m_i(x_i^{\cdot\cdot}x_i^{\cdot} + y_i^{\cdot\cdot}y_i^{\cdot} + z_i^{\cdot\cdot}z_i^{\cdot})$. Therefore in virtue of (3)

$$E^{\cdot} = \sum_{i=1}^{n}(P_{i_x}x_i^{\cdot} + P_{i_y}y_i^{\cdot} + P_{i_z}z_i^{\cdot}).$$

Integrating both sides of this equation from t_0 to t, we obtain

$$\int_{t_0}^{t}E^{\cdot}\,dt = \int_{t_0}^{t}dt\sum_{i=1}^{n}(P_{i_x}x_i^{\cdot} + P_{i_y}y_i^{\cdot} + P_{i_z}z_i^{\cdot}). \tag{4}$$

The left side of formula (4) is equal to $E - E_0$, where E denotes the kinetic energy at the time t, and E_0 the kinetic energy at the time t_0, while the right side expresses the work L_{t_0t} of the forces acting from the time t_0 to t ((II), p. 208). Consequently

$$E - E_0 = L_{t_0t}. \tag{5}$$

Equation (5) expresses the *principle of equivalence of the work of the acting forces and of the kinetic energy.*

Let us now discard the assumption that the constraints are bilateral. Let us assume that in addition to the relations expressed by equalities, the coordinates of the points of the system have to satisfy the inequalities ((15), p. 472):

$$\Phi_r(x_1, \ldots, z_n) \leqq 0 \qquad (r = 1, 2, \ldots, \varrho). \tag{6}$$

D'Alembert's principle in this case has the form

$$\sum_{i=1}^{n}[(P_{i_x} - m_ix_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_iy_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_iz_i^{\cdot\cdot})\,\delta z_i] \leqq 0. \tag{7}$$

Let us assume that the velocity changes in a continuous manner during the motion.

If the position of the system at a certain time t' (where $t_0 \leqq t' \leqq t$) is not a boundary position, i. e. if the $<$ signs hold in the inequalities (6), then the inequalities (6) do not give any conditions on the virtual displacements (p. 471). In this case the virtual displacements are reversible, consequently equation (1) holds and then (2) holds. On the other hand; if the system occupies a boundary position at the time t' (where $t_0 < t' < t$), i. e. if the equality

$$\Phi_r(x_1, \ldots, z_n) = 0$$

holds for a certain r, then according to the assumption that the functions $x_1, \ldots, z_n$, have continuous derivatives with respect to the time t, the function Φ_r will also have a continuous derivative. Moreover, since $\Phi_r \leqq 0$ constantly, the function Φ_r attains a maximum at the time t'. It follows from this that $\Phi_r^{\cdot} = 0$ for $t = t'$; hence

$$\frac{\partial\Phi_r}{\partial x_1}x_1^{\cdot} + \ldots + \frac{\partial\Phi_r}{\partial z_n}z_n^{\cdot} = 0. \tag{8}$$

In view of (8) the virtual displacement $\delta x_1 = x_1^{\cdot}, \ldots, \delta z_n = z_n^{\cdot}$, is a reversible displacement. Hence for this displacement equation (1) holds, and consequently equation (2) holds.

We have therefore proved that equation (2) is satisfied for each instant t' (where $t_0 < t' < t$). From equation (2) — reasoning as before — we obtain formula (4).

Therefore: *the principle of equivalence of the work of the acting forces and of the kinetic energy applies to holonomo-scleronomic systems* (while for unilateral constraints this holds when the velocities of the points vary in a continuous manner).

If the acting forces have a potential V, then $L_{t_0 t} = V - V_0$, where V and V_0 denote the corresponding potentials at the instants t and t_0. From (5) we therefore have $E - E_0 = V - V_0$ or $E - V = E_0 - V_0$. Denoting the constant $E_0 - V_0$ by h we obtain

$$E - V = h. \tag{9}$$

We have called $-V$ the potential energy and denoted it by U (p. 216). Consequently

$$E + U = h. \tag{10}$$

We have called the sum $E + U$ the *total energy of the system* (p. 216).

Therefore: *the principle of conservation of total energy applies to holonomo-scleronomic systems* (under the assumption that in the case of unilateral constraints the velocities vary in a continuous manner).

Remark. In general, the principle of equivalence of work and kinetic energy does not hold for rheonomic systems.

For example, if a point is constrained to remain on a moving curve and no forces act on the point, then in spite of this the kinetic energy of the point can change depending on the motion of the curve.

In rheonomic systems the increase in kinetic energy also depends on the work of the forces of reaction, which in general is not zero.

§ 6. Lagrange's equations of the first kind. Let a holonomic system of n material points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, be given. Let us denote by $\mathbf{P}_1, \ldots, \mathbf{P}_n$ the forces acting on the points of the system, and by $m_1, \ldots, m_n$, the masses of these points. Let us assume that the constraints are bilateral, defined by the equations ((I), p. 466):

$$F_j(x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t) = 0 \qquad (j = 1, 2, \ldots, m). \tag{1}$$

The virtual displacements of the system satisfy the equations:

$$\sum_{i=1}^{n} \left(\frac{\partial F_j}{\partial x_i} \delta x_i + \frac{\partial F_j}{\partial y_i} \delta y_i + \frac{\partial F_j}{\partial z_i} \delta z_i \right) = 0 \qquad (j = 1, 2, \ldots, m). \tag{2}$$

In virtue of d'Alembert's principle ((II'), p. 475) we have;

$$\sum_{i=1}^{n} [(P_{i_x} - m_i x_i^{\cdot\cdot}) \delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot}) \delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot}) \delta z_i] = 0. \tag{3}$$

Equation (3) holds for every set of numbers $\delta x_i, \delta y_i, \delta z_i$, satisfying the system of equations (2). From the considerations on p. 447 it follows that there exist numbers $\lambda_1, \ldots, \lambda_m$, such that equations (I), p. 448 are satisfied at each moment t (where it is necessary to substitute $P_{i_x} - m_i x_i^{\cdot\cdot}$ for P_{i_x}, etc.):

$$\begin{aligned} P_{i_x} - m_i x_i^{\cdot\cdot} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial x_i} &= 0, \\ P_{i_y} - m_i y_i^{\cdot\cdot} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i} &= 0, \\ P_{i_z} - m_i z_i^{\cdot\cdot} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} &= 0 \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{4}$$

The numbers $\lambda_1, \ldots, \lambda_m$, depend on t and hence are functions of time; consequently $\lambda_1 = \lambda_1(t), \ldots, \lambda_m = \lambda_m(t)$. From (4) we get:

$$\begin{aligned} m_i x_i^{\cdot\cdot} &= P_{i_x} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial x_i}, \\ m_i y_i^{\cdot\cdot} &= P_{i_y} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i}, \\ m_i z_i^{\cdot\cdot} &= P_{i_z} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{I}$$

Equations (I) are called *Lagrange's equations of the first kind.*

Let the forces $\mathbf{P}_i$ be given as functions of the variables $x_1, \ldots, z_n, x_1^{\cdot}, \ldots, z_n^{\cdot}, t$, defining the position of the system at the time t. From equations (I) and (1) we can therefore determine the unknown functions of time $x_1, \ldots, z_n$, defining the motion of the system, as well as the functions $\lambda_1 = \lambda_1(t), \ldots, \lambda_m = \lambda_m(t)$. There are as many unknown functions as there are equations, i. e. $3n + m$.

Let us denote by $\mathbf{R}_1, \ldots, \mathbf{R}_n$, the forces whose projections are defined by the equalities:

$$R_{i_x} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial x_i}, \quad R_{i_y} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i}, \quad R_{i_z} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} \qquad (i = 1, 2, \ldots, n). \tag{II}$$

By (I) and (II) we have:

$$m_i x_i^{\cdot\cdot} = P_{i_x} + R_{i_x}, \quad m_i y_i^{\cdot\cdot} = P_{i_y} + R_{i_y}, \quad m_i z_i^{\cdot\cdot} = P_{i_z} + R_{i_z} \qquad (i = 1, 2, \ldots, n). \tag{5}$$

Denoting by $\mathbf{p}_i$ the acceleration of the point A_i, we can write (5) in the form

$$m_i\mathbf{p}_i = \mathbf{P}_i + \mathbf{R}_i \qquad (i = 1, 2, \ldots, n). \qquad (6)$$

From (6) it follows that the forces $\mathbf{R}_i$ are reactions. For, if we add them to the acting forces, then by (6) we shall be able to regard the system as free. Therefore the reactions are defined by relations (II).

Example 1. Let a point of mass m, subjected to the action of the force $\mathbf{P}$, be constrained to remain on the surface whose equation is

$$F(x, y, z) = 0. \qquad (7)$$

Lagrange's equations (I) assume the form:

$$mx^{\cdot\cdot} = P_x + \lambda\frac{\partial F}{\partial x}, \quad my^{\cdot\cdot} = P_y + \lambda\frac{\partial F}{\partial y}, \quad mz^{\cdot\cdot} = P_z + \lambda\frac{\partial F}{\partial z}. \qquad (8)$$

From equations (7) and (8) we can determine the unknown functions of time x, y, z, and λ.

Equations (8) were obtained in another way (cf. (I), p. 127).

Now let the point be constrained to lie on a moving surface having the equation

$$F(x, y, z, t) = 0. \qquad (9)$$

Equations (I) of Lagrange will then have the form (8), too.

Example 2. Let a material point of mass m be constrained to remain on a sphere whose centre is the origin of a coordinate system, and whose radius r varies together with the time t. Let

$$r = at + r_0, \qquad (10)$$

where the numbers a and r_0 are given. Consequently the coordinates of the point satisfy the equation

$$x^2 + y^2 + z^2 - (at + r_0)^2 = 0. \qquad (11)$$

Let us assume that no forces act on the point. Equations (8) will then assume the form:

$$mx^{\cdot\cdot} = 2\lambda x, \quad my^{\cdot\cdot} = 2\lambda y, \quad mz^{\cdot\cdot} = 2\lambda z. \qquad (12)$$

From equations (12) it follows that the direction of the acceleration passes through the origin of the coordinate system. Hence the motion will be a central motion (p. 85), and it will therefore take place in a plane passing through the origin of the coordinate system (p. 86).

Let us assume that the plane of motion is the xz-plane. Consequently

$$y = 0. \qquad (13)$$

Let us introduce in the xz-plane the polar coordinates r, φ:

$$x = r\cos\varphi, \quad z = r\sin\varphi. \qquad (14)$$

Since the areal velocity is constant, by (I), p. 47,

$$r^2\varphi^{\cdot} = \text{const} = c, \qquad (15)$$

from which by (10) $\varphi^{\cdot} = c\,/\,(at + r_0)^2$. Integrating, we get

$$\varphi = -c\,/\,(at + r_0)\,a + c_1. \qquad (16)$$

Assuming that $\varphi = 0$ for $t = 0$, we obtain from (16) $c_1 = c\,/\,r_0 a$ or

$$\varphi = ct\,/\,r_0(at + r_0). \qquad (17)$$

Equations (14), (10), and (17), define the motion of the point. The constant c is obtained from (15) if one knows, for example, the angular velocity $\varphi^{\cdot}$ for $t = 0$.

§ 7. Lagrange's equations of the second kind. We shall now consider equations of motion in which only the generalized coordinates will appear.

Let there be given a holonomic system of n material points whose natural coordinates $x_1, \ldots, z_n$, are defined in terms of the parameters $q_1, \ldots, q_k$, by means of the functions:

$$x_i = f_i(q_1, \ldots, q_k, t), \quad y_i = \varphi_i(q_1, \ldots, q_k, t), \quad z_i = \psi_i(q_1, \ldots, q_k, t) \qquad (1)$$
$$(i = 1, 2, \ldots, n).$$

Let us denote the masses of the points of the system by $m_1, \ldots, m_n$.

Let us assume that the acting forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, depend on the position of the system, on the velocities of the points, and on the time t.

By d'Alembert's principle we have

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i] \leqq 0. \qquad (2)$$

Let us note that

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i] =$$
$$= \sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) - \sum_{i=1}^{n}m_i(x_i^{\cdot\cdot}\,\delta x_i + y_i^{\cdot\cdot}\,\delta y_i + z_i^{\cdot\cdot}\,\delta z_i). \qquad (3)$$

The first sum on the right side of equation (3) represents the virtual work $\delta' L$ of the acting forces. In virtue of (VII), p. 456, we can write it in the form

$$\delta' L = \sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) = \sum_{j=1}^{k}Q_j\,\delta q_j, \qquad (4)$$

where Q_j (for $j = 1, 2, \ldots, k$) are the components of the generalized force. By (VI′), p. 456, we therefore have

$$Q_j = \sum_{i=1}^{n}\left(P_{i_x}\frac{\partial x_i}{\partial q_j} + P_{i_y}\frac{\partial y_i}{\partial q_j} + P_{i_z}\frac{\partial z_i}{\partial q_j}\right) \qquad (j = 1, 2, \ldots, k). \quad (5)$$

Taking the derivatives of equations (1) with respect to the time t, we obtain

$$\dot{x}_i = \frac{\partial x_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t} \qquad (i = 1, 2, \ldots, n) \quad (6)$$

and similar formulae for $\dot{y}_i$, $\dot{z}_i$.

By hypothesis, the projections $P_{i_x}, P_{i_y}, P_{i_z}$ are functions of the time t as well as of the variables $x_1, \ldots, z_n, \dot{x}_1, \ldots, \dot{z}_n$, which we can by (1) and (6) express in terms of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k$. Therefore from (5) it follows that Q_j can also be regarded as functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$:

$$Q_j = Q_j(q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t) \qquad (j = 1, 2, \ldots, k). \quad (7)$$

By (4) and (7) the first sum on the right side of equation (3) can therefore be expressed in terms of the generalized coordinates.

We shall now consider the second sum on the right side of equation (3). From (1) we obtain ((III), p. 473):

$$\delta x_i = \sum_{j=1}^{k}\frac{\partial x_i}{\partial q_j}\delta q_j, \quad \delta y_i = \sum_{j=1}^{k}\frac{\partial y_i}{\partial q_j}\delta q_j, \quad \delta z_i = \sum_{j=1}^{k}\frac{\partial z_i}{\partial q_j}\delta q_j, \quad (i = 1, 2, \ldots, n). \quad (8)$$

Consequently

$$\begin{aligned}\sum_{i=1}^{n} m_i(\ddot{x}_i\,\delta x_i + \ddot{y}_i\,\delta y_i + \ddot{z}_i\,\delta z_i) &= \\ = \sum_{i=1}^{n} m_i\left(\ddot{x}_i\sum_{j=1}^{k}\frac{\partial x_i}{\partial q_j}\delta q_j + \ddot{y}_i\sum_{j=1}^{k}\frac{\partial y_i}{\partial q_j}\delta q_j + \ddot{z}_i\sum_{j=1}^{k}\frac{\partial z_i}{\partial q_j}\delta q_j\right) &= \quad (9)\\ = \sum_{j=1}^{k}\delta q_j\sum_{i=1}^{n} m_i\left(\ddot{x}_i\frac{\partial x_i}{\partial q_j} + \ddot{y}_i\frac{\partial y_i}{\partial q_j} + \ddot{z}_i\frac{\partial z_i}{\partial q_j}\right).\end{aligned}$$

In virtue of (6) we can regard $\dot{x}_i$ (and similarly $\dot{y}_i$, $\dot{z}_i$) as functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k$, as well as of the time t. Assuming that $\dot{x}_i$ denotes the right side of the equality (6) and that $\dot{q}_1, \ldots, \dot{q}_k$, are independent variables, let us calculate the partial derivatives with respect to $\dot{q}_1, \ldots, \dot{q}_k$. We obtain:

$$\partial\dot{x}_i / \partial\dot{q}_1 = \partial x_i / \partial q_1, \ldots, \partial\dot{x}_i / \partial\dot{q}_k = \partial x_i / \partial q_k, \quad (10)$$

i. e.

$$\partial\dot{x}_i / \partial\dot{q}_j = \partial x_i / \partial q_j \qquad (j = 1, 2, \ldots, k). \quad (11)$$

Calculating the partial derivatives of equations (6) with respect to q_j (and at the same time regarding $\dot{q}_1, \ldots, \dot{q}_k$, as independent variables), we obtain

$$\frac{\partial\dot{x}_i}{\partial q_j} = \frac{\partial^2 x_i}{\partial q_1\,\partial q_j}\dot{q}_1 + \ldots + \frac{\partial^2 x_i}{\partial q_k\,\partial q_j}\dot{q}_k + \frac{\partial^2 x_i}{\partial t\,\partial q_j}. \quad (12)$$

On the other hand we have

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial^2 x_i}{\partial q_j\,\partial q_1}\dot{q}_1 + \ldots + \frac{\partial^2 x_i}{\partial q_j\,\partial q_k}\dot{q}_k + \frac{\partial^2 x_i}{\partial q_j\,\partial t}. \quad (13)$$

Since the order of differentiation does not affect the result, from (12) and (13) we get

$$\frac{\partial\dot{x}_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right). \quad (14)$$

Let us note that

$$\frac{d}{dt}\left(\dot{x}_i\frac{\partial x_i}{\partial q_j}\right) = \ddot{x}_i\frac{\partial x_i}{\partial q_j} + \dot{x}_i\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right); \quad (15)$$

hence, in virtue of (11) and (14), we obtain from this the formula

$$\ddot{x}_i\frac{\partial x_i}{\partial q_j} = \frac{d}{dt}\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial\dot{q}_j}\right) - \dot{x}_i\frac{\partial\dot{x}_i}{\partial q_j} \quad (16)$$

and similar formulae for the variables y_i, z_i. From these formulae we get for arbitrary j:

$$\begin{aligned}\sum_{i=1}^{n} m_i\left(\ddot{x}_i\frac{\partial x_i}{\partial q_j} + \ddot{y}_i\frac{\partial y_i}{\partial q_j} + \ddot{z}_i\frac{\partial z_i}{\partial q_j}\right) &= \\ = \frac{d}{dt}\sum_{i=1}^{n} m_i\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial\dot{q}_j} + \dot{y}_i\frac{\partial\dot{y}_i}{\partial\dot{q}_j} + \dot{z}_i\frac{\partial\dot{z}_i}{\partial\dot{q}_j}\right) - \sum_{i=1}^{n} m_i\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial q_j} + \dot{y}_i\frac{\partial\dot{y}_i}{\partial q_j} + \dot{z}_i\frac{\partial\dot{z}_i}{\partial q_j}\right). \quad (17)\end{aligned}$$

The kinetic energy of the system is

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2). \quad (18)$$

In (18) let us substitute for $\dot{x}_i, \dot{y}_i, \dot{z}_i$, the right sides of the equalities (6) and of the analogous equalities for $\dot{y}_i$, $\dot{z}_i$. In this way we shall represent E as a function of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$:

$$E = E(q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t). \quad (19)$$

Regarding q_j, $\dot{q}_j$, t, as independent variables, we obtain from (19) and (18)

$$\frac{\partial E}{\partial q_j} = \sum_{i=1}^{n} m_i\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial q_j} + \dot{y}_i\frac{\partial\dot{y}_i}{\partial q_j} + \dot{z}_i\frac{\partial\dot{z}_i}{\partial q_j}\right), \quad (20)$$

$$\frac{\partial E}{\partial \dot{q}_j} = \sum_{i=1}^{n} m_i \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} + \dot{y}_i \frac{\partial \dot{y}_i}{\partial \dot{q}_j} + \dot{z}_i \frac{\partial \dot{z}_i}{\partial \dot{q}_j} \right). \tag{21}$$

From (17) we get in virtue of (20) and (21)

$$\sum_{i=1}^{n} m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \ddot{y}_i \frac{\partial y_i}{\partial q_j} + \ddot{z}_i \frac{\partial z_i}{\partial q_j} \right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j}, \tag{22}$$

whence by (9)

$$\sum_{i=1}^{n} m_i (\ddot{x}_i \, \delta x_i + \ddot{y}_i \, \delta y_i + \ddot{z}_i \, \delta z_i) = \sum_{j=1}^{k} \delta q_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j} \right]. \tag{23}$$

From equations (3), (4), and (23), we obtain

$$\sum_{i=1}^{n} [(P_{i_x} - m_i \ddot{x}_i) \, \delta x_i + (P_{i_y} - m_i \ddot{y}_i) \, \delta y_i + (P_{i_z} - m_i \ddot{z}_i) \, \delta z_i] = \\ = \sum_{j=1}^{k} \delta q_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} \right]. \tag{24}$$

From d'Alembert's principle (2) it therefore follows that

$$\sum_{j=1}^{k} \delta q_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} \right] \leq 0. \tag{I}$$

Relations (24) and (I) hold whether the parameters $q_1, \ldots, q_k$, are dependent or not, and whether the constraints are unilateral or bilateral.

In the case of bilateral constraints inequality (I) becomes the equality:

$$\sum_{j=1}^{k} \delta q_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} \right] = 0. \tag{I'}$$

Let us assume that the parameters are independent. Consequently $\delta q_1, \ldots, \delta q_k$, are arbitrary numbers. It follows from this that the coefficients of δq_j in (I′) are zero, and hence that

$$Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k),$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j} = Q_j \qquad (j = 1, 2, \ldots, k). \tag{II}$$

Equations (II) are called *Lagrange's equations of the second kind*.

Only the generalized coordinates appear in them.

From equations (II) we can determine $q_1, \ldots, q_k$, as functions of the time t; hence they enable one to determine the motion without passing over to the natural coordinates.

Let us now assume that the parameters are not independent, but must satisfy relations (17), p. 472:

$$\Phi_r(q_1, \ldots, q_k, t) = 0 \qquad (r = 1, 2, \ldots, s). \tag{25}$$

The virtual displacements δq_j consequently satisfy equalities (IV), p. 473,

$$\sum_{j=1}^{k} \frac{\partial \Phi_r}{\partial q_j} \delta q_j = 0 \qquad (r = 1, 2, \ldots, s). \tag{26}$$

Equalities (I′) hold for every system of numbers δq_j satisfying (26). From considerations analogous to those on p. 447 it follows that for each moment t it is possible to choose numbers $\lambda_1, \ldots, \lambda_s$, satisfying the equations:

$$Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} + \sum_{r=1}^{s} \lambda_r \frac{\partial \Phi_r}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k),$$

i. e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j} = Q_j + \sum_{r=1}^{s} \lambda_r \frac{\partial \Phi_r}{\partial q_j} \qquad (j = 1, 2, \ldots, k). \tag{II'}$$

The Lagrange's multipliers λ_r depend on the time and are therefore functions of the variable t; consequently $\lambda_r = \lambda_r(t)$.

Equations (II′) together with (25) enable one to determine the unknown functions of time $q_1, \ldots, q_k$, and $\lambda_1(t), \ldots, \lambda_s(t)$. The number of these equations is $k + s$, i. e. it is equal to the number of unknown functions.

Remark. In forming equations (II) it is first necessary to represent E and Q_j as functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$.

In order to obtain Q_j we substitute in the formula for virtual work $\delta' L = \Sigma(P_{i_x} \, \delta x_i + P_{i_y} \, \delta y_i + P_{i_z} \, \delta z_i)$ the expression obtained from (1) for $\delta x_i, \delta y_i, \delta z_i, x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i$, and then we arrange the terms according to $\delta q_1, \ldots, \delta q_k$. The coefficients of $\delta q_1, \ldots, \delta q_k$, will be the components Q_j of the generalized force.

Substituting next for $\dot{x}_i, \dot{y}_i, \dot{z}_i$, in the formula for kinetic energy $E = \frac{1}{2}\Sigma m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$ the derivatives obtained by differentiating (1) with respect to t, we get E as a function of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$.

Having determined E and Q_j as functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$, we form the derivatives: $\partial E / \partial q_j$ as well as $\partial E / \partial \dot{q}_j$ and finally

$$\frac{\mathrm{d}}{\mathrm{d}t}(\partial E / \partial \dot{q}_j).$$

Substituting in (II), we obtain Lagrange's equations.

Lagrange's equations in a potential field. Let us assume that the acting forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, have a potential V at each moment t. The potential V is therefore a function of the variables $x_1, \ldots, z_n, t$; in addition:

$$P_{i_x} = \partial V / \partial x_i,\ P_{i_y} = \partial V / \partial y_i,\ P_{i_z} = \partial V / \partial z_i,\ (i = 1, 2, \ldots, n), \quad (27)$$

whence by (5)

$$Q_j = \sum_{i=1}^{n} \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_j} \right), \qquad (j = 1, 2, \ldots, k). \quad (28)$$

Expressing the coordinates $x_1, \ldots, z_n$, in V in terms of $q_1, \ldots, q_k$, by means of (1), we can regard V as a function of the variables $q_1, \ldots, q_k$, and of the time t. From (28) we therefore get

$$Q_j = \partial V / \partial q_j \qquad (j = 1, 2, \ldots, k). \quad (29)$$

From (29) and (27) we see that the components Q_j of the generalized force are expressed in the same way as the coordinates of the forces $\mathbf{P}_i$. Hence we can regard V as the *generalized potential* of the forces $\mathbf{Q}_j$.

From (II) and (29) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot q_j}\right) - \frac{\partial E}{\partial q_j} = \frac{\partial V}{\partial q_j},$$

i. e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot q_j}\right) - \frac{\partial (E+V)}{\partial q_j} = 0 \quad (j = 1, 2, \ldots, k). \quad (30)$$

Since V does not depend on the derivatives $\dot q_1, \ldots, \dot q_k$, it follows that $\partial V / \partial \dot q_j = 0$. Consequently

$$\partial E / \partial \dot q_j = \partial (E+V) / \partial \dot q_j \qquad (j = 1, 2, \ldots, k). \quad (31)$$

From (30) and (31) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial (E+V)}{\partial \dot q_j}\right) - \frac{\partial (E+V)}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k). \quad (32)$$

The sum of the kinetic and potential energies, i. e. $E + V$ is called the *kinetic potential*.

Putting

$$W = E + V, \quad (33)$$

we obtain by (32)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot q_j}\right) - \frac{\partial W}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k). \quad \text{(III)}$$

Lagrange's equations of the second kind therefore assume form (III) when the forces have a potential (or — which amounts to the same thing — a kinetic potential) at every moment.

Cyclic coordinates. The coordinate q_j (where j is a certain number) is called *cyclic* if the kinetic potential W does not depend on q_j, i. e. if

$$\partial W / \partial q_j = 0. \quad (34)$$

If q_j is a cyclic coordinate, then from equations (III) and (34) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot q_j}\right) = 0,$$

whence

$$\partial W / \partial \dot q_j = \text{const.} = c. \quad (35)$$

Equation (35) is a differential equation of the first order. Therefore, if some coordinate q_j is cyclic, then its corresponding equation in Lagrange's equations (III) can be replaced by a differential equation (35) of the first order.

Example I. Two pulleys of radii R and r are fastened to a common axis. Two heavy material points A_1 and A_2 of masses m_1 and m_2 hang on inextensible strings passing over the pulleys (Fig. 321). Determine the motion of the system, assuming that there is no friction.

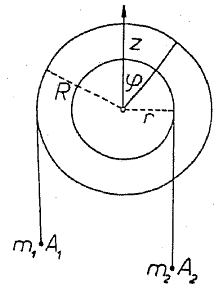

Fig. 321.

Let us assume that the motion takes place in a vertical plane. Let us give the z-axis a direction vertically upwards. Denote by z_1 and z_2 the coordinates of the points m_1 and m_2 at the time t, and by z_1^0 and z_2^0 those at $t = 0$. Let φ denote the angle of rotation of the pulleys, reckoning from the initial position. Assuming the angle of rotation as positive when clockwise, we obtain:

$$z_1 = z_1^0 + R\varphi, \quad z_2 = z_2^0 - r\varphi. \quad (36)$$

The angle φ therefore determines the position of the system; hence we can take φ as the parameter.

Let I_1 and I_2 be the moments of inertia of the pulleys with respect to the common axis. The kinetic energy is

$$E = \tfrac{1}{2} m_1 \dot z_1^2 + \tfrac{1}{2} m_2 \dot z_2^2 + \tfrac{1}{2} I_1 \omega^2 + \tfrac{1}{2} I_2 \omega^2, \quad (37)$$

where ω denotes the angular velocity of the pulleys. By (36) we have $\dot z_1 = R\dot\varphi$ and $\dot z_2 = -r\dot\varphi$, and in addition $\omega = \dot\varphi$. Putting $I = I_1 + I_2$, we obtain from (37)

$$E = \tfrac{1}{2}(m_1 R^2 + m_2 r^2 + I)\,\dot\varphi^2.$$

The potential of the force of gravity is

$$V = -m_1 g z_1 - m_2 g z_2 = -m_1 g(z_1^0 + R\varphi) - m_2 g(z_2^0 - r\varphi),$$

and therefore the kinetic potential $W = E + V$:

$$W = \tfrac{1}{2}(m_1 R^2 + m_2 r^2 + I)\,\varphi^{\cdot 2} - m_1 g(z_1^0 + R\varphi) - m_2 g(z_2^0 - r\varphi),$$

from which:

$$\partial W / \partial\varphi = -(m_1 R - m_2 r)\,g,\quad \partial W / \partial\varphi^{\cdot} = (m_1 R^2 + m_2 r^2 + I)\,\varphi^{\cdot}. \quad (38)$$

Lagrange's equation (III), p. 488, in our case has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \varphi^{\cdot}}\right) - \frac{\partial W}{\partial \varphi} = 0;$$

hence in view of (38) $(m_1 R^2 + m_2 r^2 + I)\,\varphi^{\cdot\cdot} + (m_1 R - m_2 r)\,g = 0$, whence

$$\varphi^{\cdot\cdot} = (m_2 r - m_1 R)\,g \,/\, (m_1 R^2 + m_2 r^2 + I). \quad (39)$$

Therefore the angular acceleration is constant.

From (36) we have $z_1^{\cdot\cdot} = R\varphi^{\cdot\cdot}$, and $z_2^{\cdot\cdot} = -r\varphi^{\cdot\cdot}$, consequently the material points will move with a uniformly accelerated motion.

In particular, when $R = r$, we have *Atwood's machine* (p. 193 and 375).

Example 2. A system composed of three rigid rods OA, AB, BC, of equal length l and equal mass m moves under the influence of the force of gravity in a vertical plane Π (Fig. 322). The rods are pinned at A and B, and fixed at O and C, where O and C lie on the horizontal line $OC = l$.

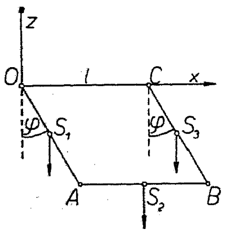

Fig. 322.

Let us choose axes x and z in the plane Π taking O as the origin of the system and giving the x-axis the horizontal direction OC and the z-axis an upward sense. Let us denote by φ the angle which the rods OA and CB make with the vertical, and by S_1, S_2, S_3, the centres of gravity of the rods (assuming that they lie at the geometric centres of these rods).

The angle φ defines the position of the system of rods; consequently φ is a parameter.

The instantaneous motion of the rods OA and BC is an instantaneous rotation about O and C with an angular velocity $\varphi^{\cdot}$. The rod AB moves with an advancing motion (cf. example 1, p. 321) with a velocity $\mathbf{v}$ of the point A, where $|\mathbf{v}| = l|\varphi^{\cdot}|$. The kinetic energy of the system is therefore

$$E = \tfrac{1}{2}I\varphi^{\cdot 2} + \tfrac{1}{2}ml^2\varphi^{\cdot 2} + \tfrac{1}{2}I\varphi^{\cdot 2} = (I + \tfrac{1}{2}ml^2)\,\varphi^{\cdot 2}, \quad (40)$$

where I denotes the moment of inertia of the rod with respect to an end. The coordinates z_1, z_2, z_3, of the centres of gravity S_1, S_2, S_3, are:

$$z_1 = -\tfrac{1}{2}l\cos\varphi,\quad z_2 = -l\cos\varphi,\quad z_3 = -\tfrac{1}{2}l\cos\varphi,$$

consequently the potential of the force of gravity

$$V = -mg(z_1 + z_2 + z_3) = 2mgl\cos\varphi. \quad (41)$$

In virtue of (40) and (41) the kinetic potential $W = E + V$ will therefore be

$$W = (I + \tfrac{1}{2}ml^2)\,\varphi^{\cdot 2} + 2mgl\cos\varphi, \quad (42)$$

whence

$$\partial W / \partial\varphi = -2mgl\sin\varphi,\quad \partial W / \partial\varphi^{\cdot} = 2(I + \tfrac{1}{2}ml^2)\,\varphi^{\cdot}. \quad (43)$$

Lagrange's equations (III), p. 488, will assume the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \varphi^{\cdot}}\right) - \frac{\partial W}{\partial \varphi} = 0;$$

hence in virtue of (43) $2(I + \tfrac{1}{2}ml^2)\,\varphi^{\cdot\cdot} + 2mgl\sin\varphi = 0$, whence

$$\varphi^{\cdot\cdot} = -\frac{mgl}{I + \tfrac{1}{2}ml^2}\sin\varphi. \quad (44)$$

Comparing equation (44) with the equation of the simple pendulum ((I), p. 130) we see that the given system of rods will oscillate like a simple pendulum of length $(I + \tfrac{1}{2}ml^2)\,/\,ml$.

Example 3. A line l lies in the vertical plane xz and rotates about the centre O of the coordinate system with a constant angular velocity ω. A heavy point A of mass m is constrained to the line l (Fig. 323). Determine the motion of the point A.

Let us denote by O' the projection of the point O on the line l, by φ the angle between OO' and the x-axis, and let us put $p = OO' = \text{const.}$ Let us assume that the line l has the direction of the z-axis at $t = 0$. Consequently

$$\varphi = \omega t. \quad (45)$$

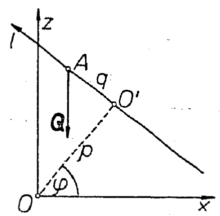

Fig. 323.

Let us give the line l an arbitrary sense and denote by q the coordinate of the point A on the line l, taking the point O' as the origin of this axis. Therefore the coordinates x and z of the point A are:

$$x = p\cos\omega t - q\sin\omega t,\quad z = p\sin\omega t + q\cos\omega t. \quad (46)$$

The system is consequently rheonomic and q is the parameter.

The virtual work is expressed by the formula $\delta' L = -mg\delta z$ (the z-axis has a sense vertically upwards). Since $\delta z = \delta q \cos \omega t$ by (46), $\delta' L = -mg\, \delta q \cos \omega t$. Therefore the generalized force is

$$Q = -mg \cos \omega t. \tag{47}$$

Let us now calculate the kinetic energy E. Differentiating (46), we obtain:

$$x^{\cdot} = -(p\omega + q^{\cdot}) \sin \omega t - q\omega \cos \omega t, \quad z^{\cdot} = (p\omega + q^{\cdot}) \cos \omega t - q\omega \sin \omega t;$$

consequently

$$E = \tfrac{1}{2} m (x^{\cdot 2} + z^{\cdot 2}) = \tfrac{1}{2} m [(p\omega + q^{\cdot})^2 + q^2\omega^2], \tag{48}$$

whence

$$\partial E / \partial q = mq\omega^2, \quad \partial E / \partial q^{\cdot} = m(p\omega + q^{\cdot}). \tag{49}$$

By (II), p. 486, Lagrange's equation has in our case the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial q^{\cdot}}\right) - \frac{\partial E}{\partial q} = Q,$$

from which by (49) and (47) we obtain $mq^{\cdot\cdot} - mq\omega^2 = -mg \cos \omega t$, i. e.

$$q^{\cdot\cdot} - q\omega^2 = -g \cos \omega t. \tag{50}$$

The homogeneous equation $q^{\cdot\cdot} - q\omega^2 = 0$ has the general solution $q = c_1 e^{\omega t} + c_2 e^{-\omega t}$, where c_1 and c_2 are arbitrary constants. A particular solution of equation (50) is, as is easily verified, $q = g \cos \omega t / 2\omega^2$. Therefore the general solution of equation (50) is

$$q = c_1 e^{\omega t} + c_2 e^{-\omega t} + \frac{g}{2\omega^2} \cos \omega t. \tag{51}$$

The constans c_1 and c_2 are determined from initial conditions.

Equations (46) and (51) determine the motion of the point.

Remark. Weight has the potential $V = -mgz$; hence, by (46), $V = -mg(p \sin \omega t + q \cos \omega t)$. In virtue of (48), therefore, the kinetic potential $W = E + V$ is equal to

$$W = \tfrac{1}{2} m [(p\omega + q^{\cdot})^2 + q^2\omega^2] - mg(p \sin \omega t + q \cos \omega t). \tag{52}$$

By (III), p. 488, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial q^{\cdot}}\right) - \frac{\partial W}{\partial q} = 0,$$

whence by (52) we obtain equation (50).

Example 4. Two heavy material points A and B of masses $m + \mu$ and m hang at the ends of a weightless and inextensible string passing over a

pulley (Atwood's machine, p. 193 and 375) (Fig. 324). An insect C of mass μ crawls along the string on the side of the weight B. Denoting by h the projection of the vector $\overline{BC}$ on the z-axis having its origin at the centre of the pulley and a direction vertically downwards, we have

$$h = f(t), \tag{53}$$

where $f(t)$ is a given function. Determine the motion of the system of points A, B, C.

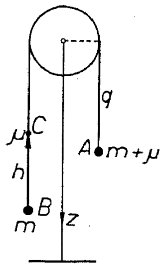

Fig. 324.

Let z_1, z_2, z_3, be the coordinates of the points A, B, C; l the length of the string, r the radius of the pulley, and I the moment of inertia of the pulley with respect to the centre. Taking the coordinate z_1 as the parameter q, we have

$$z_1 = q, \quad z_2 = l - q - r\pi, \quad z_3 = l - q - r\pi + f(t), \tag{54}$$

whence $\delta z_1 = \delta q$, $\delta z_2 = -\delta q$, $\delta z_3 = -\delta q$.

The virtual work of the weights is equal to

$$\delta' L = (m + \mu) g\, \delta z_1 + mg\, \delta z_2 + \mu g\, \delta z_3 = (m + \mu) g\, \delta q - mg\, \delta q - \mu g\, \delta q = 0;$$

hence the generalized force is

$$Q = 0. \tag{55}$$

The kinetic energy E is

$$E = \tfrac{1}{2}(m + \mu) z_1^{\cdot 2} + \tfrac{1}{2} m z_2^{\cdot 2} + \tfrac{1}{2} \mu z_3^{\cdot 2} + \tfrac{1}{2} I \omega^2, \tag{56}$$

where ω denotes the angular velocity of the pulley. From (54) we have:

$$z_1^{\cdot} = q^{\cdot}, \quad z_2^{\cdot} = -q^{\cdot}, \quad z_3^{\cdot} = -q^{\cdot} + f^{\cdot}. \tag{57}$$

Since $r|\omega| = |z_1^{\cdot}| = |q^{\cdot}|$ it follows that $\omega^2 = q^{\cdot 2} / r^2$, whence by (56) and (57)

$$E = \tfrac{1}{2}(m + \mu)\, q^{\cdot 2} + \tfrac{1}{2} m q^{\cdot 2} + \tfrac{1}{2} \mu (q^{\cdot} - f^{\cdot})^2 + \tfrac{1}{2} I q^{\cdot 2} / r^2.$$

From this

$$\frac{\partial E}{\partial q} = 0, \quad \frac{\partial E}{\partial q^{\cdot}} = (2m + 2\mu + I / r^2)\, q^{\cdot} - \mu f^{\cdot}. \tag{58}$$

Lagrange's equations (II), p. 486, will assume the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial q^{\cdot}}\right) - \frac{\partial E}{\partial q} = Q.$$

By (55) and (58) we obtain from this

$$(2m + 2\mu + I / r^2)\, q^{\cdot\cdot} - \mu f^{\cdot\cdot} = 0 \tag{59}$$

and after integration

$$(2m + 2\mu + I / r^2)\, q - \mu f(t) = c_1 t + c_2. \tag{60}$$

The constants c_1 and c_2 are determined from the initial conolitions.

Let us assume that at $t = 0$:

$$f(0) = 0, \quad f^{\cdot}(0) = 0, \quad z_1 = q = q_0, \quad z_1^{\cdot} = q^{\cdot} = 0.$$

From equation (60) and its derivative we obtain:

$$c_2 = (2m + 2\mu + I / r^2)\, q_0, \quad c_1 = 0. \tag{61}$$

Putting $k = 2m + 2\mu + I / r^2$, we get from formulae (60) and (61) $q = \frac{\mu}{k} \cdot f(t) + q_0$, and consequently by (54):

$$z_1 = \frac{\mu}{k} f(t) + q_0, \quad z_3 = l - r\pi - q_0 + \frac{k-\mu}{k} f(t). \tag{62}$$

Since $k - \mu > 0$, it follows from (62) that if the insect C crawls up the string, then the weight A will also go up. At the moment the insect reaches the pulley, i. e. the height $z_3 = 0$, we shall have, as follows from (62),

$$z_1 = q_0 - \frac{\mu}{k-\mu}(l - r\pi - q_0).$$

Example 5. A material point A of mass m moves along the xy-plane under the action of a central force **P** whose projection P on the radius vector $\overline{OA}$ (where O denotes the origin of the coordinate system) is a function of the distance $r = OA$ and

$$P = f(r). \tag{63}$$

Let us introduce the polar coordinates r, φ. The coordinates x, y, of the point A will therefore be expressed by the formulae:

$$x = r\cos\varphi, \quad y = r\sin\varphi. \tag{64}$$

The polar coordinates r, φ, are consequently independent parameters.

From (64) we obtain after differentiating:

$$x^{\cdot} = r^{\cdot}\cos\varphi - r\varphi^{\cdot}\sin\varphi, \quad y^{\cdot} = r^{\cdot}\sin\varphi + r\varphi^{\cdot}\cos\varphi.$$

Therefore the kinetic energy is

$$E = \tfrac{1}{2}m(x^{\cdot 2} + y^{\cdot 2}) = \tfrac{1}{2}m(r^{\cdot 2} + r^2\varphi^{\cdot 2}). \tag{65}$$

Since the field is a central field and the force depends on the distance,

the field is a potential field (p. 101). Consequently by (63) and (3), p. 101, the potential is

$$V = \int P\, dr = \int f(r)\, dr, \tag{66}$$

and the kinetic potential $W = E + V$ is by (65) and (66),

$$W = \tfrac{1}{2}m(r^{\cdot 2} + r^2\varphi^{\cdot 2}) + \int P\, dr. \tag{67}$$

Since the kinetic potential W does not depend on φ, it follows that φ is a cyclic coordinate, whence (p. 489) $\partial W / \partial\varphi^{\cdot} = \text{const.}$, i. e.

$$mr^2\varphi^{\cdot} = \text{const.} \tag{68}$$

Lagrange's equation for the coordinate r has the form ((III), p. 488, with r instead of q_j)

$$\frac{d}{dt}\left(\frac{\partial W}{\partial r^{\cdot}}\right) - \frac{\partial W}{\partial r} = 0. \tag{69}$$

From (67) we get

$$mr^{\cdot\cdot} - mr\varphi^{\cdot 2} - P = 0. \tag{70}$$

From (68) we have $\varphi^{\cdot} = \text{const} / mr^2 = c / r^2$, where c is a certain constant. Substituting this value of $\varphi^{\cdot}$ in (70), we get $m(r^{\cdot\cdot} - c^2 / r^3) = P$, i. e.

$$r^{\cdot\cdot} - c^2 / r^3 = f(r) / m.$$

From this equation we can determine r as a function of the time t.

Example 6. Motion of a point on a surface of revolution. A curve lying in the xz-plane and having the equation

$$z = f(x), \tag{71}$$

generates a surface of revolution S by rotating about the z-axis. The equation of the surface S is therefore the equation

$$z = f(\sqrt{x^2 + y^2}). \tag{72}$$

Let us introduce polar coordinates r, φ, in the xy-plane.
Then

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = f(r). \tag{73}$$

The variables r, φ, are therefore independent parameters.

A material point of mass m, constrained to remain on the surface S, is subjected to the action of a force **P**. Determine Lagrange's equations of the second kind.

We shall first determine the generalized forces. From equations (73) we have:

$$\delta x = \delta r\cos\varphi - r\,\delta\varphi\sin\varphi, \quad \delta y = \delta r\sin\varphi + r\,\delta\varphi\cos\varphi,$$
$$\delta z = f'(r)\,\delta r. \tag{74}$$

The virtual work is

$$\delta' L = P_x\,\delta x + P_y\,\delta y + P_z\,\delta z. \tag{75}$$

Substituting (74) in (75), we obtain

$$\begin{aligned}\delta' L = {} & (P_x \cos\varphi + P_y \sin\varphi + P_z f'(r))\,\delta r + \\ & + (-P_x \sin\varphi + P_y \cos\varphi)\, r\,\delta\varphi.\end{aligned} \tag{76}$$

The coefficients of δr and $\delta\varphi$ are the generalized forces. Let us denote them by Q_r and Q_φ. Consequently:

$$Q_r = P_x \cos\varphi + P_y \sin\varphi + P_z f'(r),\quad Q_\varphi = (-P_x \sin\varphi + P_y \cos\varphi)\, r. \tag{77}$$

Let us now determine the kinetic energy. Differentiating (73), we get:

$$x^\cdot = r^\cdot \cos\varphi - r\varphi^\cdot \sin\varphi,\quad y^\cdot = r^\cdot \sin\varphi + r\varphi^\cdot \cos\varphi,\quad z^\cdot = f'(r)\, r^\cdot. \tag{78}$$

The kinetic energy $E = \frac{1}{2}m(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2})$, whence by (78)

$$E = \tfrac{1}{2}m[(1 + f'^2(r))\, r^{\cdot 2} + r^2\varphi^{\cdot 2}], \tag{79}$$

and from this

$$\partial E / \partial r = m[f'(r)\, f''(r)\, r^{\cdot 2} + r\varphi^{\cdot 2}],\qquad \partial E / \partial\varphi = 0, \tag{80}$$

$$\partial E / \partial r^\cdot = m[1 + f'^2(r)]\, r^\cdot,\qquad \partial E / \partial\varphi^\cdot = mr^2\varphi^\cdot. \tag{81}$$

From equations (II), p. 486, putting $q_1 = r$, $q_2 = \varphi$, $Q_1 = Q_r$, $Q_2 = Q_\varphi$, we obtain by (80) and (81):

$$m\frac{\mathrm{d}}{\mathrm{d}t}[(1 + f'^2(r))\, r^\cdot] - m[f'(r)\, f''(r)\, r^{\cdot 2} + r\varphi^{\cdot 2}] = Q_r, \tag{82}$$

$$m\mathrm{d}(r^2\varphi^\cdot) / \mathrm{d}t = Q_\varphi. \tag{83}$$

The generalized forces Q_r and Q_φ are given by formulae (77).

Let us assume that the motion takes place in a potential field, e. g. in a gravitational field. The potential will then be $V = -mgz$ (when the z-axis has a sense vertically upwards). Hence by (73) we have

$$V = -mg\, f(r), \tag{84}$$

whence for the kinetic potential $W = E + V$:

$$\frac{\partial W}{\partial r} = \frac{\partial E}{\partial r} - mg\, f'(r),\quad \frac{\partial W}{\partial r^\cdot} = \frac{\partial E}{\partial r^\cdot},\quad \frac{\partial W}{\partial\varphi} = 0,\quad \frac{\partial W}{\partial\varphi^\cdot} = \frac{\partial E}{\partial\varphi^\cdot}.$$

It follows from this that the coordinate φ is cyclic. For the coordinate r equations (III), p. 488, assume the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[(1 + f'^2(r))\, r^\cdot] - [f'(r)\, f''(r)\, r^{\cdot 2} + r\varphi^{\cdot 2} - g\, f'(r)] = 0. \tag{85}$$

Since φ is a cyclic coordinate, $\partial W / \partial\varphi^\cdot = \text{const}$, i. e.

$$r^2\varphi^\cdot = \text{const} = c. \tag{86}$$

From the theorem on the conservation of total energy (p. 105) it follows that $E - V = \text{const}$; hence by (79) and (84)

$$[1 + f'^2(r)]\, r^{\cdot 2} + r^2\varphi^{\cdot 2} + g\, f(r) = \text{const} = c_1. \tag{87}$$

From the equations of the first order (86) and (87) we can determine the motion of the point.

Example 7. Spherical coordinates. We shall investigate the motion of a free material point $A(x, y, z)$ moving under the influence of a force **P** in a spherical coordinate system r, ϑ, φ, where $r = OA$, O denotes the origin of the coordinate system (x, y, z), ϑ is the angle between OA and the z-axis, and φ the angle between the x-axis and the projection OA' of the segment OA on the xy-plane (Fig. 325).

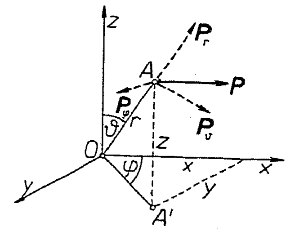

Fig. 325.

We have:

$$x = r\sin\vartheta\cos\varphi,\quad y = r\sin\vartheta\sin\varphi,\quad z = r\cos\vartheta. \tag{88}$$

Since the material point is free, the parameters r, ϑ, φ, are independent. From (88) we get:

$$\begin{aligned}\delta x &= \delta r\,\sin\vartheta\,\sin\varphi + r\,\delta\vartheta\,\cos\vartheta\,\cos\varphi - r\,\delta\varphi\,\sin\vartheta\,\sin\varphi,\\ \delta y &= \delta r\,\sin\vartheta\,\sin\varphi + r\,\delta\vartheta\,\cos\vartheta\,\sin\varphi + r\,\delta\varphi\,\sin\vartheta\,\cos\varphi,\\ \delta z &= \delta r\,\cos\vartheta - r\,\delta\vartheta\,\sin\vartheta.\end{aligned} \tag{89}$$

The virtual work is equal to $\delta' L = P_x\,\delta x + P_y\,\delta y + P_z\,\delta z$, whence by (89)

$$\begin{aligned}\delta' L = {} & (P_x \sin\vartheta\cos\varphi + P_y \sin\vartheta\sin\varphi + P_z \cos\vartheta)\,\delta r + \\ & + r(P_x \cos\vartheta\cos\varphi + P_y \cos\vartheta\sin\varphi - P_z \sin\vartheta)\,\delta\vartheta + \\ & + r\,.\,\sin\vartheta(-P_x \sin\varphi + P_y \cos\varphi)\,\delta\varphi.\end{aligned} \tag{90}$$

The coefficients of δr, $\delta\vartheta$, and $\delta\varphi$, are the components of the generalized force. Let us denote them by Q_r, Q_ϑ, and Q_φ. Consequently:

$$\begin{aligned}Q_r &= P_x \sin\vartheta\cos\varphi + P_y \sin\vartheta\sin\varphi + P_z \cos\vartheta,\\ Q_\vartheta &= r(P_x \cos\vartheta\cos\varphi + P_y \cos\vartheta\sin\varphi - P_z \sin\vartheta),\\ Q_\varphi &= r\sin\vartheta(-P_x \sin\varphi + P_y \cos\varphi).\end{aligned} \tag{91}$$

Let Π be a plane passing through OA and the z-axis. From the point A let us draw the axes Θ, Φ, perpendicular to OA: the axis Θ in the plane Π, and the axis Φ perpendicular to Π. Let us give the axes senses in the direction of the increase of the angles ϑ, φ, and let us denote by $P_r, P_\vartheta, P_\varphi$, the components of the force **P** in the directions $\overline{OA}$, Θ, Φ, respectively. It is easy to show that in virtue of (91):

$$Q_r = P_r,\quad Q_\vartheta = rP_\vartheta,\quad Q_\varphi = r\sin\vartheta P_\varphi. \tag{92}$$

The kinetic energy is $E=\frac{1}{2}m(x^{\cdot 2}+y^{\cdot 2}+z^{\cdot 2})$. The derivatives $x^{\cdot}, y^{\cdot}, z^{\cdot}$, are obtained from (89) by writing $r^{\cdot}, \vartheta^{\cdot}, \varphi^{\cdot}$, instead of $\delta r, \delta\vartheta, \delta\varphi$. Substituting the values obtained, we get

$$E=\tfrac{1}{2}m(r^{\cdot 2}+r^2\varphi^{\cdot 2}\sin^2\vartheta+r^2\vartheta^{\cdot 2}), \tag{93}$$

whence:

$$\partial E\,/\,\partial r=mr(\varphi^{\cdot 2}\sin^2\vartheta+\vartheta^{\cdot 2}),\quad \partial E\,/\,\partial\vartheta=mr^2\varphi^{\cdot 2}\sin\vartheta\cos\vartheta,\quad \partial E\,/\,\partial\varphi=0, \tag{94}$$

$$\partial E\,/\,\partial r^{\cdot}=mr^{\cdot},\quad \partial E\,/\,\partial\vartheta^{\cdot}=mr^2\vartheta^{\cdot},\quad \partial E\,/\,\partial\varphi^{\cdot}=mr^2\varphi^{\cdot}\sin^2\vartheta. \tag{95}$$

Putting in Lagrange's equations (II), p. 486:

$$q_1=r,\quad q_2=\varphi,\quad q_3=\vartheta,$$

we get by (94) and (95):

$$mr^{\cdot\cdot}-mr(\varphi^{\cdot 2}\sin^2\vartheta+\vartheta^{\cdot 2})=Q_r,$$
$$m\frac{\mathrm{d}}{\mathrm{d}t}(r^2\varphi^{\cdot}\sin^2\vartheta)=Q_\varphi, \tag{96}$$
$$m\frac{\mathrm{d}}{\mathrm{d}t}(r^2\vartheta^{\cdot})-mr^2\varphi^{\cdot 2}\sin\vartheta\cos\vartheta=Q_\vartheta\,.$$

Equations (96) are the equations of motion in spherical coordinates.

§ 8. Hamilton's canonical equations. Let $q_1,\ldots,q_k$, be independent parameters. The kinetic energy E is in the general case a function of the variables $q_1,\ldots,q_k, q_1^{\cdot},\ldots,q_k^{\cdot}$, and of the time t. Regarding these variables as independent let us put

$$\frac{\partial E}{\partial q_j^{\cdot}}=p_j \qquad (j=1,2,\ldots,k). \tag{I}$$

The expressions (I) are called *generalized impulses*.

In virtue of (I), p_j are functions of the variables $q_1,\ldots,q_k, q_1^{\cdot},\ldots,q_k^{\cdot}, t$; we can therefore write

$$p_j=\Phi_j(q_1,\ldots,q_k,q_1^{\cdot},\ldots,q_k^{\cdot},t) \qquad (j=1,2,\ldots,k). \tag{1}$$

It can be proved under rather general assumptions that equations (1) can be solved for the variables $q_1^{\cdot},\ldots,q_k^{\cdot}$. Consequently

$$q_j^{\cdot}=\boldsymbol{\Phi}_j(q_1,\ldots,q_k,p_1,\ldots,p_k,t) \qquad (j=1,2,\ldots,k). \tag{2}$$

The kinetic potential $W=E+V$ is a function of the variables $q_1,\ldots,q_k,q_1^{\cdot},\ldots,q_k^{\cdot},t$:

$$W=F(q_1,\ldots,q_k,q_1^{\cdot},\ldots,q_k^{\cdot},t). \tag{3}$$

Substituting (2) in (3), we get

$$W=\mathbf{F}(q_1,\ldots,q_k,p_1,\ldots,p_k,t). \tag{4}$$

The function $\mathbf{F}$ is therefore a function compounded of the function F by means of the functions $\boldsymbol{\Phi}_j$. From the theorem on the derivative of a compound function we obtain:

$$\frac{\partial\mathbf{F}}{\partial q_i}=\frac{\partial F}{\partial q_i}+\sum_{j=1}^{k}\frac{\partial F}{\partial q_j^{\cdot}}\frac{\partial q_j^{\cdot}}{\partial q_i},\quad \frac{\partial\mathbf{F}}{\partial p_i}=\sum_{j=1}^{k}\frac{\partial F}{\partial q_j^{\cdot}}\frac{\partial q_j^{\cdot}}{\partial p_i} \qquad (i=1,2,\ldots,k). \tag{5}$$

Since V does not depend on the derivatives $q_1^{\cdot},\ldots,q_k^{\cdot}$, it follows that $\partial V\,/\,\partial q_j^{\cdot}=0$; from $W=V+E$ we have

$$\partial W\,/\,\partial q_j^{\cdot}=\partial E\,/\,\partial q_j^{\cdot}. \tag{6}$$

Hence by (I) and (3) we have $\partial F\,/\,\partial q_j^{\cdot}=p_j$. From (5) we consequently get:

$$\frac{\partial\mathbf{F}}{\partial q_i}=\frac{\partial F}{\partial q_i}+\sum_{j=1}^{k}p_j\frac{\partial q_j^{\cdot}}{\partial q_i},\quad \frac{\partial\mathbf{F}}{\partial p_i}=\sum_{j=1}^{k}p_j\frac{\partial q_j^{\cdot}}{\partial p_i} \qquad (i=1,2,\ldots,k). \tag{7}$$

Let us put

$$H=\sum_{j=1}^{k}p_jq_j^{\cdot}-W \tag{II}$$

and assume that $q_j^{\cdot}$ and W are functions of the variables $q_1,\ldots,q_k$, $p_1,\ldots,p_k,t$, i. e. that they denote the functions (2) and (4). Then

$$H=\sum_{j=1}^{k}p_jq_j^{\cdot}-\mathbf{F}. \tag{8}$$

Forming partial derivatives, we obtain from (8):

$$\frac{\partial H}{\partial q_i}=\sum_{j=1}^{k}p_j\frac{\partial q_j^{\cdot}}{\partial q_i}-\frac{\partial\mathbf{F}}{\partial q_i},\quad \frac{\partial H}{\partial p_i}=q_i^{\cdot}+\sum_{j=1}^{k}p_j\frac{\partial q_j^{\cdot}}{\partial p_i}-\frac{\partial\mathbf{F}}{\partial p_i}; \tag{9}$$

hence in virtue of (7):

$$\partial H\,/\,\partial q_i=-\,\partial F\,/\,\partial q_i,\quad \partial H\,/\,\partial p_i=q_i^{\cdot}. \tag{10}$$

Lagrange's equations (III), p. 488, have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial q_j^{\cdot}}\right)-\frac{\partial W}{\partial q_j}=0 \qquad (j=1,2,\ldots,k). \tag{11}$$

By (I) and (6) $\partial W\,/\,\partial q_j^{\cdot}=p_j$. From equation (11) we get

$$p_j^{\cdot}=\partial W\,/\,\partial q_j \qquad (j=1,2,\ldots,k), \tag{12}$$

whence by (3) $\partial F\,/\,\partial q_j=p_j^{\cdot}$, and hence by (10):

$$p_i^{\cdot}=-\frac{\partial H}{\partial q_i},\quad q_i^{\cdot}=\frac{\partial H}{\partial p_i} \qquad (i=1,2,\ldots,k). \tag{III}$$

The function H is called *Hamilton's function*, and equations (III) are called *Hamilton's canonical equations*.

The variables p_i or the generalized impulses are therefore defined by equations (I), and the function H by equations (II). In equations (III) the function H is a function of the variables q_i, p_i, t. Equations (III) consequently form a system of differential equations of the first order, where the unknown functions are q_i and p_i as functions of the time t.

The investigation of motions of systems having a potential is therefore reduced to the examination of differential equations of the form (III). Hence the name *canonical equations*.

Scleronomic systems. Let us assume that a system is scleronomic. By (I), p. 498, we therefore have

$$\sum_{j=1}^{k} p_j \dot{q}_j = \sum_{j=1}^{k} \frac{\partial E}{\partial \dot{q}_j} \dot{q}_j. \tag{13}$$

Let the natural coordinates be expressed by the functions:

$$x_i = f_i(q_1, \ldots, q_k), \quad y_i = \varphi_i(q_1, \ldots, q_k), \quad z_i = \psi_i(q_1, \ldots, q_k) \quad (i = 1, 2, \ldots, n). \tag{14}$$

Consequently

$$\dot{x}_i = \frac{\partial x_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial x_i}{\partial q_k}\dot{q}_k. \tag{15}$$

Forming partial derivatives with respect to $\dot{q}_j$, we get:

$$\delta \dot{x}_i / \delta \dot{q}_j = \delta x_i / \delta q_j, \quad \delta \dot{y}_i / \delta \dot{q}_j = \delta y_i / \delta q_j, \quad \delta \dot{z}_i / \delta \dot{q}_j = \delta z_i / \delta q_j. \tag{16}$$

We have

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2); \tag{17}$$

hence

$$\frac{\partial E}{\partial \dot{q}_j} = \sum_{i=1}^{n} m_i\left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} + \dot{y}_i \frac{\partial \dot{y}_i}{\partial \dot{q}_j} + \dot{z}_i \frac{\partial \dot{z}_i}{\partial \dot{q}_j}\right),$$

whence by (16)

$$\frac{\partial E}{\partial \dot{q}_j}\dot{q}_j = \sum_{i=1}^{n} m_i\left(\dot{x}_i \frac{\partial x_i}{\partial q_j}\dot{q}_j + \dot{y}_i \frac{\partial y_i}{\partial q_j}\dot{q}_j + \dot{z}_i \frac{\partial z_i}{\partial q_j}\dot{q}_j\right),$$

and from this

$$\sum_{j=1}^{k} \frac{\partial E}{\partial \dot{q}_j}\dot{q}_j = \sum_{i=1}^{n} m_i\left[\dot{x}_i\left(\frac{\partial x_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial x_i}{\partial q_k}\dot{q}_k\right) + \right.$$

$$\left. + \dot{y}_i\left(\frac{\partial y_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial y_i}{\partial q_k}\dot{q}_k\right) + \dot{z}_i\left(\frac{\partial z_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial z_i}{\partial q_k}\dot{q}_k\right)\right].$$

Therefore in virtue of (15)

$$\sum_{j=1}^{k} \frac{\partial E}{\partial \dot{q}_j}\dot{q}_j = \sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = 2E,$$

whence by (13)

$$\sum_{j=1}^{k} p_j \dot{q}_j = 2E. \tag{18}$$

From (II), p. 499, and (18) we get

$$H = 2E - W. \tag{19}$$

According to the definition ((33), p. 488), we have $W = E + V$, where V is the potential. From (19) it therefore follows that

$$H = E - V. \tag{20}$$

Now, $E - V$ is the total energy of the system.

Therefore: *in scleronomic systems Hamilton's function H denotes the total energy of the system.*

Let us assume that the potential V does not depend on the time. H is then a function of the variables $q_1, \ldots, q_k, p_1, \ldots, p_k$ only. Consequently

$$\frac{dH}{dt} = \sum_{i=1}^{k}\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right). \tag{21}$$

From equations (III) we obtain

$$\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = -\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i = 0,$$

whence by (21) $dH / dt = 0$, i. e. $H = \text{const.}$

We have therefore proved that if a scleronomic system moves in a potential field, then it is subject to the principle of the conservation of total energy.

Example 1. A free material point of mass m moves in a potential field having a potential V.

Let us take the natural coordinates x, y, z, as parameters. The generalized impulses will be defined by relations (I), p. 498, if we substitute x, y, z, for q_1, q_2, q_3. Consequently:

$$p_1 = \partial E / \partial \dot{x}, \quad p_2 = \partial E / \partial \dot{y}, \quad p_3 = \partial E / \partial \dot{z}. \tag{22}$$

Since $E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$,

$$p_1 = m\dot{x}, \quad p_2 = m\dot{y}, \quad p_3 = m\dot{z}. \tag{23}$$

We see from this that p_1, p_2, p_3, are the projections of the momentum on the coordinate axes.

Determining from (23) $x^\cdot, y^\cdot, z^\cdot$, we obtain

$$E = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2). \tag{24}$$

Since the system is scleronomic, Hamilton's function H denotes its total energy. Consequently $H = E - V$, whence by (24)

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - V.$$

From this $\partial H / \partial p_1 = p_1 / m$, etc., $\partial H / \partial x = -\partial V / \partial x$, etc. Hamilton's equations (III) therefore assume the form:

$$p_1^\cdot = \partial V / \partial x, \quad p_2^\cdot = \partial V / \partial y, \quad p_3^\cdot = \partial V / \partial z, \tag{25}$$

$$x^\cdot = p_1 / m, \quad y^\cdot = p_2 / m, \quad z^\cdot = p_3 / m. \tag{26}$$

Determining p_1, p_2, p_3 from (26) and substituting in (25), we obtain Newton's equations:

$$mx^{\cdot\cdot} = \partial V / \partial x, \quad my^{\cdot\cdot} = \partial V / \partial y, \quad mz^{\cdot\cdot} = \partial V / \partial z.$$

Example 2. A material point of mass m is constrained to remain on the surface of a cylinder of revolution $x^2 + y^2 = r^2$. The point is acted on by an elastic force $\boldsymbol{P}$ whose projections are:

$$P_x = -k^2mx, \quad P_y = -k^2my, \quad P_z = -k^2mz, \tag{27}$$

where k is a certain constant.

The elastic force — as is easily verified — has the potential

$$V = -\tfrac{1}{2}k^2m(x^2 + y^2 + z^2). \tag{28}$$

Let us introduce the polar coordinates r, φ, in the xy-plane. Therefore $x = r\cos\varphi$, and $y = r\sin\varphi$, whence (because $r = \text{const}$) we have $x^{\cdot 2} + y^{\cdot 2} = r^2\varphi^{\cdot 2}$.

The kinetic energy is consequently equal to

$$E = \tfrac{1}{2}m(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) = \tfrac{1}{2}m(r^2\varphi^{\cdot 2} + z^{\cdot 2}). \tag{29}$$

The variables φ, z, can be taken as independent parameters. Denoting by p_1 and p_2 the corresponding generalized impulses and writing φ, z, instead of q_1, q_2, we obtain from (I) $\partial E / \partial\varphi^\cdot = p_1$ and $\partial E / \partial z^\cdot = p_2$, whence by (29):

$$p_1 = mr^2\varphi^\cdot, \quad p_2 = mz^\cdot. \tag{30}$$

Determining $\varphi^\cdot$ and $z^\cdot$ from (30) and substituting in (29) we obtain

$$E = \frac{1}{2m}[p_1^2 / r^2 + p_2^2].$$

By (28) $V = -\frac{1}{2}k^2m(r^2 + z^2)$; hence Hamilton's function ((20), p. 501) assumes the form

$$H = E - V = \frac{1}{2m}[p_1^2 / r^2 + p_2^2] + \tfrac{1}{2}k^2m(r^2 + z^2). \tag{31}$$

Consequently Hamilton's equations (III) are:

$$p_1^\cdot = -\partial H / \partial\varphi, \quad p_2^\cdot = -\partial H / \partial z, \quad \varphi^\cdot = \partial H / \partial p_1, \quad z^\cdot = \partial H / \partial p_2,$$

and hence by (31):

$$p_1^\cdot = 0, \quad p_2^\cdot = -k^2mz, \quad \varphi^\cdot = p_1 / mr^2, \quad z^\cdot = p_2 : m. \tag{32}$$

The last two of the equations (32) are equivalent to equations (30).

The first of the equations (32) gives $p_1 = \text{const}$; hence by (30) $mr^2\varphi^\cdot = \text{const}$, i. e. $\varphi^\cdot = \text{const}$. The projections of the point on the horizontal plane will therefore go around the base of the cylinder with a uniform motion.

In virtue of (30) the second of the equations (32) gives $mz^{\cdot\cdot} = -k^2mz$, i. e. $z^{\cdot\cdot} + k^2z = 0$. Comparing it with equation (2), p. 110, we see that the projection of the point on the z-axis will execute a simple harmonic motion.