Loaded cable. Let a force **P** directed vertically downwards be applied at a point C of a cable. As we already know, parts CB and AC of the cable are catenaries. Let us denote the constants for the curves BC and CA in equations (14) and (15) by a_1, c_1, c_1', c_1'' and a_2, c_2, c_2', c_2'', respectively, and the tensions in the parts BC and CA at the point C by $\mathbf{T}_1$ and $\mathbf{T}_2$ (Fig. 233).

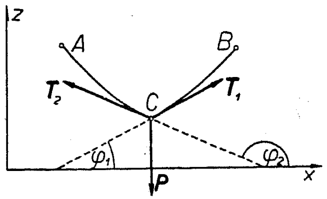

Fig. 233.

Considering the cable as a chain consisting of many small links, and the point C as a joint, we have in the position of equilibrium $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{P} = 0$. Forming projections on the x and z axes and putting $P = |\mathbf{P}|$, we obtain:

$$T_{1_x} + T_{2_x} = 0, \quad T_{1_z} + T_{2_z} - P = 0. \tag{21}$$

Denoting the right-hand and the left-hand derivatives at C by z_1' and z_2', we obtain by (18) and (19):

$$T_{1_x} = \delta / a_1, \; T_{1_z} = \delta z_1' / a_1, \; T_{2_x} = -\delta / a_2, \; T_{2_z} = -\delta z_2' / a_2,$$

whence by (21)

$$\frac{\delta}{a_1} - \frac{\delta}{a_2} = 0, \qquad \frac{\delta z_1'}{a_1} - \frac{\delta z_2'}{a_2} - P = 0.$$

Hence we get:

$$a_1 = a_2, \qquad z_1' - z_2' = \frac{Pa_1}{\delta} = \frac{Pa_2}{\delta}. \tag{22}$$

Knowing the lengths l_1 and l_2 of the arcs BC and CA we can obtain the equations of the curves CB and AC. In this case it is necessary to determine ten constants a_1, c_1, c_1', c_1'', a_2, c_2, c_2', c_2'' and x_0, z_0, where x_0 and z_0 are the coordinates of the point C. To determine these constants for CB and AC we have two sets of four equations analogous to (16) and (17), and in addition two equations (22), i. e. ten altogether.

CHAPTER VII

KINEMATICS OF A RIGID BODY

§ 1. Displacement and rotation of a body about an axis. According to the definition of a rigid body (p. 231), its points do not change their mutual distances during motion. When the point A moved to the point B, the vector $\overline{AB}$ was called the *displacement* of the point (p. 34). During a change of position of a rigid body, the points of this body undergo, in general, various displacements.

We shall first become acquainted with certain theorems from geometry which give the resolution of the displacements of the points of a body. These theorems will be helpful to us in determining the velocities of these points.

Parallel displacement or translation. A body is said to undergo a *parallel displacement* or a *translation* if the displacements of all the points of the body during a change of its position are equal.

The displacement common to all points of the body is called the *displacement vector* or the *displacement of the body.*

The position of the body after a displacement is therefore determined by the initial position and the displacement vector.

Let us assume that the points A_1, B_1 moved to the points A_2, B_2 after a translation. Since the displacements of both points are equal, $\overline{A_1A_2} = \overline{B_1B_2}$. It follows from this that $\overline{A_1B_1} = \overline{A_2B_2}$.

Therefore: *the vectors attached to a body do not change either their sense or direction during a translation.*

Conversely, it is easy to prove that *if the vectors in a body maintain their sense and direction during a displacement of the body, then the displacement is a translation.*

For let us assume that two arbitrary points A_1, B_1 moved to the points A_2, B_2 (Fig. 234). By hypothesis, $\overline{A_1B_1} = \overline{A_2B_2}$; hence $\overline{A_1A_2} = \overline{B_1B_2}$.

The points A_1, B_1 therefore have equal displacements, i. e. the change of position of the body is a translation.

It is easy to see that lines and planes in a body remain parallel to one another after a translation.

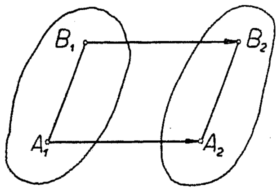

Fig. 234.

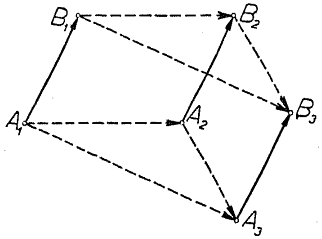

Fig. 235.

Let us suppose that a body has made two successive translations: first, from position I to position II, and next, from position II to position III. Let A_1, B_1 be two arbitrary points in position I, and A_2, B_2 and A_3, B_3 their corresponding points in positions II and III (Fig. 235). By hypothesis, $\overline{A_1A_2} = \overline{B_1B_2}$ and $\overline{A_2A_3} = \overline{B_2B_3}$. Since $\overline{A_1A_3} = \overline{A_1A_2} + \overline{A_2A_3}$ and $\overline{B_1B_3} = \overline{B_1B_2} + \overline{B_2B_3}$, it follows that $\overline{A_1A_3} = \overline{B_1B_3}$. Consequently we can go directly from position I to position III by means of one translation. Denoting the displacements in passing from I to II, from II to III, and from I to III, by $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}$, we obviously obtain

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2.$$

Therefore: *if a body has made several successive translations, then the final position can be obtained from the initial position by means of one translation; the displacement of the body from the initial position to the final position is equal to the sum of the displacements of the separate translations.*

This theorem can be called the *law of composition of displacements.*

Since the resultant displacement is the sum of the component displacements, then (in virtue of the commutativity of the sum of vectors) the resultant displacement does not depend on the order in which the body made the component displacements.

Rotation about an axis. If two points of a body, e. g. K and M, remained fixed during a change of position of the body, then, obviously, all the points of the line l passing through K and M will also remain fixed. We then say that the body *rotated about the line* l; this line is called the *axis of rotation.*

If some plane Π_1 in the initial position of the body passes through the axis of rotation, then the corresponding plane Π_2 in the final position will also pass through this axis.

Let us give the axis l an arbitrary sense. The angle φ through which it is necessary to rotate the plane Π_1 (counterclockwise with respect to axis l) in order that it fall on Π_2 and in order that the corresponding points coincide is called the *angle of rotation.*

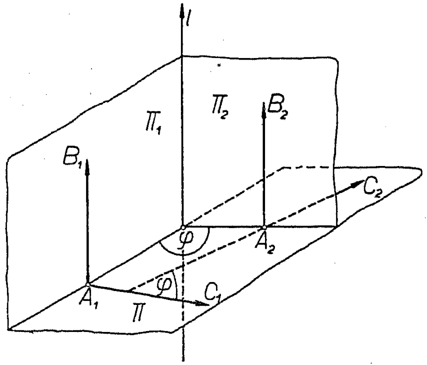

Fig. 236.

The rotation of a body about an axis is determined by giving the axis and the angle of rotation. During a rotation the points of a body remain in planes perpendicular to the axis of rotation.

During a rotation every vector $\overline{A_1B_1}$ parallel to the axis of rotation falls on a vector $\overline{A_2B_2}$ parallel to the vector $\overline{A_1B_1}$. It is easy to prove that only vectors parallel to the axis do not change either direction or sense during a rotation.

Let us note that if a vector $\overline{A_1C_1}$ lies in a plane Π perpendicular to the axis of rotation, then the angle which this vector makes with the corresponding vector $\overline{A_2C_2}$ is (relative to the chosen sense of the axis of rotation) equal to the angle of rotation (Fig. 236).

If the body makes several rotations about this same axis l through the angles $\varphi_1, \varphi_2, \ldots$, then the final displacement is obviously also a rotation about the axis l through the angle $\varphi = \varphi_1 + \varphi_2 + \ldots$ It follows from this, in virtue of the commutativity of the sum, that the final position does not depend on the order in which the partial rotations of the body were made.

The situation is quite different when the body makes successive rotations about various axes as the example on p. 314 shows.

Example. A rigid body made two successive rotations about two parallel lines l and m which are rigidly attached to the body. The rotations had opposite senses, but the angles of rotation were equal. Prove that the body can be displaced from its initial position to its final position by means of a translation.

Let C be an arbitrary point of the body. Through C let us pass a plane Π perpendicular to the given axes of rotation l and m. Let L and M denote the corresponding points of intersection and φ the angle of rotation (Fig. 237).

During a rotation about the axis l through an angle φ the axis m will assume the position of the line m'; let us denote the point of intersection of this line with the plane Π by M'. Next, after a rotation about the axis m' through an angle φ the axis l will assume the position of the line l'; let us denote its point of intersection with the plane Π by L'.

Finally, let C_1 be the position of the point C after a rotation about the line l, and C' the position of the point C_1 after a rotation about the line m'.

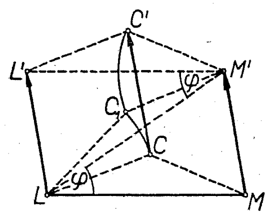

Fig. 237.

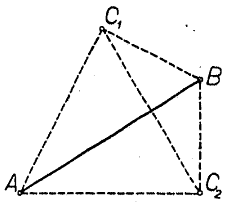

Fig. 238.

The triangle LMC assumed ultimately the position $L'M'C'$, while $\overline{LL'} = \overline{MM'} = \overline{CC'}$ (as in Fig. 237). Since the displacements of the points situated on the axis l (or m) are equal, the above relation indicates that the displacements of all the points are equal. It is easy to verify that

$$LL' = 2LM \sin \tfrac{1}{2}\varphi. \tag{1}$$

§ 2. Displacements of points of a body in plane motion. The motion of a plane figure moving in a plane is called a *plane motion.*

The position of a figure in plane motion is determined by the position of two of its arbitrary points.

For suppose that there are two possible positions of the figure at which the two points A and B would occupy the same positions. Let us consider an arbitrary point C of the figure which in one position is at C_1, and in the another at C_2 (Fig. 238). The triangles ABC_1 and ABC_2 are congruent and are situated symmetrically with respect to AB. Therefore

they cannot be made to coincide without taking them out of the plane. This, however, is contrary to the hypothesis that in the plane motion the triangle ABC once occupied the position ABC_1, and the second time the position ABC_2.

Rotation about a point. If a figure lying in the plane Π is rotated about a line l perpendicular to this plane, then the figure will continue to remain in the plane Π. Such a rotation is called a *rotation* of the figure *about a point* (the point of intersection of the line l with the plane Π).

Theorem. *Every figure in plane motion can be displaced from one arbitrary position to another by means of one translation and one rotation.*

For let A_1, B_1 be two points of this figure in the first position, and A_2, B_2 the corresponding points in the second position. Let us first translate the figure so that the point A_1 falls on A_2. After this displacement the point B_1 will fall on a certain point B_1'. Let us now rotate the figure about A_2 so that B_1' falls on B_2. Since the two points A_1, B_1 coincided with the corresponding points A_2, B_2 after a translation and a rotation, the remaining points will also coincide. Thus, we have displaced the figure from one position to another by means of one translation and one rotation, q. e. d.

We shall now prove that the most general displacement of a figure in plane motion is either a translation or a rotation.

I Theorem of Euler. *A figure can be displaced from one position to another* (which it occupies in plane motion) *either by means of a translation or a rotation.*

Proof. Let A_1B_1 be an arbitrary segment of the figure in the initial position I, and A_2B_2 the same segment in the final position II.

In the case when the vectors $\overline{A_1B_1}$ and $\overline{A_2B_2}$ are equal, we have $\overline{A_1A_2} = \overline{B_1B_2}$. Therefore, by a parallel displacement of the figure so that the point A_1 falls on A_2, the point B_1 will fall on B_2. Since the figure will have the points A_2 and B_2 in common with position II after this displacement, it will have all points in common. In this case, therefore, it is possible to displace the figure from position I to position II by means of a translation.

In the case when the vectors $\overline{A_1B_1}$ and $\overline{A_2B_2}$ are not equal, let us draw the perpendicular bisectors l_1 and l_2 of the segments A_1A_2 and B_1B_2.

Let us assume at first that these bisectors are distinct and intersect at the point O (Fig. 239). The triangles OA_1B_1 and OA_2B_2 are congruent. Denote by B_1' the point symmetrical to B_1 with respect to the bisector l_1.

The triangles OA_1B_1 and OA_2B_1' are obviously situated symmetrically with respect to l_1. They are consequently congruent and not superposable without taking them out of the plane. Hence if the figure is rotated about O so that the point A_1 falls on A_2, then, since the point B_1 cannot fall on B_1', the point B_1 will fall on B_2. After this rotation the figure will therefore have the points A_2 and B_2 in common with position II, and consequently all the other points in common.

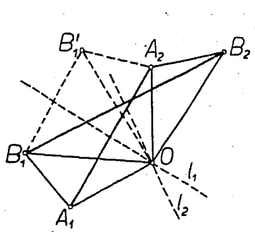

Fig. 239.

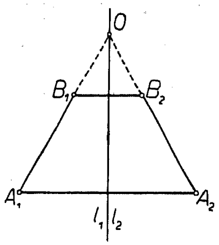

Fig. 240.

Next, let us assume that the bisectors of the segments A_1A_2 and B_1B_2 are identical (Fig. 240). In this case the segments A_1B_1 and A_2B_2 are situated symmetrically with respect to l_1 (or l_2); the centre of rotation will be the point of intersection of the line A_1B_1 with l_1 (or A_2B_2 with l_2). In this case, therefore, the displacement of the figure from position I to position II can be made by means of a rotation, q. e. d.

Plane motion of a body. If a body can move only in such a way that its points remain constantly in planes parallel to a certain fixed plane Π, then the body is said to be in *plane motion*, and the plane Π is called the *directional plane* (cf. the definition and example on p. 272).

Let us cut a body in plane motion by a plane Π' parallel to the directional plane Π. Let C be the plane section. The position of the plane section C obviously determines the position of the entire body. Since the plane section C must remain constantly in one and the same plane Π', by I Theorem of Euler we can displace this figure from the arbitrary position it occupies to another arbitrary position, either by means of a translation, or by means of a rotation about a point lying in Π'.

It follows from this that *a body in plane motion can be displaced from one position to another, either by means of a translation, or by means of a rotation about an axis perpendicular to the directional plane.*

§ 3. Displacements of the points of a body. If a rigid body has one fixed point, then it can rotate about this point, and if it has two fixed

points, it can rotate about an axis passing through these points. Giving the position of one or two points of a body is, therefore, not sufficient to determine the positions of all the other points of the body. But we have the following

Theorem I. *The position of all the points of a rigid body is determined by the position of three of its points, provided the points are not collinear.*

Proof. Let us suppose that there exist two distinct positions of the body at which three non-collinear points A, B, C, would occupy the same positions. Let us consider an arbitrary point D of this body which in the first position is at D_1, and in the second at D_2. The tetrahedrons $ABCD_1$ and $ABCD_2$ have a common base and correspondingly equal edges (Fig. 241). It follows from this that they are symmetrically placed with respect to the plane ABC. Therefore they cannot be brought into complete coincidence. This is, however, contrary to the assumption that the first position of the tetrahedron $ABCD$ is $ABCD_1$ and the other one $ABCD_2$.

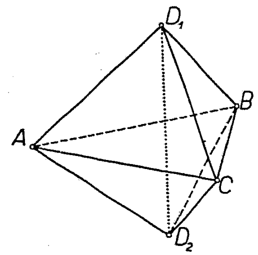

Fig. 241.

Rotation about a point of a body. If one point of a body remains fixed during a displacement of the body, then the body is said to have been *rotated* about this point.

II Theorem of Euler. *A rotation about a point is equivalent to a rotation about a line passing through this point.*

Proof. Let us suppose that a body has been rotated about the point O. In the initial position I let us select in the body an arbitrary segment A_1B_1 (not passing through O) and let A_2B_2 be the corresponding segment in the final position II. Let us draw the planes of symmetry Π_1 and Π_2 of the segments A_1A_2 and B_1B_2.

Let us assume at first that Π_1 and Π_2 are distinct and that they intersect in the line l (Fig. 242). The line l passes through O because l is the locus of points equidistant from A_1 and A_2, as well as from B_1 and B_2, while $OA_1 = OA_2$ and $OB_1 = OB_2$. Let C be an arbitrary point (different from O) on the line l.

The tetrahedrons OCA_1B_1 and OCA_2B_2 are equal. It is easy to show that they are also superposable. The vertices O, C, A_1 and O, C, A_2 are placed symmetrically with respect to Π_1; hence, if the tetrahedrons OCA_1B_1 and OCA_2B_2 were not superposable, the vertices B_1 and B_2

would have to be placed symmetrically with respect to Π_1, which is impossible, since B_1 and B_2 are placed symmetrically with respect to Π_2, and $\Pi_1 \neq \Pi_2$. We have proved, therefore, that the tetrahedrons OCA_1B_1 and OCA_2B_2 are equal and superposable.

If we now rotate the body about the axis l so that A_1 falls on A_2, then the tetrahedron OCA_1B_1 will fall on the tetrahedron OCA_2B_2. After this rotation the body will therefore have three points in common (namely O, A_2 and B_2) with the body in position II, and consequently all other points. We have thus displaced the body from position I to position II by means of a rotation about the line l.

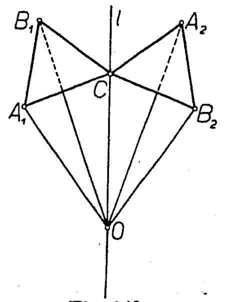

Fig. 242.

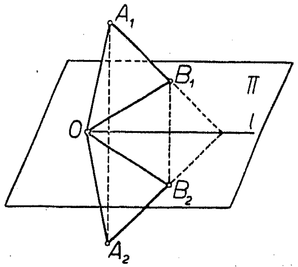

Fig. 243.

Let us now assume that the segments A_1A_2 and B_1B_2 have a common plane of symmetry Π (Fig. 243). Then the triangles OA_1B_1 and OA_2B_2 are situated symmetrically with respect to Π. The axis of rotation in this case will be the line of intersection of the plane OA_1B_1 (or OA_2B_2) with the plane Π.

Remark I. From II Theorem of Euler it follows that during a rotation of a body about a point, there exists in the body a certain line having the property that its points do not change their position.

Remark 2. If a body makes two successive rotations about two axes passing through one point O, then the body can be displaced from its initial position to its final position by means of one rotation about an axis passing through O, because the point O did not change its position. Therefore the composition of two rotations about an axis passing through one point is a rotation about an axis passing through the same point.

Example. A body made two successive rotations about the axes of a fixed coordinate system: first about the z-axis and then about the x-axis, both rotations counterclockwise through a right angle.

Since the origin O of the system remained fixed during both rotations, we can displace the body from its initial position to its final position by means of one rotation about a certain axis l passing through O (Fig. 244).

Let us consider the point $A(0, 0, 1)$ on the z-axis in the initial position. After a rotation about this axis the point A did not change its position, and after a rotation about the x-axis it assumed the position $A'(0, 1, 0)$.

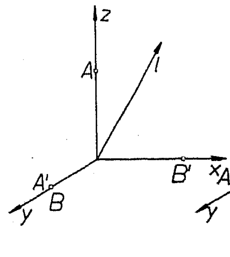

Fig. 244.

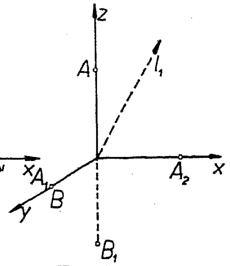

Fig. 245.

Let us next consider the point $B(0,1, 0)$ on the y-axis in the initial position. After a rotation about the z-axis the point B occupied the position $B'(1, 0, 0)$ on the x-axis, and then during a rotation about this axis the point B' did not change its position any more. The sought for axis l will therefore be the intersection of the planes of symmetry of the segments AA' and BB'.

The plane of symmetry of the segment AA' has the equation $y = z$, and the plane of symmetry of the segment BB' has the equation $x = y$. The axis l, being the intersection of both planes, consequently has the equation

$$x = y = z.$$

Let us now suppose that the body had made the same rotations in the reverse order, i. e. first about the x-axis, and then about the z-axis (Fig. 245). The points $A(0, 0, 1)$ and $B(0, 1, 0)$ after a rotation about the x-axis will occupy the positions $A_1(0, 1, 0)$ and $B_1(0, 0, -1)$, and then, after a rotation about the z-axis, they will assume the positions $A_2(1, 0, 0)$ and $B_2(0, 0, -1)$. The planes of symmetry of the segments AA_2 and BB_2 have the equations $x = z$ and $y = -z$. The axis l_1 about which it is necessary to rotate the body in order that it go from its initial position to its final position will therefore have the equation

$$x = -y = z.$$

It follows from this that the final position depends on the order in which the rotations were made.

Chasles' theorem. *A body can be displaced from one arbitrary position to another by means of one translation and one rotation about an axis.*

In general, this can be done in infinitely many ways, but the axes *of rotation will always be parallel and the angles of rotation equal (if the axes have the same sense).*

Proof. Let O_1 be an arbitrary point of the body in the initial position I, and O_2 the corresponding point in the final position II. Let us first translate the body to the position I′ so that O_1 falls on O_2. If the position I′ is identical with II, then the body has been displaced from the position I to the position II by means of one translation, conformably to the requirements of the theorem.

Let us assume, therefore, that the position I′ is different from II. Since the positions II and I′, of the body have the point O_2 in common (Fig. 246), it follows that, by II Theorem of Euler, we can displace it from the position I′ to the position II by means of a rotation about a certain axis l passing through the point O_2.

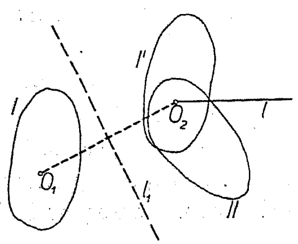

Fig. 246.

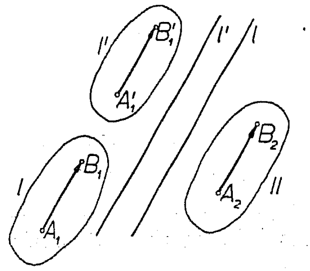

Fig. 247.

In each case we have therefore displaced the body from the position I to the position II by means of one translation and one rotation; thus we have proved the first part of the theorem.

Had we chosen a different point O_1 in the beginning, then in general we would have obtained a different translation and a different rotation about a different axis. We shall show, however, that in every case the axes of rotation would be parallel.

In the position I let us consider in the body an arbitrary vector $\overline{A_1B_1}$ parallel to the axis of rotation l (Fig. 247). The corresponding vector $\overline{A_2B_2}$ in the position II will have the same direction and sense as the vector $\overline{A_1B_1}$. For neither a translation nor a rotation (about a parallel axis) changes the direction or sense of a vector.

Let us now assume that the body has been displaced from the position I to the position II by means of a different translation and rotation about a different axis l'. After this displacement let the vector $\overline{A_1B_1}$ fall on the vector $\overline{A_1'B_1'}$ (Fig. 247). Obviously, after a rotation about the new axis l', the vector $\overline{A_1'B_1'}$ will fall on the vector $\overline{A_2B_2}$. Since the displacement changes neither the direction nor sense of the vector, $\overline{A_1'B_1'}$ has the same sense and direction as the vector $\overline{A_1B_1}$. It follows from this that the vector $\overline{A_1'B_1'}$ will also have the same sense and direction as the vector $\overline{A_2B_2}$. Hence the axis of rotation l' must be parallel to $\overline{A_2B_2}$, i. e. to the axis l.

Finally, we shall show that the angles of rotation about the axes l and l' are equal, provided that the axes l and l' are given the same senses. In the position I let us select in the body an arbitrary vector $\boldsymbol{a}_1$ perpendicular to l and obviously to l' at the same time. The angle which vector $\boldsymbol{a}_1$ makes with the corresponding vector $\boldsymbol{a}_2$ in the position II relative to the chosen sense of the axis is the angle of rotation (p. 309). The angle of rotation is therefore in both cases the same, q. e. d.

Theorem 2. *A body can be displaced from one arbitrary position to another by means of two successive rotations.*

Proof. Let O_1 be an arbitrary point of the body in the position I, and O_2 the corresponding point of the body in the position II. Let us rotate the body through 180° about the axis l_1, which is the axis of symmetry of the segment O_1O_2. By means of this rotation the point O_1 will fall on the point O_2. The body will assume the position II′ which has the point O_2 in common with the position II. Consequently we can go from position II′ to position II by means of a rotation about a certain axis l passing through O_2. In this manner we have displaced a body from position I to position II by means of two rotations about the lines l_1 and l, q. e. d.

Twist. If a body makes a translation and then a rotation about an axis parallel to the translation, then the body is said to have made a *twist*.

In particular, a translation or a rotation is also called a twist.

Theorem 3. *A body can always be displaced from one arbitrary position to another by means of a twist and this can be done in only one way.*

Proof. Let us consider in the body in position I an arbitrary triangle $A_1B_1C_1$ lying in the plane Π_1 perpendicular to the possible axes of rotation. Let us denote by $A_2B_2C_2$ the corresponding triangle and by Π_2 the corresponding plane in the position II. The planes Π_1 and Π_2 are therefore parallel. Let us now displace the body from position I to position I′ by giving the body a displacement perpendicular to Π_1 so that the plane Π_1 assumes the position of the plane Π_2 (Fig. 248). By means of this displacement the triangle $A_1B_1C_1$ coincides with the triangle $A_1'B_1'C_1'$ lying in the plane Π_2 in which the triangle $A_2B_2C_2$ also lies.

We can displace the body from position I′ to position II first by means of a translation so that A_1' falls on A_2, and next by means of a rotation about a line perpendicular to Π_2 and passing through A_2. It follows from this that the triangle $A_1'B_1'C_1'$, remaining constantly in the plane Π_2, will fall on $A_2B_2C_2$. Hence by II Theorem of Euler we can displace the triangle $A_1'B_1'C_1'$ to $A_2B_2C_2$ by means of a translation or of a rotation (about a certain point O lying in Π_2, i. e. by means of a rotation about an axis l perpendicular to Π_2 at the point O). By this translation (or rotation) the body is displaced from position I′ to position II.

It follows from this that a body is displaced from position I to position II either by means of a translation (if the displacement from I′ to II is a translation), or by means of a translation and of a rotation about the axis l parallel to the translation, q. e. d.

The axis l is called the *axis of twist.*

During a twist the axis of twist slides along itself. Every other line changes its position during a twist. It follows from this that a body cannot be displaced from position I to position II by means of a twist along any other axis l'.

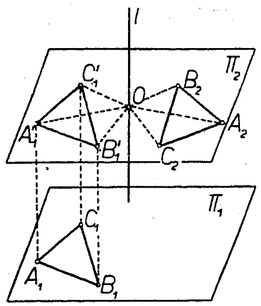

Fig. 248.

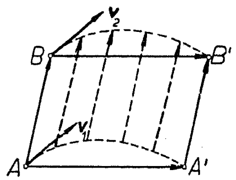

Fig. 249.

§ 4. Advancing motion and rotation about an axis.

Advancing motion. If every position that a body assumes during motion can be obtained from the initial position by means of a translation, then the body is said to move with an *advancing motion.*

From the definition of an advancing motion it follows that every two positions of a body in an advancing motion can be obtained from each other by means of a translation.

Let A and B be two arbitrary points of a body. In an advancing motion the vector $\overline{AB}$ changes neither its direction nor its sense. Therefore, if the path of the point A is translated so that the displacement of all its points is equal to the vector $\overline{AB}$, then the path of the point A will coincide with the path of the point B (Fig. 249).

Therefore: *in an advancing motion the paths of all points are congruent and they can cover one another by means of a translation.*

The advancing motion of a body is therefore determined by the motion of one of its points and the position of the body at the initial moment. For if we know the motion of the point A, for example, then the displacement vector of the body from its initial position to its position at the time t will also be known, because it is equal to the known displacement of the point A.

Conversely: *if the vectors attached to a rigid body do not change their direction, then the body moves with an advancing motion.*

Indeed, let us consider a fixed system of coordinates (x, y, z) and two points in the body $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. By hypothesis, the vector $\overline{AB}$ does not change its direction (that it does not change its length is obvious); we shall show that it also does not change its sense. During the motion the projections of the vector $\overline{AB}$ on the coordinate axes are constant in absolute value. Hence $|x_2-x_1|$, $|y_2-y_1|$, and $|z_2-z_1|$, are certain constants. Since x_2-x_1 is a continuous function of the time t, therefore from the fact that $|x_2-x_1| = \text{const.}$ it follows that x_2-x_1 is also a constant. Similarly $y_2-y_1 = \text{const.}$ and $z_2-z_1 = \text{const.}$ Consequently the vector $\overline{AB}$ does not change its sense.

Therefore, denoting by A, B and A', B' the positions of the given points at two arbitrary moments t and t', we obtain $\overline{AB} = \overline{A'B'}$, whence $\overline{AA'} = \overline{BB'}$. The displacements of two arbitrary points are consequently equal, and we can displace the body from the position at the moment t to the position at the moment t' by means of a translation. Hence the motion of the body is an advancing motion.

In a body moving with an advancing motion let us consider two arbitrary points A_1, A_2 at the moment t and the positions A_1', A_2' of these points at the moment $t+\Delta t$. Denoting by $\mathbf{v}_1$ and $\mathbf{v}_2$ the velocities of the points A_1 and A_2, we obtain (p. 35):

$$\mathbf{v}_1 = \lim_{\Delta t\to 0}\frac{\overline{A_1A_1'}}{\Delta t},\quad \mathbf{v}_2 = \lim_{\Delta t\to 0}\frac{\overline{A_2A_2'}}{\Delta t}.$$

Since the body moves with an advancing motion, $\overline{A_1A_1'} = \overline{A_2A_2'}$, whence $\mathbf{v}_1 = \mathbf{v}_2$.

Therefore: *in an advancing motion the velocities of all the points of a body at an arbitrary moment t are equal to one another.*

The velocity of an advancing motion at the moment t is called the common velocity of all the points of a body at this moment.

Let the body now move in such a way that at each moment t the velocities of all the points of the body are equal to one another. Let us select two arbitrary points of the body $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Since the velocity of the point A at each moment is always equal to the velocity of the point B,

$$x_1^{\cdot} = x_2^{\cdot}, \quad y_1^{\cdot} = y_2^{\cdot}, \quad z_1^{\cdot} = z_2^{\cdot}, \quad \text{whence} \quad x_2^{\cdot} - x_1^{\cdot} = 0.$$

Consequently $x_2 - x_1 = c_1$ and similarly $y_2 - y_1 = c_2$, $z_2 - z_1 = c_3$, where c_1, c_2, and c_3 are certain constants. It follows from this that the vector $\overline{AB}$ has constant projections on the coordinate axes, and hence does not change either its direction or sense during motion. The motion is consequently an advancing motion.

Therefore: *if all the points of a body have equal velocities at each moment, then the body moves with an advancing motion.*

Rotation about an axis. If a body moves so that all the points on a certain line l remain at rest, then we say that the body *rotates about the axis* l (p. 308).

During a rotation all the points move in circles lying in planes perpendicular to the axis of rotation; the centres of these circles lie on the axis of rotation.

The radii joining the points of the body with the centres of the circles along which these points move sweep out equal angles in equal times. It follows from this that during a rotation about an axis all the points have equal angular velocities at each moment. Their common angular velocity is called the *angular velocity* of the rotation about the axis.

From the definition of the angular velocity vector (p. 45), it follows that during a rotation of a body about an axis all the points have the same angular velocity vector lying on the axis of rotation. This vector is called the *angular velocity vector* of the rotating body.

Example 1. If the vertices A and D of a parallelogram $ABCD$ are fixed, then the sides AB and DC can rotate about the vertices A and D. During these rotations the side BC remains constantly parallel to AD. Consequently BC moves with an advancing motion.

Example 2. A circle with centre O' and radius r moves with an advancing motion and remains constantly tangent to the circle K with centre O and radius R. Determine the path of an arbitrary point A (*vide* Fig. 284 on p. 368).

The centre O' obviously moves along a circle K' with centre O and radius $a = R - r$. The path of the point A (dotted in Fig. 284) will consequently be a circle of radius a. The centre O_1 of this circle is obtained by giving the centre O a displacement $\overline{OO_1}$ equal to the vector $\overline{O'A}$.

§ 5. Distribution of velocities in a rigid body. When a rigid body moves, its points can in general have various velocities at a given moment.

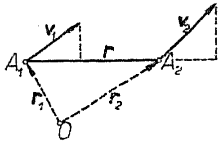

Fig. 250.

Relations among the velocities of the points of a body. Let us consider in the body two points A_1 and A_2 whose velocities are $\mathbf{v}_1$ and $\mathbf{v}_2$. Let O be the origin of the coordinate system (Fig. 250). Let us put:

$$\mathbf{r}_1 = \overline{OA_1}, \quad \mathbf{r}_2 = \overline{OA_2}, \quad \mathbf{r} = \overline{A_1A_2} = \mathbf{r}_2 - \mathbf{r}_1.$$

Consequently (p. 35, (III)):

$$\mathbf{r}_1^{\cdot} = \mathbf{v}_1, \quad \mathbf{r}_2^{\cdot} = \mathbf{v}_2, \quad \mathbf{r}^{\cdot} = \mathbf{r}_2^{\cdot} - \mathbf{r}_1^{\cdot} = \mathbf{v}_2 - \mathbf{v}_1. \tag{1}$$

We have $\mathbf{r}^2 = |\mathbf{r}|^2$. Since $|\mathbf{r}| = \text{const.}$, forming the derivative, we obtain $2\mathbf{r}\mathbf{r}^{\cdot} = 0$, i. e. $\mathbf{r}^{\cdot}\mathbf{r} = 0$; hence by (1) $\mathbf{r}(\mathbf{v}_2 - \mathbf{v}_1) = 0$, whence

$$\mathbf{r}\mathbf{v}_2 = \mathbf{r}\mathbf{v}_1. \tag{2}$$

From the definition of a scalar product it follows that

$$\mathbf{r}\mathbf{v}_1 = |\mathbf{r}| \operatorname{Proj}_{\overline{A_1A_2}}\mathbf{v}_1 \quad \text{and} \quad \mathbf{r}\mathbf{v}_2 = |\mathbf{r}| \operatorname{Proj}_{\overline{A_1A_2}}\mathbf{v}_2.$$

Hence by (2) we obtain, after dividing by $|\mathbf{r}|$,

$$\operatorname{Proj}_{\overline{A_1A_2}}\mathbf{v}_1 = \operatorname{Proj}_{\overline{A_1A_2}}\mathbf{v}_2. \tag{3}$$

We have proved, therefore, that *in a rigid body the projections of the velocities of two points on the segment joining these points are equal.*

We can also say that the components of the velocities of two points with respect to the segment joining these points are equal.

Example 1. Let the velocities $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of three non-collinear points A_1, A_2, A_3 of a body be given. Let us choose an arbitrary point C, not lying in the plane $A_1A_2A_3$, and denote its velocity by $\mathbf{v}$ (Fig. 251). From the point C let us draw the vectors $\overline{CB_1}, \overline{CB_2}, \overline{CB_3}$, equal to the projections of the velocities $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ on the lines A_1C, A_2C, A_3C.

According to the theorem proved, these vectors will also be the projections of the vector $\mathbf{v}$ with its origin C on the lines A_1C, A_2C, and A_3C. If planes perpendicular to these lines are passed through the points B_1, B_2, and B_3, then the point of intersection of these planes will be the terminus of the vector $\mathbf{v}$.

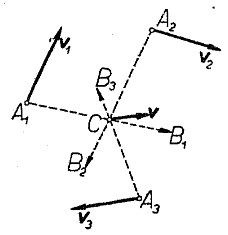

Fig. 251.

In order to determine the velocity of a point D lying in the plane $A_1A_2A_3$, we first determine the velocity of an arbitrary point C not lying in the plane $A_1A_2A_3$, and then the velocity of the point C as before (by taking from among the points A_1, A_2, A_3, and C, three points which are not complanar with D).

Therefore: *the velocities of all the points of a rigid body are determined by the velocities of three of its non-collinear points.*

***Example* 2.** Velocities of points of a straight line and a plane. Let us give an arbitrary sense to a moving line l and take on it a point O whose coordinates are x_0, y_0, z_0. Denote the angles which the axis l makes with the coordinate axes by α, β, γ, and put:

$$a = \cos\alpha, \quad b = \cos\beta, \quad c = \cos\gamma.$$

For an arbitrary point $A(x, y, z)$ of the axis l, having the coordinate r on this axis, we have:

$$x = x_0 + ar, \; y = y_0 + br, \; z = z_0 + cr. \tag{4}$$

Denoting the velocity of an arbitrary point A by $\mathbf{v}$ and calculating the derivative of (4) with respect to time, we obtain (because $r = \text{const.}$):

$$v_x = x^{\cdot} = x_0^{\cdot} + a^{\cdot}r, \quad v_y = y^{\cdot} = y_0^{\cdot} + b^{\cdot}r, \quad v_z = z^{\cdot} = z_0^{\cdot} + c^{\cdot}r. \tag{5}$$

From the point A let us draw the vector $\overline{AA'}$ equal to the velocity $\mathbf{v}$ of the point A, and denote by ξ, η, ζ, the coordinates of the point A'. Since $\xi = x + v_x$ etc., we get in virtue of (4) and (5) the equations

$$\xi = x_0 + x_0^{\cdot} + (a + a^{\cdot})\,r, \quad \eta = y_0 + y_0^{\cdot} + (b + b^{\cdot})\,r,$$
$$\zeta = z_0 + z_0^{\cdot} + (c + c^{\cdot})\,r.$$

Being equations of the first degree of the parameter r, they are the equations of a line.

Therefore: *if velocity vectors are drawn from points lying on a straight line, then the ends of these vectors lie on a straight line.*

By making use of this theorem similar theorem for a plane can easily be proved.

Namely: *if velocity vectors are drawn from points lying in a plane, then the ends of these vectors lie in one plane.*

Knowing the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of two points A_1 and A_2 of the line l, the velocity $\mathbf{v}$ of an arbitrary point A of this line can be determined in the following manner (*vide* Fig. 252):

We pass a line m through the ends of the velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ drawn from the points A_1 and A_2. The end of the velocity vector $\mathbf{v}$ drawn from A lies on this line. On the line l we determine a point B so that the vector $\overline{AB}$ is equal to the projection of $\mathbf{v}_1$ (or $\mathbf{v}_2$) on the axis l. According to the theorem proved on p. 321, $\overline{AB}$ is the projection of $\mathbf{v}$ on l. If we pass a plane through B and perpendicular to l, then the point A' in which this plane cuts m will be the end of the vector $\mathbf{v}$ drawn from A.

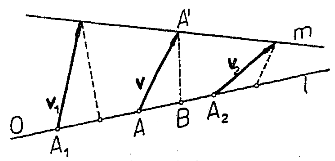

Fig. 252.

Instantaneous motion of a rigid body. Let us consider in a rigid body an arbitrary point A with coordinates x, y, z at a certain time t and denote by $\mathbf{v}$ its velocity. Since $\mathbf{v}$ depends on the point A, $\mathbf{v}$ is a function of the coordinates x, y, z at the time t. We can therefore assume that

$$\mathbf{v} = \mathbf{F}(x, y, z). \tag{6}$$

The vector function (6) defines the velocity at the time t at every point of the body.

The distribution of the velocities in a body at a certain instant is called the *instantaneous motion* of the body.

The instantaneous motion of a body is consequently determined by giving the vector function (6).

In an advancing motion all the points have the same velocity. Function (6) will therefore assume the form $\mathbf{v} = \text{const.}$

Denoting by ω the angular velocity vector, and by O an arbitrary point on the axis l we have for the point A, in a rotating motion about the axis l (cf. formulae (2) and (III), p. 46):

$$\mathbf{v} = \text{Mom}_A\,\boldsymbol{\omega} \quad \text{or} \quad \mathbf{v} = \overline{OA} \times \boldsymbol{\omega}. \tag{7}$$

The instantaneous motion of a body at the moment t is called the *instantaneous advancing motion* if all the points of the body have the same velocity. This velocity is called the *instantaneous velocity of the advancing motion.*

The instantaneous motion of a body at the moment t is called an *instantaneous rotation* about an axis l if the velocities of the points of the body are expressed by formula (7), i. e. if the velocities of the points of a body at the moment t are such as if the body were rotating about the axis l. The velocity $\boldsymbol{\omega}$ is called the *instantaneous angular velocity vector*, and the axis l the *instantaneous axis of rotation*.

If the instantaneous motion at each instant of time is a rotation about a fixed line l, then it is a rotation about the axis l.

For let us choose a fixed coordinate system $O(x, y, z)$, taking the axis l as the z-axis. Let A be an arbitrary point of the body whose coordinates are x, y, z, and $\boldsymbol{\omega}$ the instantaneous velocity vector. Then $\omega_x = 0$ and $\omega_y = 0$. According to formula (7) the projections of the velocity $\mathbf{v}$ of the point A are:

$$v_x = x^{\cdot} = y\omega_z, \quad v_y = y^{\cdot} = -x\omega_z, \quad v_z = z^{\cdot} = 0. \tag{8}$$

As $z^{\cdot} = 0$, $z = \text{const}$. The points therefore move in planes perpendicular to the axis l. From equations (8) we get $xx^{\cdot} + yy^{\cdot} = 0$, whence after integration $\frac{1}{2}x^2 + \frac{1}{2}y^2 = \text{const}$. The points of the body therefore move in planes perpendicular to the axis l at a constant distance from l, and hence the body rotates about l.

§ 6. Instantaneous plane motion. Let a plane figure move in the plane Π. Let us select two arbitrary coordinate systems in this plane: the one (x, y) fixed, and the other (ξ, η) moving with an advancing motion and having its origin at an arbitrarily chosen point M of the given figure (Fig. 253).

The instantaneous motion of the figure with respect to the system (ξ, η) will then be a rotation about the point M (i. e. about the axis l perpendicular to Π at the point M). The relative velocities of the points at the moment t will therefore be such as if the figure were rotating about the point M with a certain angular velocity ω. Consequently we can say that the instantaneous relative motion is a rotation about the point M. Since the system (ξ, η) moves with an advancing motion, the velocity of transport is the same for all points of the figure (p. 58) and equal to the velocity of the point M. Denoting by $\mathbf{v}$ the absolute velocity of an arbitrary point A, by $\mathbf{v}_r$ the relative velocity of the point A, and by $\mathbf{v}_t$ the velocity of transport (i. e. the velocity of the point M), we therefore obtain (p. 58).

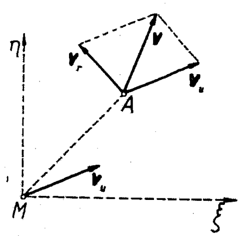

Fig. 253.

$$\mathbf{v} = \mathbf{v}_t + \mathbf{v}_r. \tag{1}$$

In view of this we can say that the velocities of the points of the figure are such as if the figure were executing two motions simultaneously: an advancing motion with the velocity $\mathbf{v}_t$ of an arbitrary point M of the figure, and a rotation about this point M.

Therefore: *the instantaneous motion of a figure in plane motion is composed of an instantaneous advancing motion and an instantaneous rotation; the advancing motion has the velocity of an arbitrary point of the figure, and the rotation is about this point.*

In general, the instantaneous motion of a figure can be represented in infinitely many ways as the composition of an instantaneous advancing motion and an instantaneous rotation, for this depends on the choice of the point M.

In all these representations, however, the instantaneous angular velocities are equal. For let us choose in the figure an arbitrary axis k at the time t and let k' denote the position of this axis at the time $t + \Delta t$. The angle $\Delta\varphi$ which the axis k' makes with the axis k is equal to the angle through which the figure turned about M in the relative motion. It follows from this that $\Delta\varphi$ does not depend on the choice of the point M, and the same is true of ω, because $\omega = \lim_{\Delta t \to 0} \Delta\varphi / \Delta t$.

In particular, an instantaneous motion can be an instantaneous advancing motion (if the instantaneous angular velocity ω is zero) or an instantaneous rotation (if the point M has the velocity $\mathbf{v}_t = 0$).

For each point A of the figure the velocity $\mathbf{v}_r$ (of the instantaneous rotation) is perpendicular to the segment MA, where $|\mathbf{v}_r| = MA \cdot \omega$. Knowing the sense of the instantaneous rotation, we can therefore determine the sense of $\mathbf{v}_r$ and then obtain the velocity $\mathbf{v}$ of the point A from formula (1).

Now let $\mathbf{v}_t \neq 0$ and $\omega \neq 0$. On a line l perpendicular to $\mathbf{v}_t$ and passing through M let us consider two points O and O' at a distance r from M, where $r = |\mathbf{v}_t| / \omega$ (Fig. 254).

The velocities of the instantaneous rotation $\mathbf{v}_r$ and $\mathbf{v}_r'$ of the points O and O' are perpendicular to l and therefore parallel to $\mathbf{v}_t$. In addition we have $|\mathbf{v}_r| = MO \cdot \omega = r\omega = |\mathbf{v}_t|$ and similarly $|\mathbf{v}_r'| = |\mathbf{v}_t|$. Since $\mathbf{v}_r$ and $\mathbf{v}_r'$ have opposite senses, it follows that for one of the points O and O', e. g. for O, we have $\mathbf{v}_r = -\mathbf{v}_t$. Consequently the velocity of the point O is $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_t = 0$. Therefore, if the origin of the system (ξ, η) is placed at the point O, then the velocity of transport will be zero and hence the instantaneous motion will be an instantaneous rotation about O.

The point O is called the *instantaneous centre of rotation.*

The velocity $\mathbf{v}$ of an arbitrary point A is perpendicular to the segment OA and has a sense which depends on the sense of the instantaneous rotation. Moreover

$$|\mathbf{v}| = OA \cdot \omega. \tag{2}$$

Therefore: *an instantaneous plane motion is either an instantaneous advancing motion or an instantaneous rotation about the instantaneous centre of rotation.*

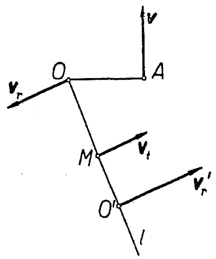

Fig. 254.

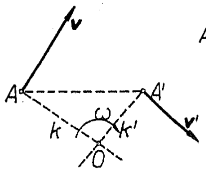

Fig. 255.

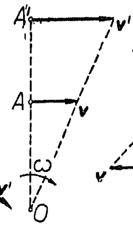

Fig. 256.

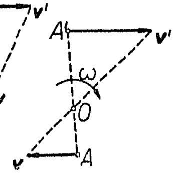

Fig. 257.

Determination of the instantaneous centre of rotation. The instantaneous centre of rotation obviously has a zero velocity. According to (2) every other point has a velocity $\mathbf{v} \neq 0$ (if $\omega \neq 0$). Consequently there exists only one instantaneous centre of rotation (when $\omega \neq 0$).

If at an arbitrary point A we draw a perpendicular to the velocity $\mathbf{v}$ of this point, then the instantaneous centre of rotation will lie on this line (Fig. 255).

The instantaneous centre of rotation can in general be determined if the velocity $\mathbf{v}$ of one point, e. g. A, is known, as well as the direction of the velocity $\mathbf{v}'$ of another point A'. Let us draw k and k' perpendicular to $\mathbf{v}$ and $\mathbf{v}'$ at the points A and A'. The point O of intersection of these lines is the instantaneous centre of rotation. From formula (2) we obtain $\omega = |\mathbf{v}| / OA$. The sense of the instantaneous rotation is obtained from the sense of the velocity $\mathbf{v}$.

When the lines k and k' are parallel, the instantaneous motion is an instantaneous advancing motion.

When the lines k and k' coincide, in order to determine the instantaneous motion, we must know in addition the velocity $\mathbf{v}'$ of the point A'. Knowing only the direction of the velocity $\mathbf{v}'$, then, is not sufficient.

When $\mathbf{v} = \mathbf{v}'$, the instantaneous motion is an advancing motion.

However, when $\mathbf{v} \neq \mathbf{v}'$, the instantaneous motion is an instantaneous rotation. Denoting, then, by O the instantaneous centre of rotation (obviously lying on the lines k and k'), we have by (2) $|\mathbf{v}| = OA \cdot \omega$ and $|\mathbf{v}'| = OA' \cdot \omega$. Consequently

$$\frac{OA}{OA'} = \frac{|\mathbf{v}|}{|\mathbf{v}'|}. \tag{3}$$

If $\mathbf{v}$ and $\mathbf{v}'$ have the same senses (Fig. 256) and $|\mathbf{v}| < |\mathbf{v}'|$, then the point O lies on the prolongation of the segment $A'A$ beyond the point A. We then have $OA' - OA = A'A$, whence by (3)

$$OA = A'A \cdot \frac{|\mathbf{v}|}{|\mathbf{v}'| - |\mathbf{v}|}. \tag{4}$$

On the other hand, when $\mathbf{v}$ and $\mathbf{v}'$ have opposite senses (Fig. 257), the point O lies on the segment AA'. We then have $OA + OA' = AA'$, whence by (3)

$$OA = AA' \cdot \frac{|\mathbf{v}|}{|\mathbf{v}| + |\mathbf{v}'|}. \tag{5}$$

Example 1. A rod AB moves in a plane in such a way that both of its ends move constantly along the curves C and C'. The velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the points A and B are tangent to the curves C and C' (Fig. 258).

Drawing perpendiculars to the tangents at the points A and B, we obtain the instantaneous centre of rotation O of the rod AB as the point of intersection of these perpendiculars. Denoting the instantaneous angular velocity by ω, we obtain:

$$|\mathbf{v}_1| = OA \cdot \omega, \quad |\mathbf{v}_2| = OB \cdot \omega \quad \text{and} \quad |\mathbf{v}_1| / |\mathbf{v}_2| = OA / OB.$$

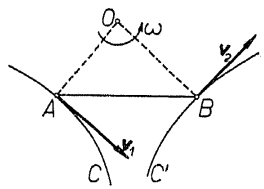

Fig. 258.

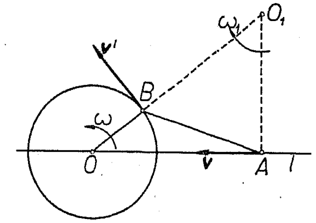

Fig. 259.

Example 2. In a crank-mechanism the rod AB moves in such a way that its end B is pin-connected with the rod OB fixed at O, and the other end A moves along a line l passing through O. The rod OB revolves about the point O with an angular velocity ω (Fig. 259). What is the velocity of the point A?

The velocity $\mathbf{v}'_1$ of the point B is perpendicular to OB, and the velocity $\mathbf{v}$ of the point A has the direction of the line l. Drawing the perpendiculars to $\mathbf{v}'$ and $\mathbf{v}$, we obtain the instantaneous centre of rotation O_1 of the rod AB. Now, $OB \cdot \omega = |\mathbf{v}'| = O_1B \cdot \omega_1$; consequently $\omega_1 = OB \cdot \omega / O_1B$, whence

$$|\mathbf{v}| = O_1A \cdot \omega_1 = O_1A \cdot OB \cdot \omega / O_1B.$$

Example 3. A system of pin-connected rods AO and OB moves in a plane (Fig. 260). The velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the points A and B are given. Determine the velocity of the point O.

Let us denote by α and β the angles which the two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ make with the rods OA and OB, by δ the angle which the velocity $\mathbf{v}$ of the point O makes with the rod OA, and by φ the angle AOB.

Since the projections of the velocity $\mathbf{v}$ on the rods OA and OB are equal to the corresponding projections of $\mathbf{v}_1$ and $\mathbf{v}_2$ on these rods, denoting the absolute values of these velocities by v, v_1 and v_2, we obtain

$$v_1 \cos\alpha = v \cos\delta, \quad v_2 \cos\beta = v \cos(\varphi - \delta), \tag{6}$$

whence

$$\cos(\varphi - \delta) / \cos\delta = v_2 \cos\beta / v_1 \cos\alpha;$$

therefore $\cos\varphi + \tan\delta \sin\varphi = v_2 \cos\beta / v_1 \cos\alpha$, whence

$$\tan\delta = (v_2 \cos\beta - v_1 \cos\alpha \cos\varphi) / v_1 \cos\alpha \sin\varphi.$$

Knowing δ, we determine v from equation (6).

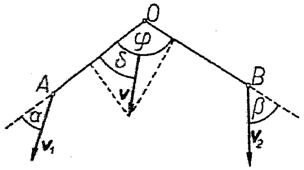

Fig. 260.

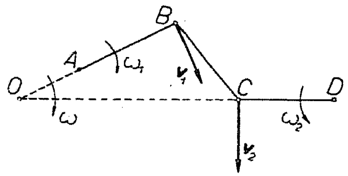

Fig. 261.

Example 4. A system of three rods AB, BC, and CD, pin-connected and fixed at A and D, lies in a plane (Fig. 261). The rod AB rotates about A with an angular velocity ω_1. Determine the angular velocity ω_2 of the rod CD about D.

The points B and C move in circle with centres at A and D; their velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are therefore perpendicular to AB and CD. The instantaneous centre of rotation of the rod BC is obtained by drawing perpendiculars to $\mathbf{v}_1$ and $\mathbf{v}_2$, or by prolonging the segments AB and DC to their point of intersection O. The point O is the instantaneous centre of rotation of the rod BC.

Let us denote by ω the instantaneous angular velocity of the rod BC. Putting $v_1 = |\mathbf{v}_1|$ and $v_2 = |\mathbf{v}_2|$, we therefore obtain:

$$v_1 = OB \cdot \omega \quad \text{and} \quad v_2 = OC \cdot \omega. \tag{7}$$

The rod AB rotates about A with an angular velocity ω_1; consequently

$$v_1 = AB \cdot \omega_1 \quad \text{and similarly} \quad v_2 = CD \cdot \omega_2. \tag{8}$$

From (7) and (8) we obtain $AB \cdot \omega_1 = OB \cdot \omega$, i.e. $\omega = AB \cdot \omega_1 / OB$, whence by (7) $v_2 = AB \cdot OC \cdot \omega_1 / OB$, and since by (8) $\omega_2 = v_2 / CD$,

$$\omega_2 = \frac{AB \cdot OC}{OB \cdot CD} \omega_1.$$

Example 5. A system of rods, pin-connected at the joints B, C, D, and E, is given (Fig. 262). The rods AB, FD, and EG, can rotate about the points A, F, and G, which are fixed. The rod GE rotates about G with an angular velocity ω. Determine the angular velocities ω' and ω'' of the rods FD and AB about F and A.

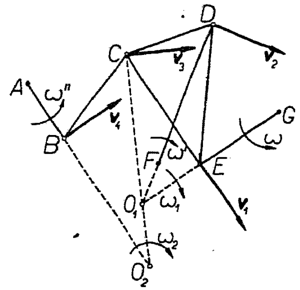

Fig. 262.

Let us denote the velocities of the points E, D, C, B by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and their absolute values by v_1, v_2, v_3, v_4. The points E and D move along the circumferences of circles with centres at G and F. The velocities of these points are perpendicular to EG and FD.

Let O_1 be the point of intersection of the perpendiculars to $\mathbf{v}_1$ and $\mathbf{v}_2$ (i. e. of the prolongations of the rods GE and DF). The point O_1 will be the instantaneous centre of rotation of the rod ED and at the same time of the triangle EDC, because the rods ED, DC, and CE, form a rigid system.

Denoting by ω_1 the instantaneous velocity of rotation about O_1, whose sense is determined from the sense of $\mathbf{v}_1$, we have:

$$v_1 = O_1E \cdot \omega_1, \quad v_2 = O_1D \cdot \omega_1. \tag{9}$$

The sense of $\mathbf{v}_2$ is determined from the sense of the angular velocity ω_1.

Since $v_1 = GE \cdot \omega$, in virtue of (9):

$$\omega_1 = \frac{GE}{O_1E}\omega, \quad v_2 = \frac{O_1D \cdot GE}{O_1E}\omega_1 \tag{10}$$

The rod FD rotates about F with an angular velocity ω'; hence $v_2 = FD \cdot \omega'$, whence $\omega' = v_2 / FD$. Consequently by (10)

$$\omega' = \frac{O_1D \cdot GE}{O_1E \cdot FD}\,\omega. \tag{11}$$

The sense of ω' is obtained from the sense of $\mathbf{v}_2$.

The velocity $\mathbf{v}_3$ of the point C is perpendicular to O_1C; we therefore have

$$v_3 = O_1C \cdot \omega_1. \tag{12}$$

The sense of $\mathbf{v}_3$ is determined from the sense of the rotation about O_1.

In order to determine the centre O_2 of the instantaneous rotation of the rod BC, let us note that the velocity $\mathbf{v}_4$ of the point B is perpendicular to the rod AB. We therefore draw perpendiculars to the velocities $\mathbf{v}_4$ and $\mathbf{v}_3$, i. e. we prolong AB and CO_1 to their intersection at O_2.

Denoting by ω_2 the angular velocity of the instantaneous rotation about O_2, we have:

$$v_3 = O_2C \cdot \omega_2, \quad v_4 = O_2B \cdot \omega_2. \tag{13}$$

The sense of the instantaneous rotation is obtained from the sense of $\mathbf{v}_3$.

From (13) we get:

$$\omega_2 = v_3 / O_2C, \quad v_4 = O_2B \cdot v_3 / O_2C. \tag{14}$$

The sense of $\mathbf{v}_4$ is obtained from the sense of the rotation about O_2.

Since the rod AB rotates about A with an angular velocity ω'', $v_4 = AB \cdot \omega''$, whence $\omega'' = v_4 / AB$. Hence by (14), (12) and (10) we get

$$\omega'' = \frac{O_2B \cdot O_1C \cdot GE}{AB \cdot O_2C \cdot O_1E}\,\omega.$$

The sense of the angular velocity ω'' is obtained from the sense of $\mathbf{v}_4$.

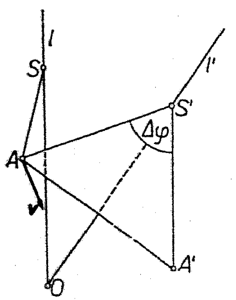

Fig. 263.

§ 7. Instantaneous space motion. We shall first consider a particular case.

Rotation about a point. Let us suppose that a body rotating about a fixed point O occupied position I at time t and position II at time $t + \Delta t$. The body can therefore be displaced from position I to position II by means of a rotation about a certain line l' through an angle $\Delta\varphi$ (Fig. 263). Let us assume that the line l' tends to a certain line l when Δt tends to zero.

Let us set

$$\lim_{\Delta t \to 0} \Delta\varphi / \Delta t = \omega. \tag{1}$$

Let us denote by A an arbitrary point of the body in position I, and by A' the corresponding point in position II. Let S' be the centre of the circle along which the point A moves during its rotation about the line l', and S the limiting position of the point S' as $\Delta t \to 0$. Finally, let us denote by $\mathbf{v}$ the velocity of the point A. We have

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\overline{AA'}}{\Delta t};$$

consequently

$$|\mathbf{v}| = \lim_{\Delta t \to 0} \left|\frac{AA'}{\Delta t}\right| = \lim_{\Delta t \to 0} \frac{2AS' \sin\frac{1}{2}\Delta\varphi}{\Delta t} = \lim_{\Delta t \to 0} \frac{AS' \sin\frac{1}{2}\Delta\varphi}{\frac{1}{2}\Delta\varphi} \cdot \frac{\Delta\varphi}{\Delta t},$$

and since $\lim_{\Delta\varphi \to 0}(\sin\frac{1}{2}\Delta\varphi / \frac{1}{2}\Delta\varphi) = 1$, it follows by (1) that

$$|\mathbf{v}| = AS \cdot \omega. \tag{2}$$

Let us note that the vector $\overline{AA'} / \Delta t$ is perpendicular to l' and makes an angle of $90° - \frac{1}{2}\Delta\varphi$ with the segment $S'A$; the vector $\mathbf{v}$ is then in the limit perpendicular to l and AS. The velocity of the point A at the time t is therefore such as if the body were rotating about the axis l with an angular velocity ω. Thus we have proved the following theorem:

The instantaneous motion of a body rotating about a certain point O is an instantaneous rotation about an axis passing through O.

The velocity $\mathbf{v}$ of the point A is perpendicular to the plane Π passing through l and A, whence $\mathbf{v} \perp OA$, since OA lies in Π.

Hence: *during the rotation of a body about the point O, the velocities of the points of the body are perpendicular to the lines connecting these points with the point O.*

The axis l of the instantaneous rotation lies in a plane passing through the point A and perpendicular to the velocity of the point A. Therefore, knowing the directions of the velocities of two points of the body, we obtain the instantaneous axis of rotation as the line of intersection of the planes passing through these points and perpendicular to the directions of the velocities. The instantaneous angular velocity is obtained from equation (2).

Example I. The sphere $x^2 + y^2 + z^2 = 1$ rotates about the centre O. The velocity $\mathbf{v}$ of the point $A(1, 0, 0)$ at a certain instant t and the direction of the velocity $\mathbf{w}$ of the point $B(0, 1, 0)$ at the same instant are given.

Determine the instantaneous axis of rotation, the angular velocity, and the velocity $\mathbf{w}$ (at the moment t).

Since the velocity $\mathbf{v}$ is perpendicular to OA, i. e. to the x-axis, $v_x = 0$. Let us denote the cosines of the angles which the velocity $\mathbf{w}$ makes with the coordinate axes by a, b, c. We obviously have $b = 0$, because $\mathbf{w}$ is perpendicular to OB, i. e. to the y-axis.

The instantaneous axis of rotation is the intersection of the planes passing through O as well as through A and B, and perpendicular to the velocities $\mathbf{v}$ and $\mathbf{w}$. The equations of these planes are the following:

$$yv_y + zv_z = 0, \quad ax + cz = 0, \tag{3}$$

from which

$$\frac{x}{c\,/\,a} = \frac{y}{v_z\,/\,v_y} = \frac{z}{-1}. \tag{4}$$

Equations (4) are the equations of the instantaneous axis of rotation.

Let $\boldsymbol{\omega}$ denote the instantaneous angular velocity. The projections of $\boldsymbol{\omega}$ on the coordinate axes are proportional to the direction numbers of the axis of rotation, i. e. to the numbers $c\,/\,a$, $v_z\,/\,v_y$, and -1. Denoting by λ the factor of proportionality we get:

$$\omega_x = \lambda c\,/\,a, \quad \omega_y = \lambda v_z\,/\,v_y, \quad \omega_z = -\lambda. \tag{5}$$

In order to determine λ we calculate the velocity $\mathbf{v}$ by making use of the formula $\mathbf{v} = \overline{OA} \times \boldsymbol{\omega}$. We obtain $v_x = 0$, $v_y = \lambda$, and $v_z = \lambda v_z\,/\,v_y$, whence $\lambda = v_y$, and hence by (5):

$$\omega_x = cv_y\,/\,a, \quad \omega_y = v_z, \quad \omega_z = -v_y. \tag{6}$$

Since $\mathbf{w} = \overline{OB} \times \boldsymbol{\omega}$, we get by (6):

$$w_x = -v_y, \quad w_y = 0, \quad w_z = -cv_y\,/\,a.$$

Instantaneous motion in the general case. Let us place the origin of the coordinate system (ξ, η, ζ), moving with an advancing motion, at an arbitrary point M of a given body. Since the point M is fixed relative to the system (ξ, η, ζ), the motion of the body relative to the system (ξ, η, ζ) is a rotation about the point M. The instantaneous relative motion will therefore be an instantaneous rotation about the axis l passing through M (Fig. 264).

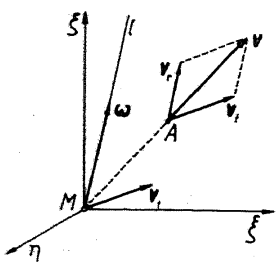

Fig. 264.

Let A be an arbitrary point of the body. Let us denote by $\mathbf{v}$ the absolute velocity of

the point A, by $\mathbf{v}_t$ the velocity of transport, and by $\mathbf{v}_r$ the relative velocity. Consequently

$$\mathbf{v} = \mathbf{v}_t + \mathbf{v}_r. \tag{7}$$

The velocity of transport $\mathbf{v}_t$ is equal to the velocity of the point M, because the system (ξ, η, ζ) moves with an advancing motion. The velocities of the points of the body are therefore such as if the body were executing two motions simultaneously: an advancing motion with the velocity $\mathbf{v}_t$ of an arbitrary point M of the body, and a rotation about a certain axis passing through M.

Hence: *an instantaneous motion of a body is composed of an instantaneous advancing motion with the velocity of an arbitrary point M of this body, and an instantaneous rotation about the instantaneous axis of rotation passing through the point M.*

The instantaneous motion of a body can in general be represented in infinitely many ways as the composition of an instantaneous advancing motion and an instantaneous rotation, for this depends on the choice of the point M.

We shall show that *the instantaneous angular velocity vectors are equal for all possible representations* (i. e. *that the instantaneous axes of rotation are parallel and the instantaneous angular velocities are equal*).

Let us suppose that a point M of the body at the time t coincided with the point M' at the time $t + \Delta t$. Relative to the fixed system, the change of position of the body in the time Δt is the composition of the displacement $\overline{MM'}$ and the rotation through an angle $\Delta\varphi$ about a certain axis l'. Relative to the system (ξ, η, ζ) the change of position is only a rotation about the axis l' through the angle $\Delta\varphi$, because the system (ξ, η, ζ) made the displacement $\overline{MM'}$ in the time Δt. Consequently the limiting position of the axis l' is the instantaneous axis of rotation l, and the instantaneous angular velocity is $\omega = \lim\limits_{\Delta t \to 0} \Delta\varphi\,/\,\Delta t$.

From the theorem given on p. 315 it follows that had we chosen another point M_1 in the body, then with similar notations the axis l_1' would be parallel to l', while $\Delta\varphi_1 = \Delta\varphi$. Consequently the instantaneous axis of rotation l_1 passing through M_1 is parallel to l, and the instantaneous angular velocity ω_1 is equal to ω. Therefore the angular velocity vectors for M and for M_1 will be equal.

Remark. Denoting the instantaneous angular velocity vector by $\boldsymbol{\omega}$, we obtain by (III), p. 46, $\mathbf{v}_r = \overline{MA} \times \boldsymbol{\omega}$ for an arbitrary point A. Hence by (7)

$$\mathbf{v} = \mathbf{v}_t + \overline{MA} \times \boldsymbol{\omega}. \tag{I}$$

Velocity of transport. Let a system of coordinates (ξ, η, ζ) with origin M move in space relative to a fixed coordinate system (x, y, z) (Fig. 265). The system (ξ, η, ζ) (together with the points rigidly attached to it) can be considered as a rigid body. The instantaneous motion of the system (ξ, η, ζ) will therefore be the composition of the instantaneous advancing motion with a velocity $\mathbf{v}_0$ of the origin M of the system, and the rotation with an instantaneous angular velocity $\boldsymbol{\omega}$ about an instantaneous axis of rotation passing through M. The velocity of transport $\mathbf{v}_t$ of an arbitrary point A in motion relative to the system (ξ, η, ζ) is the velocity the point A would possess were it rigidly attached to the system (ξ, η, ζ). Consequently $\mathbf{v}_t$ is the sum of the velocity $\mathbf{v}_0$ and the velocity $\mathbf{w}$ of the instantaneous rotation. In view of the preceding we have by (I)

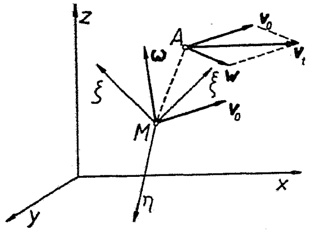

Fig. 265.

$$\mathbf{v}_t = \mathbf{v}_0 + \overline{MA} \times \boldsymbol{\omega}. \tag{8}$$

Therefore: *the velocity of transport of an arbitrary point is such as if this point were rigidly attached to the moving coordinate system, and this system executed two simultaneous motions: an advancing motion with a velocity of the origin of the system, and a rotation about an axis passing through the origin of the system.*

This theorem was given without proof on p. 62.

Instantaneous twist. An instantaneous motion of a body is called an *instantaneous twist* or an *instantaneous screw motion* if the velocity of the instantaneous advancing motion is parallel to the instantaneous axis of rotation.

In particular, an instantaneous advancing motion or an instantaneous rotation is also called an instantaneous twist.

The instantaneous axis of twist is called the *central axis* of the instantaneous rotation.

By means of the theorem given on p. 317, we shall prove the following theorem:

An instantaneous motion of a rigid body can be represented as an instantaneous twist and this can be done in only one way.

Proof. Let us assume that we have displaced the body from position I at the instant t to the position II at the instant $t + \Delta t$ by means of a twist about the axis l' (p. 317). Let l be the limiting position of the line

l' as $\Delta t \to 0$ (Fig. 266). Let us consider an arbitrary point O on l at the instant t and denote the position of the point O at the instant $t + \Delta t$ by O'. During the displacement of the body from position I to position II by means of a twist, the point O will occupy the position O_1 after the displacement, and then it will go to the position O' after a rotation about l' through an angle $\Delta\varphi$. Denoting by $\mathbf{u}$ the velocity of the point O, we therefore have

$$\mathbf{u} = \lim_{\Delta t \to 0} \frac{\overline{OO'}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{OO_1} + \overline{O_1O'}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{OO_1}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\overline{O_1O'}}{\Delta t}. \tag{9}$$

Let S be the centre of the circle along which the point O moved during the rotation about the axis l' through the angle $\Delta\varphi$. Consequently $|\overline{O_1O'}| = 2SO_1 \cdot \sin\frac{1}{2}\Delta\varphi$, whence

$$\lim_{\Delta t \to 0} \left| \frac{\overline{O_1O'}}{\Delta t} \right| = \lim_{\Delta t \to 0} 2SO_1 \cdot \left| \frac{\sin(\frac{1}{2}\Delta\varphi)}{\Delta t} \right| = \lim_{\Delta t \to 0} SO_1 \cdot \left| \frac{\sin(\frac{1}{2}\Delta\varphi)}{\frac{1}{2}\Delta\varphi} \right| \cdot \left| \frac{\Delta\varphi}{\Delta t} \right|.$$

Since $\lim_{\Delta t \to 0} SO_1 = 0$, and

$$\lim_{\Delta\varphi \to 0} \left| \frac{\sin(\frac{1}{2}\Delta\varphi)}{\frac{1}{2}\Delta\varphi} \right| = 1, \qquad \lim_{\Delta t \to 0} \left| \frac{\overline{O_1O'}}{\Delta t} \right| = 0.$$

Therefore, by (9), $\mathbf{u} = \lim_{\Delta t \to 0} \overline{OO_1} / \Delta t$. But $\overline{OO_1} \parallel l'$; hence because l' tends to l as $\Delta t \to 0$, the velocity $\mathbf{u}$ has the direction of the axis l, which is the instantaneous axis of rotation passing through O. The instantaneous motion is therefore a twist, because the velocity $\mathbf{u}$ of the instantaneous advancing motion has the direction of the instantaneous axis of rotation l.

The points lying on the instantaneous axis of twist have velocities equal to $\mathbf{u}$, and hence parallel to the axis of twist. The points situated outside the axis of twist have velocities which are not parallel to the axis of twist, for the velocity of such a point is the sum of the velocity $\mathbf{u}$ and the velocity of rotation $\mathbf{w}$ perpendicular to $\mathbf{u}$. Consequently the sum $\mathbf{u} + \mathbf{w}$ is never parallel to $\mathbf{u}$ (and hence also to l), except when $\mathbf{w} = 0$, i. e. when the point lies on the axis of twist.

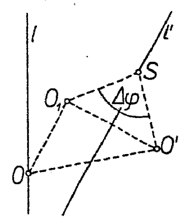

Fig. 266.

It follows from this that an instantaneous motion can be represented as an instantaneous twist in only one way. For, if we had represented this instantaneous motion as a twist about another axis l_1, then by the theorem on p. 333, the lines l and l_1 would be parallel and in that case the velocities of the points lying on l_1 would be parallel to l_1 and l, which is impossible, since — as we have just proved — only points situated on the axis l have velocities parallel to l.

Example 2. Let a body move in such a way that its instantaneous motion is a twist of constant advancing velocity $\mathbf{u}$ and angular velocity $\boldsymbol{\omega}$ about a fixed axis l.

Let us choose the z-axis as the instantaneous axis of rotation. Let us denote by ω and u the components of $\boldsymbol{\omega}$ and $\mathbf{u}$ with respect to the z-axis. The velocity $\mathbf{v}$ of a point $A(x, y, z)$ of the body is expressed, in virtue of (I), p. 333, by the formula

$$\mathbf{v} = \mathbf{u} + \overline{OA} \times \boldsymbol{\omega},$$

where O is the origin of the system. Since $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$, and $u_x = 0$, $u_y = 0$, $u_z = u$:

$$\dot{x} = \omega y, \quad \dot{y} = -\omega x, \quad \dot{z} = u. \tag{10}$$

The last of the equations (10) gives after integration,

$$z = ut + c, \tag{11}$$

where c is an arbitrary constant. Differentiating the first of the equations (10), we obtain $\ddot{x} = \omega \dot{y}$. Substituting for $\dot{y}$ the value from the second equation, we get the equation $\ddot{x} + \omega^2 x = 0$, whose general solution has the form

$$x = a \sin \omega t + b \cos \omega t, \tag{12}$$

where a and b are arbitrary constants. Since $y = \dot{x} / \omega$,

$$y = a \cos \omega t - b \sin \omega t. \tag{13}$$

Let us assume that the point A had the coordinates $x_0 = r$, $y_0 = 0$, and $z_0 = 0$, at $t = 0$. Substituting $t = 0$ in (11)—(13), we get $a = 0$, $b = r$, and $c = 0$, whence:

$$x = r \cos \omega t, \quad y = -r \sin \omega t, \quad z = ut.$$

The point will therefore move with a screw motion (p. 55) on a cylindrical surface whose axis is the z-axis (because $x^2 + y^2 = r^2$), describing the so-called *helix*. If we develop the lateral surface of the cylinder, the helix will appear as a straight line. The helix consequently cuts all the generatrices at the same angle α. The distance of two neighbouring points of a helix on the same generatrix is called the *lead of the helix* and we denote it by h. We therefore have

$$\tan \alpha = 2r\pi / h, \tag{14}$$

where r is the radius of the base of the cylinder. Since the time for one revolution is $2\pi / |\omega|$ or $h / |\mathbf{u}|$,

$$h = 2\pi |\mathbf{u}| / |\boldsymbol{\omega}|, \quad \tan\alpha = r|\boldsymbol{\omega}| / |\mathbf{u}|. \tag{15}$$

Determination of the motion of a body. Let us select in the body an arbitrary point O whose coordinates are x_0, y_0, z_0 with respect to a certain fixed coordinate system (x, y, z). Let us denote the instantaneous angular velocity by $\boldsymbol{\omega}$ and the velocity of the point O by $\mathbf{u}$. A point A of the body whose coordinates are x, y, z, has the velocity ((I), p. 333)

$$\mathbf{v} = \mathbf{u} + \overline{OA} \times \boldsymbol{\omega}. \tag{16}$$

Since: $v_x = \dot{x}$, $v_y = \dot{y}$, $v_z = \dot{z}$, and $u_x = \dot{x}_0$, $u_y = \dot{y}_0$, $u_z = \dot{z}_0$, we get from formula (16):

$$\begin{aligned} \dot{x} &= \dot{x}_0 + (y - y_0)\,\omega_z - (z - z_0)\,\omega_y, \\ \dot{y} &= \dot{y}_0 + (z - z_0)\,\omega_x - (x - x_0)\,\omega_z, \\ \dot{z} &= \dot{z}_0 + (x - x_0)\,\omega_y - (y - y_0)\,\omega_x. \end{aligned} \tag{17}$$

If the motion of the point O and the angular velocity $\boldsymbol{\omega}$ are given by the functions:

$$x_0 = f(t), \; y_0 = \varphi(t), \; z_0 = \psi(t); \quad \omega_x = \alpha(t), \; \omega_y = \beta(t), \; \omega_z = \gamma(t),$$

then (17) is a system of differential equations in which the unknown functions are the functions $x = F(t)$, $y = \Phi(t)$, and $z = \Psi(t)$, defining the motion of the point A. From equations (17) we can determine the functions F, Φ, Ψ, if we know the initial position of the point A, the instantaneous angular velocity $\boldsymbol{\omega}$, and the motion of the point O.

Therefore: *the motion of a rigid body is determined by giving the following*:

a) *the initial position of the body,*
b) *the motion of one of its points,*
c) *the instantaneous angular velocity of rotation* $\boldsymbol{\omega}$ *at each instant.*

§ 8. Rolling and sliding. Let a plane curve C, moving in its own plane Π, be in contact at each moment with a certain fixed curve C' lying in Π (Fig. 267).

If the instantaneous motion of the curve C is an instantaneous rotation about the point of contact O, then its instantaneous motion is called a *rolling* motion of the curve C on C'.

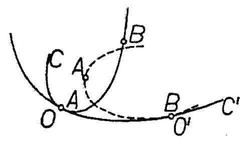

Fig. 267.

It follows from this that during a rolling motion the point of contact has a zero velocity. The point of contact is the instantaneous centre of rotation.

If the instantaneous motion of the curve C is an advancing motion, then its instantaneous motion is called a *sliding* motion of the curve C on C'.

The velocity of the advancing motion during sliding is equal to the velocity of the point of contact.

In the general case, the instantaneous motion of the curve C can be considered as the composition of two motions: a rotation about the point of contact O, and an advancing motion with a velocity of the point of contact.

Therefore: *an instantaneous motion is the composition of a rolling motion and of a sliding motion.*

It can be proved that during the rolling of curve C on curve C', the points of contact describe arcs of equal length on both curves (arcs OO' and AB in the Fig. 267). For example, when a circle rolls along a line l, the distance between the points of contact after one complete revolution is equal to the circumference of the circle.

The rolling and sliding of one surface on another is defined analogously.

Namely, let a surface Σ move in such a way that it is constantly tangent to a certain fixed surface Σ'.

If the point of contact has a zero velocity, then we say that the instantaneous motion of the surface Σ is a *rolling* motion on the surface Σ'.

During rolling the instantaneous motion is a rotation about an axis passing through the point of contact. In particular, if the surfaces Σ and Σ' are cylinders or cones, tangent along their generatrices, then during rolling the instantaneous axis of rotation is the generatrix along which the surfaces are in contact.

If the instantaneous motion of the surface Σ is an advancing motion, we say that the instantaneous motion of the surface Σ is a *sliding* motion on the surface Σ'.

Curve of instantaneous centres. Let a figure K move in the plane Π. At each instant let us consider the instantaneous centre of rotation in the plane Π. These centres will describe in Π a certain curve C', called the *fixed curve of instantaneous centres.*

At each instant let us now take under consideration on the figure K a point which coincides with the instantaneous centre of rotation at the given moment. These points will describe on the figure K a certain curve C which moves together with the figure. This curve is called the *moving curve of instantaneous centres.*

In general, the curves of instantaneous centres: the moving curve C and the fixed curve C' are tangent to each other at each instant, and their point of contact is the instantaneous centre of rotation. The moving curve therefore moves together with the figure K in such a way that its instantaneous motion is at each instant a rotation about its point of contact with the fixed curve C'.

Hence: *in a plane motion the moving curve of instantaneous centres rolls on the fixed curve of instantaneous centres.*

Cone of instantaneous axes. Let a body K rotate about the point O. Let us consider in space an instantaneous axis of rotation at each instant. These axes will generate a certain conical surface Σ' with vertex at O; it is called the *fixed cone of instantaneous axes.*

Let us next consider in the body K at each instant that line which coincides with the instantaneous axis of rotation at a given instant. The surface Σ which these lines form is called the *moving cone of instantaneous axes.*

The surface Σ moves together with the body and is in general tangent to Σ'.

Therefore: *during a rotation of a body about a point the moving surface of instantaneous axes rolls on the fixed surface of instantaneous axes.*

Surface of central axes. Let a body K move arbitrarily in space. Let us consider in space the axis of twist, i. e. the central axis at each instant. These axes form a certain surface Σ', called the *fixed surface of central axes.*

Let us also consider in the body at each moment a line which coincides with the central axis at a given moment. The surface Σ generated by these axes in the body is called the *moving surface of central axes.*

In general, the moving surface of central axes is tangent to the fixed surface at each moment along the central axis. The instantaneous motion of the moving surface is a twist about the axis of tangency.

Example 1. A circle K of radius r, whose centre O moves with a uniform velocity $\mathbf{u}$, rolls along the line l (Fig. 268). Since the point of contact S is the centre of instantaneous rotation, denoting by ω the instantaneous angular velocity, we have $u = r\omega$ (where $u = |\mathbf{u}|$), whence

$$\omega = u / r. \tag{1}$$

The point A lying on the diameter SO has a velocity equal in magnitude to $2r\omega = 2u$; the point A therefore has the velocity $2\mathbf{u}$.

The wheels of a railway carriage have flanges on the inner sides of the rails in order to prevent the carriage from derailing. Therefore the lowest point of a railway carriage wheel (in the figure the point B) is below the point of tangency S. Its instantaneous velocity consequently has a sense opposite to the velocity of the train (i. e. that of the centre O of the wheel) and is $SB \cdot \omega = u \cdot SB / r$.

For example, if a train moves with a velocity of 50 km/h, then at each instant there exist points on the wheels of the train having instantaneous velocities of 100 km/h (the point A in the figure), and even such that move in a direction opposite to that of the train (e. g. the point B).

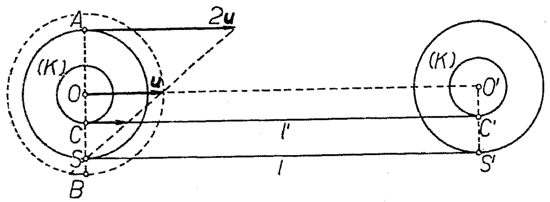

Fig. 268.

On the circle K let us consider a circle K' with centre O and radius $OC = r'$. The circle remains constantly tangent to the line $l' \parallel l$. Since the point C is not an instantaneous centre of rotation, this circle does not roll on l'. The motion of the wheel K' is a composition of the rolling motion and the sliding motion on the line l'. The rolling motion takes place with an angular velocity ω, and the sliding motion has the velocity of the point C, i. e. $SC \cdot \omega$.

Since the entire wheel K of radius r rolls on l, the segment SS' described by the points of contact during one revolution of the wheel is equal to the circumference of the wheel, i. e. $2\pi r$. The corresponding segment CC' for the wheel K is also $2\pi r$. It is not equal to the circumference $2\pi r'$ of the wheel K', since the motion of the circle K along the line l' is not a rolling motion, but a composition of the rolling motion and the sliding motion along this line.

Example 2. A circle (C) of radius r rolls on a fixed circle (C') of radius $2r$. The circle (C) is within the circle (C') (Fig. 269). Determine the paths of the points of circle (C).

Let A be an arbitrary point on the circumference of the circle (C). Since the point of contact S of both circles is an instantaneous centre of rotation, the velocity $\mathbf{v}$ of the point A is perpendicular to SA. The direction of the velocity of the point A therefore passes through the point O which lies on the circle (C) and on the diameter passing through S. Since $SO = 2r$, O is the centre of the circle (C'). The velocity of the point A is therefore constantly directed towards the fixed point O. Consequently the

point A moves along the line OA. Hence the points on the circumference of the circle (C) move along the diameters of the circle (C').

Now let P be an arbitrary point within the circle (C). Let us pass an arbitrary chord AB through P. The points A and B move along the lines l and l' passing through O. Since the segment AB does not change its length, by a well-known theorem from analytic geometry the point P describes an ellipse.

Example 3. A cone of revolution rolls on a plane Π (Fig. 270). The instantaneous motion of the cone is therefore an instantaneous rotation about a generatrix along which it is tangent to the plane Π.

The vertex W of the cone always lies on the instantaneous axis of rotation. Hence it constantly has a zero velocity, i. e. it remains at rest.

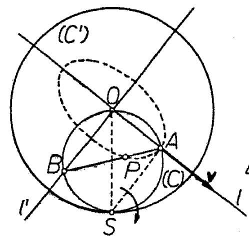

Fig. 269.

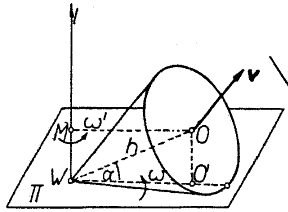

Fig. 270.

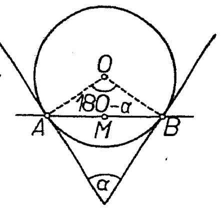

Fig. 271.

Let us denote by α the angle between the generatrices and the altitude h of the cone, by O the centre of the base of the cone, and by O' the projection of O on the plane Π. Since $O'O = h \sin\alpha = \text{const}$, the point O moves in a plane parallel to Π.

The distance of the point O from the line l, perpendicular to Π at the point W, is $MO = WO' = h\cos\alpha = \text{const}$. It follows from this that the point O moves in a circle with centre at M in a plane perpendicular to l, and hence it rotates about the line l.

Let ω be the angular velocity of the rolling cone, and ω' the angular velocity of the centre O during the rotation about the axis l. Finally, let $\mathbf{v}$ be the velocity of the point O. We therefore have $|\mathbf{v}| = O'O \cdot \omega$ and $|\mathbf{v}| = MO \cdot \omega'$, whence $\omega' = \omega \cdot O'O / MO = \omega \cdot O'O / WO'$, and hence

$$\omega' = \omega \tan\alpha. \qquad (2)$$

Example 4. A sphere rolls in a trough formed by two planes (Fig. 271).

Since the points of contact A and B have a zero velocity, the instantaneous axis of rotation is the line AB.

Let us denote by r the radius of the sphere, by α the angle between the planes of the trough, by ω the angular velocity of the rolling motion, and finally by $\mathbf{v}$ the velocity of the centre O of the sphere.

The distance of the centre of the sphere from the axis of rotation is $OM = r \sin\frac{1}{2}\alpha$. Putting $v = |\mathbf{v}|$ we therefore obtain

$$v = r\omega \sin\tfrac{1}{2}\alpha. \tag{3}$$

Example 5. A segment AB of length d moves in such a way that its ends remain constantly on the lines l and m, perpendicular to each other and intersecting at the point M (Fig. 272).

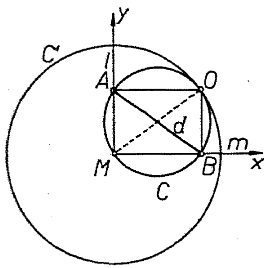

Fig. 272.

The centre of instantaneous rotation O is obtained by drawing perpendiculars l and m at the points A and B. Since $MO = AB = d$, the centres of instantaneous rotation form a circle C' with centre at M and radius d. The circle C' is therefore the fixed curve of instantaneous centres. Since the angle AOB is equal to $\frac{1}{2}\pi$ at every position of the segment AB, the moving curve of instantaneous centres will be the circle C of diameter AB (cf. example 2).

§ 9. Composition of motions of a body. Two simultaneous rotations. Let the instantaneous motion of a body K relative to the body K_1 (i. e. relative to the coordinate system attached rigidly to the body K_1) be a rotation about the axis l_1 with an angular velocity $\boldsymbol{\omega}_1$, and the instantaneous motion of the body K_1 relative to a body K' a rotation about the axis l_2 with an angular velocity $\boldsymbol{\omega}_2$. We then say that the body K makes, relative to the body K', two simultaneous instantaneous rotations about the axes l_1 and l_2 with angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$.

The instantaneous motion of the body K relative to the body K' is called the *resultant motion* of these two simultaneous rotations; we also say that it is *equivalent to the system of both rotations.*

Let us take a system of coordinates (ξ, η, ζ) in the body K_1, and a system of coordinates (x, y, z) in the body K' (Fig. 273). Let A be an arbitrary point of the body K. The velocity $\mathbf{v}$ of point A relative to the system (x, y, z) is the sum of its relative velocity $\mathbf{v}_r$ with respect to the system (ξ, η, ζ) and the velocity of transport $\mathbf{v}_t$:

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_t. \tag{1}$$

Since the instantaneous motion of the body K relative to the system (ξ, η, ζ) is a rotation about the axis l_1 with an angular velocity $\boldsymbol{\omega}_1$ (p. 323)

$$\mathbf{v}_r = \mathrm{Mom}_A \boldsymbol{\omega}_1. \tag{2}$$

The velocity of transport $\mathbf{v}_t$ of the point A is obtained by assuming that the point A is attached rigidly to the system (ξ, η, ζ). The system (ξ, η, ζ) rotates about the axis l_2 with an angular velocity $\boldsymbol{\omega}_2$; hence

$$\mathbf{v}_t = \mathrm{Mom}_A \boldsymbol{\omega}_2. \tag{3}$$

From (1), (2) and (3) we obtain

$$\mathbf{v} = \mathrm{Mom}_A \boldsymbol{\omega}_1 + \mathrm{Mom}_A \boldsymbol{\omega}_2. \tag{4}$$

As it is seen from this formula, the instantaneous velocity of the body K relative to K' is determined by the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. It is not

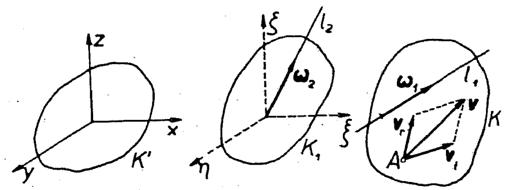

Fig. 273.

necessary to state, in addition, which of them is the velocity of rotation of the body K relative to K_1, and which the velocity of rotation of the body K_1 relative to K'.

Let us suppose that the axes l_1 and l_2 intersect at the point O. Let us consider the vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ with its origin at O. We have (p. 17)

$$\mathrm{Mom}_A \boldsymbol{\omega} = \mathrm{Mom}_A \boldsymbol{\omega}_1 + \mathrm{Mom}_A \boldsymbol{\omega}_2, \tag{5}$$

whence by (4)

$$\mathbf{v} = \mathrm{Mom}_A \boldsymbol{\omega}. \tag{6}$$

The instantaneous motion of the body K relative to K' is consequently an instantaneous rotation about an axis passing through the point O, with an angular velocity equal to the sum of the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ of the component rotations.

Therefore: *a system of two simultaneous instantaneous rotations about the axes which intersect at the point O is equivalent to a rotation about an axis passing through O, with an angular velocity equal to the angular velocities of the component rotations.*

Let us suppose now that the axes l_1 and l_2 are parallel, where $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \neq 0$. Then the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ have a resultant vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ for which the equalities (5) and (6) hold.

Therefore: *a system of two simultaneous rotations about parallel axes, with angular velocities whose sum is different from zero, is equivalent to a rotation about an axis parallel to the preceding axes; the position of this axis and the angular velocity of rotation about it is determined by the resultant of the angular velocities of the component rotations.*

Finally, let us suppose that the axes l_1 and l_2 are parallel, but $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = 0$, i. e. that the senses of the rotations are opposite, and the absolute values of the angular velocities are equal. In this case the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ form a couple. Since the moment of the couple is a constant vector, by (4) all the points of the body K have one and the same velocity equal to the moment of the couple $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$. The instantaneous motion of the body K relative to the body K' is consequently an advancing motion.

Therefore: *a system of two simultaneous instantaneous rotations about parallel axes with angular velocities equal in magnitude, but opposite in sense, is equivalent to an instantaneous advancing motion.*

Let us now pass to the general case. Let O be an arbitrary point of the body K. In virtue of the theorem on reduction (p. 24), the system of vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ is equipollent to a system composed of the vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ with its origin at O and a couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$ of moment equal to the moment of the system $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ with respect to O. For each point A of the body K we therefore have $\mathrm{Mom}_A\boldsymbol{\omega}_1 + \mathrm{Mom}_A\boldsymbol{\omega}_2 = \mathrm{Mom}_A\boldsymbol{\omega} + \boldsymbol{u}$, where $\boldsymbol{u}$ is the moment of the couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$. By (4) we then have

$$\boldsymbol{v} = \mathrm{Mom}_A\boldsymbol{\omega} + \boldsymbol{u}. \tag{7}$$

Consequently the instantaneous motion of the body K relative to K' is the composition of an advancing motion with a velocity $\boldsymbol{u}$ of the point O and a rotation with an angular velocity $\boldsymbol{\omega}$ about an axis passing through O.

Therefore: *a system of two simultaneous instantaneous rotations is equivalent to the composition of a rotation about an axis passing through an arbitrary point O of the body and an advancing motion with a velocity of the point O; the vector of angular velocity of the resultant motion is equal to the sum of the vectors of angular velocities of the component rotations.*

Composition of several simultaneous rotations. The results obtained can be generalized to the case of several simultaneous rotations. The resultant motion is defined in a manner similar to that for two rotations.

We shall therefore say, for instance, that the body K makes simultaneous rotations about the axes l_1, l_2, and l_3, with the velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_3$, relative to a certain body K', if the body K rotates about the axis l_1 with an angular velocity $\boldsymbol{\omega}_1$ relative to a certain body K_1, while K_1

rotates about the axis l_2 with an angular velocity $\boldsymbol{\omega}_2$ relative to a certain body K_2, and K_2 rotates about the axis l_3 with an angular velocity $\boldsymbol{\omega}_3$ relative to the body K'.

Similarly, the instantaneous motion of the body K relative to K' is defined as the resultant motion of the instantaneous rotations about the axes l_1, l_2, l_3, with the angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$.

Let a body K make several simultaneous rotations with the angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ As in the case of two rotations, one proves that the velocity $\boldsymbol{v}$ of an arbitrary point A of the body K relative to the fixed body K' is

$$\boldsymbol{v} = \mathrm{Mom}_A\boldsymbol{\omega}_1 + \mathrm{Mom}_A\boldsymbol{\omega}_2 + \ldots \tag{8}$$

The velocity $\boldsymbol{v}$ of the point A is the total moment of the system of angular velocity vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$:

$$\boldsymbol{v} = \mathrm{Mom}_A(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots). \tag{9}$$

Therefore, if the systems of angular velocities

$$\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots \quad \text{and} \quad \boldsymbol{\omega}_1', \boldsymbol{\omega}_2', \ldots$$

for two systems of simultaneous rotations are equipollent (p. 22), then the resultant motions of these rotations are the same.

This enables us to interpret the theorems of chapter I on systems of vectors as theorems on systems of simultaneous rotations.

The theorem on the reduction of a system of vectors (p. 26) can therefore be formulated as follows:

A system of several simultaneous rotations with angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ is the composition of an advancing motion with a velocity of an arbitrary point O of the body and a rotation with an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots$ about an axis passing through O.

According to this interpretation the theorems 1—4 on p. 26 will assume the form:

1. *A system of simultaneous rotations with angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ about axes passing through a point O is equivalent to one rotation with an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots$ about an axis passing through O.*

2. *A system of simultaneous rotations with angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ about parallel axes (about axes lying in one plane Π) is equivalent to a rotation about a parallel axis (about an axis lying in the plane Π), when $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots \neq 0$, and to an advancing motion, when $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots = 0$.*

According to the definition of a parameter of a system of vectors (p. 20), the parameter of a system of angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ is expressed as the scalar product $K = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots) \cdot \mathrm{Mom}_A(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots)$, where A is an arbitrary point of the body. Denoting by $\boldsymbol{\omega}$ the sum of the angular velocities, we have by (9)

$$K = \boldsymbol{\omega} \cdot \mathbf{v}, \tag{10}$$

where $\mathbf{v}$ is the resultant velocity of the point A.

From theorem 4, p. 20, it follows that the scalar product $\boldsymbol{\omega} \cdot \mathbf{v}$ has a constant value. In particular, for $K = 0$ the system of simultaneous rotations is equivalent to a rotation or to an advancing motion (cf. table on p. 25).

As we know, every motion of a body is the composition of an advancing motion with a velocity $\mathbf{u}$ of an arbitrary point O of the body and of a rotation with an angular velocity $\boldsymbol{\omega}$ (p. 333). Therefore the instantaneous motion can be represented as the composition of a rotation $\boldsymbol{\omega}$ and of a couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$ of moment equal to $\mathbf{u}$.

Suppose that the motion of the body has been represented in another way as the composition of the rotation $\boldsymbol{\omega}_1$ and of the couple of rotations $\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$. Since the systems $\boldsymbol{\omega}, \boldsymbol{\omega}', -\boldsymbol{\omega}'$, and $\boldsymbol{\omega}_1, \boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$, are equipollent, because they represent the same resolution of the velocities in the body, their sums are equal, i. e. $\boldsymbol{\omega} = \boldsymbol{\omega}_1$. Thus, we also obtain in this way the theorem (proved on p. 333), according to which the instantaneous axes of rotation are parallel and the instantaneous angular velocities are equal for of all representations of the instantaneous motion.

Let us notice in this connection that the parameter of the system $\boldsymbol{\omega}, \boldsymbol{\omega}', -\boldsymbol{\omega}'$ is $K = \boldsymbol{\omega} \cdot \mathrm{Mom}_O(\boldsymbol{\omega}', -\boldsymbol{\omega}') = \boldsymbol{\omega} \cdot \mathbf{u}$. Therefore, if $\boldsymbol{\omega} \cdot \mathbf{u} = 0$, then the instantaneous motion is an instantaneous rotation.

In particular, if $\boldsymbol{\omega} \perp \mathbf{u}$, and hence *if the instantaneous axis of rotation is perpendicular to the velocity of the advancing motion, then the instantaneous motion is equivalent to an instantaneous rotation.*

On p. 27 we proved that every system of vectors is equivalent to a wrench. From the definition of a wrench it follows that the instantaneous motion of a rigid body is a twist. We have therefore obtained a new proof of the theorem given on p. 334.

Relative motion of a body. Let the instantaneous motions of the two bodies K_1 and K_2 relative to a fixed system of coordinates (x, y, z) be given. We shall determine the instantaneous motion of the body K_2 relative to K_1, i. e. relative to a moving system of coordinates (ξ, η, ζ) attached rigidly to the body K_1.

Let us denote by $\boldsymbol{\omega}_1$ the instantaneous angular velocity vector of the body K_1, and let us represent the advancing motion as the composition of a couple of rotations with angular velocities $\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$. Similarly, we shall represent the instantaneous motion of the body K_2 as the composition of the rotation $\boldsymbol{\omega}_2$ and of the couple of rotations $\boldsymbol{\omega}_2', -\boldsymbol{\omega}_2'$.

The absolute velocity $\mathbf{v}_a$ and the velocity of transport $\mathbf{v}_t$ of an arbitrary point A of the body K_2 are:

$$\mathbf{v}_a = \mathrm{Mom}_A(\boldsymbol{\omega}_2, \boldsymbol{\omega}_2', -\boldsymbol{\omega}_2'), \qquad \mathbf{v}_t = \mathrm{Mom}_A(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1', -\boldsymbol{\omega}_1').$$

Since $\mathbf{v}_r = \mathbf{v}_a - \mathbf{v}_t$ ((I), p. 57),

$$\mathbf{v}_r = \mathrm{Mom}_A(\boldsymbol{\omega}_2, \boldsymbol{\omega}_2', -\boldsymbol{\omega}_2', -\boldsymbol{\omega}_1, -\boldsymbol{\omega}_1', \boldsymbol{\omega}_1'). \tag{11}$$

The instantaneous motion of the body K_2 relative to the body K_1 is the composition of six simultaneous rotations. By the theorem on reduction (p. 24) the system of vectors referred to is equipollent to the vector $\boldsymbol{\omega}$ and the couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$. The vector $\boldsymbol{\omega}$ is the instantaneous angular velocity of the relative motion, and the moment of the couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$ is equal to the velocity of the instantaneous advancing motion.

Therefore: *the instantaneous relative motion of the body K_2 with respect to the body K_1 is obtained by adding the system of angular velocities with opposite senses, which determine the instantaneous motion of the body K_1, to the system of angular velocities, which determine the instantaneous motion of the body K_2.*

This theorem is usually stated more briefly by saying that the motion of the body K_2 relative to K_1 is obtained by c o m p o u n d i n g the instantaneous motion of the body K_2 with the instantaneous motion of the body K_1 with an opposite sense.

Example I. A cube makes two simultaneous twists about two skew edges. Let $\mathbf{u}_1, \boldsymbol{\omega}_1$ and $\mathbf{u}_2, \boldsymbol{\omega}_2$ denote the instantaneous angular velocities of the advancing and rotating motions of these twists. We shall determine the resultant twist.

Let us choose a system of coordinates in such a way that one of the edges (about which the twists take place) lies on the x-axis, and the other lies in the yz-plane and is parallel to the y-axis (Fig. 274).

The advancing motions can be replaced by the couples of rotations $\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2', -\boldsymbol{\omega}_2'$ of moments $\mathbf{u}_1$ and $\mathbf{u}_2$. Therefore the instantaneous resultant motion is equivalent to the composition of six simultaneous rotations:

$$\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1', \boldsymbol{\omega}_1, \quad \boldsymbol{\omega}_2', -\boldsymbol{\omega}_2', \boldsymbol{\omega}_2. \tag{12}$$

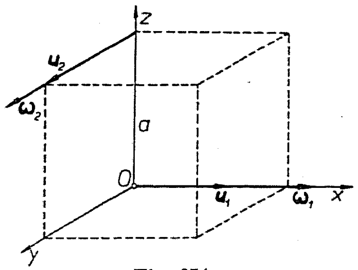

Fig. 274.

The sum of the system (12) is equal to $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$; consequently:

$$\omega_x = \omega_1, \quad \omega_y = \omega_2, \quad \omega_z = 0, \tag{13}$$

where ω_1 and ω_2 denote the corresponding projections of the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ on the x and y axes

In order to determine the moment of the rotations (12) with respect to the origin O of the system, let us note that the moments of the couples $\boldsymbol{\omega}_1'$, $-\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2'$, $-\boldsymbol{\omega}_2'$ are equal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, respectively. The moment $\boldsymbol{\omega}_1$ is zero because $\boldsymbol{\omega}_1$ lies on the x-axis. The moment $\boldsymbol{\omega}_2$ is perpendicular to the yz-plane and its projection on the x-axis is $a\omega_2$, where a is the length of an edge of the cube. Therefore, denoting the moment of the rotations (12) with respect to O by $\boldsymbol{u}$, we obtain:

$$u_x = u_1 + a\omega_2, \quad u_y = u_2, \quad u_z = 0, \tag{14}$$

where u_1 denotes the projection of $\boldsymbol{u}_1$ on the x-axis and u_2 the projection of $\boldsymbol{u}_2$ on the y-axis. Equations (14) obviously represent the projections of the velocity of the origin of the system.

The instantaneous motion is therefore the composition of an advancing motion with a velocity $\boldsymbol{u}$ and a rotation with a velocity $\boldsymbol{\omega}$ about an axis passing through O.

According to formula (I), p. 333, the velocity of an arbitrary point $A(x, y, z)$ is $\boldsymbol{v} = \boldsymbol{u} + \overline{OA} \times \boldsymbol{\omega}$, whence by (13) and (14)

$$v_x = u_1 + a\omega_2 - z\omega_2, \; v_y = u_2 + z\omega_1, \; v_z = x\omega_2 - y\omega_1. \tag{15}$$

In order to determine the central axis of twist we must find a point A such that its velocity has the direction of the vector $\boldsymbol{\omega}$, i. e. so that $\boldsymbol{v} = \lambda\boldsymbol{\omega}$ for a suitable numerical value of the factor λ. Hence the equations $v_x = \lambda\omega_1$, $v_y = \lambda\omega_2$, and $v_z = 0$, i. e. $v_x / \omega_1 = v_y / \omega_2$ and $v_z = 0$ must be satisfied. In view of (15) we therefore obtain the following equations of the central axis:

$$(u_1 + a\omega_2 - z\omega_2) / \omega_1 = (u_2 + z\omega_1) / \omega_2, \quad x\omega_2 - y\omega_1 = 0. \tag{16}$$

The velocity of the advancing screw motion is equal to the velocity of an arbitrary point of the central axis, e. g. of the point $A(0, 0, z)$; the value of z is calculated from the first of the equations (16), and then the velocity of the point A from formulae (15). We get:

$$v_x = k\omega_1, \quad v_y = k\omega_2, \quad v_z = 0,$$

where

$$k = [(u_1 + a\omega_2)\,\omega_1 + u_2\omega_2] / (\omega_1^2 + \omega_2^2) = \boldsymbol{u} \cdot \boldsymbol{\omega} / |\boldsymbol{\omega}|^2.$$

***Example* 2.** Steady precession. If a body moves in such a way that the instantaneous motion at each instant is the composition of two simultaneous rotations about two intersecting axes, of which the first l is fixed in space and the other m has a fixed position in the body, while the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ of these rotations are constant in magnitude, then the motion of the body is called *steady precession.*

Since by hypothesis the axes l and m intersect, the instantaneous motion of the body is a rotation with an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ about an axis p passing through the point of intersection of l and m (Fig. 275). Let us note that the instantaneous motion of the axis m is an instantaneous rotation about the axis l with an angular velocity $\boldsymbol{\omega}_1$ (for the rotation of the axis m about itself is left unconsidered). Consequently the axis m rotates about the axis l with a constant angular velocity ω_1. The point of intersection O of both axes is therefore fixed and the angle between the axes l and m is constant. It follows from this that the vector $\boldsymbol{\omega}$, and hence also the instantaneous axis of rotation p, make constant angles with the axes l and m.

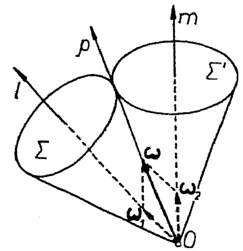

Fig. 275.

The axis p describes a cone of revolution Σ in space. The trace of the axis p in the body is also a cone of revolution Σ'.

Therefore: *the cones of instantaneous axes of rotation, fixed and moving, are cones of revolution with axes l and m.*

The earth's axis does not maintain a fixed direction in space, but describes a cone of revolution about an axis perpendicular to the ecliptic and passing through the centre of the earth. The time for a complete circuit of the earth's axis lasts about 26 000 years, and the angle between the earth's axis and the axis perpendicular to the ecliptic is $23\frac{1}{2}°$.

Let us take the centre of the earth as the origin of a system of coordinates moving with an advancing motion, and let us give the z-axis a direction perpendicular to the ecliptic. The x and y axes will then lie in the ecliptic. With respect to this system of coordinates the earth executes a steady precessional motion. The axis fixed in space is the z-axis and the axis fixed in the body is the earth's axis.

***Example* 3.** Two circles K_1 and K_2, lying in the plane Π, rotate about their centres with angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. Determine the instantaneous relative motion of the circle K_2 with respect to K_1 (Fig. 276).

Let $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ denote the angular velocity vectors, obviously perpendicular to Π (Fig. 277). The instantaneous motion of the circle K_2 relative to K_1 is the composition of two simultaneous rotations $\boldsymbol{\omega}_2$ and $-\boldsymbol{\omega}_1$.

If $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 = 0$, then the instantaneous relative motion of the circle K_2 is an advancing motion with a velocity $\boldsymbol{u}$ equal to the moment of the couple $(\omega_2, -\omega_1)$. Denoting by d the distance between the centres, we have $|\boldsymbol{u}| = d\omega_1 = d\omega_2$. The velocity $\boldsymbol{u}$ is perpendicular to the line O_1O_2 joining the centres of the circles K_1 and K_2.

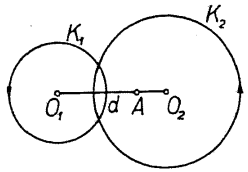

Fig. 276.

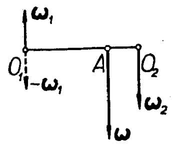

Fig. 277.

If $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \neq 0$, then the vectors $\boldsymbol{\omega}_2$ and $-\boldsymbol{\omega}_1$ have a resultant $\boldsymbol{\omega} = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1$ whose origin is at A lying on the line O_1O_2 at the point with respect to which the moment of the system $\boldsymbol{\omega}_2, -\boldsymbol{\omega}_1$ is zero. The instantaneous relative motion of the circle K_2 is therefore an instantaneous rotation about the point A with an angular velocity $\boldsymbol{\omega}$.

If $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ have opposite senses (as in Fig. 277), then denoting by ω_1, ω_2, and ω, the absolute values of the angular velocities, we obtain:

$$\omega = \omega_1 + \omega_2, \quad O_1A = O_1O_2 \cdot \omega_2 / (\omega_1 + \omega_2).$$

§ 10. Analytic representation of the motion of a rigid body. Instantaneous angular velocity. Let us suppose that we are considering the motion of a rigid body relative to the system of coordinates (x, y, z). Let us choose a system of coordinates (ξ, η, ζ) with origin at M and attached rigidly to the body. The position of the body relative to the system (x, y, z) will be determined if the position of the system (ξ, η, ζ) is given, i. e. the coordinates x_0, y_0, z_0, of the point M and the angles $\alpha_1, \alpha_2, \alpha_3$, $\beta_1, \beta_2, \beta_3$, and $\gamma_1, \gamma_2, \gamma_3$, which the axes ξ, η, ζ make with the axes x, y, z.

Let A be an arbitrary point of the body. Let us denote its coordinates with respect to the moving system by ξ, η, ζ and with respect to the fixed system by x, y, z.

Knowing the coordinates ξ, η, ζ and the position of the moving frame, we can determine the coordinates x, y, z by means of the formulae (II), p. 54. If $\cos\alpha_i$, $\cos\beta_i$, and $\cos\gamma_i$, are denoted by a_i, b_i, and c_i (where $i = 1, 2, 3$), then these formulae will assume the form:

$$x = x_0 + a_1\xi + a_2\eta + a_3\zeta, \quad y = y_0 + b_1\xi + b_2\eta + b_3\zeta, \qquad z = z_0 + c_1\xi + c_2\eta + c_3\zeta. \tag{1}$$

Let $\boldsymbol{v}$ be the velocity of the point A relative to the system (x, y, z). Since by hypothesis the system (ξ, η, ζ) is attached rigidly to the body, the coordinates ξ, η, ζ of the point A are constant (independent of the time). Differentiating (1) (and remembering that $a_1, a_2, \ldots, c_3$ are functions of the time t), we obtain:

$$\begin{aligned} v_x &= \dot{x} = \dot{x}_0 + \dot{a}_1\xi + \dot{a}_2\eta + \dot{a}_3\zeta, \\ v_y &= \dot{y} = \dot{y}_0 + \dot{b}_1\xi + \dot{b}_2\eta + \dot{b}_3\zeta, \\ v_z &= \dot{z} = \dot{z}_0 + \dot{c}_1\xi + \dot{c}_2\eta + \dot{c}_3\zeta. \end{aligned} \tag{2}$$

From analytic geometry it is known that:

$$a_1^2 + a_2^2 + a_3^2 = 1, \quad b_1^2 + b_2^2 + b_3^2 = 1, \quad c_1^2 + c_2^2 + c_3^2 = 1, \tag{3}$$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0, \quad a_1a_3 + b_1b_3 + c_1c_3 = 0, \qquad a_2a_3 + b_2b_3 + c_2c_3 = 0. \tag{4}$$

Differentiating equations (3) and (4), we obtain:

$$a_1\dot{a}_1 + a_2\dot{a}_2 + a_3\dot{a}_3 = 0, \quad b_1\dot{b}_1 + b_2\dot{b}_2 + b_3\dot{b}_3 = 0, \qquad c_1\dot{c}_1 + c_2\dot{c}_2 + c_3\dot{c}_3 = 0, \tag{5}$$

$$\begin{aligned} a_1\dot{a}_2 + b_1\dot{b}_2 + c_1\dot{c}_2 &= -\dot{a}_1a_2 - \dot{b}_1b_2 - \dot{c}_1c_2, \\ a_1\dot{a}_3 + b_1\dot{b}_3 + c_1\dot{c}_3 &= -\dot{a}_1a_3 - \dot{b}_1b_3 - \dot{c}_1c_3, \\ a_2\dot{a}_3 + b_2\dot{b}_3 + c_2\dot{c}_3 &= -\dot{a}_2a_3 - \dot{b}_2b_3 - \dot{c}_2c_3. \end{aligned} \tag{6}$$

Let $\boldsymbol{\omega}$ denote a vector whose projections on the axes of the system (ξ, η, ζ) are expressed by the formulae:

$$\omega_\xi = a_2\dot{a}_3 + b_2\dot{b}_3 + c_2\dot{c}_3, \quad \omega_\eta = a_3\dot{a}_1 + b_3\dot{b}_1 + c_3\dot{c}_1, \qquad \omega_\zeta = a_1\dot{a}_2 + b_1\dot{b}_2 + c_1\dot{c}_2. \tag{7}$$

Let us form the projections v_ξ, v_η, v_ζ of the velocity $\boldsymbol{v}$ on the axes ξ, η, ζ; we get $v_\xi = a_1v_x + b_1v_y + c_1v_z$, whence by substituting the values of v_x, v_y, v_z, from formulae (2):

$$v_\xi = (a_1\dot{x}_0 + b_1\dot{y}_0 + c_1\dot{z}_0) + (a_1\dot{a}_1 + b_1\dot{b}_1 + c_1\dot{c}_1)\,\xi + {} + (a_1\dot{a}_2 + b_1\dot{b}_2 + c_1\dot{c}_2)\,\eta + (a_1\dot{a}_3 + b_1\dot{b}_3 + c_1\dot{c}_3)\,\zeta.$$

The coefficient of ξ is equal to zero by (5). The coefficients of η and ζ are by (6) and (7) equal to ω_ζ and $-\omega_\eta$, respectively. Consequently

$$v_\xi = (a_1\dot{x}_0 + b_1\dot{y}_0 + c_1\dot{z}_0) + \omega_\zeta\eta - \omega_\eta\zeta. \tag{8}$$

For the projections of the velocity $\boldsymbol{u}$ of the point M on the axes

x, y, z we get $u_x = \dot{x_0}$, $u_y = \dot{y_0}$, $u_z = \dot{z_0}$, and for the projections of this velocity on the axes ξ, η, ζ

$$u_\xi = a_1 u_x + b_1 u_y + c_1 u_z = a_1 \dot{x_0} + b_1 \dot{y_0} + c_1 \dot{z_0}, \qquad \text{etc.}$$

In virtue of (8), therefore, we obtain for v_ξ (and similarly for v_η, v_ζ) the formulae:

$$v_\xi = u_\xi + \omega_\zeta \eta - \omega_\eta \zeta, \quad v_\eta = u_\eta + \omega_\xi \zeta - \omega_\zeta \xi, \quad v_\zeta = u_\zeta + \omega_\eta \xi - \omega_\xi \eta. \tag{9}$$

From formulae (9) it follows that the velocity $\mathbf{v}$ is the sum of two velocities: $\mathbf{v} = \mathbf{u} + \mathbf{w}$, of which the first is the velocity of the point M, and the other has the projections on the axes ξ, η, ζ:

$$w_\xi = \omega_\zeta \eta - \omega_\eta \zeta, \quad w_\eta = \omega_\xi \zeta - \omega_\zeta \xi, \quad w_\zeta = \omega_\eta \xi - \omega_\xi \eta. \tag{10}$$

Comparing these formulae with formulae (V), p. 46, we see that $\mathbf{w}$ is the velocity the point A would have if the body were rotating with an angular velocity $\boldsymbol{\omega}$ about an axis passing through M. It follows from this that the vector $\boldsymbol{\omega}$ defined by formulae (7) is the instantaneous angular velocity vector.

Remark. Formulae (7) become simpler if we assume that the coordinate systems (x, y, z) and (ξ, η, ζ) coincide at a given moment t. Under this assumption we have $\alpha_1 = \beta_2 = \gamma_3 = 0$, and the remaining angles are equal to $\frac{1}{2}\pi$. Hence $a_1 = b_2 = c_3 = 1$, and the remaining cosines are zero. Then by (7) and (6):

$$\omega_\xi = -\dot{c_2}, \quad \omega_\eta = -\dot{a_3}, \quad \omega_\zeta = -\dot{b_1}. \tag{11}$$

Since $c_2 = \cos\gamma_2$, $\dot{c_2} = -(\sin\gamma_2)\dot{\gamma_2} = -\dot{\gamma_2}$; consequently $\omega_\xi = \dot{\gamma_2}$. Proceeding similarly, we obtain

$$\omega_\xi = \omega_x = \dot{\gamma_2}, \quad \omega_\eta = \omega_y = \dot{\alpha_3}, \quad \omega_y = \omega_z = \dot{\beta_1}.$$

Therefore: *if the axes of a moving system of coordinates coincide at the instant t with the axes of a fixed coordinate system, then the projections of the instantaneous angular velocity vector on the axes of the moving system are the derivatives of the angles* $\measuredangle \eta z$, $\measuredangle \zeta x$, *and* $\measuredangle \xi y$.

Central axis. In order to obtain the central axis it is necessary to determine the points whose velocities have the direction of the vector $\boldsymbol{\omega}$. Therefore, if the point $A(\xi, \eta, \zeta)$ lies on the central axis, then its velocity is equal to $\mathbf{v} = \lambda\boldsymbol{\omega}$, where λ is a certain constant. By substituting in equations (9), we consequently obtain the equations of the central axis:

$$\lambda\omega_\xi = u_\xi + \omega_\zeta \eta - \omega_\eta \zeta, \quad \lambda\omega_\eta = u_\eta + \omega_\xi \zeta - \omega_\zeta \xi, \quad \lambda\omega_\zeta = u_\zeta + \omega_\eta \xi - \omega_\xi \eta, \tag{12}$$

whence

$$\frac{u_\xi + \omega_\zeta \eta - \omega_\eta \zeta}{\omega_\xi} = \frac{u_\eta + \omega_\xi \zeta - \omega_\zeta \xi}{\omega_\eta} = \frac{u_\zeta + \omega_\eta \xi - \omega_\xi \eta}{\omega_\zeta}. \tag{13}$$

In order to obtain the velocity $\mathbf{v}$ of the instantaneous advancing motion during a twist, let us multiply both sides of equations (12) by $\omega_\xi, \omega_\eta, \omega_\zeta$, respectively. Adding them, we obtain

$$\lambda(\omega_\xi^2 + \omega_\eta^2 + \omega_\zeta^2) = u_\xi \omega_\xi + u_\eta \omega_\eta + u_\zeta \omega_\zeta = \mathbf{u} \cdot \boldsymbol{\omega}. \tag{14}$$

Since $\mathbf{v} = \lambda\boldsymbol{\omega}$, putting $\omega = |\boldsymbol{\omega}| = \sqrt{\omega_\xi^2 + \omega_\eta^2 + \omega_\zeta^2}$, we get by (14)

$$\mathbf{v} = \frac{\mathbf{u} \cdot \boldsymbol{\omega}}{\omega^2}\,\boldsymbol{\omega},$$

whence

$$|\mathbf{v}| = |\mathbf{u} \cdot \boldsymbol{\omega}| \,/\, \omega. \tag{15}$$

If $\mathbf{u} \perp \boldsymbol{\omega}$, then $\mathbf{u} \cdot \boldsymbol{\omega} = 0$, whence by (15) $\mathbf{v} = 0$.

Therefore: *if the velocity of an instantaneous advancing motion is perpendicular to the instantaneous axis of rotation, then the instantaneous motion is an instantaneous rotation about the central axis.*

Plane motion. Let us suppose that we are considering the motion of a figure in the plane Π. Let us select a fixed coordinate system (x, y) as well as a moving system (ξ, η) with origin at M and rigidly attached to the figure. Denote by x_0, y_0 the coordinates of the point M and by φ the angle between the axes ξ and x. Finally, let A be an arbitrary point of the figure having the coordinates x, y with respect to the fixed frame, and ξ, η with respect to the moving frame. The relations among these coordinates are given by formulae (II), p. 54:

$$x = x_0 + \xi\cos\varphi - \eta\sin\varphi, \quad y = y_0 + \xi\sin\varphi + \eta\cos\varphi. \tag{16}$$

Let $\mathbf{v}$ be the velocity of the point A. Since the point A is attached rigidly to the moving system, ξ and η are constants. Differentiating (16), we obtain

$$v_x = \dot{x} = \dot{x_0} - (\xi\sin\varphi + \eta\cos\varphi)\dot{\varphi}, \quad v_y = \dot{y} = \dot{y_0} + (\xi\cos\varphi - \eta\sin\varphi)\dot{\varphi}. \tag{17}$$

Comparing (17) with formulae (16) and putting

$$\omega = \dot{\varphi}, \tag{18}$$

we get:

$$v_x = \dot{x_0} - (y - y_0)\,\omega, \quad v_y = \dot{y_0} + (x - x_0)\,\omega. \tag{19}$$

Therefore: *the velocity* $\mathbf{v}$ *is the sum of two velocities:* $\mathbf{v} = \mathbf{u} + \mathbf{w}$, *of* *which the first is the velocity of the point M, and the other has the projections*:

$$w_x = -(y - y_0)\,\omega, \quad w_y = (x - x_0)\,\omega \qquad (20)$$

on the axes of the fixed system.

We see from this that $\mathbf{w}$ is the velocity the point A would have if the figure were rotating about the point $M(x_0, y_0)$ with an angular velocity ω (where the positive sense of the rotation agrees with the positive sense of the angle). Consequently $\omega = \varphi^{\cdot}$ is the instantaneous angular velocity.

In order to obtain the instantaneous centre of rotation it is necessary to determine the point whose velocity $\mathbf{v} = 0$. Denoting the coordinates of the instantaneous centre of rotation by x' and y', we obtain from (4):

$$0 = x_0^{\cdot} - (y' - y_0)\,\omega, \quad 0 = y_0^{\cdot} + (x' - x_0)\,\omega,$$

whence

$$x' = x_0 - y_0^{\cdot} / \omega, \quad y' = y_0 + x_0^{\cdot} / \omega. \qquad (21)$$

Euler's angles. In some considerations it is convenient to define the position of the body by means of the so-called Euler's angles.

Let a body rotate about the point O. Let us choose two systems of coordinates with a common origin O: a fixed (x, y, z) and a moving (ξ, η, ζ) attached rigidly to the body (Fig. 278). The position of the moving system is determined as follows.

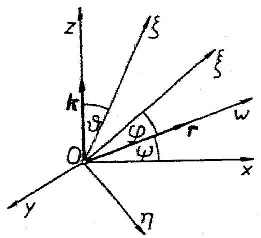

Fig. 278.

Let w denote the line of intersection of the planes xy and $\xi\eta$. This line is called the *line of nodes*.

The line w is perpendicular to the axes z and ζ. Let us give the line w such a sense that the system of axes (ζ, z, w) is left-handed, i. e. agrees with the assumed senses of the systems (x, y, z) and (ξ, η, ζ).

Let us denote by ϑ the angle through which it is necessary to rotate the z-axis about the w-axis in the positive direction (i. e. from right to left), in order that the positive direction of the z-axis coincides with the positive direction of the ζ-axis. Similarly, we denote by φ the angle through which it is necessary to rotate the w-axis about the ζ-axis in the positive direction, in order that the positive direction of the w-axis coincides with the positive direction of the ξ-axis. Finally, we denote by ψ the angle through which it is necessary to rotate the x-axis about the z-axis in the positive direction, in order that the positive directions of the x and w-axes coincide.

The angles ϑ, φ, ψ, are called *Euler's angles*.

These angles define the positions of the axes ξ, η, ζ, with the exception of the case when $\vartheta = 0$ or $\vartheta = \pi$, for then the position of the w-axis, and hence also the angles φ, ψ, are undefined.

However, if $\vartheta \neq 0$ and $\vartheta \neq \pi$, then the angle ψ defines the position of the w-axis in the xy-plane. Knowing the position of the w-axis already, we obtain the position of the ζ-axis by rotating the z-axis through the angle ϑ (in the positive direction) about the w-axis. Finally, we obtain the ξ-axis by rotating the w-axis about the ζ-axis in the positive direction through the angle φ.

Euler's angles vary between the following limits:

$$0 < \vartheta < \pi, \quad 0 \leqq \varphi \leqq 2\pi, \quad 0 \leqq \psi \leqq 2\pi.$$

The instantaneous motion of the system (ξ, η, ζ) is a rotation about a certain axis. Let $\boldsymbol{\omega}$ be its instantaneous angular velocity vector. Let us resolve $\boldsymbol{\omega}$ into three component vectors $\mathbf{o}_z$, $\mathbf{o}_w$, and $\mathbf{o}_\zeta$, in the direction of the axes z, w, and ζ. Consequently

$$\boldsymbol{\omega} = \mathbf{o}_z + \mathbf{o}_w + \mathbf{o}_\zeta. \qquad (21)$$

Let us denote by o_z, o_w, and o_ζ, the coordinates of the corresponding vectors with respect to the axes z, w, and ζ. Let us note that if the system (ξ, η, ζ) rotates about the z-axis, then the angles ϑ and φ do not vary and the angle ψ is the angle of rotation. Therefore, during a rotation about the z-axis the magnitude of the instantaneous angular velocity of the system (ξ, η, ζ) is $\psi^{\cdot}$. Since $\mathbf{o}_z$ is the angular velocity vector for a rotation about the z-axis,

$$o_z = \psi^{\cdot} \quad \text{and similarly} \quad o_w = \vartheta^{\cdot}, \; o_\zeta = \varphi^{\cdot}. \qquad (22)$$

Consequently the derivatives $\psi^{\cdot}, \vartheta^{\cdot}, \varphi^{\cdot}$ define the instantaneous angular velocity vector $\boldsymbol{\omega}$ if we know the position of the system (ξ, η, ζ), i. e. the angles ψ, ϑ, and φ.

We shall now derive formulae for the projections of the vector on the axes of the system (ξ, η, ζ).

Let us choose the unit vectors $\mathbf{k}$ and $\mathbf{r}$ on the axes z and w. Consequently $\mathbf{o}_z = o_z\mathbf{k}$ and $\mathbf{o}_w = o_w\mathbf{r}$; hence according to (22):

$$\mathbf{o}_z = \psi^{\cdot}\mathbf{k}, \quad \mathbf{o}_w = \vartheta^{\cdot}\mathbf{r}. \qquad (23)$$

The projection of the vector $\mathbf{k}$ on the ζ-axis is $\cos\vartheta$. The vector $\mathbf{k}$ makes an angle $\frac{1}{2}\pi - \vartheta$ with the $\xi\eta$-plane; therefore the projection of $\mathbf{k}$

on the $\xi\eta$-plane has the length $\sin\vartheta$ and is perpendicular to w (since the z-axis is perpendicular to w). The projection of **k** makes angles $\frac{1}{2}\pi - \varphi$ and $\pi - \vartheta$ with the axes ξ and η; hence:

$$k_\xi = \sin\vartheta \sin\varphi, \quad k_\eta = -\sin\vartheta\cos\varphi, \quad k_\zeta = \cos\vartheta. \tag{24}$$

The projections of the vector **r** on the axes ξ and η are $\cos\varphi$ and $\sin\varphi$, while the projection on the ζ-axis is zero. Consequently:

$$r_\xi = \cos\varphi, \quad r_\eta = \sin\varphi, \quad r_\zeta = 0. \tag{25}$$

By (23) and (24) the projections of the vector $\mathbf{o}_z$ on the axes ξ, η, and ζ, are: $\psi^\cdot \sin\vartheta\sin\varphi$, $-\psi^\cdot \sin\vartheta\cos\varphi$, and $\psi^\cdot\cos\vartheta$, while the projections of the vector $\mathbf{o}_w$ on the axes ξ, η, and ζ, are by (23) and (25) equal to $\vartheta^\cdot\cos\varphi$, $\vartheta^\cdot\sin\varphi$, and 0, respectively; finally, the projections of the vector **o** on the axes ξ, η, and ζ, are obviously 0, 0 and $\varphi^\cdot$. From this we obtain in virtue of (21):

$$\omega_\xi = \vartheta^\cdot\cos\varphi + \psi^\cdot\sin\vartheta\sin\varphi, \quad \omega_\eta = \vartheta^\cdot\sin\varphi - \psi^\cdot\sin\vartheta\cos\varphi, \quad \omega_\zeta = \psi^\cdot\cos\vartheta + \varphi^\cdot. \tag{I}$$

Determining $\vartheta^\cdot$, $\psi^\cdot$, and $\varphi^\cdot$, from (I), we obtain:

$$\vartheta^\cdot = \omega_\xi\cos\varphi + \omega_\eta\sin\varphi, \quad \psi^\cdot = (\omega_\xi\sin\varphi - \omega_\eta\cos\varphi) / \sin\vartheta, \quad \varphi^\cdot = \omega_\zeta - (\omega_\xi\sin\varphi - \omega_\eta\cos\varphi)\cot\vartheta. \tag{II}$$

Knowing the projections ω_ξ, ω_η, and ω_ζ, of the angular velocity $\boldsymbol{\omega}$ on the axes ξ, η, and ζ, of the moving system at each instant, we can therefore determine ϑ, φ, and ψ, as a function of time by solving the system of differential equations (II).

Proceeding similarly, we obtain the following formulae for the projections of the angular velocity $\boldsymbol{\omega}$ on the axes x, y, z, of the fixed system:

$$\omega_x = \varphi^\cdot\sin\vartheta\sin\psi + \vartheta^\cdot\cos\psi, \quad \omega_y = \varphi^\cdot\sin\vartheta\cos\psi - \vartheta^\cdot\sin\psi, \quad \omega_z = \psi^\cdot + \varphi^\cdot\cos\vartheta, \tag{I'}$$

$$\vartheta^\cdot = \omega_x\cos\psi - \omega_y\sin\psi, \quad \varphi^\cdot = (\omega_x\sin\psi + \omega_y\cos\psi) / \sin\vartheta, \quad \psi^\cdot = \omega_z - (\omega_x\sin\psi + \omega_y\cos\psi)\cot\vartheta. \tag{II'}$$

Euler's angles in a steady precession. Let us assume that during the motion of a body we constantly have:

$$\vartheta^\cdot = 0, \quad \psi^\cdot = \text{const}, \quad \varphi^\cdot = \text{const}. \tag{26}$$

The instantaneous motion of the body is at each instant, therefore, the composition of two simultaneous rotations about the axes z and ζ with angular velocities $\psi^\cdot$ and $\varphi^\cdot$ of constant magnitudes. The z-axis has a fixed position in space, and the ζ-axis is rigidly attached to the body.

Under these conditions the motion of the body is a steady precession (cf. p. 349).

§ 11. Resolution of accelerations. Plane motion. If a plane figure rotates about a fixed point M with a variable angular velocity ω, then each point of the figure moves along the periphery of a circle. The acceleration $\mathbf{p}$ of an arbitrary point A of the figure is therefore the sum of the normal acceleration $\mathbf{p}_n$ directed towards M, and the tangential acceleration $\mathbf{p}_t$ perpendicular to MA. The normal and tangential accelerations are defined by the formulae (I) and (II), p. 45:

$$p_n = r\omega^2, \quad p_t = r\varepsilon, \tag{1}$$

where $r = MA$, and $\varepsilon = \omega^\cdot$ is the angular acceleration.

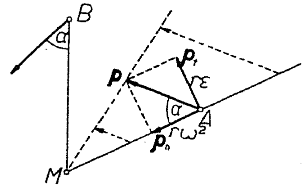

Fig. 279.

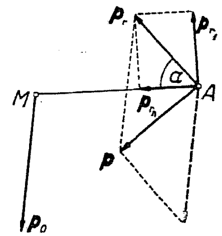

Fig. 280.

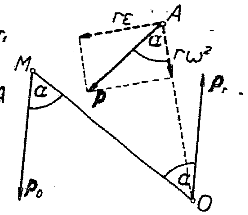

Fig. 281.

Let α denote the angle which the acceleration $\mathbf{p}$ of the point A makes with the line MA (Fig. 279). Then

$$\tan\alpha = p_t / p_n = \varepsilon / \omega^2. \tag{2}$$

We see from this that α is the same for all points.

Therefore: *in a rotation of a plane figure about a fixed point the accelerations of the points of the figure are proportional to their distances from the centre of rotation and make equal angles with the line joining these points with the centre of rotation.*

Let us assume now that the figure moves in the plane entirely arbitrarily (Fig. 280). Let us take an arbitrary point M of the figure as the origin of a system of coordinates (ξ, η), which moves with an advancing motion. The acceleration $\mathbf{p}$ of an arbitrary point A of the figure will be the sum of the accelerations: relative $\mathbf{p}_r$ and transport $\mathbf{p}_t$. The acceleration of transport is equal to the acceleration p_0 of the point M. The relative motion of the figure is a rotation about M with a variable angular velocity ω. We can therefore resolve the relative acceleration $\mathbf{p}_r$ into the sum of the (relative) accelerations: normal $\mathbf{p}_{r_n}$ and tangential $\mathbf{p}_{r_t}$. Consequently

$$\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_r = \mathbf{p}_0 + \mathbf{p}_{r_n} + \mathbf{p}_{r_t}. \tag{3}$$

Therefore: *in a plane motion the accelerations of the points of a figure are the sums of the acceleration of an arbitrary point M of the figure and of the accelerations which these points would have during a rotation of the figure about the point M* (as a fixed point) *with an angular velocity* (of the instantaneous rotation of the figure) ω *and with an angular acceleration* $\varepsilon = \omega^{\cdot}$.

If $\omega \neq 0$ or $\varepsilon \neq 0$, the relative acceleration $\mathbf{p}_r$ makes with the line MA an angle α defined by formula (2) and this angle is constant for all points of the figure.

In this case let us pass through the point M a line l making an angle α with the direction of the acceleration $\mathbf{p}_0$ of the point M (Fig. 281). The relative accelerations of the points lying on the line l will have the direction of the acceleration $\mathbf{p}_0$. Since the relative accelerations are proportional to the distances from M, we shall find on l a point O whose relative acceleration will be equal to $-\mathbf{p}_0$. The acceleration of the point O will therefore be zero.

The point O is called the *centre of instantaneous accelerations*.

On the other hand, if $\omega = 0$ and $\varepsilon = 0$, then the accelerations of all the points of the figure are equal (namely, equal to the acceleration of the point M).

Therefore: *if the accelerations of the points of a figure in plane motion are not equal, then there exists a point whose acceleration is equal to zero.*

The accelerations of the points are hence such as if the figure were rotating about the instantaneous centre of accelerations (as a fixed point) with an angular velocity ω and with an angular acceleration $\varepsilon = \omega^{\cdot}$.

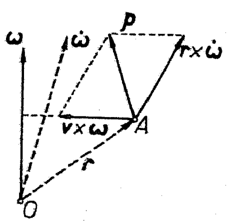

Fig. 282.

The accelerations of the points of the figure are proportional to the distances from the centre of instantaneous accelerations and make equal angles with the lines joining these points with the centre of instantaneous accelerations.

Motion in space. If a body rotates about a fixed point O with an instantaneous angular velocity $\boldsymbol{\omega}$ (Fig. 282), then the velocity of an arbitrary point A of the body is

$$\mathbf{v} = \mathbf{r} \times \boldsymbol{\omega}, \tag{4}$$

where $\mathbf{r} = \overline{OA}$. Calculating the derivative and denoting the acceleration of the point A by $\mathbf{p}$, we obtain

$$\mathbf{p} = \mathbf{r}^{\cdot} \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\omega}^{\cdot}.$$

Since O is by hypothesis a fixed point, $\mathbf{r}^{\cdot} = \mathbf{v}$; consequently

$$\mathbf{p} = \mathbf{v} \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\omega}^{\cdot}. \tag{5}$$

If a body rotates about a fixed axis with a constant angular velocity ω, then $\boldsymbol{\omega}^{\cdot} = 0$, whence by (5) $\mathbf{p} = \mathbf{v} \times \boldsymbol{\omega}$. On the other hand, each point then has a centripetal acceleration $\varrho|\boldsymbol{\omega}|^2$, where ϱ denotes the distance of the point from the axis of rotation. The product $\mathbf{v} \times \boldsymbol{\omega}$ is therefore the acceleration with which the points of the body would move if the body were rotating about a fixed axis with a constant angular velocity $\boldsymbol{\omega}$. The product $\mathbf{r} \times \boldsymbol{\omega}^{\cdot} = \mathrm{Mom}_A \boldsymbol{\omega}^{\cdot}$ represents the velocity which the point A would have if the body were rotating about a fixed axis with an angular velocity $\boldsymbol{\omega}^{\cdot}$. In general, the derivative $\boldsymbol{\omega}^{\cdot}$ has a direction different from $\boldsymbol{\omega}$. If $\boldsymbol{\omega}$ has a fixed direction (i. e. if the axis of rotation is fixed), then $\boldsymbol{\omega}^{\cdot}$ has the direction of $\boldsymbol{\omega}$; consequently $\mathbf{r} \times \boldsymbol{\omega}^{\cdot}$ has the direction of the velocity $\mathbf{v}$. In this case $\mathbf{r} \times \boldsymbol{\omega}^{\cdot}$ is the tangential acceleration and $\mathbf{v} \times \boldsymbol{\omega}$ the normal acceleration.

Let us now assume that the body moves arbitrarily in space. Then the resolution of the accelerations is obtained by taking an arbitrary point O of the body as the origin of the system of coordinates (ξ, η, ζ) moving with an advancing motion. The accelerations of the points will be the sums of the acceleration of transport (i. e. of the acceleration of the point O) and of the relative acceleration. The relative motion will be a rotation about the point O. Therefore the relative acceleration of an arbitrary point A is expressed according to (5) by the formula

$$\mathbf{p}_r = \mathbf{v}_r \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\omega}^{\cdot}, \tag{6}$$

where $\mathbf{r} = \overline{OA}$, and $\mathbf{v}_r$ denotes the relative velocity of the point A.

The interpretation of the products $\mathbf{v}_r \times \boldsymbol{\omega}$ and $\mathbf{r} \times \boldsymbol{\omega}^{\cdot}$ is similar to that used in the case of the rotation of a body about a fixed point. Denoting the velocity of the point A by $\mathbf{v}$, the acceleration and the velocity of the point O by $\mathbf{p}_0$ and $\mathbf{v}_0$, respectively, we obtain $\mathbf{v}_r = \mathbf{v} - \mathbf{v}_0$, and hence by (6) the acceleration of the point A is

$$\mathbf{p} = \mathbf{p}_0 + (\mathbf{v} - \mathbf{v}_0) \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\omega}^{\cdot}. \tag{7}$$