Hence in order to make an interplanetary journey in a rocket having together with its passengers a mass of one ton, it would be necessary to take along 160000 tons of fuel — which is obviously impossible. This shows that at the present state of technical sciences such a journey cannot be made. The matter would be pushed forward if w (the velocity of the escaping gases), which to-day is close to 2000 m/sec, could be markedly increased.

CHAPTER VI[1])

STATICS OF A RIGID BODY

I FREE BODY

§ I. Rigid body. A material body which despite the action of forces does not sustain any deformations (i. e. in which the mutual distances of the points of the body do not undergo a change) is called a *rigid body*.

Rigid bodies are not found in nature, since every body becomes deformed more or less under the influence of the action of forces. However, if some body under the influence of forces experiences only small deformations not exceeding a certain limit, then we can take as a model of such a body a rigid body, and the conclusions that we shall draw will be approximately in agreement with experience (provided the forces are not large). From this arises the great importance of the theory of a rigid body for practical applications.

We shall consider in turn statics, kinematics and dynamics of a rigid body.

In the theory of a rigid body we shall meet, in addition to rigid material solids, rigid material surfaces and lines (p. 168) as models of bodies in which one or two dimensions are small in comparison with those remaining. Examples of such bodies are plates, rods, wires, etc.

Rigid systems of material points. It often proves useful to look upon a rigid body as a collection (system) of a large number of material points. We assume then, that the material points act on each other with certain forces which ensure that the system of points is rigid, i. e. that the mutual distances of its points do not undergo a change. These forces are called *internal forces*.

We assume that Newton's law of action and reaction (p. 173) applies to internal forces, i. e. that two points act on each other with

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[1]) For the understanding of this chapter the information included in chapters I and III (from p. 69 to 75) and the theorems on centre of gravity in chapter IV, §§ 1, 2, 6, 7 and 8, are sufficient.

forces equal in magnitude and oppositely directed along the straight line joining these points.

In addition to internal forces, other forces, called *external forces*, can act on the points of a system.

Therefore, if a rigid body is considered as a rigid system of material points, then the forces acting on a rigid body are external forces acting on the points of the system

One might question whether it is admissible to consider a rigid body as a system of material points. This assumption can be justified, however, in the following manner: by subdividing the rigid body into very many small pieces and replacing each one of them by a material point of the same mass, we obtain a rigid system of material points representing the given body with considerable approximation.

Although the assumption that a rigid body is a collection of material points is not correct, we shall make use of it since it simplifies reasoning and leads to satisfactory results. Properly, however, the theory of a rigid body and the theory of rigid systems of material points should be treated separately.

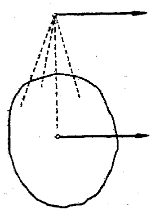


Fig. 148.

§ 2. Force. Point of application of a force. In the theory of a rigid body we assume that the point of application (origin) of the force acting on a rigid body may belong to the body or not; in the latter case we assume, however, that the point of application is rigidly attached to the body (we can imagine e. g., that the point of application is joined to the body by means of rigid massless rods) (Fig. 148). The action of the force will therefore be such as if the point of application belonged to the body.

Moment with respect to a point. If the force **P** acts at the point A whose coordinates are x, y, z, then the moment of the force with respect to the point O whose coordinates are x_0, y_0, z_0 has the projections:

$$M_x = P_y(z - z_0) - P_z(y - y_0); \quad M_y = P_z(x - x_0) - P_x(z - z_0), \quad M_z = P_x(y - y_0) - P_y(x - x_0), \tag{1}$$

on the axes of the system (p. 17, (I)).

In particular, if O is the origin of the system, i. e. if $x_0 = y_0 = z_0 = 0$, we get:

$$M_x = P_y z - P_z y, \quad M_y = P_z x - P_x z, \quad M_z = P_x y - P_y x. \tag{2}$$

From the definition of the moment (p. 15) it follows that

$$|\mathbf{M}| = |\mathbf{P}|h, \tag{3}$$

where h denotes the distance of the point O from the position of the force **P** (i. e. from the line on which **P** lies); this distance is called the *arm of the force P with respect to the point O*.

The moment of the force **P** with respect to the axis l is obtained by selecting an arbitrary point O on l and then forming the projection on the axis l of the moment of the force P with respect to O (p. 18).

If a sense is given on the line l, then the moment of the force **P** with respect to the axis l will be defined by giving its component with respect to this axis. This component is also called (if an error is precluded) the moment of the force **P** with respect to the axis l.

If the axis l passes through the point $O(x_0, y_0, z_0)$ and forms with the axes of the coordinate system the angles α, β, γ, then denoting by **M** the moment of the force **P** with respect to O, and by M_l the moment with respect to the axis l, we get

$$M_l = M_x \cos\alpha + M_y \cos\beta + M_z \cos\gamma \tag{4}$$

or, in virtue of (1),

$$M_l = P_x[(y - y_0)\cos\gamma - (z - z_0)\cos\beta] + P_y[(z - z_0)\cos\alpha - (x - x_0)\cos\gamma] + P_z[(x - x_0)\cos\beta - (y - y_0)\cos\alpha]. \tag{5}$$

In particular, if the point O, through with the axis l passes, is the origin of the coordinate system, i. e. if $x_0 = y_0 = z_0 = 0$, we obtain

$$M_l = P_x[y\cos\gamma - z\cos\beta] + P_y[z\cos\alpha - x\cos\gamma] + P_z[x\cos\beta - y\cos\alpha]. \tag{6}$$

The projections M_x, M_y, M_z, in formulae (1) and (4) are the moments of the force **P** with respect to axes parallel to the axes x, y, z, and passing through O, whereas in formulae (2) they are the moments with respect to the axes x, y, z.

If we denote the distance of the axis l from the force **P** by d (more exactly: from the position of the force **P**, i. e. the line on which **P** lies), and the angle between l and **P** by α (Fig. 149), we obtain (p. 18, formula (III))

$$|M_l| = |\mathbf{P}|d \sin\alpha. \tag{7}$$

If, in particular, $\mathbf{P} \perp l$, or $\alpha = \frac{1}{2}\pi$, then

$$|M_l| = |\mathbf{P}|d. \tag{8}$$

The sign of the moment M_l is obtained from the following rule:

$M_l > 0$ if the force **P** tries to turn the body about the axis l counterclockwise (with respect to a person whose feet are at an arbitrary point O of the axis l, and whose head points in the direction of the axis l); in the contrary case $M_l < 0$.

By means of the above rule and formula (7) we can determine M_l, knowing $|\mathbf{P}|$, d, and α.

If the force $\mathbf{P}$ and the point O lie in a certain plane Π (Fig. 150), then the moment $\mathbf{M}$ of the force $\mathbf{P}$ with respect to O is perpendicular to the plane Π. Consequently $\mathbf{M}$ is equal to the moment of the force $\mathbf{P}$ with respect to the axis l, perpendicular to Π and passing through O:

$$|\mathbf{M}| = |M_l|.$$

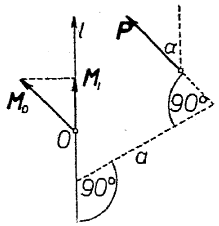


Fig. 149.

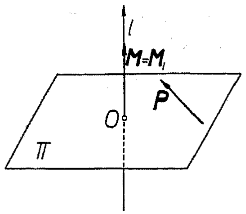


Fig. 150.

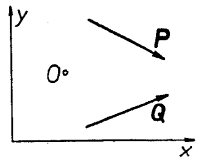


Fig. 151.

If we consider, for example, a system of forces lying in the xy-plane, then assuming that O also lies in xy, we have $M_x = 0$ and $M_y = 0$. The moment with respect to an axis parallel to z, i. e. M_z, is then called briefly the *moment with respect to* O and we denote it simply by M. Therefore

$$M = P_x(y - y_0) - P_y(x - x_0) \quad \text{or} \quad M = P_x y - P_y x. \tag{8}$$

Let us suppose, for example, that we have drawn the x and y axes as in Fig. 151. Therefore the z-axis should be taken directed vertically downwards. Hence if we want to determine the moment of the force $\mathbf{P}$ with respect to some point O, it is necessary to remember that $M > 0$ if the force tries to turn the piece of paper about O clockwise (i. e. as in Fig. 151); in the contrary case $M < 0$, as for the force $\mathbf{Q}$.

Given the arm h, we can therefore obtain M from formula (3), determining the sign in the manner given above.

Equilibrium of forces. If a rigid body is at rest we say that it is *in equilibrium*. The forces acting on a rigid body which remains in equilibrium are said to *balance one another* (to be in equilibrium) or to *annul one another*.

Statics is concerned with the investigation of conditions which forces in equilibrium must satisfy.

It is necessary to note the difference that exists between the equilibrium of a body and the equilibrium of forces. A body is in equilibrium then, and only then, when it is at rest. If a body is in equilibrium, then the system of forces acting on it is in equilibrium. Conversely, however, if a system of forces acting on a body is in equilibrium, it does not follow necessarily that the body is in equilibrium, since it can move e. g. with a uniformly advancing motion.

At this time we shall deduce conditions for the equilibrium of forces independently of the principles of dynamics by assuming certain hypotheses which are rather obvious. We shall show later (in chapter IX) that the conditions for equilibrium follow from the so-called *principle of virtual work*.

§ 3. Hypotheses for the equilibrium of forces. In order to deduce the conditions for the equilibrium of a rigid body, we shall assume the following hypotheses:

I. *To a system of forces acting on a rigid body which is in equilibrium we can add* (*or remove from the system*) *without disturbing equilibrium*:

a) *two forces equal in magnitude and acting along the same line, but oppositely directed* (Fig. 152a);

b) *several forces having a common point of application and whose sum is zero* (Fig. 152b).

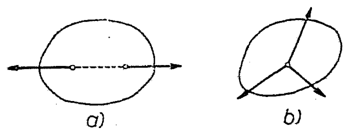


Fig. 152.

II. *Zero forces balance one another*; in other words: *if no forces act on a rigid body, then the body can remain in equilibrium.*

These hypotheses can be verified experimentally. We shall deduce from them the necessary and sufficient conditions for the equilibrium of forces. For the time being we shall be concerned with certain corollaries resulting from the assumed hypotheses.

§ 4. Transformation of systems of forces. Making use of the definition of elementary transformations (p. 28), we can formulate hypothesis I as follows:

I′. *If a rigid body is in equilibrium, we can perform arbitrary elementary transformations on the system of acting forces without disturbing equilibrium.*

Change of the point of application of a force. From theorem 1, p. 28 it follows that

1° *the point of application of a force can be chosen anywhere on its line of action.*

In the case of equilibrium the action of a force will therefore be defined if we give its magnitude, direction, sense, and position; the point of application of the force is immaterial. In virtue of the theorem on p. 18. we conclude from this that the action of the force $\boldsymbol{P}$ will be determined if we give its projections and the projections of its moment $\boldsymbol{M}$ with respect to an arbitrary point. The projections:

$$P_x, P_y, P_z, \quad M_x, M_y, M_z, \tag{1}$$

therefore define the action of a force on a rigid body. Let us note that since $\boldsymbol{M} \perp \boldsymbol{P}$, the scalar product $\boldsymbol{M} \cdot \boldsymbol{P}$ is zero, whence

$$M_x P_x + M_y P_y + M_z P_z = 0. \tag{2}$$

In general, therefore, five of the numbers (1) are sufficient to define a force; the sixth can be determined from equation (2).

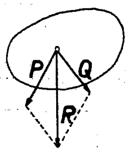


Fig. 153.

Law of composition and resolution of forces. From theorems 2 and 3, p. 30, we conclude that:

2° *several forces acting at one point can be replaced by their sum acting at this same point;*

3° *each force can be replaced by several forces having the same origin as the given force and a sum equal to the given force.*

These theorems are known as the *law of composition and resolution of forces.*

Equipollent systems. From hypothesis I and theorem 3, p. 30, we conclude that:

4° *a system of forces acting on a rigid body can be replaced by an arbitrary equipollent system.*

In other words: *equipollent systems of forces act on a rigid body in the same manner*; hence the importance of the notion of the equipollence of systems. It is easy to see that theorem 4° includes theorems 1°, 2°, and 3°.

As we know, two systems of forces are equipollent if they have equal sums and equal total moments with respect to one point (p. 22). By theorem 4° the action of a system of forces on a rigid body will therefore be defined if we give the sum $\boldsymbol{R}$ and the total moment $\boldsymbol{M}$ of the system of forces with respect to an arbitrary point.

Let the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$, whose points of application $A_1, A_2, \ldots$ have coordinates $x_1, y_1, z_1, \ x_2, y_2, z_2, \ldots$, act on a rigid body. Denoting the sum by $\boldsymbol{R}$ and the total moment with respect to the origin of tho system by $\boldsymbol{M}$, we obtain from formula (2), p. 232,

$$R_x = \Sigma P_{i_x}, \quad R_y = \Sigma P_{i_y}, \quad R_z = \Sigma P_{i_z}, \tag{3}$$

$$M_x = \Sigma(P_{i_y} z_i - P_{i_z} y_i), \; M_y = \Sigma(P_{i_z} x_i - P_{i_x} z_i), \; M_z = \Sigma(P_{i_x} y_i - P_{i_y} x_i).$$

The action of a system of forces is hence defined by means of the six numbers $R_x, R_y, R_z, M_x, M_y, M_z$.

The parameter of the system (p. 21) is $K = \mathbf{R} \cdot \mathbf{M}$, i. e.

$$K = R_x M_x + R_y M_y + R_z M_z. \tag{4}$$

Force couple. A system consisting of two forces equal in magnitude, parallel, but oppositely directed, is called a *force couple* (p. 23). The moment of a couple does not depend on the choice of the point with respect to which the moment is determined (p. 23). Since the sum of the forces of a couple is zero, two couples are equipollent if they have equal moments. Therefore the action of a force couple on a rigid body is defined by giving its moment.

A force couple tries to turn a body. The action of a couple does not undergo a change if the couple is arbitrarily translated and rotated in its plane (without changing the sense of the moment). A couple can also be arbitrarily translated in space without a change of the sense of its moment so that in every position it remains in a parallel plane (Fig. 154).

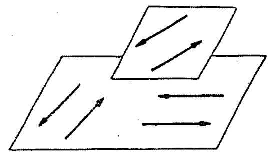


Fig. 154.

A couple whose moment is equal to zero is equipollent to a zero vector. Such a couple is also called a *zero couple*.

Reduction of a system of forces. The theorems concerning the reduction of systems (§ 16, p. 23) enable one to determine the simplest system of forces equipollent to the given one (i. e. the simplest system of forces by which one can replace the given system). In particular, the theorem on reduction can be stated as follows:

Every system of forces acting on a rigid body can be replaced:

a) *either by one force equal to the sum of the forces of the system and acting at an arbitrary point O, and a force couple whose moment is equal to the moment of the system with respect to O,*

b) *or by two forces, one of which acts at an arbitrarily chosen point.*

The theorems given on pp. 25 and 26 can be stated in a similar manner.

Let a force $\boldsymbol{P}$ whose origin is at the point A act on a rigid body. Let us choose an arbitrary point O. From the theorem on reduction it follows (if the system is assumed to be the force **P**) that *the force* **P** *can be replaced by an equal force acting at* O, *and by a force couple whose moment is equal to the moment of the force* **P** *with respect to* O.

Plane system of forces. If a system of forces lies in one plane, then their system is called a *plane system*. By theorem 3, p. 26, *a plane system of forces either has a resultant or is equipollent to a force couple.*

From the table given on p. 25 we see that a plane system has a resultant if the sum of the forces of a system is different from zero, or if the sum as well as the total moment are equal to zero; on the other hand, if the sum is zero and the total moment is different from zero, then the system is equipollent to a couple.

In the xy-plane let there be given a plane system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$, acting at the points $A_1, A_2, \ldots$ whose coordinates are $x_1, y_1, x_2, y_2, \ldots$

The projections of the forces P_{i_z} on the z-axis as well as the coordinates z_i of the points A_i are zero. Therefore, denoting the sum of the forces by **R**, and the total moment with respect to the origin of the system by **M**, we obtain from formulae (3), p. 237:

$$R_z = 0, \;\; M_x = 0, \;\; M_y = 0.$$

Hence the action of a plane system of forces is determined by three numbers: R_x, R_y, and M_z.

From formulae (3), p. 237, we also obtain (writing M instead of M_z):

$$R_x = \Sigma P_{i_x}, \;\; R_y = \Sigma P_{i_y}, \;\; M = \Sigma(P_{i_x} y_i - P_{i_y} x_i). \tag{5}$$

Parallel system of forces. From theorem 4, p. 26, it follows that *a parallel system of forces has a resultant or is equipollent to a force couple.*

Let the parallel forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ (Fig. 155) have origins at the points $A_1, A_2, \ldots$ whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ Let us assume that the sum of the forces **R** is different from zero. Consequently the given system has a resultant.

Let us select the sense of an arbitrary force of the system, e. g. the sense of the force $\mathbf{P}_1$ as positive. Let us denote for $i = 1, 2, \ldots$ by P_i the number whose absolute value is equal to $|\mathbf{P}_i|$ and whose sign is positive or negative depending on whether $\mathbf{P}_i$ has a positive sense (i. e. agreeing with the sense of $\mathbf{P}_1$) or not. We define R similarly. We have $R = \Sigma P_i$.

On p. 28 we proved that the resultant **R** passes through a certain point O called the *centre of forces*. The coordinates x_0, y_0, z_0, of the centre of forces are obtained from formula (4), p. 28, by putting $a_i = P_i$:

$$x_0 = \Sigma P_i x_i \,/\, R, \quad y_0 = \Sigma P_i y_i \,/\, R, \quad z_0 = \Sigma P_i z_i \,/\, R. \tag{6}$$

If the forces are rotated about their points of application through the same angle so that they still remain parallel (as e. g. the dotted vectors in Fig. 155), then the centre of forces does not undergo a change. This follows from formulae (6) because the coordinates x_0, y_0, z_0, depend only on P_i, x_i, y_i, and z_i, and do not depend on the direction of the forces. The new resultant will therefore also pass through O.

If the points of application of the forces lie in one plane (or on one line), then the centre of forces also lies on this plane (or on this line).

For assuming that the points of application lie in the plane Π and choosing this plane as the xy-plane, we obtain $z_1 = z_2 = \ldots = 0$; from formulae (6) we therefore get $z_0 = 0$, which means that the centre of forces lies in the plane Π.

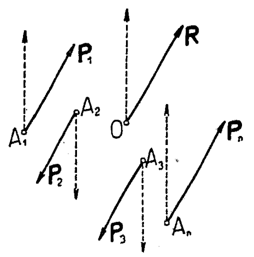


Fig. 155.

Similarly, if the points of application lie on one line l, then choosing it as the x-axis, we have $y_1 = y_2 = \ldots = 0$, and $z_1 = z_2 = \ldots = 0$; hence by (6) $y_0 = 0$ and $z_0 = 0$; consequently the centre of forces lies on the line l.

Let the material points whose masses are $m_1, m_2, \ldots$ be acted upon by forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ which are parallel, have the same sense, and are in magnitude proportional to the masses of the individual points. Putting $P_1 = |\mathbf{P}_1|$, $P_2 = |\mathbf{P}_2|, \ldots$, we obtain:

$$P_1 = km_1, \;\; P_2 = km_2, \ldots \quad R = P_1 + P_2 + \ldots = km, \tag{7}$$

where k is the factor of proportionality, and $m = m_1 + m_2 + \ldots$ From formulae (6) and (7) we get after substitution:

$$x_0 = \Sigma m_i x_i \,/\, m, \quad y_0 = \Sigma m_i y_i \,/\, m, \quad z_0 = \Sigma m_i z_i \,/\, m.$$

Comparing these equalities with formulae (I), p. 152, we see that the centre of forces is the centre of mass of a given system of material points.

Therefore: *the centre of mass of a system of material points is the centre of forces which are parallel, have the same sense and are in magnitude proportional to the masses of the points on which they act.*

Gravitational forces. Let a rigid body be situated in a gravitational field. Consider the body as a system of material points of masses $m_1, m_2, \ldots$, we can assume that the weights of the separate points are parallel forces, having the same sense (vertically downwards). The weights therefore have a resultant (Fig. 156).

The magnitudes of the weights of the separate points are $Q_1 = m_1 g$, $Q_2 = m_2 g, \ldots$ (where g denotes the acceleration of gravity). Consequently the magnitudes of the weights are proportional to the masses of the points. Therefore in virtue of the preceding theorem, the centre of the gravitational forces is the centre of mass of the body. The magnitude of the resultant is

$$Q = m_1 g + m_2 g + \ldots = (m_1 + m_2 + \ldots)\, g = mg,$$

where m denotes the mass of the body.

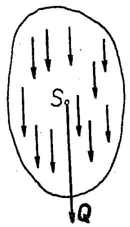


Fig. 156.

Therefore: *in every position of a body the resultant of the gravitational forces passes through the centre of gravity of the body. The weight of the body* (i. e. the resultant of the gravitational forces acting on its separate points) *is*

$$Q = mg, \tag{8}$$

where m denotes the mass of the body, and g the acceleration of gravity.

On the basis of the above theorem we can replace the action of the force of gravity by one force situated at the centre of gravity of the body.

Systems of couples. A system consisting of several couples has a zero sum. From the table on p. 25 it follows that such a system is equipollent to a couple or to a zero vector (i. e. to a zero couple). Let $\mathbf{M}_1, \mathbf{M}_2, \ldots$ denote the moments of the individual couples. Then the total moment will be $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \ldots$ From the theorem on reduction (p. 24) we therefore obtain the following theorem:

A system consisting of several couples is equipollent to one couple whose moment is equal to the total moment of the system.

Let us note that a force couple (whose moment is different from zero) cannot be equipollent to one force. For in view of the fact that the sum of the forces of the couple is zero, this force would have to be zero and its moment different from zero, which is impossible.

Example I. The centres of the sides of a plane polygon $A_1, A_2, \ldots, A_n$ (Fig. 157) are acted upon by forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, lying in the plane of the polygon, forming with its sides $\overline{A_1A_2}, \overline{A_2A_3}, \ldots, \overline{A_nA_1}$ an angle φ, and directed towards the exterior and proportional in magnitude to the sides of the polygon.

It is easy to see that the sum of the forces is zero, because forming the sum $\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$, we obtain a polygon similar to the given one, but turned through an angle φ relative to it.

The system of forces is therefore equipollent to a couple or zero.

Let us select an arbitrary system of coordinates $O(x, y)$ and denote by $x_1, y_1, x_2, y_2, \ldots$ the coordinates of the points $A_1, A_2, \ldots$ The point of application of the force $\mathbf{P}_1$ has the coordinates $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$. Therefore the moment of the force $\mathbf{P}_1$ with respect to O is ((8), p. 234)

$$M_1 = \tfrac{1}{2}[P_{1_x}(y_1 + y_2)] - \tfrac{1}{2}[P_{1_y}(x_1 + x_2)]. \tag{9}$$

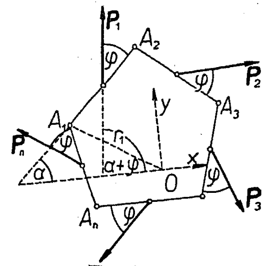


Fig. 157.

By hypothesis

$$|\mathbf{P}_1| = \lambda d_1, \tag{10}$$

where $d_1 = \overline{A_1A_2}$ and λ is the factor of proportionality. If $\overline{A_1A_2}$ forms an angle α with the x-axis, then the force $\mathbf{P}_1$ forms an angle $\alpha + \varphi$ with the x-axis. Therefore:

$$P_{1_x} = |\mathbf{P}_1| \cos(\alpha + \varphi), \quad P_{1_y} = |\mathbf{P}_1| \sin(\alpha + \varphi).$$

Hence in virtue of (10), $P_{1_x} = \lambda d_1(\cos\alpha\cos\varphi - \sin\alpha\sin\varphi)$. But $d_1\cos\alpha = x_2 - x_1$, and $d_1 \sin\alpha = y_2 - y_1$; consequently

$$P_{1_x} = \lambda[(x_2 - x_1)\cos\varphi - (y_2 - y_1)\sin\varphi].$$

Similarly

$$P_{1_y} = \lambda[(y_2 - y_1)\cos\varphi + (x_2 - x_1)\sin\varphi].$$

Substituting in (9), we obtain

$$M_1 = \tfrac{1}{2}\lambda[2(y_1x_2 - y_2x_1)\cos\varphi + (y_1^2 + x_1^2 - y_2^2 - x_2^2)\sin\varphi]. \tag{11}$$

Putting $OA_1 = r_1$, $OA_2 = r_2, \ldots$ and denoting by $p_1, p_2, \ldots$ the areas of the triangles $OA_1A_2, OA_2A_3, \ldots$, we get $r_1^2 = x_1^2 + y_1^2$, $r_2^2 = x_2^2 + y_2^2, \ldots$, $p_1 = \frac{1}{2}(y_1x_2 - y_2x_1)$, etc. Hence by (11)

$$M_1 = \tfrac{1}{2}\lambda[4p_1\cos\varphi + (r_1^2 - r_2^2)\sin\varphi]. \tag{12}$$

Similar expressions are obtained for the moments of the remaining forces.

The total moment of the forces with respect to O is $M = M_1 + M_2 + \ldots$ Hence according to (12)

$$M = \tfrac{1}{2}\lambda[4(p_1 + p_2 + \ldots + p_n)\cos\varphi + (r_1^2 - r_2^2 + r_2^2 - r_3^2 + \ldots + r_n^2 - r_1^2)\sin\varphi].$$

Since $p = p_1 + p_2 + \ldots + p_n$ is the area of the polygon $A_1A_2\ldots A_n$,

$$M = 2\lambda p\cos\varphi. \tag{13}$$

The total moment is therefore proportional to the area of the polygon and to the cosine of the angle φ.

In particular, if the forces are perpendicular to the sides of the polygon, then $\varphi = \frac{1}{2}\pi$ and $\cos\varphi = 0$, whence in virtue of (13) $M = 0$, i. e. the forces form a system equipollent to zero.

On the other hand, if the forces are directed along the sides (i. e. if $\varphi = 0$), we have by (13) $M = 2\lambda p$, and hence the moment is then proportional to the area of the polygon.

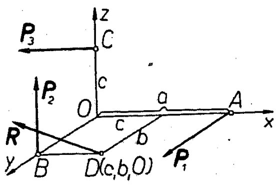


Fig. 158.

Example 2. The points $A(a, 0, 0)$, $B(0, b, 0)$, and $C(0, 0, c)$, on the axes of the coordinate system (x, y, z) are the points of application of the forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, and $\boldsymbol{P}_3$, parallel to the axes of the system, equal in magnitude and having senses as shown in the Fig. 158. What relation exists among the coordinates a, b, c if the system has a resultant?

Let us put $P = |\boldsymbol{P}_1| = |\boldsymbol{P}_2| = |\boldsymbol{P}_3|$. The sum of the forces $\boldsymbol{R}$ therefore has the projections

$$R_x = -P, \quad R_y = P, \quad R_z = P. \tag{14}$$

Let us calculate the total moment $\boldsymbol{M}$ with respect to O. The moment of the forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_3$ with respect to the x-axis is zero; the moment of the force $\boldsymbol{P}_2$ with respect to the x-axis is $-Pb$. Hence $M_x = -Pb$; similarly $M_y = Pc$ and $M_z = -Pa$. The parameter of the system is $K = \boldsymbol{R} \cdot \boldsymbol{M} = R_x M_x + R_y M_y + R_z M_z$; therefore

$$K = P^2(b + c - a).$$

If the system has a resultant, then $K = 0$ (p. 26). Consequently

$$b + c - a = 0. \tag{15}$$

Equation (15) constitutes the s u f f i c i e n t condition and, as is easily seen from the table on p. 25, also the n e c e s s a r y condition that the system have a resultant, because $\boldsymbol{R} \neq 0$.

As the point of application of the resultant we can take the point $D(x, y, z)$ with respect to which the total moment is zero.

Let us denote the total moment with respect to D by $\boldsymbol{M}'$. We have:

$$M'_x = -Pz - P(b - y), \quad M'_y = -Px + P(c - z),$$
$$M'_z = -P(a - x) + Py.$$

Assuming that the moment with respect to D is zero, we get:

$$y - z = b, \quad z + x = c, \quad x + y = a.$$

On account of (15) these equations are dependent. Two of them are the equations of the line on which the resultant lies. Putting $z = 0$, for example, we obtain $x = c$, and $y = b$. Therefore we can take the point $D(c, b, 0)$ as the point of application of the resultant.

Example 3. Parallel forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ act at the points A and B, $\boldsymbol{P} + \boldsymbol{Q} \neq 0$. Determine the center of forces.

The center of forces lies on the line AB (p. 239). Let us choose it as the x-axis, taking the point A as the origin of the x-axis and giving it a sense such that the point B lies on its positive part. Let us put $P = |\boldsymbol{P}|$ and denote by Q the number whose absolute value is equal to $|\boldsymbol{Q}|$, while the sign is + or — depending on whether $\boldsymbol{Q}$ has a sense which agrees, or does not agree, with that of the force $\boldsymbol{P}$. Putting $AB = d$ and denoting the coordinate of the centre of forces O by x_0, we get from formula (6), p. 238,

$$x_0 = Qd / R, \quad \text{where } R = P + Q.$$

If the forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ have the same sense (Fig. 159a), then $Q > 0$, and consequently $0 < Q/R < 1$, whence $0 < x_0 < d$. The centre of forces is therefore situated between the points A and B.

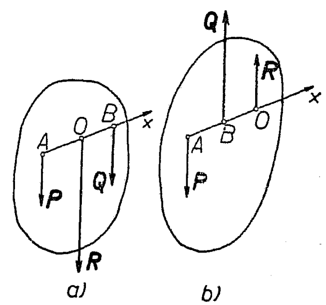


Fig. 159.

On the other hand, if the forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ have opposite senses and e. g. $|\boldsymbol{P}| < |\boldsymbol{Q}|$ (Fig. 159b), then $Q < 0$, and $R < 0$, whence $x_0 > 0$. Furthermore $|R| < |Q|$; consequently $x_0 > d$. The centre of forces hence lies beyond the point B.

It is easy to verify that in both cases $AO / BO = |\boldsymbol{Q}| / |\boldsymbol{P}|$.

Therefore: *the centre of two parallel forces* (whose sum $\neq 0$) *is situated on the line joining the points of application of these forces.*

If the forces have the same senses, then their centre lies between the points of application; in the opposite case it lies beyond the point of application of that force whose absolute value is greater.

The distances of the centre of forces from the points of application are inversely proportional to the magnitudes of these forces.

Example 4. Forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ whose origins are $A_1, A_2, \ldots$, and forces $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots$ whose origins are $B_1, B_2, \ldots$, act on a rigid rod AB. All forces are parallel to each other and perpendicular to the rod, and the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ have a sense opposite to that of the forces $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots$ (Fig. 160).

Let $\mathbf{R} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{Q}_1 + \mathbf{Q}_2 + \ldots$ Let us denote by $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ the absolute values of the forces, and by $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ the corresponding lengths of the segments $AA_1, AA_2, \ldots$ and $AB_1, AB_2, \ldots$ Let us assume the sense of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ as positive. Put

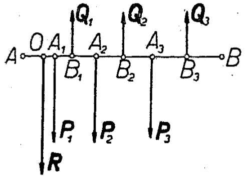


Fig. 160.

$$R = P_1 + P_2 + \ldots - Q_1 - Q_2 - \ldots \quad (16)$$

Obviously $|R| = |\mathbf{R}|$. If $R > 0$, the sum $\mathbf{R}$ has a sense agreeing with the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ However, if $\mathbf{R} < 0$, then $\mathbf{R}$ has a sense agreeing with the forces $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$

Let us calculate the total moment M of the forces with respect to A. Denoting by $M_1, M_2, \ldots$ and $M_1^1, M_2^1, \ldots$ the moments of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ and $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$ with respect to A, we have (according to the agreement concerning the sign of the moment assumed on p. 233):

$$M_1 = P_1 a_1, \; M_2 = P_2 a_2, \; \ldots, \quad M_1^1 = -Q_1 b_1, \; M_2^1 = -Q_2 b_2, \; \ldots$$

Consequently

$$M = P_1 a_1 + P_2 a_2 + \ldots - Q_1 b_1 - Q_2 b_2 - \ldots \quad (17)$$

Let us assume that $\mathbf{R} = 0$. The system of forces is therefore equipollent to a couple of moment M according to formula (17).

If $M > 0$, the couple will tend to turn the rod clockwise, if $M < 0$ — counterclockwise. Finally, if $M = 0$, the system will be equipollent to zero.

Let us now assume that $\mathbf{R} \neq 0$. The system therefore has a resultant.

Let O be the origin of the resultant $\mathbf{R}$ lying on the line AB. Set $d = \pm AO$, taking the $+$ sign if the point O is on the same side of the point A as the origin of the force, and the $-$ sign in the contrary case. The moment of the resultant with respect to O, as is easily verified, is Rd according to our convention. Since the moment of the resultant is equal to the total moment, we get from (17)

$$d = \frac{1}{R}(P_1 a_1 + P_2 a_2 + \ldots - Q_1 b_1 - Q_2 b_2 - \ldots).$$

§ 5. Conditions for equilibrium of forces. We shall now prove the following

Theorem I. *In order that a system of forces acting on a rigid body be in equilibrium, it is necessary and sufficient that the sum of the forces and the total moment be zero, i. e. that the system of forces be equipollent to zero.*

Proof. We shall prove at first that the condition is necessary. Let us assume that the rigid body is a rigid system of material points $A_1, A_2, \ldots$ and that it is in equilibrium under the action of a given system of forces. Let us consider an arbitrary point A_i. Denote by $\mathbf{P}_i$ the sum of the external forces, and by $\mathbf{W}_i$ the sum of the internal forces acting at A_i. Since the point A_i is in equilibrium (because the entire rigid system is in equilibrium), it follows that $\mathbf{P}_i + \mathbf{W}_i = 0$. Consequently

$$\Sigma(\mathbf{P}_i + \mathbf{W}_i) = 0, \quad (1)$$

where the sum Σ extends over all the points A_i of the given rigid system.

From the law of action and reaction it follows that the sum of the forces with which two points react on each other is zero. Since all the internal forces can be grouped in pairs, the sum of the internal forces is zero or $\Sigma \mathbf{W}_i = 0$, whence by (1)

$$\Sigma \mathbf{P}_i = 0. \quad (2)$$

It follows from this that the sum of the external forces, i. e. the forces acting on the rigid body in equilibrium, is zero.

Let us now choose an arbitrary point O. Since the forces $\mathbf{P}_i$ and $\mathbf{W}_i$ have the common origin A_i, and moreover $\mathbf{P}_i + \mathbf{W}_i = 0$, it follows that (p. 17) $\mathrm{Mom}_O \mathbf{P}_i + \mathrm{Mom}_O \mathbf{W}_i = 0$, whence

$$\Sigma(\mathrm{Mom}_O \mathbf{P}_i + \mathrm{Mom}_O \mathbf{W}_i) = 0. \quad (3)$$

The total moment of the internal forces with which two points react on each other is — as is easily verified — zero. Consequently the total moment of all the internal forces is zero. Therefore $\Sigma \mathrm{Mom}_O \mathbf{W}_i = 0$, whence by (3)

$$\Sigma \mathrm{Mom}_O \mathbf{P}_i = 0. \quad (4)$$

We have proved, therefore, that the sum as well as the total moment of the forces acting on a rigid body is zero. This proves the necessity of the condition.

Let us now assume that the given system of forces is equipollent to zero. Since a system equipollent to zero is equipollent to a zero force, it is in equilibrium by hypothesis II (p. 235). The condition is therefore at the same time sufficient, q. e. d.

From theorem I it follows that if a system of forces acting on a rigid body is not equipollent to zero, then the body cannot be in equilibrium. In particular, a rigid body cannot remain in equilibrium under the action of a system of forces consisting of:

a) one force different from zero,

b) one force couple of moment different from zero,

c) one force different from zero and one couple of moment different from zero.

As we know (p. 22), a system of forces is equipollent to zero if the total moments with respect to three non-collinear points are zero. On the basis of theorem I we obtain from this the following

Theorem II. *In order that a system of forces acting on a rigid body be in equilibrium, it is necessary and sufficient that the moments of the system with respect to three non-collinear points be equal to zero.*

In applications we frequently find the following theorem useful:

Theorem III. *If a system consisting of three forces is in equilibrium, then these forces lie in one plane and are either parallel or their prolongations intersect in one point.*

Theorem III follows from the theorem of chapter I, § 14 (p. 22), in the proof of which it was shown that the vectors lie in one plane.

Analytic form of the conditions for equilibrium. Let us choose an arbitrary system of coordinates $O(x, y, z)$. Let the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ act on a rigid body at the points $A_1, A_2, \ldots$ whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ Let us denote by $\mathbf{R}$ the sum, and by $\mathbf{M}$ the total moment, of the system with respect to O. According to theorem I (p. 244) the equations:

$$\mathbf{R} = 0, \quad \mathbf{M} = 0 \tag{5}$$

constitute the necessary and sufficient conditions for equilibrium. Forming the projections on the axes of the system, we obtain from (5) and from formulae (3), p. 237:

$$\Sigma P_{i_x} = 0, \quad \Sigma P_{i_y} = 0, \quad \Sigma P_{i_z} = 0, \tag{I}$$

$$\Sigma(P_{i_y} z_i - P_{i_z} y_i) = 0, \ \Sigma(P_{i_z} x_i - P_{i_x} z_i) = 0, \ \Sigma(P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{II}$$

Equations (I) are called the *condition of projections* and equations (II) the *condition of moment.*

Equations (I) and (II) are the analytic form of the conditions for the equilibrium of a system of forces. From these equations we can determine in general six unknowns.

Plane systems of forces. The conditions for equilibrium obviously apply also to a plane system of forces.

Let the forces lie in the xy-plane. Since $P_{i_z} = 0$ and $z_i = 0$, the conditions for equilibrium (I) and (II) assume the form:

$$\Sigma P_{i_x} = 0, \quad \Sigma P_{i_y} = 0, \tag{I'}$$

$$\Sigma(P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{II'}$$

In the case of a plane system we therefore obtain three equations. From them we can in general determine three unknowns.

Example 1. A heavy sphere is in equilibrium under the action of three forces (Fig. 161): the weight $\mathbf{Q}$ (acting at the centre of the sphere O), the horizontal force $\mathbf{P}$ (acting at the point A situated on the surface of the sphere at the end of the vertical diameter) and the force $\mathbf{R}$ (acting at the point B situated on the surface of the sphere at the end of the horizontal diameter). The weight $\mathbf{Q}$ is given. Determine the forces $\mathbf{P}$ and $\mathbf{R}$.

The forces $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$, are in equilibrium; therefore by theorem III they lie in a plane and their directions intersect in one point which is the point A. Consequently the force $\mathbf{R}$ has the direction of the line BA which forms an angle of 45° with the horizontal. Since $\mathbf{Q}+\mathbf{P}+\mathbf{R} = 0$, knowing the force $\mathbf{Q}$ and the directions of the forces $\mathbf{P}$ and $\mathbf{R}$, we can form a triangle of forces (Fig. 162). From this triangle we obtain

$$P = Q, \quad R = Q / \cos 45° = \sqrt{2}Q,$$

where P, Q and R, denote the absolute values of the forces.

Example 2. The vertices of a square $ABCD$ of side a are the points of application of four forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$, lying in the plane of the square and forming with the sides the angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (Fig. 163). Give the conditions for equilibrium.

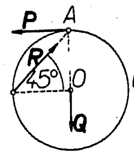


Fig. 161.

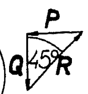


Fig. 162.

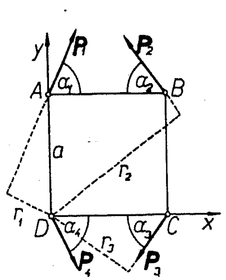


Fig. 163.

Let us denote the absolute values of the forces by P_1, P_2, P_3, P_4. Let us select the x and y axes along the sides of the square. Forming the projections of the forces on the x and y axes, we get in the case of equilibrium (when the forces have senses as shown in Fig. 163):

$$P_1 \cos\alpha_1 - P_2 \cos\alpha_2 - P_3 \cos\alpha_3 + P_4 \cos\alpha_4 = 0, \tag{6}$$

$$P_1 \sin\alpha_1 + P_2 \sin\alpha_2 - P_3 \sin\alpha_3 - P_4 \sin\alpha_4 = 0. \tag{7}$$

Denoting the arms of the forces with respect to the vertex D by r_1, r_2, r_3, r_4, we obtain:

$$r_1 = a\cos\alpha_1, \quad r_2 = \sqrt{2}a\sin(\alpha_2 + 45^\circ), \quad r_3 = a\sin\alpha_3.$$

Moreover $r_4 = 0$. Since in the case of equilibrium the total moment with respect to D is zero (taking the sign of the moment according to the rule given on p. 233) we get after dividing by a

$$P_1\cos\alpha_1 - P_2\sqrt{2}\sin(\alpha_2 + 45^\circ) + P_3\sin\alpha_3 = 0. \tag{8}$$

Equations (6), (7), and (8), constitute the necessary and sufficient condition for equilibrium.

Example 3. A rod AB lying in the horizontal xy-plane is acted upon by the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, lying in this plane and acting at the points $A_1, A_2, \ldots, A_n$ (Fig. 164). Give the conditions which the forces must satisfy in order that the rod be in equilibrium for every position in the xy-plane, if we assume that the forces do not change their magnitudes, directions, senses, or points of application (on the rod).

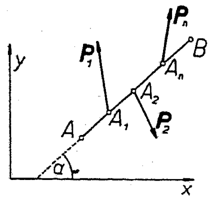


Fig. 164.

Let us consider an arbitrary position of the rod AB in the xy-plane. Denote by x_0, y_0 the coordinates of the point A, by $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ the coordinates of the points $A_1, A_2, \ldots$, and by α the angle which the rod AB makes with the x-axis. Let us put: $d_1 = AA_1, d_2 = AA_2, \ldots, d_n = AA_n$. We have

$$x_i = x_0 + d_i\cos\alpha, \ y_i = y_0 + d_i\sin\alpha \quad \text{for } i = 1, 2, \ldots, n. \tag{9}$$

From the conditions of equilibrium (I′) and (II′), p. 247, we obtain

$$\Sigma P_{i_x} = 0, \quad \Sigma P_{i_y} = 0, \tag{10}$$

$$\Sigma(P_{i_x}y_i - P_{i_y}x_i) = \Sigma[P_{i_x}(y_0 + d_i\sin\alpha) - P_{i_y}(x_0 + d_i\cos\alpha)] = 0. \tag{11}$$

Condition (11) can be written in the form

$$y_0\,\Sigma P_{i_x} - x_0\,\Sigma P_{i_y} + \sin\alpha\,\Sigma P_{i_x}d_i - \cos\alpha\,\Sigma P_{i_y}d_i = 0, \tag{12}$$

whence by (10)

$$\sin\alpha\,\Sigma P_{i_x}d_i - \cos\alpha\,\Sigma P_{i_y}d_i = 0. \tag{13}$$

Since relation (13) has to hold for every angle α, we get for $\alpha = \frac{1}{2}\pi$ and then for $\alpha = 0$

$$\Sigma P_{i_x}d_i = 0, \quad \Sigma P_{i_y}d_i = 0. \tag{14}$$

Equations (10) and (14) are the necessary and sufficient conditions in order that the rod be in equilibrium for every position in the plane.

For if conditions (10) and (14) hold, it is easy to see that condition (13) holds, and consequently by (10) conditions (12) and (11) also hold.

Conditions (10) and (11) are, as we have seen, the necessary and sufficient conditions for equilibrium by (I′) and (II′).

§ 6. Graphical statics. String polygon. The problems which one meets in statics often lead to long and tedious computations. However, there exist graphical methods which enable one to obtain in many cases approximate solutions which are sufficiently accurate for applications.

These methods are of great importance in engineering for they lead more rapidly to the goal omitting intricate computations.

That part of theoretical statics which deals with graphical methods is called *graphical statics*.

Here we shall become acquainted only with some graphical methods as, for example, the graphical determination (by means of a string polygon) of the resultant of a plane system of forces and certain applications of these methods. Later (in § 16) we shall become acquainted with graphical methods serving to determine the stress in the bars of a frame.

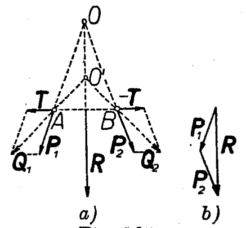


Fig. 165.

Composition of forces. Having two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ whose directions intersect at the point O, we determine the sum $\mathbf{R}$ (Fig. 165a), and then we draw the resultant through the point O (Fig. 165b).

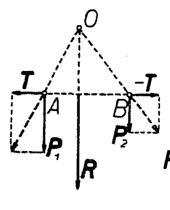
If the point O lies outside the limits of the drawing, we can proceed as follows: we add two forces $\mathbf{T}$ and $-\mathbf{T}$ acting at the points A and B (i. e. at the initial points of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$) and along the line AB. The system $\mathbf{T}, -\mathbf{T}, \mathbf{P}_1, \mathbf{P}_2$, is obviously equipollent to the system $\mathbf{P}_1, \mathbf{P}_2$, because the forces $\mathbf{T}$ and $-\mathbf{T}$ annul each other, and consequently the resultant of the new system of four forces is the same as before.

The forces $\mathbf{T}$ and $\mathbf{P}_1$ are replaced by the force $\mathbf{Q}_1 = \mathbf{T} + \mathbf{P}_1$ with its origin at A; similarly the forces $-\mathbf{T}$ and $\mathbf{P}_2$ are replaced by the force $\mathbf{Q}_2 = -\mathbf{T} + \mathbf{P}_2$ with its origin at B. The resultant $\mathbf{R}$ passes through the point of intersection O' of the forces $\mathbf{Q}_1, \mathbf{Q}_2$.

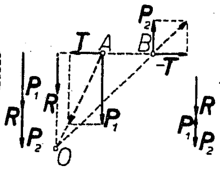
This construction can also be applied to the case of two parallel forces not forming a couple (Fig. 166a, 166b and 167a, 167b).

In this way we can obtain the resultant (or the resultant couple) of the system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ (Fig. 168a and 168b). We first form the resultant $\mathbf{R}_1$ of two of these forces (e. g. the forces $\mathbf{P}_1$ and $\mathbf{P}_2$) and we obtain a system consisting of only $n-1$ forces.

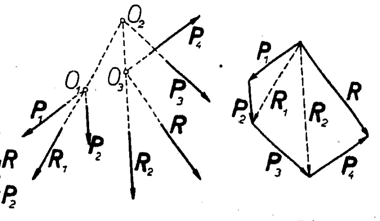
A method that we shall become acquainted with presently will lead us to the goal more quickly.



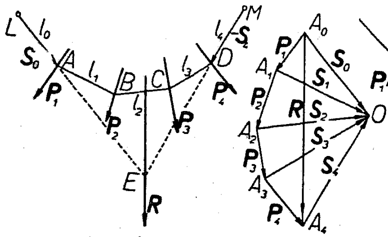
a) b) Fig. 166.



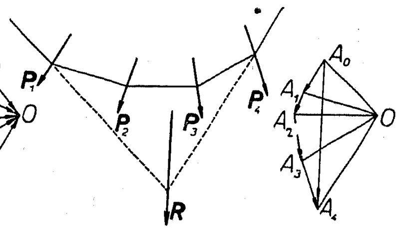
a) b) Fig. 167.



a) b) Fig. 168.



a) b) Fig. 169.



a) b) Fig. 170.

String polygon. Let us assume that we have to find the resultant of the system of forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ (Fig. 169a).

We first form the sum $\mathbf{R} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4$. The polygon obtained is called the *polygon of forces*.[1])

Let us denote (in the polygon of forces) by A_0 the origin of the force $\mathbf{P}_1$, and by A_1, A_2, A_3, A_4, the termini of the forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$. Let us now select an arbitrary point O outside the polygon of forces. This point is called the *pole*.

We connect the pole O with the points $A_0, A_1, \ldots, A_4$. From an arbitrary point A situated on the direction of the force $\mathbf{P}_1$ we draw the lines l_0 and l_1 parallel to the lines OA_0 and OA_1, respectively. The line l_1 is prolonged to the point B of its intersection with the direction of the force $\mathbf{P}_2$. The line l_1 will cut the direction of the force $\mathbf{P}_2$, because l_1 is parallel to OA_1, and OA_1 is not parallel to $\mathbf{P}_2$ (Fig. 169b).

From the point B we draw the line $l_2 \parallel OA_2$ to the point C of its intersection with the direction of the force $\mathbf{P}_3$. From the point C we draw the line $l_3 \parallel OA_3$ to the point D of its intersection with the direction of the force $\mathbf{P}_4$. From the point D we draw the line $l_4 \parallel OA_4$.

We now determine the point of intersection E of the lines l_0 and l_4. The resultant $\mathbf{R}$ passes through the point E. Since $\mathbf{R}$ is known from the polygon of forces, this resultant can be drawn.

We shall now justify the above construction.

Let us denote the vectors $\overline{A_0O}, \overline{A_1O_1}, \ldots, \overline{A_4O}$ by $\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_4$, respectively. From the force polygon (Fig. 169b) we obtain:

$$\begin{aligned} &\mathbf{P}_1 + \mathbf{S}_1 + (-\mathbf{S}_0) = 0, \quad \mathbf{P}_2 + \mathbf{S}_2 + (-\mathbf{S}_1) = 0, \\ &\mathbf{P}_3 + \mathbf{S}_3 + (-\mathbf{S}_2) = 0, \quad \mathbf{P}_4 + \mathbf{S}_4 + (-\mathbf{S}_3) = 0. \end{aligned} \tag{1}$$

Let us add to the system of forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ the forces $\mathbf{S}_0$ and $-\mathbf{S}_0$ lying on the line l_0, the forces $\mathbf{S}_1$ and $-\mathbf{S}_1$ lying on l_1, etc., finally the forces $\mathbf{S}_4$ and $-\mathbf{S}_4$ lying on l_4. The system added is obviously equipollent to zero, because the forces $\mathbf{S}_0, -\mathbf{S}_0, \mathbf{S}_1, -\mathbf{S}_1$, etc. annul each other in pairs. The resultant of the enlarged system is therefore the same as before.

In virtue of (1) the forces $\mathbf{P}_1, \mathbf{S}_1$, and $-\mathbf{S}_0$, annul one another because their sum is zero and their directions intersect at A. These forces can therefore be removed. Similarly we can remove the forces $\mathbf{P}_2, \mathbf{S}_2$, and $-\mathbf{S}_1$, next, $\mathbf{P}_3, \mathbf{S}_3$, and $-\mathbf{S}_2$, etc., and finally the forces $\mathbf{P}_4, \mathbf{S}_4$, and $-\mathbf{S}_3$. The remaining forces $\mathbf{S}_0$ and $-\mathbf{S}_4$ consequently form a system equipollent to the given one. The resultant $\mathbf{R}$ therefore passes through the point of intersection E of the forces $\mathbf{S}_0$ and $-\mathbf{S}_4$ (i. e. of the lines l_0, l_4).

The segments l_0, l_1, l_2, l_3, l_4 form a so-called *string polygon*; l_0 and l_4 are called its *extreme sides*.

Therefore: *the resultant passes through the point of intersection of the extreme sides of the string polygon.*

The name of string polygon arises from the fact that a weightless and inextensible string fastened at the points L and M on the lines l_0 and l_4 in the directions $-S_0$ and S_4 (in other respects arbitrary), and assuming the position of the polygon $LABCDM$, will be in equilibrium under the action of the forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$, whose points of application are at the points A, B, C, D, respectively.

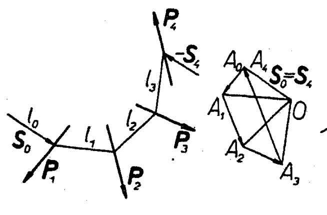
In practice superfluous notations are omitted and the drawing is as in Fig. 170 as well as in the Fig. 171 and 172.

[1]) In Fig. 169a, 170a, and those appearing farther on, only the positions of the forces are given. The magnitudes of the forces are indicated in the force polygons (Fig. 169b, 170b, etc.).

Fig. 171a represents a system of forces whose sum is zero. We then say that *the polygon of forces is closed* (Fig. 171b).

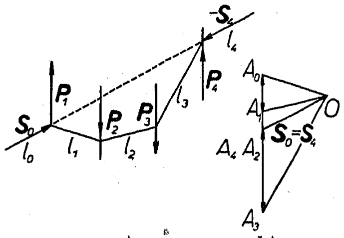
Drawing the string polygon we see that its extreme sides do not intersect (are parallel). We then say that *the string polygon is not closed.*

The system of forces in this case is equipollent to the system of forces $\mathbf{S}_0$ and $-\mathbf{S}_4$, which, as is seen from the polygon of forces, form a couple. The system is then equipollent to the force couple $\mathbf{S}_0$ and $-\mathbf{S}_4$ lying on the extreme sides of the string polygon.



a) b)

Fig. 171.



a) b)

Fig. 172.

Therefore: *if the force polygon is closed and the string polygon is not closed, then the system is equipollent to a force couple.*

In Fig. 172a we see a system of forces for which the polygon of forces (Fig. 172b) is closed and the extreme sides of the string polygon lie on one line. We then say that *the string polygon is closed.*

A system of forces in this case is equipollent to the system of forces $\mathbf{S}_0$ and $-\mathbf{S}_4$ lying on the extreme sides of the string polygon and hence on one straight line. Since (as is seen from the polygon of forces) $\mathbf{S}_0 = \mathbf{S}_4$, the forces $\mathbf{S}_0$ and $-\mathbf{S}_4$ balance each other; the given system is then equipollent to zero.

Therefore: *if the polygon of forces and the string polygon are closed, then the system of forces is equipollent to zero.*

Resultant of a part of a system. Having a string polygon of a certain system of forces, one can easily determine the resultant of an arbitrary part of the system consisting of the forces following each other in the polygon of forces.

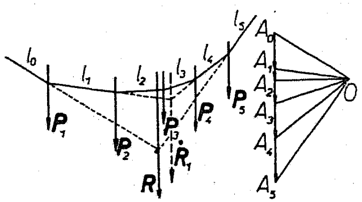


Fig. 173.

For example, let the system of parallel forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5$, be given (Fig. 173). Let the resultant $\mathbf{R}$ of the entire system and the resultant

$\mathbf{R}_1$ of the forces $\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ be determined. From Fig. 173 we see that the string polygon for the system $\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ is a part of the string polygon for the entire system.

§ 7. Some applications of the string polygon. Determination of the reactions at the points of support of a beam. A system of parallel forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$, is given. Determine two forces $\mathbf{R}_1, \mathbf{R}_2$, parallel to the preceding and forming together with them a system equipollent to zero. The lines k_1 and k_2 on which the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ are to lie are given.

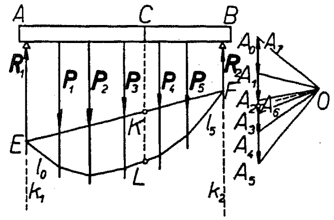


Fig. 174.

A problem like this occurs in the case of a rigid horizontal beam supported at the points A and B and acted upon by vertical forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$ (Fig. 174). If there is no friction, the reactions at the points A and B are vertical and in equilibrium with the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$ (p. 263).

In order to determine the forces $\mathbf{R}_1$ and $\mathbf{R}_2$, we draw the string polygon for the given system of forces in the order $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5, \mathbf{R}_2, \mathbf{R}_1$. In the polygon of forces the line A_6O joining the terminus of the force $\mathbf{R}_2$ with the pole O is for the moment unknown. Since the polygon of forces is closed, the point A_7, i. e. the terminus of the force $\mathbf{R}_1$, coincides with the initial point of the force $\mathbf{P}_1$, i. e. with the point A_0.

We draw the string polygon starting from the line $l_0 \parallel A_0O$ until we get to the line $l_5 \parallel A_5O$.

Let us denote by E, F the points of intersection of the lines l_0 and l_5 with the directions of the forces $\mathbf{R}_1$ and $\mathbf{R}_2$, i. e. with the given lines k_1 and k_2. Drawing in the polygon of forces the line OA_6 parallel to the line EF, we obtain the forces $\mathbf{R}_1 = \overline{A_5A_6}$ and $\mathbf{R}_2 = \overline{A_6A_7}$. For it is easy to see that by continuing the drawing of the string polygon for the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ so determined, we obtain a closed string polygon.

Determination of the moment of forces. If we have to determine the moment of the force $\mathbf{P}$ with respect to a certain point A, we first draw the string polygon from an arbitrary point B situated on the direction of the force $\mathbf{P}$, as in Fig. 175.

Next, we pass through A a line parallel to $\mathbf{P}$. We denote the points of intersection of this line with the sides of the string polygon by L, K. From the similarity of triangles A_0A_1O and KLB we obtain $KL : |\mathbf{P}| = h : w$, where h and w are the altitudes of these triangles. From this $|\mathbf{P}|h = KL \cdot w$. Denoting the moment of the force $\mathbf{P}$ with respect to A by $\mathbf{M}$, we have $|\mathbf{M}| = |\mathbf{P}|h$, or

$$|\mathbf{M}| = KL \cdot w. \tag{1}$$

Therefore: *the moment of the force* $\mathbf{P}$ *with respect to the point* A *is* (in absolute value) *proportional to the segment which the sides of the string polygon cut off from the line passing through* A *and parallel to* $\mathbf{P}$; *the factor of proportionality is the distance of the pole from the force in the polygon of forces.*

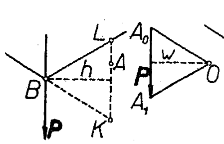


Fig. 175.

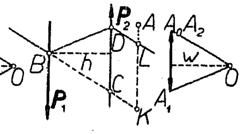


Fig. 176.

Let there now be given a force couple $\mathbf{P}_1$, $\mathbf{P}_2$ (where $\mathbf{P}_1 = -\mathbf{P}_2$). We draw the string polygon of this couple as in Fig. 176. Let us pass through an arbitrary point A a line parallel to the forces and denote by K and L the points of intersection of this line with the extreme sides of the string polygon.

We shall prove that if we denote the moment of the couple by $\mathbf{M}$ and the distance of the pole from the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ in the polygon of forces by w, then formula (1) will hold.

For let us consider the triangle BCD, where B is a point on the direction of the force $\mathbf{P}_1$ from which we started to draw the string polygon, while C and D are the points of intersection of the direction of the force $\mathbf{P}_2$ with the extreme sides of the string polygon.

Let h denote the altitude of the triangle BCD. From the similarity of the triangles BCD and A_0A_1O we have $CD : |\mathbf{P}_1| = h : w$, from which $|\mathbf{P}_1|h = CD \cdot w$. Since $|\mathbf{M}| = |\mathbf{P}_1|h$ and $CD = KL$, we get formula (1).

Consequently: *the moment of a force couple is* (in magnitude) *proportional to the segment which the extreme sides of the string polygon cut off from an arbitrary line parallel to the force couple; the factor of proportionality is the distance of the pole from the force in the force polygon.*

Similar methods of determining the moment are useful when we are dealing with several parallel forces, because then we can take the same w for all the forces (Fig. 177). If we have to determine the moment of a system of parallel forces we determine at first the resultant (or the resultant force couple) and then its moment.

Having drawn the string polygon of a system of parallel forces, we can determine with respect to A the moment of an arbitrary part of the system consisting of the forces following each other in the same order as in the force polygon. This can be done because the string polygon of this part is included in the string polygon of the entire system. In the drawing the segment $K'L'$ is proportional (in absolute value) to the moment of the system of forces $\mathbf{P}_2$, $\mathbf{P}_3$ with respect to A.

Finally, let the system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5, \mathbf{R}_1, \mathbf{R}_2$ be given, equipollent to zero (Fig. 174). Through an arbitrary point C we pass a line parallel to the forces. The segment LK of this line lying between the sides of the string polygon is proportional (in absolute value) to the total moment with respect to C of the forces situated on one side of this line (in our case to the moment of the forces $\mathbf{R}_1$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ or to that of the forces $\mathbf{P}_4$, $\mathbf{P}_5$, $\mathbf{R}_2$; the moments of both parts of the system with respect to C are equal in absolute value, since their sum is zero as a consequence of the assumption that the system is equipollent to zero). The factor of proportionality is w, i. e. the distance of the pole O from the forces in the force polygon.

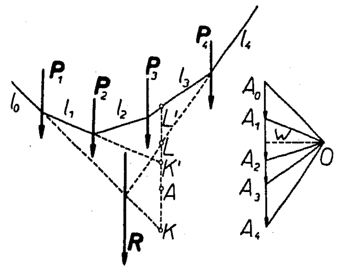


Fig. 177.

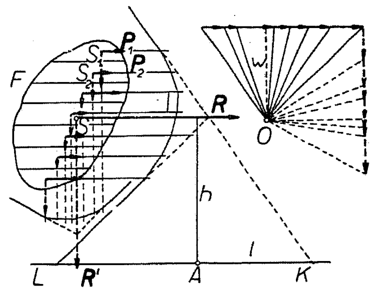


Fig. 178.

Determination of the centre of gravity and of the statical moment of plane figures. In order to determine the centre of gravity of a plane figure F, we divide it into strips by means of parallel lines. If at the centres of gravity $S_1, S_2, \ldots$ of the strips obtained we attach the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ which are parallel, have the same sense, and are proportional in magnitude to the areas $F_1, F_2, \ldots$ of these strips, then the centre of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ will be the center of gravity of the figure F.

For let us note that the centre of gravity of the figure F is the centre of mass of the system of material points which is obtained if each strip is replaced by a material point whose mass is equal to the mass of the strip

(p. 154); and by the theorem proved on p. 239 the centre of mass of the system of material points obtained is the centre of the system of parallel forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$

Strips that are sufficiently narrow can be considered as trapezoids; the centres of gravity of the trapezoids can be determined according to the construction given on p. 177, Fig. 118. The lines of division of the figure are usually drawn at equal intervals (Fig. 178). Hence the areas of the trapezoids will be proportional to their medians. The magnitudes of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ can therefore be considered as proportional to the medians of the trapezoids.

The resultant $\mathbf{R}$ passes through the centre of gravity S; we determine it by means of the string polygon (Fig. 178).

Changing the direction of the forces and determining a new resultant $\mathbf{R}'$, we obtain the centre of gravity S as the point of intersection of both resultants $\mathbf{R}$ and $\mathbf{R}'$.

In order to determine the statical moment of the figure F with respect to a certain line l, we draw the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ parallel to l.

The moment of the resultant $\mathbf{R}$ with respect to an arbitrary point A of the line l is in magnitude proportional to the statical moment of the figure F with respect to l.

For denoting by $\mathbf{M}$ the moment of the force $\mathbf{R}$ with respect to A, by h the distance of A from the direction of the force $\mathbf{R}$, by M_s the statical moment of the given figure with respect to l, finally by F the area of the figure, we have

$$|\mathbf{M}| = h|\mathbf{R}|, \quad |M_s| = hF, \tag{2}$$

where the centre of gravity lies on the direction of the resultant.

Since the magnitudes of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ have been chosen as proportional to the areas $\mathbf{F}_1, \mathbf{F}_2, \ldots$ of the individual strips,

$$|\mathbf{P}_1| = \lambda F_1, \quad |\mathbf{P}_2| = \lambda F_2, \quad \text{etc.}, \tag{3}$$

where λ is the factor of proportionality. But

$$|\mathbf{R}| = |\mathbf{P}_1| + |\mathbf{P}_2| + \ldots,$$

whence

$$|\mathbf{R}| = \lambda(F_1 + F_2 + \ldots) = \lambda F.$$

In virtue of (2), therefore, $|\mathbf{M}| = \lambda hF = \lambda |M_s|$, whence

$$|M_s| = |\mathbf{M}| \,/\, \lambda. \tag{4}$$

Consequently: *the statical moment of a figure is* (in absolute value) *proportional to the moment of the resultant.*

Statical moments of plane figures can therefore be determined by means of a string polygon.

In Fig. 178, $|\mathbf{M}| = w \cdot KL$, whence by (4)

$$|M_s| = w \cdot KL \,/\, \lambda \tag{5}$$

Let us denote by $d_1, d_2, \ldots$ the lengths of the medians of the trapezoids and by a the distance between the lines of division. Therefore $F_1 = ad_1, F_2 = ad_2, \ldots$ Since it was assumed in the drawing that $|\mathbf{P}_1| = kd_1, |\mathbf{P}_2| = kd_2, \ldots$, where $k = \frac{1}{3}$, it follows that, $|\mathbf{P}_1| = kF_1 \,/\, a, |\mathbf{P}_2| = kF_2 \,/\, a, \ldots$ By (3) we then have $\lambda = k \,/\, a$, whence by (5)

$$|M_s| = aw \cdot KL \,/\, k = 3aw \cdot KL.$$

Measuring a, w, and KL, in the drawing, we obtain $|M_s|$ from the above formula.

II. CONSTRAINED BODY

§ 8. Conditions of equilibrium. A rigid body is said to be *constrained* if the positions or the motions of this body are subject to certain conditions. These conditions are called *constraints*.

For example, if one point of a body is fixed, the body can turn only about this point. If two points A and B are fixed, the body can turn only about the line AB. Later we shall learn of still other examples of constrained rigid bodies.

When a constrained rigid body is in equilibrium we say that the forces acting on this body *balance one another* or *are in equilibrium*.

A rigid body fixed at the two points A, B (Fig. 179) and being in equilibrium, will remain in equilibrium when we add an arbitrary force $\mathbf{P}$ whose origin is at the point C lying on the line AB. This is evident intuitively because the body can turn only about the axis AB, and hence the force $\mathbf{P}$ acting on the fixed axis cannot move the body. If the body were free, then it would remain in equilibrium only in the case when $\mathbf{P} = 0$.

We see from this that the conditions for the equilibrium of a free body are different from those for a constrained body.

The investigation of the conditions for equilibrium in the case of a constrained rigid body can be reduced to the case of a free body. With this in view we shall assume that besides the given forces the constrained rigid body is acted upon by additional forces called *reactions* which cause the body to maintain the constraints. The reactions arise from those bodies which limit the freedom of the motions of the given constrained rigid body.

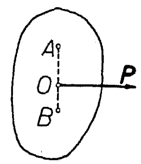


Fig. 179.

For example, if a heavy body rests on a table, then it is not free because it cannot pass through the surface of the table. In this case the reactions are the forces with which the surface of the table presses on the body.

The other forces acting on a constrained rigid body will be called *acting forces* (in order to differentiate them from the reactions). If we introduce the reactions to the acting forces, then we can consider the constrained rigid body as free.

It follows from this that *the necessary and sufficient condition for the equilibrium of the acting forces is that the acting forces balance the reactions.*

However, this condition is not convenient because it involves the forces of reaction which are in general unknown. In some instances, as in the case of a body fixed at one point or two points, we can nevertheless give conditions for the equilibrium of the acting forces without reference to the reactions (p. 270). The condition for equilibrium in which the reactions do not occur is the so-called *principle of virtual work* which we shall consider in chapter IX.

§ 9. Reactions of bodies in contact. Every two rigid bodies (solids, surfaces or lines) which are in contact with each other act on each other with certain forces. These forces are reactions and they arise from the actions of the points of both bodies on each other. Reactions conform to the law of action and reaction.

By the theorem on reduction (p. 237) the forces with which one body acts on the other can always be replaced by a force $\boldsymbol{R}$ and a couple of moment $\boldsymbol{M}$. Conversely, by the law of action and reaction, the second body acts on the first body with forces equipollent to the force $-\boldsymbol{R}$ (with the same origin as $\boldsymbol{R}$) and a couple of moment $-\boldsymbol{M}$.

The determination of reactions is very important in problems connected with engineering. So far we do not yet have a theory which solves this matter in its entirety. In practice we make use of certain hypotheses agreeing approximately with experience. We shall consider here only certain problems concerning the reactions of bodies in contact. This matter is taken up fully in textbooks on engineering mechanics.

Experience reveals that in rigid bodies in contact, only those points which are situated near the points of contact act on one another. Let us assume here the simplifying hypothesis that only the points of contact of both bodies act on one another; the reactions will then be the forces acting at the points of contact.

This hypothesis does not hold in all generality. According to this hypothesis, the reactions of two rigid bodies in contact only at one point would be reduced to one force having its origin at the point of contact. On the other hand, experience teaches that in addition to this force there can still appear a force couple whose moment is different from zero, which is contrary to the hypothesis.

For example, if a heavy rigid sphere rests on a rigid horizontal plate, then it can remain in equilibrium even if it is acted upon by a force couple (lying in the horizontal plane) of small moment. In the state of equilibrium the reactions of the plate balance the weight of the sphere as well as the force couple, which would be impossible were the reactions of the plate reduced to only one force acting at the point of tangency.

Normal and tangential reactions. Let two rigid bodies I and II be in contact at the point A (Fig. 180). Let us denote by $\boldsymbol{R}$ the force with which body II acts on body I at the point A. The force $\boldsymbol{R}$ has its origin at A. By the law of action and reaction body I acts on body II with a force $-\boldsymbol{R}$, whose origin is also at A.

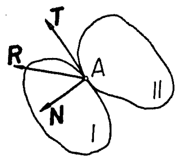


Fig. 180.

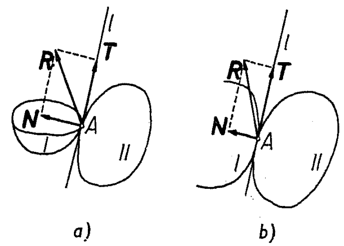


Fig. 181.

Let body I be a solid or a surface having a tangent plane Π at the point A.

Let us resolve the reaction $\boldsymbol{R}$ into two components: a component $\boldsymbol{N}$ perpendicular to the body, i. e. to the plane Π, and a component $\boldsymbol{T}$ tangent to the body, i. e. lying in Π.

The component $\boldsymbol{N}$ is called the *normal* reaction, and the component $\boldsymbol{T}$ the *tangential* reaction or the *friction*. The normal reaction is usually directed with respect to body II to that side in which body I is situated; it is then called the *pressure*. If there is no friction at the points of contact, the bodies in contact are called *smooth*.

Let us consider two more cases:

1° Body I is a surface bounded by a certain curve on which the point of contact A lies, and the bounding curve has a straight line tangent l at A (Fig. 181a).

2° Body I is a curve having a straight line tangent l at A, where A is not the end of this curve (Fig. 181b).

An example of 1° can be a rigid hemisphere bounded by a circumference on which the point A lies; an example of 2° can be an arc of a circumference with the point A lying at its midpoint. In cases 1° and 2°

the normal reaction will be the component of the reaction **R** perpendicular to the tangent l; the friction will be the component of the reaction **R** lying on the line l.

For two smooth bodies in contact at the point A, the direction and sense of the reaction are determined if one of the bodies is a solid or a surface possessing a tangent plane at A. The direction and sense of the reaction are also determined for bodies 1° and 2° if the lines tangent to them at their point of contact do not coincide. For in this case the reaction must be perpendicular to both tangents. For bodies, one of which is body 1° or 2°, we know only this about the reaction, namely, that it lies in the plane perpendicular to the tangent line l.

Supports. A fixed rigid body (e. g. one attached rigidly to the earth) is called a *support*. In many applications it is necessary to determine the reactions of the supports on other rigid bodies.

If a rigid body resting on supports is in equilibrium, then the forces acting on this body balance the reactions of the supports. If a smooth body rests on smooth supports, then we assume that *reactions* (obviously normal) *are induced which balance the forces acting on the body.*

Because of this hypothesis we can in many cases give the necessary and sufficient conditions for the equilibrium of forces which act on a rigid body resting on smooth supports.

Centre of pressure. Let two smooth bodies I and II be in contact at the points lying in a certain plane Π (Fig. 182). The reactions will therefore be perpendicular to the plane Π. The reactions acting on the body I are consequently parallel; let us assume that they are pressures. Hence they have the same sense. It follows from this that they have a resultant **R** which we can assume to be acting at a certain point O of the plane Π. The point O is called the *centre of pressure.*

Obviously the reactions acting on the other body have a resultant —**R** and the same centre of pressure.

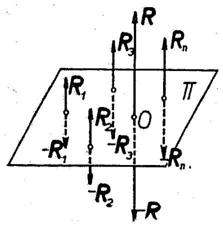


Fig. 182.

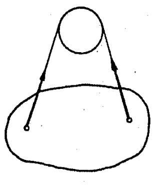


Fig. 183.

Reactions of a string. An inextensible string fastened to a body acts on it only when the string is in tension. If the mass of the string is small (so that it can be neglected) and both ends are fastned to the body, then

the string acts at both ends with forces which are equal in magnitude also in the case when it is wound around some smooth body (Fig. 183). The forces with which the string acts at its ends are tangent to the string and have senses in the direction of the string. These forces are called the *tensions of a string.*

Example 1. If a heavy body hanging on a string at the point A is in equilibrium, then the tension **T** of tbe string whose origin is A balances the weight **Q** whose origin is at the centre of gravity S.

Consequently $\mathbf{T} + \mathbf{Q} = 0$, or

$$|\mathbf{T}| = |\mathbf{Q}|. \qquad (1)$$

Moreover, the forces **T** and **Q** must act along one line. The string is therefore directed vertically and its prolongation passes through the centre of gravity (Fig. 184). Hence, hanging the body in succession from two points and drawing the directions of the string in the body, we obtain as the point of intersection the centre of gravity of the body.

Example 2. If a body hanging by two strings at the points A and B is in equilibrium, then the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ with its origins at A and B balance the weight **Q** whose origin is at the centre of gravity S (Fig. 185). Consequently

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{Q} = 0. \qquad (2)$$

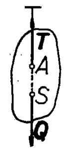


Fig. 184.

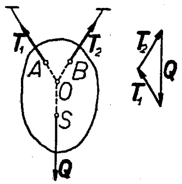


Fig. 185.

Therefore by the theorem given on p. 246 the directions of the forces either intersect at the point O or the forces are parallel. In both cases we can determine the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ by taking the moment with respect to an arbitrary point, e. g. with respect to the point A. Denoting by a_2 and d the arms of the forces $\mathbf{T}_2$ and **Q** with respect to A, we get $|\mathbf{T}_2|a = |\mathbf{Q}|d$, or

$$|\mathbf{T}_2| = |\mathbf{Q}|d\,/\,a. \qquad (3)$$

Similarly, we obtain $|\mathbf{T}_1|$ by taking the moment with respect to B. In the case when the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ are not parallel, we can determine them graphically by forming the triangle of forces (Fig. 185).

Example 3. A heavy rigid rod hangs at the ends A and B of a massless inextensible string passing through a smooth ring at the point C (Fig. 186a). Determine the tension of the string in the position of equilibrium.

Let us denote the length of the string by l, the angle ACB by φ, and the centre of gravity of the rod by S. Let us put:

$$AB = a, \quad AC = l_1, \quad BC = l_2, \quad AS = b.$$

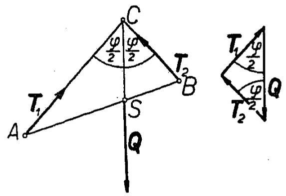


Fig. 186.

Let us suppose that a, b, l, and the weight of the rod $\mathbf{Q}$, are given.

Since the tensions $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ of the string balance the weight $\mathbf{Q}$, these forces intersect at the point C (because the forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ pass through the point C (p. 246)) and moreover

$$\boldsymbol{T}_1 + \boldsymbol{T}_2 + \mathbf{Q} = 0. \tag{4}$$

In addition to this (p. 261)

$$|\boldsymbol{T}_1| = |\boldsymbol{T}_2|. \tag{5}$$

When $\varphi = 0$, the rod has a vertical position and the forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ have vertical directions. Therefore by (5) $\boldsymbol{T}_1 = \boldsymbol{T}_2$, whence by (4)

$$|\boldsymbol{T}_1| = |\boldsymbol{T}_2| = \tfrac{1}{2}|\mathbf{Q}|.$$

Let us inquire in what case φ can be different from zero, as in Fig. 186. Let us therefore assume that $\varphi \neq 0$.

Denoting by d_1 and d_2 the distances of the directions of the forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ from S and taking the moment with respect to S, we get $|\boldsymbol{T}_1|d_1 = |\boldsymbol{T}_2|d_2$; consequently from (5), $d_1 = d_2$. The centre of gravity S is equidistant from the sides AC and BC, i. e. the line CS is the bisector of the angle φ. From a known geometrical theorem concerning the angle bisectors of a triangle, we obtain $AC : BC = AS : BS$, i. e.

$$l_1 : l_2 = b : (a - b). \tag{6}$$

Since

$$l_1 + l_2 = l, \tag{7}$$

solving the system of equations (6) and (7), we get:

$$l_1 = bl / a, \quad l_2 = (a - b)\, l / a. \tag{8}$$

In order that the sides l_1, l_2, a, form a triangle, the inequalities $l_1 + l_2 > a$, $l_1 + a > l_2$, $l_2 + a > l_1$ must hold. They can be written in the form:

$$l_1 + l_2 > a, \quad a > |l_1 - l_2|,$$

whence in virtue of (8)

$$l > a > |a - 2b|\, l / a. \tag{9}$$

The inequalities $a < l < a^2 / |a - 2b|$ must therefore be fulfilled, or, setting $k = b / a$,

$$a < l < a / |1 - 2k|. \tag{10}$$

Hence: *when $\varphi \neq 0$ equilibrium will occur if the length l satisfies condition* (10).

Let us note that if $k = b / a = \frac{1}{2}$ (i. e. if the centre of gravity S falls at the centre of the segment AB), then conditions (9) are satisfied for all $l > a$. In this case, therefore, the position of equilibrium of the rod is always possible when $\varphi \neq 0$.

Angle φ is obtained from the theorem of Carnot

$$a^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos\varphi. \tag{11}$$

From the triangle of forces we get

$$|\boldsymbol{T}_1| = |\boldsymbol{T}_2| = \tfrac{1}{2}|\mathbf{Q}| / \cos\tfrac{1}{2}\varphi. \tag{12}$$

After expressing $\cos\frac{1}{2}\varphi$ in terms of a, b, l, from formulae (8) and (11), we obtain by (12)

$$|\boldsymbol{T}_1| = |\boldsymbol{T}_2| = |\mathbf{Q}|\, \frac{l}{a} \sqrt{\frac{(a-b)\,b}{l^2 - a^2}}.$$

In particular, for $b = \frac{1}{2}a$ we get

$$|\boldsymbol{T}_1| = |\boldsymbol{T}_2| = |\mathbf{Q}|\, \frac{l}{2\sqrt{l^2 - a^2}}.$$

Example 4. A horizontal beam rests at the points A and B on two smooth supports. Vertical forces (directed downwards) $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_5$ (Fig. 174) act on the beam. Determine the reactions of the supports.

Let us denote by $x_1, x_2, \ldots, x_5$ the distances of the points of application of the forces from A. Put $AB = d$. The reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ at A and B are vertical. Taking the moment with respect to A and denoting by $P_1, P_2, \ldots, P_5, R_1, R_2$ the absolute values of the forces, we obtain $P_1x_1 + P_2x_2 + \ldots + P_5x_5 - R_2d = 0$, whence

$$R_2 = (P_1x_1 + P_2x_2 + \ldots + P_5x_5) / d. \tag{13}$$

Since $R_1 + R_2 = P_1 + P_2 + \ldots + P_5$,

$$R_1 = [P_1(d - x_1) + \ldots + P_5(d - x_5)] / d. \tag{14}$$

The reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ can also be determined by means of the string polygon as on p. 253.

Example 5. A heavy sphere of constant density touches a smooth plane Π inclined at an angle α with the horizontal (Fig. 187). Determine the horizontal force $\boldsymbol{P}$ maintaining the sphere in equilibrium.

The origin of the weight **Q** of the sphere is at its centre O, and that of the reaction **R** of the plane Π at the point of tangency A; the reaction is perpendicular to Π. The forces **Q** and **R** intersect at the point O. Since the forces **P**, **Q** and **R** balance one another, in virtue of theorem III, p. 246, they intersect at the point O. Moreover $\mathbf{P}+\mathbf{Q}+\mathbf{R}=0$; hence the forces **P** and **R** are obtained from the triangle of forces. We have

$$R = Q / \cos\alpha, \quad P = Q\tan\alpha,$$

where P, Q and R, denote the absolute values of the corresponding forces.

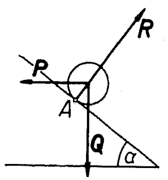


Fig. 187.

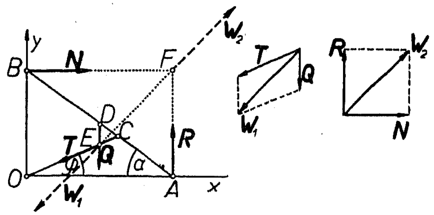


Fig. 188, 189, 190.

Example 6. A heavy rod AB of constant density lies in the vertical plane Π and rests against two smooth planes: a horizontal plane Π_1 and a vertical plane Π_2. Let Ox and Oy be the lines of intersection of the planes Π_1 and Π_2 with the plane Π. The rod AB is tied by an inextensible string OC to the point O. The rod is in equilibrium. Determine the reactions, having been given: $AB = 2l$, the angle α between AB and the x-axis, and the angle φ between OC and the x-axis (Fig. 188).

The force acting on the rod is the weight **Q** acting at the midpoint D of the rod AB. The reactions are the reactions of the planes **R** and **N**, acting at A and B and perpendicular to the planes, as well as the reaction of the string **T** acting at C and directed along the string towards the point O. The acting force balances the reactions. From the condition of projections on the x and y axes we obtain

$$N_x + T_x = 0, \quad R_y + Q_y + T_y = 0, \tag{15}$$

and from the condition of moment with respect to O:

$$-R_y\cdot 2l\cos\alpha + N_x\cdot 2l\sin\alpha - Q_y\cdot l\cos\alpha = 0. \tag{16}$$

Let us denote by R, N, T, Q, the absolute values of the corresponding forces. We obviously have $R_y = R$, $N_x = N$, $Q_y = -Q$ and $T_x = -T\cos\varphi$, $T_y = -T\sin\varphi$. Consequently from equations (15) and (16) we obtain:

$$N - T\cos\varphi = 0, \quad R - Q - T\sin\varphi = 0, \tag{17}$$

$$-2R\cos\alpha + 2N\sin\alpha + Q\cos\alpha = 0. \tag{18}$$

Determining R and N from equations (17) and substituting in (18) we obtain

$$T = \frac{Q\cos\alpha}{2\sin(\alpha-\varphi)}, \tag{19}$$

whence by equations (17)

$$N = \frac{Q\cos\alpha\cos\varphi}{2\sin(\alpha-\varphi)}, \quad R = Q\left(1+\frac{\cos\alpha\sin\varphi}{2\sin(\alpha-\varphi)}\right). \tag{20}$$

Since $T > 0$, in virtue of (19) $\alpha > \varphi$; hence the point C must lie between A and D.

The problem can also be solved graphically (Fig. 189 and 190).

Denote by E the point of intersection of the forces **T** and **Q**, and by F that of the forces **N** and **R**. The resultant $\mathbf{W}_1$ of the forces **T** and **Q** acts at E, whereas the resultant $\mathbf{W}_2$ of the forces **R** and **N** acts at F. Since the system of forces **N**, **R**, **T**, **Q** is equipollent to zero, the system of forces $\mathbf{W}_1$, $\mathbf{W}_2$ is also equipollent to zero. Hence the forces act along the line EF and $\mathbf{W}_1 + \mathbf{W}_2 = 0$. The force **Q** as well as the directions of the forces **T** and $\mathbf{W}_1 = \mathbf{T} + \mathbf{Q}$ are given; therefore we can determine the forces $\mathbf{W}_1$ and **T** as in Fig. 189. Since $\mathbf{N}+\mathbf{R} = \mathbf{W}_2 = -\mathbf{W}_1$, the forces **N** and **R** are obtained by resolving the force $\mathbf{W}_2$ into components in the directions of the x and y axes (Fig. 190).

Example 7. A heavy rod AB whose centre of gravity is at S rests on a smooth horizontal plane at the point A and on a smooth sphere at the point B (Fig. 191). An inextensible string fastened at A passes over a pulley C and sustains a weight **P** at its other end. Determine the weight **P**, the reaction **R** of the horizontal plane and the reaction **N** of the sphere in the position of equilibrium, having been given $a = AS$, $b = AB$, the angle α between the rod and the plane, and the weight **Q** of the rod.

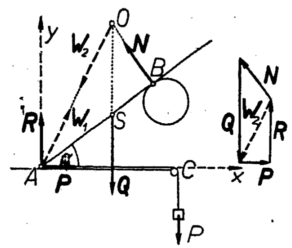


Fig. 191.

Let us choose the axes x and y as in drawing. Since the tension in the string at the point A is **P**, denoting by P, Q, R, N, the absolute values of the forces and forming projections on the x and y axes, we get:

$$P - N\sin\alpha = 0, \quad R - Q + N\cos\alpha = 0. \tag{21}$$

The total moment with respect to A is

$$Qa\cos\alpha - bN = 0. \tag{22}$$

From equations (21) and (22) we obtain

$$N = Q\frac{a\cos\alpha}{b},\quad P = Q\frac{a\sin 2\alpha}{2b},\quad R = Q\left(1-\frac{a}{b}\cos^2\alpha\right).$$

The problem can also be solved graphically. With this in view, let us denote by $\mathbf{W}_1$ the resultant of the forces $\mathbf{R}$ and $\mathbf{P}$, and by $\mathbf{W}_2$ the resultant of the forces $\mathbf{Q}$ and $\mathbf{N}$. The force $\mathbf{W}_1 = \mathbf{R} + \mathbf{P}$ acts at the point A, and $\mathbf{W}_2 = \mathbf{Q} + \mathbf{N}$ at the point O in which the directions of the forces $\mathbf{Q}$ and $\mathbf{N}$ intersect. Since we know the positions of the forces $\mathbf{Q}$ and $\mathbf{N}$, the point O can be determined.

The forces $\mathbf{P}$, $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{N}$, are in equilibrium; therefore the forces $\mathbf{W}_1$ and $\mathbf{W}_2$ balance each other. Consequently $\mathbf{W}_1 + \mathbf{W}_2 = 0$; moreover the forces $\mathbf{W}_1$ and $\mathbf{W}_2$ lie on one line. This line is obviously the line AO. Since we already know the direction of the force $\mathbf{W}_2$, we can determine $\mathbf{W}_2$ and $\mathbf{N}$ from the relation $\mathbf{W}_2 = \mathbf{Q} + \mathbf{N}$ by drawing the triangle of forces. We have $\mathbf{W}_1 = -\mathbf{W}_2$, and $\mathbf{W}_1 = \mathbf{R} + \mathbf{P}$, hence we obtain the forces $\mathbf{R}$ and $\mathbf{P}$ by resolving the force $\mathbf{W}_1$ in the directions of the forces $\mathbf{R}$ and $\mathbf{P}$.

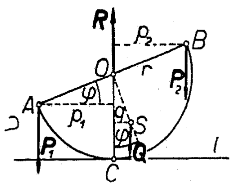


Fig. 192.

Example 8. A heavy rigid wire of constant density, in the form of a semicircle, lies in a vertical plane and rests on a horizontal line l (Fig. 192). Forces $\mathbf{P}_1$ and $\mathbf{P}_2$, directed vertically downwards, act at the ends A and B of the wire. Determine the angle φ which the diameter AB makes with the horizontal in the position of equilibrium, as well as the reaction $\mathbf{R}$ at the point of tangency C (under the assumption that there is no friction).

In the position of equilibrium the forces $\mathbf{P}_1$, $\mathbf{P}_2$, the weight $\mathbf{Q}$ acting at the centre of mass S, and the reaction $\mathbf{R}$ perpendicular to l, balance one another. Since these forces are parallel, (denoting their absolute values by P_1, P_2, Q, and R), we obtain from the condition of projections on the y-axis, which is directed vertically upwards, $-P_1 + R - Q - P_2 = 0$, whence

$$R = P_1 + P_2 + Q. \tag{23}$$

Let us calculate the total moment of the forces with respect to the point of tangency C. From the condition of moment we obtain

$$-P_1p_1 + Qq + P_2p_2 = 0, \tag{24}$$

where p_1, p_2, and q, denote the arms of the forces P_1, P_2, and Q, with respect to C. Putting $r = OB$ (where O is the centre of the diameter AB), we obtain:

$$p_1 = p_2 = r\cos\varphi,\quad q = OS\cdot\sin\varphi, \tag{25}$$

and since $OS = 2r/\pi$ (p. 176),

$$q = \frac{2r\sin\varphi}{\pi}. \tag{26}$$

From (25) and (26) we get, after substituting in (24),

$$(P_2 - P_1)\,r\cos\varphi + \frac{2rQ\sin\varphi}{\pi} = 0,$$

whence

$$\tan\varphi = \frac{(P_1 - P_2)\,\pi}{2Q}. \tag{27}$$

§ 10. Friction. Let two bodies I and II, which are at rest, be in contact at the point A and let them have a common tangent plane Π at this point (Fig. 193). Let us denote by $\mathbf{R}$ the reaction which body II exerts on body I at the point A. If the bodies are not smooth, then the reaction $\mathbf{R}$ is not perpendicular to the plane Π.

Let α be the angle which $\mathbf{R}$ makes with the normal n to Π.

Denoting the normal component by $\mathbf{N}$, the tangential component or *friction* by $\mathbf{T}$, and putting $R = |\mathbf{R}|$, $N = |\mathbf{N}|$, $T = |\mathbf{T}|$, we obtain:

$$T = R\sin\alpha,\quad N = R\cos\alpha, \tag{1}$$

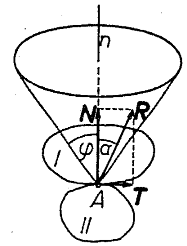


Fig. 193.

whence

$$T = N\tan\alpha. \tag{2}$$

Experiment shows that the angle α cannot exceed a certain limit which depends on the nature of the surfaces I and II.

Let us denote by φ the maximum value of the angle α at the point A for a given pair of bodies I and II in the position of equilibrium. We therefore have $0 \leqq \alpha \leqq \varphi$, i. e. $0 \leqq \tan\alpha \leqq \tan\varphi$, whence by (2)

$$T \leqq N\tan\varphi. \tag{3}$$

Putting $f = \tan\varphi$, we get

$$T \leqq Nf. \tag{4}$$

The number f is called the *static coefficient of friction* for a given pair of bodies I and II at the point of contact A.

Let us consider a cone of revolution whose vertex is at A, and whose axis is the normal n inclined at an angle φ with respect to the generatrices of the cone. This cone is called the *cone of friction* at the point A.

Since $\alpha \leqq \varphi$, *the reaction lies within the cone of friction or on its surface.*

Exactly as for smooth supports (p. 259), we also assume in the case of friction the following principle:

If a rigid body rests on supports and reactions (lying within the cones of friction), which balance the forces acting on the body, are possible, then such reactions are actually induced (if the body was initially at rest).

Example I. A heavy rod AB, lying in a vertical plane, rests against a vertical plane and a horizontal plane (Fig. 194). The coefficients of friction at A and B are f_1 and f_2. Examine the conditions for equilibrium.

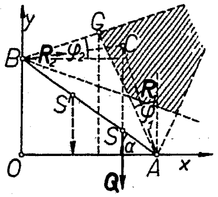


Fig. 194.

Let us consider the cones of friction at A and B. The generatrices of these cones are inclined to the normals at A and B at the angles φ_1 and φ_2, where $\tan\varphi_1 = f_1$, and $\tan\varphi_2 = f_2$. The reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at A and B must lie within or on the surface of these cones.

Three forces act on the rod AB: $\mathbf{R}_1$, $\mathbf{R}_2$, and the weight $\mathbf{Q}$ acting at the centre of gravity S.

If the rod is in equilibrium, then the directions of these forces pass through one point C (p. 246). This point must obviously lie in a region common to both cones of friction (*vide* shaded region in the figure) because the directions of there actions can only intersect in this region. The direction of the weight must therefore pass through the region common to both cones of friction.

Conversely, if the direction of the weight $\mathbf{Q}$ passes through the region common to the cones of friction, then the rod can remain in equilibrium. For let us choose on the vertical passing through S an arbitrary point C within the region common to the cones of friction. It is easy to see that reactions $\mathbf{R}_1$ and $\mathbf{R}_2$, having directions AC and BC and balancing the weight $\mathbf{Q}$, can occur. Therefore the rod can in this case remain in equilibrium.

If the centre of gravity were at a point S' such that the vertical passing through this point did not cut the region common to the cones of friction, then the equilibrium of the rod would be impossible.

Consequently: *the necessary and sufficient condition for the equilibrium of the rod is that the direction of the weight pass through the region common to the cones of friction.*

Let us put: $AB = l$, $AS = d$, $BS = d'$ and let us denote by α the angle which AB makes with the horizontal. Let us choose the x and y axes of the coordinate system as in the drawing and assume that the rod is in equilibrium. From the conditions of projections and of moment with respect to O we obtain

$$R_{1_x} + R_{2_x} = 0, \quad R_{1_y} + R_{2_y} - Q = 0, \tag{5}$$

as well as $-R_{1_y} l \cos\alpha + R_{2_x} l \sin\alpha + Qd' \cos\alpha = 0$, i. e.

$$-R_{1_y} l + R_{2_x} l \tan\alpha + Qd' = 0. \tag{6}$$

Moreover, we have the following inequalities:

$$|R_{1_x}| \leqq R_{1_y} f_1, \quad |R_{2_y}| \leqq R_{2_x} f_2. \tag{7}$$

The reactions cannot be determined from formulae (5) and (6). Relations (5)—(7) permit us only to give limits which the components of the reactions cannot exceed.

Let us denote by x_0 the abscissa of the point S, and by ξ the abscissa of the point G at which the extreme generatrices of the cones of friction intersect. Equilibrium will result if

$$\xi \leqq x_0. \tag{8}$$

In order to determine ξ, let us write the equations of the lines BG and AG:

$$y = f_2 x + l \sin\alpha, \quad y = -(x - l\cos\alpha) / f_1.$$

The point G is the point of intersection of these lines; hence

$$\xi = \frac{1 - f_1 \tan\alpha}{1 + f_1 f_2} l \cos\alpha. \tag{9}$$

Since $x_0 = d' \cos\alpha$, the inequality (8) assumes the form

$$(1 - f_1 \tan\alpha) / (1 + f_1 f_2) \leqq d' / l. \tag{10}$$

Equilibrium follows if the left side of this inequality is a negative number or zero. In this case $1 - f_1 \tan\alpha \leqq 0$; hence $1 / f_1 \leqq \tan\alpha$ or $\cot\varphi_1 \leqq \tan\alpha$; consequently $\frac{1}{2}\pi - \varphi_1 \leqq \alpha$.

Therefore if $\frac{1}{2}\pi - \varphi_1 \leqq \alpha$, then equilibrium follows. On the other hand, if $\frac{1}{2}\pi - \varphi_1 > \alpha$, then the left side of the inequality (10) will be positive and equilibrium will not occur for too small d'.

It is easy to verify these results in the drawing (p. 268).

Example 2. A beam I passes loosely through a groove in beam II. A force $\boldsymbol{P}$ parallel to beam I acts on beam II. The beams are pressed to each other at the points A and B (Fig. 195).

Let us determine the cones of friction at A and B. If the direction of the force $\boldsymbol{P}$ passes through the region common to the cones of friction, then reactions $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ balancing the force $\boldsymbol{P}$ and acting on the beam II will appear at the points A and B. Then beam II will not move. The beam *sticks fast.*

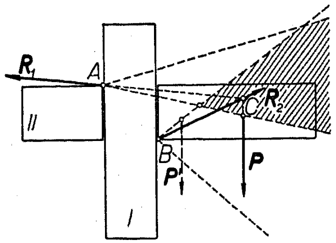


Fig. 195.

From the drawing it is easy to see that beam II will move if the force $\boldsymbol{P}'$ has its origin near beam I and is parallel to it. For then the direction of the force $\boldsymbol{P}'$ will not pass through the common part of both cones of friction; equilibrium will therefore be impossible.

§ 11. Conditions for equilibrium not involving the reaction. The condition for the equilibrium of forces acting on a rigid body given on p. 257 expresses the relation that obtains between the acting forces and the reactions. We now give several examples in which the conditions for the equilibrium of the acting forces can be made to refer only to the acting forces without including the reaction.

Body with one fixed point. Let a rigid body have one fixed point, e. g. the point O. We can therefore assume that the body is free and that the point O is acted upon by a reaction $\boldsymbol{R}$ holding the point O (Fig. 196).

Let us further assume that the body is in equilibrium under the action of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$. These forces consequently balance the reaction $\boldsymbol{R}$. From the conditions of equilibrium it follows that the sum of the forces and the moment with respect to the point O are equal to zero, i. e.

$$\boldsymbol{R} + \Sigma \boldsymbol{P}_i = 0, \tag{1}$$

$$\Sigma \mathrm{Mom}_O \boldsymbol{P}_i = 0. \tag{I}$$

The reaction $\boldsymbol{R}$ (being a force whose origin is at O) does not appear in equation (I) because its moment with respect to O is zero. From equation (1) we can determine $\boldsymbol{R}$. We have

$$\boldsymbol{R} = -\Sigma \boldsymbol{P}_i. \tag{2}$$

Equation (I) constitutes the necessary condition which the acting forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, must satisfy in the case of equilibrium.

We shall now prove that condition (I) is also a sufficient condition for equilibrium.

Let us assume that the system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, satisfies equation (I), i. e. that the moment of this system of forces with respect to O is zero. It follows from this that the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, have a resultant $\boldsymbol{P} = \Sigma \boldsymbol{P}_i$ whose origin is at O (p. 26). Now the force $\boldsymbol{P}$ acting at O cannot move the given body which is at rest, because the point O is fixed. The system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$ (equipollent to the force $\boldsymbol{P}$), is therefore in equilibrium.

Hence: *A necessary and sufficient condition that a system of forces acting on a rigid body fixed at one point O be in equilibrium is that the total moment of the acting forces with respect to the point O be zero* (or that the system have a resultant passing through O).

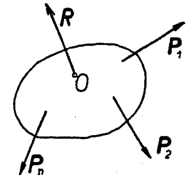


Fig. 196.

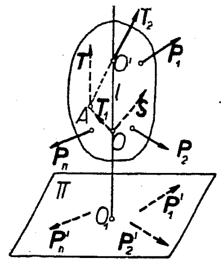


Fig. 197.

Body with a fixed axis. Let a rigid body have a certain fixed line l (it is sufficient for this purpose, for example, to fix two points of this line). We can assume that the body is free and that the points lying on the axis are acted upon by forces of reaction which cause the axis to be fixed.

Let us assume that the body is in equilibrium under the action of the forces $\{\boldsymbol{P}_i\}$. Hence the forces $\{\boldsymbol{P}_i\}$ balance the forces of reaction. From the conditions of equilibrium it follows that the total moment of these forces with respect to the axis l is equal to zero. Since the moment of the forces of reaction with respect to the axis l is zero (because the reactions are forces whose origins lie on the axis), the total moment of the forces $\{\boldsymbol{P}_i\}$ with respect to the axis l is zero, i. e.

$$\Sigma \mathrm{Mom}_l \boldsymbol{P}_i = 0. \tag{II}$$

The forces of reaction do not appear in equation (II). This equation is consequently the necessary condition that the given system of forces $\{\boldsymbol{P}_i\}$ must satisfy in order that the body be in equilibrium.

We shall now prove that condition (II) is also a sufficient condition for equilibrium.

Let us assume, then, that an arbitrary system of forces $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$ satisfies condition (II). Let us select an arbitrary point O on the axis l (Fig. 197). By the theorem on reduction (p. 237), the given system is equipollent to a system composed of two forces $\boldsymbol{S}$ and $\boldsymbol{T}$, of which $\boldsymbol{S}$ has its origin at O. Since the total moment of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, with respect to l is zero, the total moment of the forces $\boldsymbol{S}$ and $\boldsymbol{T}$ with respect to l is also zero. Since $\mathrm{Mom}_l \boldsymbol{S} = 0$ (because the force $\boldsymbol{S}$ acts at the point O lying on l), we must also have $\mathrm{Mom}_l \boldsymbol{T} = 0$. Therefore the force $\boldsymbol{T}$ either cuts l or is parallel to l.

Let us consider an arbitrary point $O' \neq O$ on the axis l. Let A be the origin of the force $\boldsymbol{T}$. Since the force $\boldsymbol{T}$ lies in the plane passing through l and A, we can resolve $\boldsymbol{T}$ into two forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ having directions OA and $O'A$, and then translate their points of application to O and O'. In this manner we have shown that the system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, is equipollent to a system of forces acting at the points of the axis l.

Since it is obvious that the forces acting at the points of the axis l, which is fixed by hypothesis, cannot move a body being at rest, the given system of acting forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, is in equilibrium.

Therefore: *the necessary and sufficient condition that a system of forces acting on a rigid body having a fixed axis be in equilibrium is that the total moment of the system of forces with respect to this axis be zero.*

Remark. Let Π be an arbitrary plane perpendicular to the line l, and O_1 the point of intersection of the plane Π with the axis l. Let us denote by $\boldsymbol{P}'_1, \boldsymbol{P}'_2, \ldots$ the projections on the plane Π of the acting forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ From the definition of a moment with respect to an axis (p. 233) it follows that the moment of the force $\boldsymbol{P}'_i$ with respect to O_1 is equal to the moment of the force $\boldsymbol{P}_i$ with respect to l. Consequently the total moment of the forces $\{\boldsymbol{P}'_i\}$ with respect to the point O_1 is equal to the total moment of the forces $\{\boldsymbol{P}_i\}$ with respect to the axis l.

Therefore: *the necessary and sufficient condition for the equilibrium of a system of forces $\{\boldsymbol{P}_i\}$ is that the total moment of the forces $\{\boldsymbol{P}'_i\}$ with respect to O_1 be zero.*

This condition is such as if the projection of a body on the plane Π had a fixed point O_1 and the projections of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ acted on the projection of the body.

In order to see whether a system of forces acting on a rigid body having a fixed axis is in equilibrium, it is therefore sufficient to know only the projections of the acting forces on a plane perpendicular to the axis and the point of intersection of the axis with this plane.

Plane motion of a body. Let it be possible for a rigid body to move only in such a way that the path of each of its points is plane and lies in a plane parallel to a certain fixed plane Π.

We then say that the body can execute only a *plane motion* and we call the plane Π a *directional plane.*

An example of a body executing a plane motion is a cylinder whose bases lie in two parallel planes Π and Π' (Fig. 198). If there is no friction the reactions of the planes Π and Π' are perpendicular to these planes.

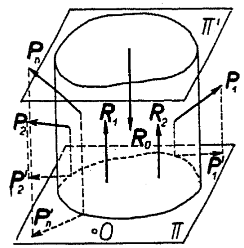


Fig. 198.

In general, let us assume that *whenever there is no friction the reactions which cause the body to execute only a plane motion are perpendicular to the directional plane Π.*

It is obvious, therefore, that a system of forces perpendicular to the directional plane is in equilibrium; this means that forces perpendicular to Π cannot move a body which can execute only a plane motion if the body is at rest.

Let a body which can execute only a plane motion be in equilibrium under the action of a system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ The acting forces therefore balance the reactions $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots$, i. e. they form a system equipollent to zero. It follows from this that the projections of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ and those of the reactions $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots$ on the directional plane Π also form a system equipollent to zero. Since the projections of the reactions are zero (because the reactions are perpendicular to Π), the projections $\boldsymbol{P}'_1, \boldsymbol{P}'_2, \ldots$ of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ on the directional plane themselves also form a system equipollent to zero.

Let O be an arbitrary point of the plane Π. If the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ are in equilibrium, we obtain:

$$\Sigma \boldsymbol{P}'_i = 0, \quad \Sigma \mathrm{Mom}_O \boldsymbol{P}'_i = 0. \tag{3}$$

Condition (3) is therefore a n e c e s s a r y condition for the equilibrium the system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ We shall prove that it is also a s u f f i c i e n t condition.

With this in view, let us assume that the system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ satisfies equations (3). In virtue of the theorem on reduction this system is equipollent to a system consisting of the force $\boldsymbol{P} = \Sigma \boldsymbol{P}_i$ and a certain couple $\boldsymbol{U}, -\boldsymbol{U}$. Let us denote by $\boldsymbol{P}', \boldsymbol{U}'$, and $-\boldsymbol{U}'$, the projections of these forces on the plane Π. By (3) we obtain:

$$\boldsymbol{P}' = 0, \quad \mathrm{Mom}(\boldsymbol{U}', -\boldsymbol{U}') = 0.$$

Consequently $\boldsymbol{P}$ is perpendicular to Π; and the couple $\boldsymbol{U}, -\boldsymbol{U}$ lies in the plane perpendicular to Π. Hence we can rotate the couple $\boldsymbol{U}, -\boldsymbol{U}$ in its plane (leaving the moment unchanged) so that in the new position the forces are perpendicular to Π. Denoting by $\mathbf{V}$, $-\mathbf{V}$ the new couple equipollent to the former, we see that the system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ is equipollent to the system of forces $\mathbf{P}$, $\mathbf{V}$, and $-\mathbf{V}$, perpendicular to Π. It follows that the system $\mathbf{P}_1, \mathbf{P}_2, \ldots$ is in equilibrium.

Therefore: *a necessary and sufficient condition that a system of forces acting on a rigid body which can execute only a plane motion be in equilibrium is that the projections of these forces on the directional plane form a system equipollent to zero.*

Hence, in order to find out whether a system of forces acting on a body which can execute only a plane motion is in equilibrium, it is sufficient to know the projections of the acting forces on the directional plane.

Example 1. Lever. A beam having a fixed horizontal axis perpendicular to it is called a *lever.*

We assume that the forces acting on a lever lie in one plane Π perpendicular to the axis of rotation and passing through the centre of gravity.

Let us denote by $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$ the forces acting on the beam at the points $A_1, A_2, \ldots$, by $\mathbf{Q}$ the weight of the beam acting at its centre of gravity S, by $Q_1, Q_2, \ldots, Q$ the absolute values, and by $q_1, q_2, \ldots, q$ the arms of these forces with respect to the point of intersection O of the plane Π with the axis of rotation (Fig. 199).

The moment of the system of forces $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}$, with respect to the axis of rotation in this case is equal to the moment of this system with respect to O. The acting forces will therefore be in equilibrium if the sum of their moments with respect to O is zero. Hence the condition of equilibrium can be written in the form

$$\pm Q_1 q_1 \pm Q_2 q_2 \pm \cdots \pm Qq = 0, \tag{4}$$

where the signs $+$ and $-$ are taken according to the rule given on p. 233.

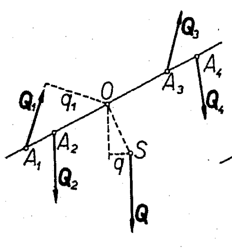


Fig. 199.

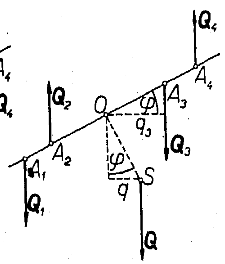


Fig. 200.

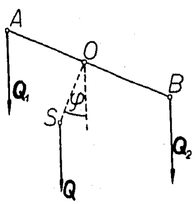


Fig. 201.

Let us assume that the center of gravity lies under the axis of rotation when the beam has a horizontal position. It follows from this, obviously, that for a beam on which no forces act (except gravity) a horizontal position is a position of equilibrium. Let us assume, in addition, that the acting forces have a vertical direction (Fig. 200).

Let φ denote the angle which the beam makes with the horizontal. Since OS is perpendicular to the beam, OS also makes an angle φ with the vertical.

Hence we have:

$$q_1 = OA_1 \cos\varphi, \quad q_2 = OA_2 \cos\varphi, \quad \ldots, \quad q = OS \sin\varphi,$$

whence by substituting in (4)

$$(\pm Q_1 \cdot OA_1 \pm Q_2 \cdot OA_2 \pm \ldots) \cos\varphi \pm Q \cdot OS \sin\varphi = 0,$$

or, dividing by $\cos\varphi$,

$$\pm Q_1 \cdot OA_1 \pm Q_2 \cdot OA_2 \pm \ldots \pm Q \cdot OS \tan\varphi = 0.$$

Knowing the forces $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}$, we can calculate from equation (5) the angle φ which the beam makes with the horizontal in the position of equilibrium.

In particular, if the beam is acted upon by two forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$, directed downwards and applied on opposite sides of the beam (as in Fig. 201), we obtain from (5) $-Q_1 \cdot OA + Q_2 \cdot OB - Q \cdot OS \tan\varphi = 0$, whence

$$\tan\varphi = (Q_2 \cdot OB - Q_1 \cdot OA) \,/\, Q \cdot OS. \tag{6}$$

If $\varphi = 0$ (i. e. if the beam is in equilibrium in a horizontal position), we obtain

$$Q_1 \cdot OA = Q_2 \cdot OB.$$

In particular, therefore, if $OA = OB$, then $Q_1 = Q_2$.

An instrument called a *balance*, which serves to compare the weights of two bodies and indirectly their masses, depends on this principle.

Example 2. A rigid body, having a fixed axis l, is in equilibrium under the action of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, whose points of application are $A_1, A_2, \ldots, A_n$ (Fig. 202). Give the necessary and sufficient conditions which these forces must satisfy in order that the body continue to be in equilibrium, if it is

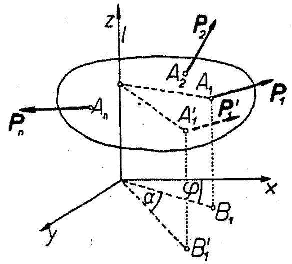


Fig. 202.

turned about the axis l through an arbitrary angle α and during this rotation the directions, senses, magnitudes, and the points of application (in the body) of these forces remain unchanged.

Let us take the axis l as the z-axis of the coordinate system; denote by $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ the coordinates of the points of application $A_1, A_2, \ldots$, and by $x_1', y_1', z_1', x_2', y_2', z_2', \ldots$ the coordinates of the points $A_1', A_2', \ldots$, into which the points $A_1, A_2, \ldots$ went when the body turned through an angle α about the axis l. Let B_1 and B_1' be the projections of the points A_1 and A_1' on the xy-plane, and φ the angle between OB and the x-axis. Putting $r_1 = OB_1 = OB_1'$, we have:

$$x_1 = r_1 \cos\varphi, \quad y_1 = r_1 \sin\varphi, \tag{7}$$

$$x_1' = r_1 \cos(\varphi + \alpha), \quad y_1' = r_1 \sin(\varphi + \alpha), \quad z_1 = z_1'. \tag{8}$$

Consequently $x_1' = r_1 \cos\varphi \cos\alpha - r_1 \sin\varphi \sin\alpha$, whence by (7)

$$x_1' = x_1 \cos\alpha - y_1 \sin\alpha, \text{ and similarly } y_1' = y_1 \cos\alpha + x_1 \sin\alpha. \tag{9}$$

Analogous formulae are obtained for the remaining points $A_2', A_3', \ldots$

Since the body has to maintain equilibrium after turning through an angle α, the moment of the forces with respect to the z-axis must be zero, i. e.

$$\Sigma(P_{i_x} y_i' - P_{i_y} x_i') = 0.$$

Substituting for x_i', y_i' the expressions from formulae (9), we obtain

$$\cos\alpha\ \Sigma(P_{i_x} y_i - P_{i_y} x_i) + \sin\alpha\ \Sigma(P_{i_x} x_i + P_{i_y} y_i) = 0. \tag{10}$$

Since equilibrium occurs for $\alpha = 0$, substituting $\alpha = 0$ in formula (10), we obtain

$$\Sigma(P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{11}$$

From (10) we have for $\alpha = \frac{1}{2}\pi$

$$\Sigma(P_{i_x} x_i + P_{i_y} y_i) = 0. \tag{12}$$

Conversely, if conditions (11) and (12) hold, then obviously condition (10) holds for every α. Equations (11) and (12) are therefore the sought for necessary and sufficient conditions.

Determination of the reactions acting on a fixed axis. Let a rigid body have a line l fixed at the two points O and O'. We can then assume that the forces of reaction act at the points O and O'.

Let us assume that a system $\{\mathbf{P}_i\}$ of forces acting on the body is in equilibrium.

Let the point O be the origin of a system of coordinates and the axis l the z-axis (Fig. 203). Let us denote by x_i, y_i, z_i, the coordinates of the points of application of the forces $\{\mathbf{P}_i\}$, by $\mathbf{R}$ and $\mathbf{N}$ the reactions at the points O and O', and by d the length of the segment OO'.

Since the system of forces $\{\mathbf{P}_i\}$ together with the reactions $\mathbf{R}$ and $\mathbf{N}$ is in equilibrium, forming the projections of the sum and total moment with respect to O on the axes of the coordinate system, we obtain six equations:

$$\Sigma P_{i_x} + R_x + N_x = 0, \tag{I}$$

$$\Sigma P_{i_y} + R_y + N_y = 0, \tag{II}$$

$$\Sigma P_{i_z} + R_z + N_z = 0, \tag{III}$$

$$\Sigma(P_{i_y} z_i - P_{i_z} y_i) + N_y d = 0, \tag{IV}$$

$$\Sigma(P_{i_z} x_i - P_{i_x} z_i) - N_x d = 0, \tag{V}$$

$$\Sigma(P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{VI}$$

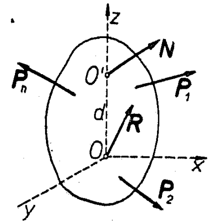


Fig. 203.

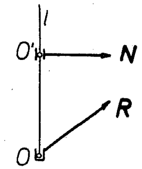


Fig. 204.

From equations (IV) and (V) we can determine N_x and N_y. Next we determine R_x and R_y from equations (I) and (II). Finally we calculate $R_z + N_z$ from equation (III).

We see, therefore, that the above equations do not permit us to determine the reactions. It is true that the number of unknowns is six (R_x, R_y, R_z and N_x, N_y, N_z), i. e. as many as there are equations, however, they appear only in five equations. Equation (VI) expresses the condition for the equilibrium of the acting forces. From equations (I)—(V) we can determine only the components of reaction perpendicular to the axis l and the sum of the components parallel to the axis l.

Problems in which the forces of reaction cannot be determined from the conditions of equilibrium are called *statically indeterminate*.

Therefore the calculation of the reactions of a rigid body which is fixed at two points is statically indeterminate.

If we assume that the given body is not a rigid body, but one that can be deformed, then the forces of reaction could be calculated by appealing to the theory of elasticity.

Our problem can be made statically determinate by assuming that the point O' is fixed in a smooth bearing (Fig. 204).

The reaction **N** is then perpendicular to the axis l. In this case we have $N_z = 0$, and therefore we can determine R_z from equation (III).

Example 3. A heavy rectangular plate has a horizontal axis l fixed at the point O and at the bearing O' (Fig. 205). The centre of the side parallel to the axis is the point of application of the force **P** perpendicular to the plate. Given are: the weight **Q** whose origin is at the centre S of the rectangle, sides a and b of the rectangle, and the length $d = OO'$. Determine in the position of equilibrium the reactions **R** and **N** (at O and $O)'$ and the angle φ which the plate makes with the horizontal.

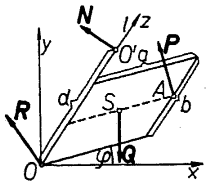


Fig. 205.

Let O be the origin and the axis l the z-axis of the coordinate system; let us give the x-axis a horizontal direction and the y-axis a direction vertically upwards. We shall be able to apply equations (I)—(VI), p. 277.

The point S has the coordinates $\frac{1}{2}a\cos\varphi$, $\frac{1}{2}a\sin\varphi$, $\frac{1}{2}b$; and the point A: $a\cos\varphi$, $a\sin\varphi$, $\frac{1}{2}b$. We have:

$$Q_x = 0, \quad Q_y = -Q, \quad Q_z = 0, \quad \text{where } Q = |\mathbf{Q}|;$$
$$P_x = -P\sin\varphi, \quad P_y = P\cos\varphi, \quad P_z = 0, \quad \text{where } P = |\mathbf{P}|.$$

Since $N_z = 0$, we get by formulae (I)—(VI), p. 277:

$$-P\sin\varphi + R_x + N_x = 0, \; P\cos\varphi - Q + R_y + N_y = 0, \; R_z = 0,$$
$$\tfrac{1}{2}bP\cos\varphi - \tfrac{1}{2}Qb + N_y d = 0, \quad \tfrac{1}{2}bP\sin\varphi - N_x d = 0,$$
$$-Pa + \tfrac{1}{2}Qa\cos\varphi = 0.$$

From the last equation we obtain $\cos\varphi = 2P/Q$, and from the remaining equations we determine R_x, R_y and N_x, N_y. We have $R_z = N_z = 0$.

Equilibrium is obviously possible if $2P/Q \leq 1$, i. e. if $P \leq Q/2$.

§ 12. Equilibrium of heavy supported bodies. If a rigid body which is not acted upon by any forces other than the force of gravity rests on a horizontal plane Π and is in equilibrium, then the forces of reaction which the plane exerts on the body (at the points of support) balance the weights of the individual points of the body.

Let us assume that the supporting plane Π is smooth. The reactions are then perpendicular to the plane; hence they have a resultant **F** acting at a certain point O of the plane Π. The point O was called the *centre of pressure* (p. 260). The weights of the individual points of the body have a resultant **Q** whose point of application is at the centre of gravity S.

If the body is in equilibrium, the forces **F** and **Q** balance each other. Consequently $\mathbf{F} + \mathbf{Q} = 0$, and moreover **F** and **Q** lie on one line. Because of this the centre of pressure lies at the point of intersection of the direction of the force **Q** with the plane Π. The centre of pressure O is therefore the projection of the centre of gravity S on the supporting plane Π.

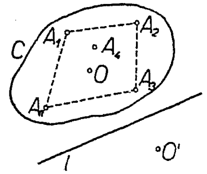


Fig. 206.

If a body rests on a plane Π only at one point O and is in equilibrium, then the reaction acts at O. Consequently the centre of gravity lies above the point of support.

Let the body now rest on the plane Π at the points $A_1, A_2, \ldots$ and let C be an arbitrary closed convex polygon enclosing all the points of support $A_1, A_2, \ldots$ (Fig. 206). We shall prove that the centre of pressure O in this case also lies either within or on the polygon C.

For let us assume that the centre of pressure lies outside the polygon C at the point O'. Let us draw an arbitrary line l in the plane Π such that the point O' and the line C lie on opposite sides of the line l. The moments of the forces of reaction with respect to l would therefore be directed opposite to the moment of the force **F**. This is impossible, however, because the total moment of the forces of reaction is equal to the moment of the force **F**. Hence the centre of pressure must lie within the convex polygon C. If K is the smallest convex polygon (in the figure the polygon A_1, A_2, A_3, A_n) within which the points of support lie,[1]) then the centre of pressure also lies within this polygon. Since we have assumed that the body is in equilibrium, the direction of the force of gravity passes through the centre of pressure and therefore also falls within the polygon K.

We shall now prove that if the weight falls within the polygon K, then reactions will appear which balance the weight.

We shall consider two cases:

1° A body is supported at two points A and B. In this case the polygon K is the line segment AB. If the direction of the force of gravity

[1]) In geometry it is proved that such a polygon always exists and lies within every convex polygon containing the points of support.

passes through the point G of the segment AB, then there exist two forces of reaction $\mathbf{R}_1$ and $\mathbf{R}_2$ directed vertically upwards and having their origins at A and B. These forces can be determined graphically as on p. 253, or calculated as in example 4, p. 263. It follows from this by the principle given on p. 260 that the reactions balance the weight.

2° A body is supported at $n > 2$ points. If the points are collinear, then denoting the extreme points of support by A and B, we can proceed as in case 1°. Suppose, then, that not all the points of support are collinear. If the force of gravity falls within the polygon K at the point O, then we can find three points of support such that the point O will lie within a triangle of which these points are the vertices (the points A_1, A_3, A_n, in the figure). As we shall show (*vide* example 4), we can then determine the reactions acting at the vertices of this triangle and balancing the force of gravity.

We therefore have the following theorem:

If a heavy rigid body rests on a smooth horizontal plane, then the necessary and sufficient condition that the reactions of the plane balance the weight of the body is that the force of gravity fall within the smallest convex polygon K containing all the points of support.

More generally: let a rigid body rest on a smooth horizontal plane and in addition to the force of gravity let other forces act on it. If the body is in equilibrium, then the resultant $\mathbf{F}$ of the reactions balances the acting forces. It follows from this that the forces acting on the body have a vertical resultant $-\mathbf{F}$ whose direction passes through the centre of pressure O and falls within the polygon K. Conversely, if the acting forces have a vertical resultant, directed downwards and falling within the polygon K, then reactions will appear which balance the forces acting on the body. The proof is carried out as before.

Example 4. A three-legged stool rests on the floor Π. Determine the reactions at the points of support A_1, A_2, A_3, under the assumption that there is no friction.

Let us denote by S the centre of gravity of the stool, by S' the projection of S on Π, by $\mathbf{Q}$ the weight of the stool, and by $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$ the reactions (Fig. 207).

The problem can be solved most simply by calculating the total moments of the forces with respect to the lines A_1A_2, A_2A_3, A_3A_1; these moments are obviously zero.

Let h_3 denote the distance of A_3 from A_1A_2, and w_3 the distance of S' from A_1A_2. Taking the moment with respect to the axis A_1A_2, we obtain:

$$R_3h_3 - Qw_3 = 0, \quad \text{where } R_3 = |\mathbf{R}_3| \text{ and } Q = |\mathbf{Q}|,$$

because the moments of the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ are zero. Consequently

$$R_3 = Qw_3 / h_3.$$

Analogously we obtain

$$R_1 = Qw_1 / h_1 \quad \text{and} \quad R_2 = Qw_2 / h_2.$$

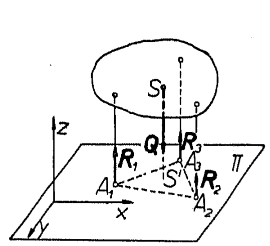


Fig. 207.

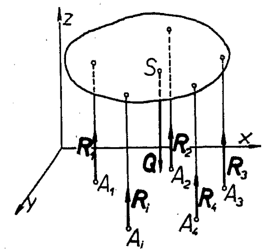


Fig. 208.

Example 5. Let a rigid body rest on a horizontal plane Π at the n points of support $A_1, A_2, \ldots, A_n$. Let us take the plane Π as the xy-plane of the coordinate system (x, y, z); denote by $x_1, y_1, 0, x_2, y_2, 0, \ldots, x_n, y_n, 0$, the coordinates of the points of support $A_1, A_2, \ldots, A_n$, and by x_0, y_0, z_0, the coordinates of the centre of gravity S (Fig. 208).

Let $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$ denote the reactions, $\mathbf{Q}$ the weight, and $R_1, R_2, \ldots, R_n, Q$, the absolute values of these forces.

Forming the projections on the coordinate axes, we obtain for the projection on the z-axis

$$R_1 + R_2 + \ldots + R_n - Q = 0 \tag{13}$$

The remaining two equations drop out because they become identically zero.

Forming next the moments with respect to the origin of the coordinate system, we obtain only two equations:

$$-R_1y_1 - R_2y_2 - \ldots + Qy_0 = 0, \quad R_1x_1 + R_2x_2 + \ldots - Qx_0 = 0, \tag{14}$$

because the moment of the forces with respect to the z-axis is zero.

We thus have only three equations for the determination of the reactions. Hence if $n > 3$, then we shall not be able to determine the reactions. The problem of determining the reactions in the case, for example, of a table standing on four legs is therefore statically indeterminate (*vide* p. 277).

However, if a body supported at $n > 3$ points is not rigid, then the reactions can be determined by appealing to the theory of elasticity. We shall show this in the next example.

Example 6. A rectangular table rests on four legs at the points A_1, A_2, A_3, A_4, of a smooth horizontal plane Π (Fig. 209).

Let us denote the reactions by $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4$, the weight by $\mathbf{Q}$, and the absolute values of these forces by R_1, R_2, R_3, R_4, Q. Let us assume that the points A_1, A_2, A_3, A_4, are the vertices of the rectangle and let us put:

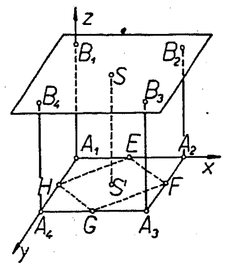


Fig. 209.

$$A_1A_2 = A_3A_4 = a, \quad A_1A_4 = A_2A_3 = b. \tag{15}$$

Let us take A_1 as the origin of the coordinate system, the lines A_1A_2 and A_1A_4 as the x and y axes, and the sense of the z-axis vertically upwards. Finally, let us denote by x_0, y_0, z_0, the coordinates of the centre of gravity S of the table top.

Forming the projections of the forces on the coordinate axes and taking the moments of the forces with respect to these axes, we obtain (cf. equations (13) and (14) of example 5):

$$R_1 + R_2 + R_3 + R_4 - Q = 0, \tag{16}$$

$$-(R_3 + R_4)\, b + Qy_0 = 0, \quad (R_2 + R_3)\, a - Qx_0 = 0. \tag{17}$$

The reactions cannot be determined from these equations.

Let us assume, however, that the table top and the plane Π (on which the legs of the table rest) are rigid, that the legs of the table are not rigid, but can be compressed, and that the reactions are proportional (in magnitude) to the contraction of the respective legs.

Therefore, if we denote the original length of the legs by l, and their lengths after compression by z_1, z_2, z_3, z_4, then the contractions are $l - z_1$, $l - z_2$, $l - z_3$, $l - z_4$, whence according to the assumption

$$R_1 = m(l - z_1), \; R_2 = m(l - z_2), \; R_3 = m(l - z_3), \; R_4 = m(l - z_4), \tag{18}$$

where m is the factor of proportionality.

Equations (16), (17) and (18) constitute a system of seven equations with eight unknowns R_1, R_2, R_3, R_4, and z_1, z_2, z_3, z_4. We obtain the eighth equation by stipulating that the points B_1, B_2, B_3, B_4, at which the table

top rests on the legs, lie in one plane; for we have assumed that the table top is rigid.

The points B_1, B_2, B_3, and B_4, have the coordinates $0, 0, z_1$, $a, 0, z_2$, a, b, z_3, and $0, b, z_4$. As is known from analytic geometry, the condition that these points lie in one plane is expressed by the formula

$$\begin{vmatrix} 0 & 0 & z_1 & 1 \\ a & 0 & z_2 & 1 \\ a & b & z_3 & 1 \\ 0 & b & z_4 & 1 \end{vmatrix} = 0.$$

Evaluating the determinant, we obtain:

$$z_1 - z_2 + z_3 - z_4 = 0. \tag{19}$$

From equations (16)—(19) we can determine the unknown reactions. We obtain:

$$z_1 = l + \frac{1}{4m} Q\left[-3 + 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right], \quad z_2 = l + \frac{1}{4m} Q\left[-1 - 2\left(\frac{x_0}{a} - \frac{y_0}{b}\right)\right],$$
$$z_3 = l + \frac{1}{4m} Q\left[1 - 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right], \quad z_4 = l + \frac{1}{4m} Q\left[-1 + 2\left(\frac{x_0}{a} - \frac{y_0}{b}\right)\right], \tag{20}$$

$$R_1 = \tfrac{1}{4}Q\left[3 - 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right], \quad R_2 = \tfrac{1}{4}Q\left[1 + 2\left(\frac{x_0}{a} - \frac{y_0}{b}\right)\right],$$
$$R_3 = \tfrac{1}{4}Q\left[-1 + 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right], \quad R_4 = \tfrac{1}{4}Q\left[1 - 2\left(\frac{x_0}{a} - \frac{y_0}{b}\right)\right]. \tag{21}$$

In order that formulae (21) give non-negative values for the reactions $R_1, \ldots, R_4$ the following relations must hold:

$$\tfrac{1}{2} \leqq \frac{x_0}{a} + \frac{y_0}{b} \leqq \tfrac{3}{2}, \quad -\tfrac{1}{2} \leqq \frac{x_0}{a} - \frac{y_0}{b} \leqq \tfrac{1}{2}. \tag{22}$$

It follows from this that the projection of the centre of gravity on the horizontal plane, i. e. the point $S'(x_0, y_0, 0)$, must lie within the parallelogram $EFGH$ whose vertices are the midpoints of the sides of the rectangle $A_1A_2A_3A_4$.

Let us suppose that the point S' falls within the triangle A_1EH (beyond the side EH). Then we would have

$$\frac{x_0}{a} + \frac{y_0}{b} < \tfrac{1}{2}, \text{ i. e. } 1 - 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right) > 0,$$

whence by (20) $z_3 > l$; this means that the leg A_3B_3 becomes elongated, which is obviously impossible.

We must therefore assume that *the table rests on only three legs*, namely, at the points A_1, A_2, A_4. Putting $R_3 = 0$, we then obtain from equations (17):

$$R_4 = Qy_0 / b, \quad R_2 = Qx_0 / a,$$

and from equation (16)

$$R_1 = Q\left[1 - \left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right].$$

Example 7. A heavy cylinder rests on a smooth horizontal plane. The cylinder is acted upon by a force couple **P** and —**P** lying in a vertical plane passing through the centre of gravity S. What can be the maximum magnitude of the moment of the couple if the cylinder is in equilibrium?

Let us denote by **M** the moment of the couple, by **Q** the weight of the cylinder, by **F** the resultant of the forces of reaction acting at O, and by r the radius of the base of the cylinder (Fig. 210).

If the cylinder is in equilibrium, the sum of the forces is zero; hence

$$\mathbf{F} + \mathbf{Q} = 0 \quad \text{or} \quad \mathbf{F} = -\mathbf{Q}.$$

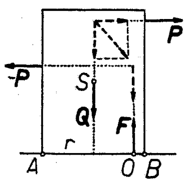


Fig. 210.

Assuming that S lies on the axis of the cylinder and putting $d = AO$, we get from the calculation of the moment with respect to A

$$|\mathbf{M}| + |\mathbf{Q}|r - |\mathbf{F}|d = 0.$$

Since $|\mathbf{Q}| = |\mathbf{F}|$,

$$|\mathbf{M}| = Q(d - r),$$

where $Q = |\mathbf{Q}|$. Since the maximum value of d is $2r$, the maximum value of $|\mathbf{M}|$ is Qr.

Hence if $|\mathbf{M}| > Qr$, equilibrium is impossible. However, if $|\mathbf{M}| \leqq Qr$, then, as is easily verified, the resultant of the forces **Q**, **P** and —**P** is equal to **Q** and intersects the horizontal plane within the base of the cylinder. The cylinder can therefore remain in equilibrium.

§ 13. Internal forces. Through an arbitrary point O of an axis chosen in a given rigid body, e. g. in a beam, let us pass a plane Π perpendicular to this axis. The plane will divide the body into two parts I and II. Let us assume that the body is acted upon by certain forces and that it is in equilibrium. In many problems of engineering mechanics it is convenient to consider the parts I and II as separate rigid bodies tangent along the intersecting plane Π (Fig. 211).

From such a conception it follows that part II acts on part I with certain forces. These forces are called *stresses*.

Taking the point O as the centre of reduction, we can replace the stresses by one force **R** with its origin at O and a force couple of moment **M**.

The component $\overline{OA}$ of the force **R**, perpendicular to the section, is called the *compressive* or *tensile* resultant at O, depending on whether the component is directed towards part I or away from it.

The component $\overline{OB}$ of the force **R**, tangent to the section, is called the resultant *bending* or *shearing* force at O.

The component $\overline{OC}$ of the moment **M**, perpendicular to the intersecting plane Π, is called the *twisting moment* at O, and the component $\overline{OD}$ tangent to the surface is called the *bending moment* at O.

The twisting moment can be considered as the moment of a certain force couple lying in the intersecting plane, and the bending moment as the moment of a force couple lying in the plane tangent to the axis at the point O. The action of these couples, of which the first tends to twist and the second to bend, explain to us the names of the moments.

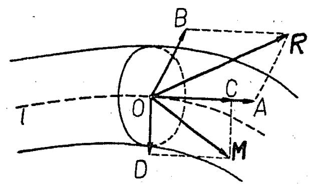


Fig. 211.

If the body is in equilibrium, then the external forces acting on part I balance the stresses. Consequently —**R** and —**M** are equal, respectively, to the sum and total moment with respect to O of the external forces acting on part I. Knowing the external forces, we can therefore determine **R** and **M**.

A knowledge of the forces **R** and of the moment **M** is of great importance in the subject of strength of materials. In general, the larger the forces **R** and **M** are, the greater is the possibility that the body will be ruptured.

According to the law of action and reaction, the stresses with which part I acts on II can be replaced by the sum —**R** with its origin at O and by a couple of moment —**M**.

Example 1. A beam supported at the points A and B carries the loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$, directed vertically downwards and situated at the distances $x_1, x_2, \ldots, x_5$, from A (Fig. 174). Assuming that the supports are smooth, we obtain (cf. formulae (13) and (14), p. 263):

$$R_2 = (P_1x_1 + P_2x_2 + \ldots + P_5x_5) / d,$$
$$R_1 = [P_1(d - x_1) + \ldots + P_5(d - x_5)] / d,$$

where $AB = d$, and P_i, R_1, R_2, denote the absolute values of the forces and reactions.

Let us cut the beam by a plane perpendicular to the axis at the point C at a distance x from A. Let us denote by $\mathbf{R}$ the sum, and by $\mathbf{M}$ the moment with respect to C, of the stresses of part CB on part AC. Assuming that the cut occurs between the forces P_3 and P_4, we obtain:

$$-\mathbf{R} = \mathbf{R}_1 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3,$$
$$-\mathbf{M} = \mathrm{Mom}_C\mathbf{R}_1 + \mathrm{Mom}_C\mathbf{P}_1 + \mathrm{Mom}_C\mathbf{P}_2 + \mathrm{Mom}_C\mathbf{P}_3.$$

Giving the y-axis a vertical direction with an upward sense, we obtain:

$$R_y = R_1 - P_1 - P_2 - P_3,$$
$$M = xR_1 - (x - x_1)P_1 - (x - x_2)P_2 - (x - x_3)P_3.$$

Since $\mathbf{R}$ and $\mathbf{M}$ lie in the intersecting plane, $\mathbf{R}$ is the bending force and $\mathbf{M}$ the bending moment. The compressive (or tensile) force and the twisting moment are zero. The force $\mathbf{R}$ and the moment $\mathbf{M}$ can also be determined by means of a string polygon as on pp. 252 and 253.

Example 2. A beam, built-in as in the Fig. 212, is loaded at A by the force $\mathbf{P}$. Let us form a section at the point C. Denote by $\mathbf{R}$ the sum, and by $\mathbf{M}$ the moment with respect to C, of the stresses of the left part on the right part. Consequently we have:

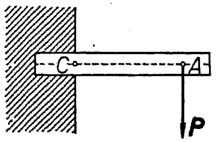


Fig. 212.

$$-\mathbf{R} = \mathbf{P}, \quad -\mathbf{M} = \mathrm{Mom}_C\mathbf{P}.$$

Hence the right part acts on the left part (built-in) with stresses of sum $-\mathbf{R}$ and moment $-\mathbf{M}$. The reactions of the wall, balancing these stresses, therefore have the sum $\mathbf{R} = -\mathbf{P}$ and the moment $\mathbf{M} = -\mathrm{Mom}_C\mathbf{P}$.

III. SYSTEMS OF BODIES

§ 14. Conditions of equilibrium. A necessary and sufficient condition for the equilibrium of a system of rigid bodies (free or not) is that each body of the system be in equilibrium. It follows from this that *the necessary and sufficient condition for the equilibrium of a system of rigid bodies is that, for each body separately, the forces acting on this body balance the reactions.*

The forces with which two bodies of a system act on each other are

called the *internal* forces of the system. The remaining forces are called *external.*

For example, if two bodies of a system touch each other, then the reactions at the points of contact are internal forces. On the other hand, those acting forces and reactions which arise from bodies not belonging to the system (e. g. from supports) are external forces.

Internal forces occur in pairs and are subject to the law of action and reaction; consequently the sum and total moment of the external forces are zero.

If a system of bodies is in equilibrium, then, for each body, the external forces acting on the body balance the internal forces. It follows from this that the external forces acting on the entire system balance the internal forces of the system. Since (as we have mentioned above) the internal forces have a sum and total moment equal to zero, *it follows that if a system of rigid bodies is in equilibrium, the sum and total moment of the external forces are zero.*

This condition is only sufficient, but not necessary for the equilibrium of a system of rigid bodies.

Each part of a system of rigid bodies which is in equilibrium is obviously itself in equilibrium. The external forces with respect to a certain part of the system are:

a) those external forces of the whole system which act on its given part,

b) the reactions exerted on this part by the remaining bodies of the system.

It follows from this that *the necessary and sufficient condition for the equilibrium of a system of rigid bodies is that the external forces acting on any part of the system balance the reactions exerted on this part by the remaining bodies of the system.*

For we can choose individual bodies of the system as the parts of the system.

Example. Two heavy rods AC and BC, lying in a vertical plane and touching at the point C, lean against vertical walls at A and B, and against horizontal plane at C (Fig. 213). Given are: the weights of the rods $\mathbf{Q}_1$ and $\mathbf{Q}_2$ acting at the centres of gravity, as well as the lengths $l_1 = AC$, $l_2 = BC$, $a_1 = S_1C$, $a_2 = S_2C$ and the distance d between the vertical walls. Determine in the posi-

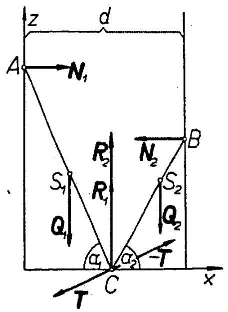


Fig. 213.

tion of equilibrium the angles α_1 and α_2 which the rods make with the horizontal under the assumption that there is no friction.

Let us denote the reactions of the vertical walls by $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ (these reactions therefore have a horizontal direction), the reactions of the horizontal wall by $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ (hence having a vertical direction), finally, the force with which the rod CB acts on the rod AC at the point C by $\boldsymbol{T}$. Then by the law of action and reaction the rod AC acts on the rod CB with a force $-\boldsymbol{T}$. Nothing can be said beforehand about the direction of the force $\boldsymbol{T}$.

Let us select the x and z axes in the plane of the rods, giving the x-axis a horizontal direction and the z-axis a direction vertically upwards. If the rod AC is in equilibrium, the forces acting on this rod balance one another. Therefore, forming their projections on the x and z axes and calculating the moment with respect to C, we obtain:

$$N_1 + T_x = 0, \quad -Q_1 + R_1 + T_y = 0, \tag{1}$$

$$N_1 l_1 \sin\alpha_1 - Q_1 a_1 \cos\alpha_1 = 0, \tag{2}$$

where N_1, R_1, and Q_1, denote the absolute values of the corresponding forces. Similarly, for the rod CB we get:

$$-N_2 - T_x = 0, \quad -Q_2 + R_2 - T_y = 0, \tag{3}$$

$$-N_2 l_2 \sin\alpha_2 + Q_2 a_2 \cos\alpha_2 = 0. \tag{4}$$

From equations (2) and (4) we have:

$$N_1 = Q_1 a_1 \cot\alpha_1 / l_1, \quad N_2 = Q_2 a_2 \cot\alpha_2 / l_2, \tag{5}$$

and from the first of the equations (1) and (3) $N_1 = N_2$, whence by (5)

$$Q_1 a_1 \cot\alpha_1 / l_1 = Q_2 a_2 \cot\alpha_2 / l_2. \tag{6}$$

Moreover, as is seen from the drawing,

$$l_1 \cos\alpha_1 + l_2 \cos\alpha_2 = d. \tag{7}$$

From equations (6) and (7) we can determine the angles α_1 and α_2.

Remark. We cannot determine the forces $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{T}$, from equations (1)—(4). However, we can obtain the forces $\boldsymbol{T}' = \boldsymbol{R}_1 + \boldsymbol{T}$ and $\boldsymbol{T}'' = \boldsymbol{R}_2 - \boldsymbol{T}$. They are the resultants of the reactions acting on the rods at C. We get:

$$T'_x = T_x = -N_1, \quad T'_y = R_1 + T_y = Q_1,$$
$$T''_x = -T_x = N_2, \quad T''_y = R_2 - T_y = Q_2.$$

§ 15. Systems of bars. If two forces act at the ends A and B of a rigid bar and the bar is in equilibrium, then these forces (because their sum and

total moment are equal to zero) act along the bar, are equal in magnitude and oppositely directed. Let us denote these forces by $\boldsymbol{P}$ and $-\boldsymbol{P}$.

Stresses in bars. Let us cut a bar at some point C and remove its right part CB (Fig. 214). Now, in order that the left part of the bar remain in equilibrium, it would be necessary to add the force $-\boldsymbol{P}$ with its initial point at C. Consequently we can assume that the right part CB acts on the left part AC with a force $-\boldsymbol{P}$. This force is called the *stress* in the bar.

If the forces $\boldsymbol{P}$ and $-\boldsymbol{P}$ are directed towards each other, the stress is called a *compression* (Fig. 215), in the opposite case a *tension* (Fig. 214).

At every point of the bar the stress has the same magnitude and direction; the sense of the stress, however, depends on whether we are considering the reaction of the right part on the left, or conversely. With reference to the end of the bar, we can only talk about the reaction of the whole bar on the end. Therefore, if we give the magnitude of the stress and its kind (i. e. whether it is a tension or a compression), then the stresses at the ends of the bar will be completely defined.

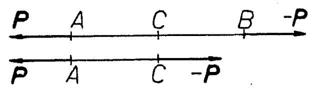


Fig. 214.

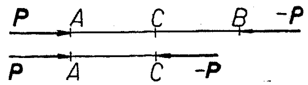


Fig. 215.

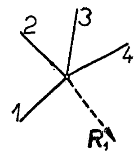


Fig. 216.

Pin-connections. Let us imagine that several rigid bars are so connected that they must constantly be in contact with each other at certain points, e. g. at the ends. If, in addition to this, the connection does not cause other limitations of the motion of the bars, we say that the bars are *pin-connected* and the points at which the bars are pinned are called *joints*.

A pin-connection can be obtained approximately by joining, for example, the ends of the bars by a very short inextensible string.

According to the theorem on reduction, the reactions which one bar exerts on the other can be replaced by one force $\boldsymbol{R}$ acting at the point of contact and a force couple of moment $\boldsymbol{M}$.

For simplicity's sake we shall assume that $\boldsymbol{M} = 0$. We then say that *the joint is smooth*.

It should be noted that not always can we assume that a joint is smooth; examples of this will be given later (p. 294).

In the case of a smooth joint, the reaction with which one bar acts on the other is a force acting at the point of contact, i. e. at the joint. In particular, if several bars come together at a smooth joint, then the reaction exerted on a certain bar by those remaining will be a force acting at the joint (e. g. the reaction $\mathbf{R}_1$ of the bars 2, 3, 4, on the bar 1, shown in Fig. 216).

Systems of bars. Let us consider a system of bars connected at their ends. If some external force acts at a joint, then it is necessary to specify clearly on which bar this force acts.

If a system of bars is in equilibrium, the forces acting on each bar must obviously balance the reactions exerted on this bar.

In Fig. 217 is shown a system of bars in which the external forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, are acting on the bar AB. The force $\mathbf{P}_1$ is acting at the end A of the bar. These forces balance the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at the ends A and B.

It is often convenient to consider a joint as a separate material point (as a separate body) connected with the ends of the bars coming together at this joint. In other words, it is assumed that the ends of the bars are not connected together directly, but by means of a joint. Under this assumption the reactions of the pinned bars are replaced by the reactions of the joint on these bars and the forces applied at a joint are considered as forces acting on the joint, and not on the bars. The only internal forces of a system of bars will then be the reactions of the joints on the bars and those of the bars on the joints.

In the case of the equilibrium of a system each bar and joint is in equilibrium; consequently (p. 286):

1° *external forces acting on an arbitrary bar* (not attached at a joint) *balance the reactions which the joints exert on this bar*;

2° *external forces acting at any joint whatsoever balance the reactions of the bars at this joint.*

In Fig. 218 the force $\mathbf{P}_2$ with its origin at A balances the reactions $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, of the bars 1, 2, 3, on the joint A. On the other hand, the external forces $\mathbf{P}_1$, $\mathbf{P}_3$, acting on the bar AB balance the reactions $\mathbf{T}_1$ and $\mathbf{T}_2$ of the joints A and B on this bar.

In general, nothing can be said in advance about the directions of the reactions of the joints on the bars. The situation is different, however, when the external forces are applied only at the joints.

Let us consider just such a system of bars which remain in equilibrium (Fig. 219). Let $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the reactions of the joints A and B on the bar connecting these joints. Since no external forces act on the bar, the reactions $\mathbf{T}_1$ and $\mathbf{T}_2$ must balance (this follows from condition 1°). Therefore the reactions act along the bar and we have $\mathbf{T}_1 = -\mathbf{T}_2$.

Hence: *if a system of pin-connected bars is in equilibrium, and the external forces are applied only at the joints, then the reactions of the joints on the bars are forces directed along the bars; the reactions which the joints exert on the bar connecting them are equal in magnitude and direction, but opposite in sense.*

The reactions of the joints A and B cause a stress in the bar which may be a tension or a compression (in our case the stress is a tension). By

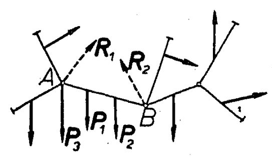


Fig. 217.

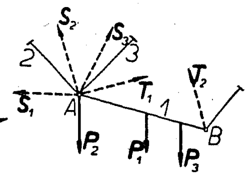


Fig. 218.

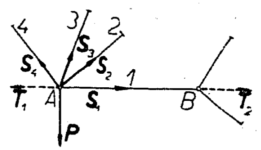


Fig. 219.

the law of action and reaction the bar AB acts on the joint A with a force $\mathbf{S}_1 = -\mathbf{T}_1$. The force $\mathbf{S}_1$ is therefore a stress in the bar at the end A. The reactions of the bars on the joints are then equal to the stresses in these bars. From condition 2°, therefore, we obtain the following theorem:

The external forces applied at a joint balance the stresses in the bars (coming together at this joint).

In Fig. 219 the stresses $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, $\mathbf{S}_4$ in the bars 1, 2, 3, 4, balance the force $\mathbf{P}$ acting at the joint A; therefore $\mathbf{P} + \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = 0$.

Example 1. Three bars AB, BC, and CD, pin-connected at the points B and C, and fixed by means of the joints at the points A and D, remain in equilibrium in a vertical plane under the action of the vertical forces $\mathbf{P}$ and $\mathbf{Q}$ whose origins are E and F. Given are: the force $\mathbf{P}$ and the points of application E and F. Determine the force $\mathbf{Q}$ (Fig. 220).

The reactions $\mathbf{R}_1$ and $\mathbf{T}_1$ act at the points A and B of the bars AB and BC, respectively. We have $\mathbf{R}_1 + \mathbf{T}_1 = 0$; the reactions $\mathbf{R}_1$ and $\mathbf{T}_1$ therefore act along the bar AB.

The reaction $-\mathbf{T}_1$, the force $\mathbf{P}$, and the reaction $\mathbf{T}_2$ of the bar CD at the point C, act on the bar BC. Since the bar BC is in equilibrium, the directions of these forces intersect at one point G, which we find as the intersection of the line AB and of the direction of the force $\mathbf{P}$. Having the point G, we can obtain the direction CG of the reaction $\mathbf{T}_2$. Since $-\mathbf{T}_1 + \mathbf{P} + \mathbf{T}_2 = 0$, knowing the

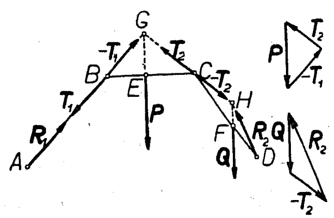


Fig. 220.

directions of the forces $\mathbf{T}_1$, $\mathbf{T}_2$, and the force $\mathbf{P}$, we can determine the forces $\mathbf{T}_1$, $\mathbf{T}_2$ from the triangle of forces.

The reactions $-\mathbf{T}_2$, $\mathbf{R}_2$, and the force $\mathbf{Q}$, act on the bar CD. These forces intersect at one point H which we obtain as the point of intersection of the directions of the forces $-\mathbf{T}_2$ and $\mathbf{Q}$. Having the point H, we obtain the direction of the force $\mathbf{R}_2$. Since $-\mathbf{T}_2+\mathbf{Q}+\mathbf{R}_2=0$, knowing $\mathbf{T}_2$, we obtain the forces $\mathbf{Q}$ and $\mathbf{R}_2$ from the triangle of forces.

Example 2. Four bars 1, 2, 3, 4, are pin-connected at A, B, and C, and fixed at the joints E and F. The bars are inclined to the horizontal at the angles α, β, γ, and δ. Vertical forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, act at the joints A, B, and C. The force $\mathbf{P}_1$ is given. Determine the forces $\mathbf{P}_2$ and $\mathbf{P}_3$ as well as the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at E and F (Fig. 221).

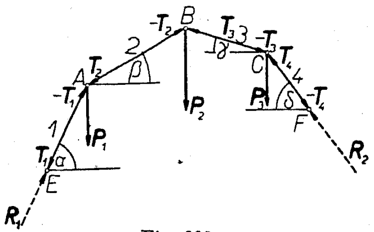


Fig. 221.

Since the external forces act only at the joints, the stresses in the bars at each joint balance the external forces acting at this joint.

At the joint E the stress $\mathbf{T}_1$ in the bar 1 balances the reaction $\mathbf{R}_1 : \mathbf{T}_1+\mathbf{R}_1=0$.

At the joint A the stresses $-\mathbf{T}_1$ in bar 1 and $\mathbf{T}_2$ in bar 2 balance the force $\mathbf{P}_1$: $-\mathbf{T}_1+\mathbf{T}_2+\mathbf{P}_1=0$. Denoting the absolute values of these forces by T_1, T_2, and P_1, and forming their projections on the horizontal and vertical directions, we obtain:

$$T_1\cos\alpha - T_2\cos\beta = 0, \quad T_1\sin\alpha - T_2\sin\beta - P_1 = 0.$$

From these equations we calculate T_1 and T_2.

At the joint B for the stress $\mathbf{T}_3$ in bar 3 we get the relation $-\mathbf{T}_2+\mathbf{T}_3+\mathbf{P}_2=0$. Forming the projections on the horizontal and vertical directions and putting $T_3=|\mathbf{T}_3|$ and $P_2=|\mathbf{P}_2|$, we obtain:

$$T_2\cos\beta - T_3\cos\gamma = 0, \quad T_2\sin\beta + T_3\sin\gamma - P_2 = 0.$$

Using an analogous notation, we get at the joint C:

$$T_3\cos\gamma - T_4\cos\delta = 0, \quad -T_3\sin\gamma + T_4\sin\delta - P_3 = 0,$$

from which we calculate T_4 and P_3.

At the joint F we finally obtain $-\mathbf{T}_4+\mathbf{R}_2=0$ or $\mathbf{R}_2=\mathbf{T}_4$.

Example 3. Decimal balance. A beam AC, supported at O, is connected at the points B and C with the beams DF and GK by means of the

rods BD and CG. The beam DF, supported at the point F, rests at the point H on the beam GK, supported at K. At the points B, C, G and D there are pin-connections (Fig. 222).

A weight $\mathbf{Q}$, which is to be weighed, is put on the beam DF and balanced by the weight $\mathbf{P}$ placed on a pan hanging from A. The weights of the beams and bars are neglected. Determine the relation between the weights $\mathbf{P}$ and $\mathbf{Q}$.

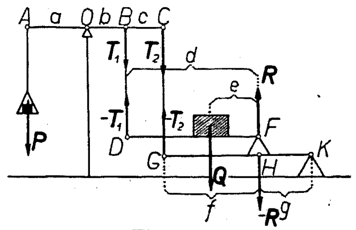


Fig. 222.

Let us denote by $\mathbf{T}_1$, $\mathbf{T}_2$, the stresses in the bars BD and CG at the points B and C.

If the beam AC is in equilibrium, the sum of the moments of the forces acting on it with respect to O is equal to zero:

$$-Pa + bT_1 + (b+c)T_2 = 0, \tag{8}$$

where $P=|\mathbf{P}|$, $T_1=|\mathbf{T}_1|$, $T_2=|\mathbf{T}_2|$, and the lengths a, b, and c, are those shown in the figure.

The forces acting on the beam DF are: the stress $-\mathbf{T}_1$ in the bar BD at the point D, the weight $\mathbf{Q}$, and the reaction $\mathbf{R}$ at the point F. Forming the projections on a vertical direction and taking the moment with respect to F, we obtain in the position of equilibrium for the beam DF:

$$T_1 + R - Q = 0, \quad T_1 d - Qe = 0, \tag{9}$$

where $R=|\mathbf{R}|$, $Q=|\mathbf{Q}|$, and the lengths d and e are those given in the figure.

The forces acting on the beam GK are: the stress $-\mathbf{T}_2$ in the bar CG at the point G and the reaction $-\mathbf{R}$ of the beam DF at the point H. Forming the moment of these forces with respect to the point of support K, we obtain

$$T_2(f+g) - Rg = 0. \tag{10}$$

From equations (9) we obtain:

$$T_1 = Q\cdot\frac{e}{d}, \quad R = Q\cdot\frac{d-e}{d},$$

whence by equation (10)

$$T_2 = Q\cdot\frac{d-e}{d}\cdot\frac{g}{f+g}.$$

Substituting the values obtained in equation (8) for T_1 and T_2, we obtain

$$P = Q\frac{(bf - cg)\,e + (b + c)\,gd}{ad(f + g)}. \tag{11}$$

If we assume that $bf - cg = 0$, whence

$$b \,/\, c = g \,/\, f, \tag{12}$$

then P will be independent of e, i. e. of the position of the weight **Q** on the beam DF. Hence in virtue of (11) and (12)

$$P = Q \cdot \frac{b}{a}.$$

For $b \,/\, a = \frac{1}{10}$ we have a *decimal balance*.

Example 4. Two bars fixed at the ends A and B and pinned at C are collinear. A force **P** whose origin is at D acts on the bar AC in a direction perpendicular to AC (Fig. 223). The system of bars is in equilibrium, since the point C cannot change its position.

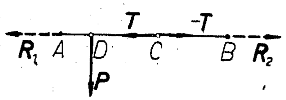


Fig. 223.

Let us suppose for a moment that the bar AC acts on CB with a force **T** whose initial point is at C. Consequently the bar CB would act on the bar AC with a force —**T** also acting at C.

Let us denote by $\mathbf{R}_1$ and $\mathbf{R}_2$ the reactions at A and B.

Since the bar CB is in equilibrium, the forces $\mathbf{R}_2$ and **T** act along the bar AB. It follows from this that the bar AC cannot be in equilibrium because the forces —**T**, $\mathbf{R}_1$, and **P**, acting on this bar do not balance one another, for their total moment with respect to A is equal to the moment of the force **P** with respect to A, which is different from zero. We have thus arrived at a contradiction.

We must therefore assume that the bar AC acts on the bar CB with a force equipollent to one force and a force couple of moment different from zero.

§ 16. Frames. A system of rigid bars, pin-connected and forming as a whole a rigid body, is called a *space frame* (or *truss*).

Examples of space frames are 1. a system of three bars, pin-connected and forming a triangle (Fig. 224a), 2. a system of six bars forming the edges of a tetrahedron (Fig. 224b). On the other hand, a system of bars forming

the edges of a rectangular parallelepiped and pin-connected at the vertices is not a frame because the bars can change their relative positions.

Joints of a frame are also called *nodes*.

Plane frame. If a system of pin-connected rigid bars is coplanar and the bars cannot change their mutual positions in this plane, then such a system is called a *plane frame*.

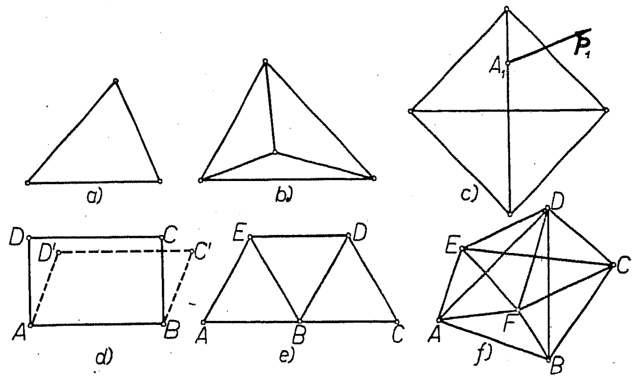


Fig. 224.

Examples of plane frames are represented in Fig. 224a, b, c, e, and f.

The system of bars in Fig. 224d does not form a frame, even a plane frame, because the bars can change their relative positions; they can assume e. g. the position indicated by the dotted lines.

A plane frame does not form a rigid system if we admit motions of the bars in space. For example, if we fix the joints B, C, D, and E, of the frame in Fig. 224e, then we can rotate the bars AE and AB in space about EB. In the plane of the frame, the bars AE and AB cannot move.

If we remove the bar AB in the frame shown in Fig. 224f, then in its plane the system of bars continues to remain a rigid system, i. e. a frame. Such a bar is called a *redundant* bar.

The frame shown in Fig. 224e does not have any redundant bars.

Analytical method of determining stresses in a frame. A plane frame has p bars and w joints $A_1, A_2, \ldots, A_w$, at which the external forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_w$ are applied. When the joint A_i is connected by a bar with the joint A_j (Fig. 225), we denote the length of this bar by d_{ij} and the number whose ab-

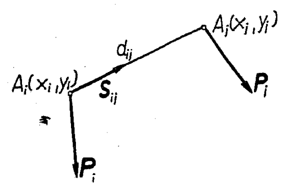


Fig. 225.

solute value is equal to the magnitude of the stress in this rod by S_{ij}; the sign is + or — depending on whether the stress is a tension or a compression (p. 289).

Let us choose an arbitrary coordinate system and denote the coordinates of the joints $A_1, A_2, \ldots, A_w$, by $x_1, y_1, x_2, y_2, \ldots, x_w, y_w$. From the definition of the number S_{ij} it follows that for $i = 1, 2, \ldots, w$, the stress in the bar at the joint A_i has the projections

$$\frac{x_j - x_i}{d_{ij}} S_{ij}, \quad \frac{y_j - y_i}{d_{ij}} S_{ij},$$

on the coordinate axes.

Since the external force $\mathbf{P}_i$ at the joint A_i balances the stresses in the bars pinned at this joint (p. 291),

$$P_{i_x} + \sum_j \frac{x_j - x_i}{d_{ij}} S_{ij} = 0, \quad P_{i_y} + \sum_j \frac{y_j - y_i}{d_{ij}} S_{ij} = 0,$$

i. e.

$$P_{i_x} = \sum_j \frac{x_i - x_j}{d_{ij}} S_{ij}, \quad P_{i_y} = \sum_j \frac{y_i - y_j}{d_{ij}} S_{ij}, \tag{1}$$

where the summation extends over all indices j for which the joint A_j is connected by a bar with the joint A_i. Since, by hypothesis, there are w joints, system (1) consists of $2w$ equations.

Equations (1) serve to determine the stresses S_{ij} when the forces $\mathbf{P}_i$ are given. It may happen, however, that system (1) does not possess a solution or there are too few equations to determine the unknowns S_{ij}.

For instance, in the frame shown in Fig. 224c, p. 295, the bars in contact at the joint A_1 are collinear and the external force $\mathbf{P}_1$ acting at this joint does not lie on this line. The stresses in these bars cannot balance the force $\mathbf{P}_1$. The system of equations (1) for this frame does not therefore have a solution (p. 294).

In the frame of Fig. 224f, p. 295, we have 13 bars and 6 vertices. The number of unknown stresses is consequently 13 and the number of equations in system (1) is only $2 \cdot 6 = 12$. Therefore there are too few equations.

In equations (1) let us denote the right sides of the first equations by E_i, and those of the second by F_i. Equations (1) then assume the form:

$$P_{i_x} = E_i, \quad P_{i_y} = F_i. \tag{2}$$

It can be easily verified by calculation that:

$$\sum_{i=1}^{w} E_i = 0, \quad \sum_{i=1}^{w} F_i = 0, \quad \sum_{i=1}^{w} (E_i y_i - F_i x_i) = 0. \tag{3}$$

The equalities (3) are identities, i. e. they hold for all values of S_{ij}.

The identities (3) can also be derived without calculation in the following manner:

Let us choose S_{ij} entirely arbitrarily and determine P_{i_x} and P_{i_y} from equations (1). Since these equations express the fact that the stresses at every joint of the bars balance the external forces, the forces so determined will be in equilibrium (p. 290). Consequently the external forces will be in equilibrium, i. e. the following equalities will hold:

$$\sum_{i=1}^{w} P_{i_x} = 0, \quad \sum_{i=1}^{w} P_{i_y} = 0, \quad \sum_{i=1}^{w} (P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{4}$$

It follows from this, in virtue of (2), that relations (3) must be satisfied identically for all values of S_{ij}.

Let us now assume that for a certain frame equations (1) have a solution for every system of forces $\{\mathbf{P}_i\}$ equipollent to zero. Let us further assume that the right sides of equations (1) satisfy identically some linear relation of the form

$$a_1 E_1 + a_2 E_2 + \ldots + b_1 F_1 + b_2 F_2 + \ldots = 0, \tag{5}$$

where $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are certain constants.

Let the system of forces $\{\mathbf{P}_i\}$ be equipollent to zero; equations (1) therefore have a solution. By (2) and (5) the forces $\{\mathbf{P}_i\}$ must consequently satisfy the relation

$$a_1 P_{1_x} + a_2 P_{2_x} + \ldots + a_w P_{w_x} + b_1 P_{1_y} + b_2 P_{2_y} + \ldots + b_w P_{w_y} = 0. \tag{6}$$

Hence, if the forces $\{\mathbf{P}_i\}$ satisfy equations (4), then they also satisfy equation (6). Relation (6) is therefore dependent on relations (4). It follows from this that relation (5) depends on relations (3).

The right sides of equations (1) consequently satisfy only three independent relations (3). Hence the system of equations (1) has $2w - 3$ independent equations (while three equations depend on the remaining ones). The unknowns must be at least as many as there are independent equations, i. e. $\geq 2w - 3$. Since there are as many unknowns S_{ij} as there are bars, namely p,

$$p \geq 2w - 3. \tag{7}$$

When $p > 2w - 3$, the number of independent equations is less than the number of unknowns; hence there exist infinitely many solutions. When $p = 2w - 3$, there are as many unknowns as linearly independent equations; consequently the stresses S_{ij} are uniquely determined.

A frame is said to be *statically determinate* if equations (1) determine uniquely the stresses S_{ij} of the bars for every system of forces $\{\mathbf{P}_i\}$ which is in equilibrium. Therefore we have proved the theorem

I. *If a frame is statically determinate, then* $p = 2w - 3$ (where p denotes the number of bars and w the number of joints).

One can prove the theorem

II. *A statically determinate frame does not possess redundant bars.*

The conditions expressed in theorems I and II are necessary, but not sufficient, in order that a frame be statically determinate (*vide* Fig. 224c, p. 295).

Determination of stresses in a frame (by means of force diagrams). We are to determine the stresses in the bars of the frame shown in the Fig. 226. Joints are denoted by the letters A, B, C, D, and bars by the numbers *1, 2, 3, 4, 5*. The frame is loaded by a vertical force $\mathbf{P}$ at the joint C and rests on smooth supports A and B.

Let us first determine the reactions at A and B. On account of symmetry each of the reactions is equal to $-\frac{1}{2}\mathbf{P}$.

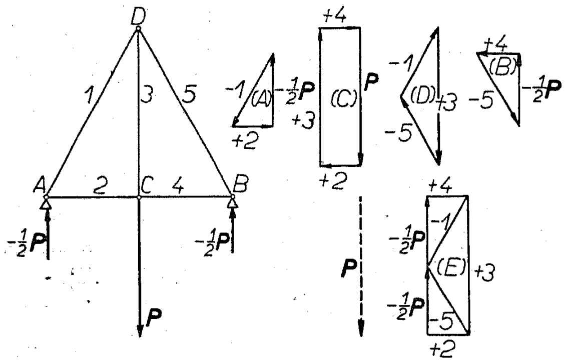


Fig. 226.

The joint A is acted upon by the external force $-\frac{1}{2}\mathbf{P}$ and the stresses in the bars *1* and *2*. Since these forces balance one another, they form a closed polygon which we can draw because we know the force $-\frac{1}{2}\mathbf{P}$ and the directions of the stresses. This polygon is shown in Fig. 226 (A); the stresses are denoted by the numbers *1* and *2*, and the signs before the numbers denote whether the stress is a tension (+) or a compression (—).

The joint C is acted upon by the stresses in the bars *2, 3*, and *4*, which balance the force $\mathbf{P}$. Since only two forces are unknown, namely, the stresses in the bars *3* and *4*, we can draw a force polygon, remembering

that the stress in bar *2* at the joint C has a sense opposite to that at the joint A. This polygon is shown in Fig. 226 (C).

No external forces act at the joint D; consequently the stresses balance one another. Therefore they form a closed polygon which can be drawn (Fig. 226 (D)), remembering that the stresses in bars *1* and *3* at the joint D have senses opposite to those at A and C.

We have determined the stresses in every bar. In order to verify our reasoning we can form another polygon for the joint B (Fig. 226 (B)).

Proceeding in this manner we have drawn each force twice. However we can simplify matters by combining all the polygons (A), (B), (C), and (D), as in Fig. 226 (E).

In Fig. 228 (E) each force appears only once; such a drawing is called a *Cremona force diagram* for the given frame.

We shall give certain directions for obtaining a Cremona force diagram in the following example:

Fig. 227 represents a frame and Fig. 228 its Cremona force diagram. The frame is loaded at the joints G and F by forces $\mathbf{P}$ and $2\mathbf{P}$. At the joints A and E it rests on smooth supports. Bars *2, 6*, and *10*, are horizontal and equal in length. Calculating the moment of the external forces with respect to E and A, we find that the reactions at A and E are $-\frac{4}{3}\mathbf{P}$ and $-\frac{5}{3}\mathbf{P}$, respectively.

We now draw the polygon of external forces in the order in which they appear on the perimeter of the frame. For instance, going clockwise we draw in turn $-\frac{4}{3}\mathbf{P}$, $-\frac{5}{3}\mathbf{P}$, $2\mathbf{P}$, and $\mathbf{P}$.

We next construct a polygon for the joint A. Let us note that bar *2* connects the joints A and G at which the external forces $-\frac{4}{3}\mathbf{P}$ and $\mathbf{P}$ act. In the force diagram the stress in bar *2* is drawn from the origin of the force $-\frac{4}{3}\mathbf{P}$ and from the terminus of the force $\mathbf{P}$. The force $-\frac{4}{3}\mathbf{P}$ defines the sense of the forces in the polygon for the joint A. We obtain the

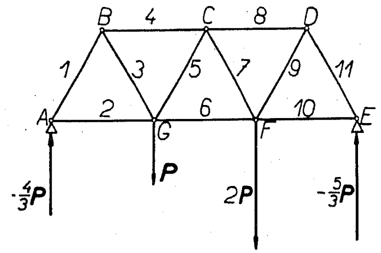


Fig. 227.

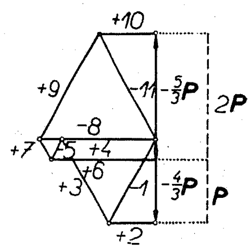


Fig. 228.

stresses in the bars *1* and *2* and indicate that bar *1* is in compression (—) while *2* is in tension (+).

Let us now proceed to consider a joint where only two stresses are unknown. Such a joint is *B*. We determine the polygon of stresses in the bars *1*, *3*, and *4*. We obtain the sense of the forces because we know from the preceding polygon that bar *1* is in compression.

Let us next consider the joint *G* at which only two stresses are unknown, namely, the stresses in bars *5* and *6*. Proceeding as before, we consider in turn the joints *C*, *F*, *D*, and *E*.

In preparing a Cremona force diagram it is necessary to adhere to the following rules:

1. The external forces in the polygon of forces are drawn on the diagram in the order in which they appear on the perimeter of the frame.

2. If a bar on the perimeter of the frame connects joints which are the origins of the external forces, then the stress in the bar is drawn on the diagram of forces from the point at which the terminus of one external force meets with the origin of the other.

Let us note that the Cremona force diagram represented in Fig. 228, p. 299, has, besides, the following two properties:

a) the forces acting at a joint form a closed polygon in the diagram,

b) if three bars form a triangle, then in the diagram their stresses have origin at one point.

A Cremona force diagram having the above two properties is called a *reciprocal force diagram*.

This name has reference to the so-called theory of reciprocal figures.

Let us note that it is not possible to construct a Cremona force diagram for every statically determinate frame.

Determination of stresses by means of sections. Let us suppose that a frame is such that it is possible to cut three of its bars whose origins are not at one joint, in such a manner that the frame is divided into two parts. If at least two of the bars cut are not parallel, then it is possible to determine the stresses in the bars cut without calculating the stresses in the remaining bars.

Let us denote the bars cut by *1*, *2*, *3*. If one part of the frame is removed, e. g. the right part, then the left part will remain in equilibrium after the addition of the stresses $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$.

Let the bars *1* and *2* intersect at the point O (Fig. 229). Since the left part of the frame is in equilibrium, the external forces acting on this part balance the stresses $\mathbf{S}_1, \mathbf{S}_2$, and $\mathbf{S}_3$. Denoting by $\mathbf{M}$ the moment of the

external forces (acting on the left part of the frame) with respect to O, and by d the distance of O from the bar *3*, we get $|\mathbf{M}| = |\mathbf{S}_3| d$, i. e. $|\mathbf{S}_3| = |\mathbf{M}| / d$.

We choose the sense of the force $\mathbf{S}_3$ such that $\mathbf{M}$ and the moment of the force $\mathbf{S}_3$ with respect to O have opposite signs. The moment $\mathbf{M}$ can be obtained by determining at first the resultant $\mathbf{R}$ (or possibly a resultant couple) of the external forces acting on the left part, and next calculating the moment of the resultant $\mathbf{R}$ (or a resultant couple) with respect to O. We calculate $\mathbf{S}_2$ and $\mathbf{S}_1$ similarly by forming the moment with respect to the point of intersection of bars *1*, *3* and *2*, *3*.

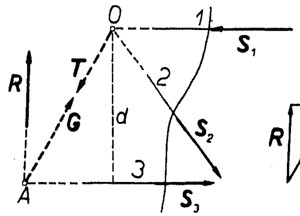


Fig. 229.

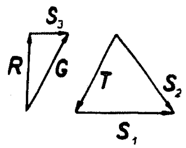


Fig. 230.

If bars *1* and *3* were parallel, we would obtain the force $\mathbf{S}_2$ by forming the projections of the forces on a line perpendicular to the bars *1* and *3* (for the projections of the forces $\mathbf{S}_1$ and $\mathbf{S}_3$ will be zero).

The method described above of calculating stresses was given by J. W. Ritter. The stresses $\mathbf{S}_1, \mathbf{S}_2$, and $\mathbf{S}_3$, can also be determined graphically by means of a method given by K. Culmann.

Let the bars *1* and *2* intersect at O, let the external forces have a resultant $\mathbf{R}$, and let $\mathbf{R}$ and $\mathbf{S}_3$ intersect at the point A. Let us denote the resultant of the forces $\mathbf{S}_1$ and $\mathbf{S}_2$ by $\mathbf{T}$, and the resultant of the forces $\mathbf{R}$ and $\mathbf{S}_3$ by $\mathbf{G}$.

Since the forces $\mathbf{R}, \mathbf{S}_1, \mathbf{S}_2$, and $\mathbf{S}_3$, are in equilibrium, the forces $\mathbf{T}$ and $\mathbf{G}$ are also in equilibrium. It follows from this that $\mathbf{T} = -\mathbf{G}$ and that the forces $\mathbf{T}$ and $\mathbf{G}$ are collinear. Since $\mathbf{T}$ has its origin at O, and $\mathbf{G}$ at A, the forces $\mathbf{T}$ and $\mathbf{G}$ lie on the line OA. Knowing already the direction of the forces $\mathbf{T}$ and $\mathbf{G}$, we determine the triangle of forces $\mathbf{R}, \mathbf{S}_3$, and $\mathbf{G}$, from which we obtain the forces $\mathbf{S}_3$ and $\mathbf{G}$. Since $\mathbf{T} = -\mathbf{G}$, we can construct the triangle of forces $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{T}$, from which we can get $\mathbf{S}_1$ and $\mathbf{S}_2$ (Fig. 230).

If the resultant $\mathbf{R}$ were parallel to the bar *3*, the force $\mathbf{G}$ would also be parallel to *3* and would pass through O. Since $\mathbf{R}$ is the resultant of the forces $\mathbf{G}$ and $-\mathbf{S}_3$, the problem would then be reduced to the resolution of the force $\mathbf{R}$ into two forces $\mathbf{G}$ and $-\mathbf{S}_3$ whose positions are given. Such a problem was solved by means of a string polygon on p. 253.

Finally, if the external forces were reduced to a couple $\mathbf{R}_1$ and $\mathbf{R}_2$, we should have $\mathbf{R}_1 = -\mathbf{R}_2$ and the resultant $\mathbf{G}$ of the forces $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{S}_3$, would be equal to $\mathbf{S}_3$ and would have its origin at O. The problem would then be reduced to the resolution of the system of forces $\mathbf{R}_1$ and $\mathbf{R}_2$ into two forces $\mathbf{G}$ and $-\mathbf{S}_3$ whose positions are given (cf. p. 253).

§ 17. Equilibrium of heavy cables. Chain. A system of rigid pin-connected rods is called a *chain* if only two rods are pinned at each joint. The rods of a chain are also called *links*.

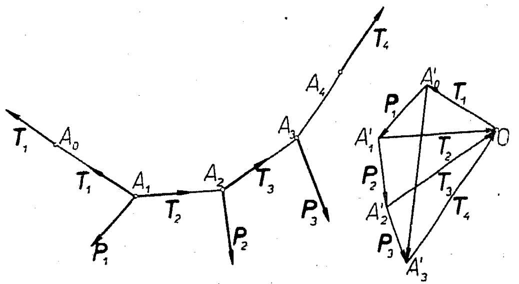


Fig. 231.

Let us assume that a chain consisting of the links A_0A_1, A_1A_2, A_2A_3, and A_3A_4, pin-connected at the joints A_1, A_2, and A_3, remains in equilibrium under the action of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, applied at the joints, and the forces $\mathbf{T}_1$ and $\mathbf{T}_4$ applied at A_0 and A_4 (Fig. 291). The forces $\mathbf{T}_1$ and $\mathbf{T}_4$ obviously have the directions of the rods A_0A_1 and A_3A_4 (p. 291).

It is easy to show that *a chain in equilibrium assumes the form of a string polygon* of the system of forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$.

To show this let us construct a polygon of forces $A'_0A'_1A'_2A'_3$ for the system $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$. For the pole O let us take the point of intersection of the lines drawn from the points A'_0, A'_3 and parallel to the extreme rods of the chain.

Since the chain is in equilibrium, the sum of the forces is zero:

$$\mathbf{T}_1 + \mathbf{T}_4 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = 0.$$

Since $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \overline{A'_0A'_3}$, and $\overline{OA'_0}$, $\overline{A'_3O}$ are parallel to $\mathbf{T}_1$, $\mathbf{T}_3$, from triangle $A'_0A'_3O$ we obtain:

$$\mathbf{T}_1 = \overline{OA'_0}, \quad \mathbf{T}_3 = \overline{A'_3O}. \tag{1}$$

Let us now consider the joint A_1. The stress of the link A_0A_1 at the joint A_1 is $\mathbf{T}_1$; let us denote by $\mathbf{T}_2$ the stress of the link A_1A_2 at the joint A_1. We obviously have

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{P}_1 = 0. \tag{2}$$

From the string polygon we obtain $\overline{OA'_0} + \mathbf{P}_1 + \overline{A'_1O} = 0$, whence by (1) $\mathbf{T}_1 + \mathbf{P}_1 + \overline{A'_1O} = 0$, and from this by (2) $\overline{A'_1O} = \mathbf{T}_2$. The segment A'_1O is therefore parallel to the rod A_1A_2.

Similarly, we ascertain that the segments A'_2O and A'_3O are parallel to the rods A_2A_3 and A_3A_4.

It follows from this that the string polygon drawn from the point A_1 will assume the form of a chain.

Cable. A *rope* or a *cable* (*flexible* and *inextensible*) is defined as a material line which can be bent arbitrarily without changing its length or that of any of its parts.

A rope can therefore assume the form of an arbitrary curve of the same length. A cable can be considered approximately as a chain consisting of very many small links.

Let a heavy cable (flexible and inextensible) be suspended from two points A and B. Let us assume that the density of the cable ϱ = constant. The weight of a portion of the cable of length s cm is therefore

$$Q = s\varrho g = s\delta, \tag{3}$$

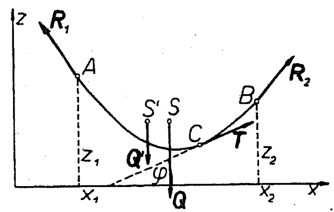


Fig. 232.

where $\delta = \varrho g$.

Let us determine the form that the cable will assume under the action of its own weight.

Let us choose a system of coordinates (x, y, z), giving the z-axis a vertical direction and an upward sense; let the xz-plane be taken vertically and passing through the points A and B (Fig. 232).

The external forces acting on the cable are: the weight, acting at the centre of gravity S of the cable, and the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at the points A and B. If the cable is in equilibrium, these forces balance each other. It follows from this that they lie in one vertical plane, namely, the xz-plane. Consequently:

$$R_{1_y} = 0, \quad R_{2_y} = 0. \tag{4}$$

Let us cut the cable at an arbitrary point C and remove the part CB. In order that the portion AC remain in equilibrium, it is necessary to add at the point C the force $\mathbf{T}$ which the portion CB exerts on the portion AC. The force $\mathbf{T}$ is the *tension of the cable.*

Since the cable is considered approximately as a chain consisting of small links, the force $\mathbf{T}$ is tangent to CB.

Let us denote the length of the arc AC by s. The external forces acting on the part AC are: the weight $\mathbf{Q}'$ of magnitude $s\delta$, acting at the centre of gravity S' of the part AC, the reaction $\mathbf{R}_1$, and the tension $\mathbf{T}$. Since the sum of these forces is zero, because the part AC remains in equilibrium, forming the projections on the coordinate axes, we obtain:

$$T_x + R_{1_x} = 0, \quad T_z + R_{1_z} - s\delta = 0, \quad T_y + R_{1_y} = 0. \tag{5}$$

From (4) $R_{1_y} = 0$; hence $T_y = 0$. Since the tension $\mathbf{T}$ is tangent to the curve, and C is an arbitrary point of this curve, the tangent at each point is parallel to the vertical xz-plane. It follows from this that the curve lies in a vertical plane, namely, the xz-plane, because it has in common with it the two points A and B. The first of the equations (5) gives

$$T_x = -R_{1_x} = \text{const.} \tag{6}$$

Therefore: *the horizontal component of the tension of the cable is the same at each point of the cable.*

Let us denote by φ the angle which $\mathbf{T}$ makes with the x-axis. We therefore have $\tan\varphi = T_z / T_x$, whence by the second of the equations (5)

$$\tan\varphi = (-\delta s / R_{1_x}) + (R_{1_z} / R_{1_x}). \tag{7}$$

Let us put:

$$a = -\delta / R_{1_x}, \quad a' = R_{1_z} / R_{1_x}. \tag{8}$$

If $z = f(x)$ is the equation of the curve, then $z' = \tan\varphi$. Hence in virtue of (7) and (8)

$$z' = as + a'. \tag{9}$$

Equation (9) is the differential equation of the curve whose form the cable assumes. Differentiating it, we obtain $z'' = a\,\mathrm{d}s / \mathrm{d}x$, and since $\mathrm{d}s = \sqrt{1+z'^2}\,\mathrm{d}x$, it follows that $z'' = a\sqrt{1+z'^2}$. Let us substitute $z' = w$. Hence $z'' = \mathrm{d}w/\mathrm{d}x$, whence $\mathrm{d}w/\mathrm{d}x = a\sqrt{1+w^2}$, i. e. $\mathrm{d}w/\sqrt{1+w^2} = a\,\mathrm{d}x$. Integrating, we obtain $\int \mathrm{d}w/\sqrt{1+w^2} = \int a\,\mathrm{d}x$; therefore $\ln(\sqrt{1+w^2}+w) = ax + c$, where c is the constant of integration. Consequently

$$\sqrt{1+w^2} + w = \sqrt{1+z'^2} + z' = e^{ax+c}. \tag{10}$$

We have

$$1 / (\sqrt{1+z'^2} + z') = \sqrt{1+z'^2} - z' = e^{-ax-c}. \tag{11}$$

From equations (10) and (11) we obtain

$$z' = \tfrac{1}{2}(e^{ax+c} - e^{-ax-c}), \tag{12}$$

$$\mathrm{d}s / \mathrm{d}x = \sqrt{1+z'^2} = \tfrac{1}{2}(e^{ax+c} + e^{-ax-c}). \tag{13}$$

Integrating equations (12) and (13), we obtain

$$z = \frac{1}{2a}(e^{ax+c} + e^{-ax-c}) + c', \tag{14}$$

$$s = \frac{1}{2a}(e^{ax+c} - e^{-ax-c}) + c'', \tag{15}$$

where c' and c'' are certain constants.

The curve defined by equation (14) is called the *catenary*.

Therefore: *a cable hangs in the form of a catenary.*

Equations (14) and (15) depend on four constants a, c, c', and c''. These constants can be determined if we know, for instance, the coordinates x_1, z_1, x_2, z_2, of the points A, B and the length l of the cable, because from the conditions that $z = z_1$, for $x = x_1$, and $z = z_2$, for $x = x_2$, we obtain by (14):

$$z_1 = \frac{1}{2a}(e^{ax_1+c} + e^{-ax_1-c}) + c', \quad z_2 = \frac{1}{2a}(e^{ax_2+c} + e^{-ax_2-c}) + c'. \tag{16}$$

While from the conditions that $s = 0$, for $x = x_1$, and $s = l$, for $x = x_2$, we obtain by (15):

$$0 = \frac{1}{2a}(e^{ax_1+c} - e^{-ax_1-c}) + c'', \quad l = \frac{1}{2a}(e^{ax_2+c} + e^{-ax_2-c}) + c''. \tag{17}$$

It can be shown that equations (16) and (17) define the constants a, c, c', and c'', uniquely.

Let us still compute the tension $\mathbf{T}$ of the cable at an arbitrary point C whose coordinates are x, z. From equations (6) and (8) we get

$$T_x = \delta / a. \tag{18}$$

Since $T_z / T_x = \tan\varphi = z'$, $T_z = T_x z'$; consequently

$$T_z = \delta z' / a. \tag{19}$$

From equations (18) and (19) we obtain

$$T = \sqrt{T_x^2 + T_z^2} = \frac{\delta}{a}\sqrt{1+z'^2} = \frac{\delta}{a}\cdot\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{\delta}{2a}(e^{ax+c} + e^{-ax-c}),$$

and hence by (14)

$$T = \delta(z - c') \tag{20}$$

Loaded cable. Let a force **P** directed vertically downwards be applied at a point C of a cable. As we already know, parts CB and AC of the cable are catenaries. Let us denote the constants for the curves BC and CA in equations (14) and (15) by a_1, c_1, c_1', c_1'' and a_2, c_2, c_2', c_2'', respectively, and the tensions in the parts BC and CA at the point C by $\mathbf{T}_1$ and $\mathbf{T}_2$ (Fig. 233).

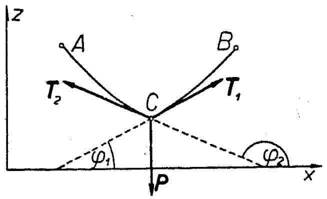


Fig. 233.

Considering the cable as a chain consisting of many small links, and the point C as a joint, we have in the position of equilibrium $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{P} = 0$. Forming projections on the x and z axes and putting $P = |\mathbf{P}|$, we obtain:

$$T_{1_x} + T_{2_x} = 0, \quad T_{1_z} + T_{2_z} - P = 0. \tag{21}$$

Denoting the right-hand and the left-hand derivatives at C by z_1' and z_2', we obtain by (18) and (19):

$$T_{1_x} = \delta / a_1, \; T_{1_z} = \delta z_1' / a_1, \; T_{2_x} = -\delta / a_2, \; T_{2_z} = -\delta z_2' / a_2,$$

whence by (21)

$$\frac{\delta}{a_1} - \frac{\delta}{a_2} = 0, \qquad \frac{\delta z_1'}{a_1} - \frac{\delta z_2'}{a_2} - P = 0.$$

Hence we get:

$$a_1 = a_2, \qquad z_1' - z_2' = \frac{Pa_1}{\delta} = \frac{Pa_2}{\delta}. \tag{22}$$

Knowing the lengths l_1 and l_2 of the arcs BC and CA we can obtain the equations of the curves CB and AC. In this case it is necessary to determine ten constants $a_1, c_1, c_1', c_1'', a_2, c_2, c_2', c_2''$ and x_0, z_0, where x_0 and z_0 are the coordinates of the point C. To determine these constants for CB and AC we have two sets of four equations analogous to (16) and (17), and in addition two equations (22), i. e. ten altogether.

CHAPTER VII

KINEMATICS OF A RIGID BODY

§ 1. Displacement and rotation of a body about an axis. According to the definition of a rigid body (p. 231), its points do not change their mutual distances during motion. When the point A moved to the point B, the vector $\overline{AB}$ was called the *displacement* of the point (p. 34). During a change of position of a rigid body, the points of this body undergo, in general, various displacements.

We shall first become acquainted with certain theorems from geometry which give the resolution of the displacements of the points of a body. These theorems will be helpful to us in determining the velocities of these points.

Parallel displacement or translation. A body is said to undergo a *parallel displacement* or a *translation* if the displacements of all the points of the body during a change of its position are equal.

The displacement common to all points of the body is called the *displacement vector* or the *displacement of the body.*

The position of the body after a displacement is therefore determined by the initial position and the displacement vector.

Let us assume that the points A_1, B_1 moved to the points A_2, B_2 after a translation. Since the displacements of both points are equal, $\overline{A_1A_2} = \overline{B_1B_2}$. It follows from this that $\overline{A_1B_1} = \overline{A_2B_2}$.

Therefore: *the vectors attached to a body do not change either their sense or direction during a translation.*

Conversely, it is easy to prove that *if the vectors in a body maintain their sense and direction during a displacement of the body, then the displacement is a translation.*

For let us assume that two arbitrary points A_1, B_1 moved to the points A_2, B_2 (Fig. 234). By hypothesis, $\overline{A_1B_1} = \overline{A_2B_2}$; hence $\overline{A_1A_2} = \overline{B_1B_2}$.