CHAPTER V

SYSTEMS OF MATERIAL POINTS

§ 1. Equations of motion. Let there be given a system of material points of masses $m_1, m_2, \ldots, m_n$. Let us denote the sums of the forces acting on the individual points by $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, and the accelerations of these points with respect to an inertial system of coordinates by $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$. Then according to Newton's law:

$$m_1\mathbf{p}_1 = \mathbf{P}_1, \quad m_2\mathbf{p}_2 = \mathbf{P}_2, \quad \ldots, \quad m_n\mathbf{p}_n = \mathbf{P}_n.$$

We write these equations compactly as

$$m_i\mathbf{p}_i = \mathbf{P}_i \qquad (i = 1, 2, \ldots, n). \quad \text{(I)}$$

Let the point m_i have the coordinates x_i, y_i, z_i. Equations (I) can be written in the form:

$$m_i\ddot{x}_i = P_{i_x}, \quad m_i\ddot{y}_i = P_{i_y}, \quad m_i\ddot{z}_i = P_{i_z} \quad (i = 1, 2, \ldots, n). \quad \text{(II)}$$

Unconstrained systems. We assume that the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, in the most general case, depend on the time, position, and acceleration, of the system of points. We shall therefore suppose that the forces are functions of: the time t, the variables $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$ defining the positions of the points, as well as the variables $\dot{x}_1, \dot{y}_1, \dot{z}_1, \ldots, \dot{x}_n, \dot{y}_n, \dot{z}_n$ defining the velocities of the points. Hence we can write:

$$P_{i_x} = F_i(t, x_1, y_1, z_1, \ldots, x_n, y_n, z_n, \dot{x}_1, \dot{y}_1, \dot{z}_1, \ldots, \dot{x}_n, \dot{y}_n, \dot{z}_n),$$

and similarly $P_{i_y} = \Phi_i$, $P_{i_z} = \Psi_i$.

We shall assume that the functions P_{i_x}, P_{i_y} and P_{i_z} are continuous and that they have continuous first partial derivatives with respect to each variable.

Equations (II) are called *Newton's equations of motion.*

They constitute a system of differential equations of the second order. From the theory of differential equations it follows that equations (II) determine the motion of the system of points if there are given at the initial moment $t = t_0$, the initial positions of the points (i. e. coordinates $x_1^0, y_1^0, z_1^0, \ldots, z_n^0$), and the initial velocities (i. e. $\dot{x}_1^0, \dot{y}_1^0, \dot{z}_1^0, \ldots, \dot{z}_n^0$).

Internal and external forces. The forces acting on the points of a system are divided into two groups.

In the first group are included those forces which arise from the mutual interactions of the points of the system. These forces are called *internal* forces.

The remaining forces are called *external* forces.

The internal forces are assumed to conform to the law of action and reaction (p. 72).

Let us consider the pair of forces which the two points m' and m'' of a system exert on each other. The sum of these forces is zero, and because they act along the line joining the points m' and m'', their moment with respect to an arbitrary point is zero. Since the internal forces can be grouped in such pairs, *the sum and the total moment of the internal forces are zero.*

Equilibrium of a system of points. A system of points is in equilibrium if each point is in equilibrium.

Therefore, if a system of points is in equilibrium, then the sum of the forces acting on each point is equal to zero. A system of forces having this property is said to be in *equilibrium* or that the *forces* of this systems *balance each other.*

Let a given system of forces be in equilibrium. Let us consider the forces of this system acting on an arbitrary material point. Since the sum of these forces as well as the total moment with respect to an arbitrary point of space are equal to zero, the given system is *equipollent to zero* (p. 22).

Hence: *if the forces acting on a material system of points balance each other, then the sum of these forces and the total moment are zero.*

This condition is a necessary condition for equilibrium of forces, but it is not a sufficient condition. Since the internal forces have a sum and total moment equal to zero, from the given condition it follows that *if a system of material points is in equilibrium, then the sum and the total moment of the external forces are zero.*

This condition, as the preceding one, is only a necessary condition for the equilibrium of a system.

D'Alembert's principle. We can write equations (I), p. 186, in the form

$$\mathbf{P}_i + (-m_i \ddot{\mathbf{p}}_i) = 0 \qquad (i = 1, 2, \ldots, n).$$

We have called the vectors $-m_i \ddot{\mathbf{p}}_i$ *forces of inertia* of the points m_i (p. 73). We can therefore say that *the forces of inertia balance the forces acting on a system of points.*

The above theorem is called *d'Alembert's principle.*

The significance of this principle appears chiefly in systems of constrained points which we shall consider further on. It is necessary to remember what we have said on p. 73, that forces of inertia are not forces, but vectors, which we have called forces only for the sake of convenience.

Example I. Two points A_1, A_2 of masses m_1, m_2, attracting each other with a force $\mathbf{P}$ according to Newton's law, move along the x-axis. Hence $|\mathbf{P}| = Km_1m_2 / r^2$, where $r = A_1A_2$. Denoting the coordinates of the points by x_1 and x_2 $(x_1 < x_2)$ (Fig. 128), we obtain the equations of motion in the form:

$$m_1\ddot{x}_1 = Km_1m_2 / (x_2 - x_1)^2, \quad m_2\ddot{x}_2 = -Km_1m_2 / (x_2 - x_1)^2. \qquad (1)$$

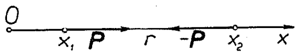

Fig. 128.

Let us suppose that at the time $t = 0$, $x_1 = x_1^0$, $x_2 = x_2^0$, $\dot{x}_1 = 0$, $\dot{x}_2 = 0$. Adding equations (1), we get $m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0$, whence after integrating

$$m_1\dot{x}_1 + m_2\dot{x}_2 = a, \quad m_1x_1 + m_2x_2 = at + b.$$

In view of the initial conditions we obtain $a = 0$, and $b = m_1x_1^0 + m_2x_2^0$. Therefore:

$$m_1\dot{x}_1 + m_2\dot{x}_2 = 0, \quad m_1x_1 + m_2x_2 = m_1x_1^0 + m_2x_2^0. \qquad (2)$$

From equations (1) we obtain in addition

$$\ddot{x}_2 - \ddot{x}_1 = -K(m_1 + m_2) / (x_2 - x_1)^2. \qquad (3)$$

Let us set $x_2 - x_1 = r$, and $K(m_1 + m_2) = h$. We get $\ddot{r} = -h / r^2$. Multiplying both sides by $\dot{r}$ and integrating, we obtain $\frac{1}{2}\dot{r}^2 = h / r + c$. Since at $t = 0$, $\dot{r} = \dot{x}_2^0 - \dot{x}_1^0 = 0$ as well as $r = x_2^0 - x_1^0 = r_0$, it follows that $c = -h / r_0$. Therefore $\frac{1}{2}\dot{r}^2 = h / r - h / r_0$, whence

$$\dot{r} = -\sqrt{2h(1 / r - 1 / r_0)}. \qquad (4)$$

We have taken the minus sign because the points will come closer to

each other, and hence r will become smaller, whence $\dot{r} < 0$. From (4) we get

$$-\frac{\mathrm{d}r}{\sqrt{2h(1 / r - 1 / r_0)}} = \mathrm{d}t; \text{ hence } -\int \frac{\mathrm{d}r}{\sqrt{2h(1 / r - 1 / r_0)}} + c' = t.$$

After integrating we obtain

$$\frac{1}{2}\frac{\sqrt{r_0}}{\sqrt{2h}}\left[2\sqrt{r_0r - r^2} + r_0 \arcsin \frac{r_0 - 2r}{r_0}\right] + c' = t. \qquad (5)$$

Since $r = r_0$ at $t = 0$,

$$c' = \frac{\pi r_0\sqrt{r_0}}{4\sqrt{2h}}. \qquad (6)$$

The time T at which the points meet is obtained from (5) by setting $r = 0$. Therefore

$$T = \tfrac{1}{2}\pi \frac{r_0\sqrt{r_0}}{\sqrt{2h}} = \frac{r_0^{\frac{3}{2}}\pi}{2\sqrt{2K(m_1 + m_2)}}.$$

We obtain $T = 3055$ sec for $m_1 = m_2 = 1$ g, $r_0 = 1$ cm and $K = 6.6 \cdot 10^{-8}$ cm^3 g^{-1} sec^{-2}.

Formulae (2), (5) and (6) (where $r = x_2 - x_1$) determine the motion of the points.

Constrained systems. If conditions exist which limit the possible motion of a system of material points, then it is called a *constrained* system and the limiting conditions are called *constraints*. As in the case of one constrained point, we assume that the constraints are the result of certain forces termed *reactions*, which cause the system to maintain the constraints.

If the forces of reaction are added to the forces acting on the points of a system, then the system can be considered as unconstrained. In this way the investigation of motions of constrained systems is reduced to the investigation of motions of unconstrained systems.

Therefore, if forces $\{\mathbf{P}_i\}$ act on a system of points of masses $\{m_i\}$, then, denoting the reactions by $\{\mathbf{R}_i\}$, we have

$$m_i\ddot{\mathbf{p}}_i = \mathbf{P}_i + \mathbf{R}_i \qquad (i = 1, 2, \ldots, n). \quad \text{(III)}$$

In particular, a constrained system is in equilibrium if the acting forces and the reactions balance each other.

The constraints of a system can be such that some points must constantly remain on certain curves or surfaces. In addition to this kind of constraints, already considered (cf. p. 121), we also meet with others.

For instance, two material points can be joined by an inextensible string of length l; consequently the distance between the points must be constantly $\leq l$. The string acts on the points only when it is in tension (Fig. 129). If we assume that the mass of the string is so small that it can be neglected, then the forces that the string exerts on the points will be of equal magnitude, even if the string is wound around some body (Fig. 129) — provided that there is no friction. These forces are obviously reactions. The reactions are tangent to the string and have a sense in the direction of the string.

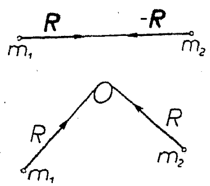

Fig. 129.

Rigid system. An important example of a constrained system of points is the so-called *rigid system*. It is a constrained system whose constraints are such that the mutual distances of the points of the system remain unchanged. Let us suppose that in this case there appear certain internal forces (i. e. forces acting between the points of the system) which cause the points to maintain constant distances in spite of the actions of external forces.

A solid physical body will in general be deformed, i. e. it will change its form under the influence of forces acting on it. It can happen, however, that when the forces do not exceed a certain limit, the deformations are so small that, practically, we can disregard them. In this case, as a model of such a body acted upon by small forces, we can choose a system of points which we have called a rigid system. The results that we shall obtain will then be approximately valid for a physical body. Thus we can apply to solid physical bodies the theorems that we shall obtain for rigid systems. Because of the important role that the theory of rigid body plays, we shall concern ourselves with this theory in all detail in chapter VI. In this chapter we shall limit ourselves to giving only a few examples based on the general theory of a system of points.

The simplest example of a rigid system is a system composed of two material points whose distance r is constant.

We can realize such a system by joining two material points with a rigid rod of a small mass, which in comparison with the masses of the points themselves, can be neglected. We then say that the points are joined by a rigid massless rod. In this manner the internal forces between the points are replaced by forces with which the rod reacts on these points. These forces are therefore equal in magnitude, have the direction of the rod, but opposite senses.

Example 2. Two heavy material points of masses m_1 and m_2 are connected by a (massless) inextensible string passing over a pulley. Point m_2 must remain on a straight vertical line l. What angle φ does the string make with the line l in the position of equilibrium if there is no friction?

The forces acting on the point m_1 are: the tension T of the string directed vertically upwards, and the weight m_1g directed vertically downwards (Fig. 130). Therefore $T - m_1g = 0$, and hence

$$T = m_1 g. \tag{7}$$

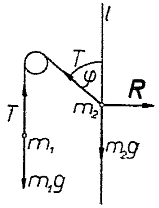

Fig. 130.

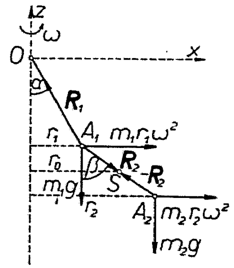

Fig. 131.

The forces acting on the point m_2 are: the tension T of the string directed along the string, the weight m_2g directed vertically downwards, and the reaction **R** perpendicular to the line l. Forming the projections of the forces on the line l, we get $T\cos\varphi - m_2g = 0$, and hence

$$T\cos\varphi = m_2 g. \tag{8}$$

From equations (7) and (8) we obtain

$$\cos\varphi = m_2 / m_1.$$

Equilibrium is therefore possible only when $m_2 < m_1$.

Example 3. Heavy material points A_1 and A_2 of masses m_1, m_2 are connected by a massless inextensible string. Point A_1 is suspended from the string OA_1 which is also massless and inextensible. The entire system rotates about a vertical axis with a constant angular velocity ω, while the angles α and β, which the strings OA_1 and A_1A_2 form with the vertical, do not undergo any change during this rotation. To determine the angles α and β.

Let us choose the point O as the origin of the moving frame (x, y, z) rotating about the vertical z-axis with an angular velocity ω (Fig. 131). We can suppose that the point A_1 is always situated in the xz-plane. Since the angle α is constant, the point A_1 is in relative equilibrium with respect to the frame (x, y, z).

We shall first show that the point A_2 is also situated in the xz-plane.

Since the point A_1 is in relative equilibrium, its force of transport $\mathbf{P}_t^{(1)}$ is in equilibrium with the acting forces, i. e. with the weight $\mathbf{Q}_1$, the reaction $\mathbf{R}_1$ of the string OA_1, and the reaction $\mathbf{R}_2$ of the string A_1A_2. Therefore

$$\mathbf{P}_t^{(1)} + \mathbf{Q}_1 + \mathbf{R}_1 + \mathbf{R}_2 = 0. \tag{9}$$

From this equation it follows that $\mathbf{R}_2 = -\mathbf{P}_t^{(1)} - \mathbf{Q}_1 - \mathbf{R}_1$. As the forces $-\mathbf{P}_t^{(1)}, -\mathbf{Q}_1$, and $-\mathbf{R}_1$, lie in the xz-plane, $\mathbf{R}_2$ also lies in the xz-plane. In addition, since $\mathbf{R}_2$ has the direction of the string A_1A_2, the string A_1A_2 also lies in the xz-plane; therefore the point A_2 likewise lies in the xz-plane.

Let us proceed to determine the angles α and β. In view of the fact that the angles are constant, and that the lengths OA_1 and A_1A_2 also remain unchanged, it follows that the point A_2 is likewise in relative equilibrium with respect to the frame (x, y, z). Denoting its force of transport by $\mathbf{P}_t^{(2)}$ and its weight by $\mathbf{Q}_2$, we obtain

$$\mathbf{P}_t^{(2)} + \mathbf{Q}_2 - \mathbf{R}_2 = 0. \tag{10}$$

Let us denote the distances of the points A_1 and A_2 from the axis of revolution by r_1 and r_2. We have:

$$r_1 = OA_1 \sin\alpha, \quad r_2 = OA_1 \sin\alpha + A_1A_2 \sin\beta, \tag{11}$$

$$|\mathbf{P}_t^{(1)}| = m_1 r_1 \omega^2, \quad |\mathbf{P}_t^{(2)}| = m_2 r_2 \omega^2. \tag{12}$$

Let us form the projections of (9) and (10) on the x and z axes. Putting $R_1 = |\mathbf{R}_1|$ and $R_2 = |\mathbf{R}_2|$, we get:

$$m_1 r_1 \omega^2 - R_1 \sin\alpha + R_2 \sin\beta = 0, \quad -m_1 g + R_1 \cos\alpha - R_2 \cos\beta = 0, \tag{13}$$

$$m_2 r_2 \omega^2 - R_2 \sin\beta = 0, \quad -m_2 g + R_2 \cos\beta = 0. \tag{14}$$

From equations (13) and (14) we obtain

$$\tan\alpha = (m_1 r + m_2 r_2)\, \omega^2 / (m_1 + m_2)\, g, \quad \tan\beta = r_2 \omega^2 / g. \tag{15}$$

If we denote the distance of the centre of mass S of system of points A_1, A_2 from the axis of rotation z by r_0, then we obtain $(m_1 + m_2)\, r_0 = m_1 r_1 + m_2 r_2$, whence by (15) $\tan\alpha = r_0\omega^2 / g$, and as $r_1 < r_0 < r_2$, we have $\tan\alpha < \tan\beta$ or $\alpha < \beta$.

Knowing OA_1 and A_1A_2, we find α and β from equations (11) and (15).

Example 4. Atwood's machine. At the ends of a string (inextensible, massless), parsing over a pulley (massless), are suspended two heavy material points of masses m_1 and m_2 (Fig. 132). Let us assume that both points move vertically. Since the string is inextensible, the paths traversed by both points are equal. Therefore the accelerations and velocities of both points are equal in magnitude, but they have opposite senses. Let us denote by p the projection of the acceleration of m_1 on the z-axis directed vertically downwards. Let R denote the absolute value of the force with which the string acts on the points m_1 and m_2. The weight and the reaction of the string acts on the point m_1. Therefore

$$m_1 p = m_1 g - R. \tag{16}$$

Similarly, for the point m_2 we obtain

$$-m_2 p = m_2 g - R. \tag{17}$$

From equations (16) and (17) we get

$$p = \frac{m_1 - m_2}{m_1 + m_2} g, \quad R = \frac{2m_1 m_2}{m_1 + m_2} g. \tag{18}$$

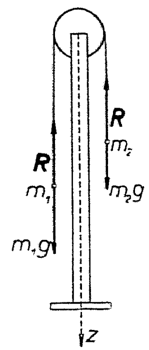

Fig. 132.

Hence the points will move with a constant acceleration.

From equation (18) we get $(m_1 + m_2)\, p = m_1 g - m_2 g$.

Therefore the acceleration is such as if a force $m_1 g - m_2 g$, i. e. a force equal to the difference of the weights, were acting on a material point of mass $m_1 + m_2$.

If $m_1 > m_2$, then $p > 0$, which means that the acceleration of the point m_1 is directed downwards and that of the point m_2 upwards.

If $m_1 = m_2$, then $p = 0$, which means that the points move with uniform motion.

Example 5. Two heavy points of masses m_1 and m_2, connected by an inextensible and massless string, move in a vertical plane along two lines l_1 and l_2. The tension of the string $\mathbf{T}_1$, the weight $m_1\mathbf{g}$, and the reaction of the line $\mathbf{R}_1$ act on the point m_1. Similarly, the forces $\mathbf{T}_2$, $m_2\mathbf{g}$, and $\mathbf{R}_2$ act on the point m_2. Let us assume that there is no friction and therefore that $\mathbf{R}_1$ and $\mathbf{R}_2$ are perpendicular to the lines l_1 and l_2, respectively (Fig. 133).

Since the string is inextensible, the paths traversed by both points will be equal, and the acceleration will then be equal in magnitude.

Let l_1 and l_2 be given downward senses. Let p denote the component (with respect to l_1) of the acceleration of the point m_1; therefore the component (with respect to l_2) of the acceleration of the point m_2 is $-p$. The forces $\mathbf{T}_1$ and $\mathbf{T}_2$ are equal in magnitude; set $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Let us denote the angles made by l_1 and l_2 with the horizontal by α_1 and α_2. Forming the projection on the lines l_1 and l_2, we obtain $m_1 p = -T + m_1 g \sin\alpha_1$, and $-m_2 p = -T + m_2 g \sin\alpha_2$, whence

$$p = \frac{m_1 \sin\alpha_1 - m_2 \sin\alpha_2}{m_1 + m_2} g.$$

Hence the points will move along the lines with uniformly accelerated motion.

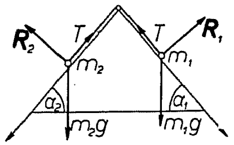

Fig. 133.

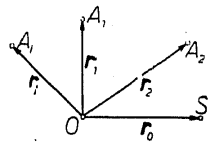

Fig. 134.

§ 2. Motion of the centre of mass. Kinematic properties of the centre of mass. Let there be given a system of material points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$, whose centre of mass is the point S (Fig. 134). Let us select an arbitrary point O. Put $m = \Sigma m_i$ as well as:

$$\mathbf{r}_0 = \overline{OS}, \quad \mathbf{r}_i = \overline{OA_i} \quad (i = 1, 2, \ldots).$$

In terms of the above notation the following equation holds

$$m\mathbf{r}_0 = \Sigma m_i \mathbf{r}_i. \tag{I}$$

Proof. Let us choose arbitrarily a system of coordinates with its origin at O. If x_i, y_i, z_i denote the coordinates of the point A_i, and x_0, y_0, z_0 the coordinates of the centre S, then by (II), p. 153,

$$mx_0 = \Sigma m_i x_i, \quad my_0 = \Sigma m_i y_i, \quad mz_0 = \Sigma m_i z_i. \tag{1}$$

Since the vector $\mathbf{r}_i$ has the projections x_i, y_i, z_i, and $\mathbf{r}_0$ has the projections x_0, y_0, z_0, equation (I) is only the vector form of the equalities (1).

Let us take the derivative of both sides of the equality (I) with respect to time; we obtain $m\mathbf{r}_0^{\cdot} = \Sigma m_i \mathbf{r}_i^{\cdot}$. Since $\mathbf{r}_i$ denotes the velocity $\mathbf{v}_i$ of the point A_i, and $\mathbf{r}_0^{\cdot}$ the velocity $\mathbf{v}_0$ of the centre of mass S with respect to the system $O(x, y, z)$,

$$m\mathbf{v}_0 = \Sigma m_i \mathbf{v}_i. \tag{II}$$

The vector $m\mathbf{v}_i$ is the momentum (p. 72) of the point A_i. The right side of the equality (II) therefore denotes the sum of the momenta of the separate points of the system. This sum is called the *(total) momentum of the system.*

The vector $m\mathbf{v}_0$ can be considered as the momentum of a material point, having a mass equal to the total mass of the system, situated at the centre of mass (and moving together with the centre of mass).

Therefore: *the (total) momentum of a system is equal to the momentum of the total mass situated at the centre of mass.*

Let us differentiate both sides of equation (II) with respect to the time t. We get $m\mathbf{v}_0^{\cdot} = \Sigma m_i \mathbf{v}_i^{\cdot}$. But $\mathbf{v}_i^{\cdot}$ denotes the acceleration $\mathbf{p}_i$ of the point A_i, and $\mathbf{v}_0^{\cdot}$ the acceleration $\mathbf{p}_0$ of the centre of mass S. Hence

$$m\mathbf{p}_0 = \Sigma m_i \mathbf{p}_i. \tag{III}$$

We have called the vector $-m_i\mathbf{p}_i$ the force of inertia of the point A_i (pp. 73 and 188).

Therefore: *the sum of the forces of inertia of the points of a system is equal to the force of inertia of the total mass of the system situated at the centre of mass.*

Remark. Forming the projections on the axes of the coordinate system, we obtain from equations (II) and (III):

$$mx_0^{\cdot} = \Sigma m_i x_i^{\cdot}, \quad my_0^{\cdot} = \Sigma m_i y_i^{\cdot}, \quad mz_0^{\cdot} = \Sigma m_i z_i^{\cdot}, \tag{II'}$$

$$mx_0^{\cdot\cdot} = \Sigma m_i x_i^{\cdot\cdot}, \quad my_0^{\cdot\cdot} = \Sigma m_i y_i^{\cdot\cdot}, \quad mz_0^{\cdot\cdot} = \Sigma m_i z_i^{\cdot\cdot}. \tag{III'}$$

Resultant of a system of weights. Let a system of points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$ be situated in a gravitational force field. Let us denote the gravitational acceleration vector by $\mathbf{g}$ and the centre of mass by S. Let O be an arbitrary point. As before, let us put $\mathbf{r}_0 = \overline{OS}$ and $\mathbf{r}_i = \overline{OA_i}$ for $i = 1, 2, \ldots$ The total moment of the weights with respect to O is $\mathbf{M} = (m_1\mathbf{g} \times \mathbf{r}_i) + (m_2\mathbf{g} \times \mathbf{r}_2) + \ldots$; therefore

$$\mathbf{M} = \mathbf{g} \times (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots),$$

whence by (I), p. 194,

$$\mathbf{M} = \mathbf{g} \times m\mathbf{r}_0 = m\mathbf{g} \times \mathbf{r}_0. \tag{2}$$

In particular, if the point O coincides with S, then $\mathbf{r}_0 = 0$, whence by (2) $\mathbf{M} = 0$.

Therefore: *the total moment of the weights of the points of a system with respect to the centre of mass is zero.*

Since the weights form a system of parallel forces having the same direction, this system has a resultant (p. 26). The resultant passes through the centre of mass because the total moment with respect to the centre of mass is zero.

Dynamic properties of the centre of mass. Let the forces $\mathbf{P}_i$ act on the material points m_i of a given system. Let us denote the acceleration of the point m_i by $\mathbf{p}_i$, and the acceleration of the centre of mass of this system of points with respect to an inertial system of coordinates by $\mathbf{p}_0$. By formula (III), p. 195, we have $m\mathbf{p}_0 = \Sigma m_i \mathbf{p}_i$, where $m = \Sigma m_i$. Since $m_i \mathbf{p}_i = \mathbf{P}_i$ it follows that $m\mathbf{p}_0 = \Sigma \mathbf{P}_i$, whence

$$m\mathbf{p}_0 = \mathbf{P}, \text{ where } \mathbf{P} = \Sigma \mathbf{P}_i. \tag{IV}$$

Therefore: *the centre of mass of a system of points moves so as if the total mass of the system were concentrated there and the sum of all the forces acted there.*

Equation (IV) can be written in the form $\mathrm{d}(m\mathbf{v}_0) / \mathrm{d}t = \mathbf{P}$.

Hence: *the derivative of the momentum of a system is equal to the sum of all the acting forces.*

If the sum of the forces acting on the points of a system is equal to zero, i. e. if $\mathbf{P} = 0$, then by (IV) we have $m\mathbf{p}_0 = 0$, i. e. $\mathbf{p}_0 = 0$. If the sum $\mathbf{P}$ is constantly zero, then $\mathbf{p}_0 = 0$ constantly, and hence the velocity $\mathbf{v}_0$ of the centre of mass is constant. The centre of mass is then at rest or in uniform motion along a straight line. Let us note that by (II), p. 194, $m\mathbf{v}_0 = \Sigma m_i \mathbf{v}_i$; hence in this case the total momentum (or quantity of motion) of the system is a constant vector.

Therefore: *if the sum of the forces acting on a system of points is constantly equal to zero, then the centre of mass is at rest or in uniform motion along a straight line and the total momentum of the system is a constant vector.*

This theorem is known as the *principle of conservation of momentum or of quantity of motion.*

As we know (p. 187), the sum of the internal forces is always zero; therefore the sum of all the forces acting on the points of a system is equal to the sum of the external forces. We can therefore replace the sum of the forces by the sum of the external forces in the theorems given.

Let us denote the sum of the external forces by $\mathbf{P}^{(e)}$. By (IV)

$$m\mathbf{p}_0 = \mathbf{P}^{(e)}. \tag{V}$$

If no external forces are acting on a system, then $\mathbf{p}_0 = 0$, and hence $\mathbf{v}_0 = \text{const}$. We can therefore say that *the internal forces cannot change the velocity of the centre of mass either as to magnitude or as to direction.*

Let us consider the solar system (i. e. the system composed of the sun and planets). The forces with which the fixed stars attract the bodies of the solar system can be neglected since these forces are very small because of the immense distances of the fixed stars from the solar system. We can therefore assume that only the internal forces with which the bodies attract each other according to Newton's law (p. 89) act on the bodies of the solar system. It follows from this that relative to the fixed stars the centre of mass of the solar system is at rest or moves with uniform motion along a straight line.

Suppose that we are inquiring into the motion of a system of points in a gravitational field. The sum of the weights is $m_1 g + m_2 g + \ldots = mg$. The centre S of mass will therefore move like a material point of mass m under the influence of the weight mg, i. e. along a straight line or a parabola until the moment when at least one of the points of the system touches the ground. For at this moment a new external force appears resulting from the collision of the point with the earth.

Examples. 1. The centre of mass of a projectile travels along a parabola even when the projectile explodes and bursts. The motion of the centre of mass will not be disturbed by this, since the explosion takes place under the influence of internal forces. Only when one of the fragments falls to the earth will the motion of the centre of mass undergo a change.

2. If a person is on a smooth horizontal plane (e. g. on ice) the external forces are the reaction of the plane and his weight; both forces are directed vertically. If the person was at rest initially, then as long as other external forces do not appear, the centre of mass will only be able to move vertically. The motions which a person executes by means of muscular action occur under the influence of internal forces, and hence cannot influence the motion of the centre of mass in the horizontal direction. Therefore, if there were no friction, walking would be impossible.

If at some moment a certain part of a system of points changes its momentum under the influence of internal forces, then the momentum of the remaining part experiences simultaneously a change equal in magnitude and direction, but opposite in sense. This is so because the internal forces cannot change the total momentum. Denoting the masses of the first and second parts by m' and m'', and the changes of the velocities of the centres of these masses by $\mathbf{v}_0'$ and $\mathbf{v}_0''$, we obtain $m'\mathbf{v}_0' + m''\mathbf{v}_0'' = 0$, whence $|\mathbf{v}_0'| / |\mathbf{v}_0''| = m'' / m'$.

Therefore: *the change of the velocities of the centres of mass is inversely proportional to the masses of both parts of the system.*

This explains the recoil of a cannon after it has been fired. Similarly, if a person starts to run along the deck of a boat, the boat begins to move in the opposite direction. The velocities of the boat and the person will be inversely proportional to their masses.

Example 1. One end of a heavy rod AB (of constant density) rests on a smooth horizontal plane Π (Fig. 135). The external forces are the weight and the reaction at the point A; both forces are vertical. Therefore, if the rod was at rest initially, then under the influence of the external forces, whose sum is directed vertically, the rod will move in such a way that the centre S of its mass will fall vertically.

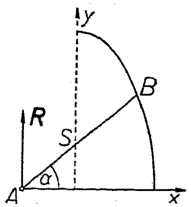

Fig. 135.

Let us choose the intersection of the vertical plane (passing through the rod) with the plane Π as the x-axis. As the y-axis let us take the vertical along which the centre of gravity moves. Put $AB = l$ and denote the angle which AB makes with the x-axis by α. Denoting the coordinates of the point B by x and y, we obtain: $x = \frac{1}{2}l\cos\alpha$, and $y = l\sin\alpha$, whence

$$(x / \tfrac{1}{2}l)^2 + (y / l)^2 = 1.$$

The end of the rod B will therefore move along an ellipse with axes l and $2l$.

***Example* 2.** Let a system of points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$ move in a central field (p. 101) of elastic forces (p. 110) proportional to the masses. Let O be the centre of the field. Set $\overline{OA}_1 = \mathbf{r}_1$, $\overline{OA}_2 = \mathbf{r}_2$, etc. Denoting the forces acting on the points $A_1, A_2, \ldots$ by $\mathbf{P}_1, \mathbf{P}_2, \ldots$ and putting $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots$, we obtain $\mathbf{P}_1 = -\lambda^2 m_1 \mathbf{r}_1$, $\mathbf{P}_2 = -\lambda^2 m_2 \mathbf{r}_2$ etc., whence $\mathbf{P} = -\lambda^2(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots)$. Denoting the center of the total mass m by S and setting $\overline{OS} = \mathbf{r}_0$, we obtain by (I), p. 194,

$$\mathbf{P} = -\lambda^2 m \mathbf{r}_0.$$

Therefore the centre of mass will move just like a material point os mass m subjected to the action of an elastic force $\mathbf{P}$. The centre of masf will therefore move with plane harmonic motion along a straight line or an ellipse (p. 113).

§ 3. Moment of momentum. Angular momentum with respect to a point. Let a system of points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$ and total mass m be given. Let us consider a system of momenta i. e. of vectors $m_1\mathbf{v}_1, m_2\mathbf{v}_2, \ldots$ with initial points at $A_1, A_2, \ldots$

The total moment of a system of momenta with respect to an arbitrary point A is called the *angular momentum* or the *moment of momentum* of the system with respect to A.

Therefore the angular momentum $\mathbf{K}$ with respect to A is

$$\mathbf{K} = \Sigma \mathrm{Mom}_A(m_i\mathbf{v}_i).$$

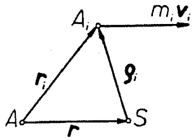

Fig. 136.

Setting $\mathbf{r}_i = \overline{AA}_i$, we obtain

$$\mathbf{K} = \Sigma(m_i\mathbf{v}_i \times \mathbf{r}_i). \qquad (1)$$

If ξ, η, ζ are the coordinates of the point A and x_i, y_i, z_i the coordinates of the point A_i, then the projections of the angular momentum on the coordinate axes are

$$\begin{aligned} K_x &= \Sigma m_i[\dot{y}_i(z_i - \zeta) - \dot{z}_i(y_i - \eta)], \\ K_y &= \Sigma m_i[\dot{z}_i(x_i - \xi) - \dot{x}_i(z_i - \zeta)], \\ K_z &= \Sigma m_i[\dot{x}_i(y_i - \eta) - \dot{y}_i(x_i - \xi)]. \end{aligned} \qquad (\mathrm{I})$$

In particular, when $\xi = \eta = \zeta = 0$, we have

$$K_x = \Sigma m_i(\dot{y}_i z_i - \dot{z}_i y_i),\quad K_y = \Sigma m_i(\dot{z}_i x_i - \dot{x}_i z_i),\quad K_z = \Sigma m_i(\dot{x}_i y_i - \dot{y}_i x_i). \qquad (\mathrm{I}')$$

Let S be the centre of mass (Fig. 136). Put $\overline{AS} = \mathbf{r}_0$ and $\overline{SA}_i = \boldsymbol{\rho}_i$ for $i = 1, 2, \ldots$ We have $\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{r}_0$. Therefore by (1) the angular momentum with respect to A is

$$\mathbf{K} = \Sigma m_i\mathbf{v}_i \times (\boldsymbol{\rho}_i + \mathbf{r}_0) = \Sigma m_i\mathbf{v}_i \times \boldsymbol{\rho}_i + (\Sigma m_i\mathbf{v}_i) \times \mathbf{r}_0.$$

The first term of the last member is the angular momentum of the centre of mass. This angular momentum we denote by $\mathbf{K}_0$. Since $\Sigma m_i\mathbf{v}_i = m\mathbf{v}_0$ (where $\mathbf{v}_0$ denotes the velocity of the centre of mass),

$$\mathbf{K} = \mathbf{K}_0 + m\mathbf{v}_0 \times \mathbf{r}_0. \qquad (2)$$

Let us note that $m\mathbf{v}_0 \times \mathbf{r}_0$ is the moment with respect to A of the total momentum whose point of application is at the centre of mass.

Formula (2) follows directly from the theorem on p. 20 concerning the change of the total moment of a system of vectors.

Angular momentum in an advancing motion. Let us assume that a system of points moves with an advancing motion with a velocity $\mathbf{v}$, i. e. that all points move with a velocity $\mathbf{v}$.

The angular momentum with respect to an arbitrary point A is therefore according to (2) $\mathbf{K} = \Sigma(m_i\mathbf{v} \times \mathbf{r}_i) = \Sigma(\mathbf{v} \times m_i\mathbf{r}_i) = \mathbf{v} \times \Sigma m_i\mathbf{r}_i$. But by (I), p. 194, $\Sigma m_i\mathbf{r}_i = m\mathbf{r}_0$. Hence $\mathbf{K} = \mathbf{v} \times m\mathbf{r}_0$, or

$$\mathbf{K} = m\mathbf{v} \times \mathbf{r}_0. \qquad (3)$$

Therefore: *the angular momentum of a system of points moving with an advancing motion relative to a certain point A is equal to the moment with respect to A of the total momentum whose point of application is at the centre of mass of the system.*

In particular, if the point A coincides with the centre of mass, then we have $\mathbf{r}_0 = 0$, and hence $\mathbf{K} = 0$. Hence: *the angular momentum with respect to the centre of mass in an advancing motion is equal to zero.*

It follows from this (p. 26) that a system of momenta in an advancing motion has a resultant whose origin is at the centre of mass of the system.

Angular momentum in a motion relative to the centre of mass. Let a system of coordinates $O(x, y, z)$ move with an advancing motion with a velocity $\mathbf{u}$. Denoting the relative velocities of the points $A_1, A_2, \ldots$ by $\mathbf{w}_1, \mathbf{w}_2, \ldots$, the relative velocity of the centre of their mass S by $\mathbf{w}_0$, the angular momentum of the relative motion by $\mathbf{K}_r$, and the angular momentum of the absolute motion of the system of these points with respect to O by $\mathbf{K}_a$, we obtain

$$\mathbf{K}_r = m_1\mathbf{w}_1 \times \overline{OA_1} + m_2\mathbf{w}_2 \times \overline{OA_2} + \ldots$$

Since $\mathbf{w}_1 = \mathbf{v}_1 - \mathbf{u}$, $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{u}$, ...,

$$\mathbf{K}_r = (m_1\mathbf{v}_1 \times \overline{OA_1} + m_2\mathbf{v}_2 \times \overline{OA_2} + \ldots) - \mathbf{u} \times (m_1\overline{OA_1} + m_2\overline{OA_2} + \ldots).$$

The expression enclosed in the first parenthesis is equal to $\mathbf{K}_a$; the expression enclosed in the second parenthesis is $m \cdot \overline{OS}$ (p. 194). Therefore

$$\mathbf{K}_r = \mathbf{K}_a - m\mathbf{u} \times \overline{OS}. \tag{4}$$

In particular, if the origin O of the moving system of coordinates is chosen at the centre of mass S of the system of points $A_1, A_2, \ldots$, then $\overline{OS} = 0$, whence by (4) $\mathbf{K}_r = \mathbf{K}_a$.

Therefore, if we are investigating the motion of a system with respect to the centre of mass, then *the angular momenta with respect to the centre of mass in relative and absolute motion are equal.*

Angular momentum with respect to an axis. The total moment of momenta of a system of points with respect to a certain axis is called the *angular momentum of a system with respect to this axis.*

Formulae (I) represent the angular momenta of a system of points with respect to axes parallel to the x, y, z axes and passing through the point A, while formulae (I′) represent the angular momenta with respect to the x, y and z axes. Let us consider the angular momentum with respect to the y-axis

$$K_y = \Sigma m_i(\dot{z}_i x_i - \dot{x}_i z_i).$$

Denoting by S_i the areal velocity (p. 47) of the motion which the projection of the point A_i executes in the vertical xz-plane, we obtain from formula (II), p. 48, $S_i = \frac{1}{2}(\dot{z}_i x_i - \dot{x}_i z_i)$; therefore

$$K_y = 2\,\Sigma m_i S_i.$$

Hence: *the angular momentum with respect to a certain axis is equal to twice the sum of the products of the masses and the areal velocities of the motions which the projections of the points execute on a plane perpendicular to this axis*:

$$K = 2\,\Sigma m_i S_i. \tag{5}$$

If we introduce the polar coordinates r, φ in the plane perpendicular to the axis, then by (I), p. 47, we obtain $S_i = \frac{1}{2}r_i^2\dot{\varphi}_i$, and hence

$$K = \Sigma m_i r_i^2 \dot{\varphi}_i. \tag{6}$$

Let a system of points rotate about a certain axis with an angular velocity ω. We then have $\dot{\varphi}_1 = \dot{\varphi}_2 = \ldots = \omega$. Therefore $K = \Sigma m_i r_i^2 \omega = \omega\,\Sigma m_i r_i^2$. Since $\Sigma m_i r_i^2 = I$, where I is the moment of inertia with respect to the axis of rotation,

$$K = I\omega. \tag{7}$$

Therefore: *if a system of points rotates about a certain axis, then the angular momentum with respect to the axis of rotation is equal to the product of the angular velocity and the moment of inertia with respect to the axis of rotation.*

Dynamic properties of angular momentum. Let A be an arbitrary point which is either fixed or in motion (Fig. 137). Denote the vectors $\overline{AA_i}$ by $\mathbf{r}_i$, the vectors $\overline{OA_i}$ by $\boldsymbol{\rho}_i$ (where O is the origin of the inertial system of coordinates), the vector $\overline{OA}$ by $\boldsymbol{\rho}$, and the velocity of the point A by $\mathbf{u}$. Therefore:

$$\dot{\boldsymbol{\rho}} = \mathbf{u} \quad \text{and} \quad \dot{\boldsymbol{\rho}}_i = \mathbf{v}_i \qquad (i = 1, 2, \ldots). \tag{8}$$

Let $\mathbf{K}$ be the angular momentum with respect to A. By (1), p. 199, $\mathbf{K} = \Sigma(m_i\mathbf{v}_i \times \mathbf{r}_i)$. Denoting the accelerations of the points A_i by $\mathbf{p}_i$, we obtain after differentiating with respect to t,

$$\dot{\mathbf{K}} = \Sigma(m_i\mathbf{p}_i \times \mathbf{r}_i) + \Sigma(m_i\mathbf{v}_i + \dot{\mathbf{r}}_i). \tag{9}$$

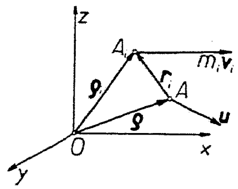

Fig. 137.

But $\mathbf{r}_i = \boldsymbol{\rho}_i - \boldsymbol{\rho}$; hence $\dot{\mathbf{r}}_i = \dot{\boldsymbol{\rho}}_i - \dot{\boldsymbol{\rho}}$. Therefore by (8) $\dot{\mathbf{r}}_i = \mathbf{v}_i - \mathbf{u}$, and $\Sigma(m_i\mathbf{v}_i \times \dot{\mathbf{r}}_i) = \Sigma(m_i\mathbf{v}_i \times \mathbf{v}_i) - \Sigma(m_i\mathbf{v}_i \times \mathbf{u})$, But $\mathbf{v}_i \times \mathbf{v}_i = 0$, and $\Sigma m_i\mathbf{v}_i = m\mathbf{v}_0$, where $\mathbf{v}_0$ denotes the velocity of the centre of mass. Therefore

$$\Sigma(m_i\mathbf{v}_i \times \dot{\mathbf{r}}_i) = -m\mathbf{v}_0 \times \mathbf{u}. \tag{10}$$

If the force $\mathbf{P}_i$ acts on the point A_i, then $m_i\mathbf{p}_i = \mathbf{P}_i$, and therefore $m_i\mathbf{p}_i \times \mathbf{r}_i = \mathbf{P}_i \times \mathbf{r}_i = \mathrm{Mom}_A\mathbf{P}_i$. Hence, if $\mathbf{M}$ is the total moment of the forces with respect to A, then $\Sigma m_i \boldsymbol{p}_i \times \boldsymbol{r}_i = \Sigma \boldsymbol{P}_i \times \boldsymbol{r}_i = \mathrm{Mom}_A \boldsymbol{P}_i = \boldsymbol{M}$. This formula together with formula (10) gives by (9)

$$\boldsymbol{K}^{\bullet} = \boldsymbol{M} - m\boldsymbol{v}_0 \times \boldsymbol{u}. \tag{II}$$

The expression $\boldsymbol{v}_0 \times \boldsymbol{u}$ will be zero if we assume that the point A is at rest (hence that $\boldsymbol{u} = 0$), or that A coincides with the centre of mass (hence that $\boldsymbol{u} = \boldsymbol{v}_0$). In both instances we obtain

$$\boldsymbol{K}^{\bullet} = \boldsymbol{M}. \tag{III}$$

Therefore: *the derivative of the angular momentum (with respect to a fixed point or a centre of mass) is equal to the total moment of the acting forces.*

Forming the projections on a fixed axis or on one passing through the centre of mass and not changing its direction, we conclude from formula (III) that *the derivative of the angular momentum with respect to a fixed axis (or to one passing through the centre of mass and not changing its direction) is equal to the total moment of the forces with respect to this axis.*

In particular, if the total moment of the forces with respect to a certain fixed point or with respect to the centre of mass is constantly zero, then the derivative of the angular momentum is zero, i. e. the angular momentum is a constant vector.

Therefore: *if the total moment of the forces (with respect to a certain fixed point or the centre of mass) is constantly zero, then the angular momentum is a constant vector.*

The preceding theorem is known as *the principle of conservation of angular momentum.*

A similar theorem is obtained for the angular momentum with respect to an axis.

Therefore: *if the moment of the forces with respect to a certain fixed axis (or one passing through the centre of mass and not changing its direction) is constantly zero, then the angular momentum with respect to this axis is constant.*

The angular momentum by formula (5), p. 201, is $K = 2\Sigma m_i S_i$, where S_i denote the areal velocities of the motions executed by the projections on the plane Π perpendicular to the axis. Therefore, if the angular momentum is constant, then

$$\Sigma m_i S_i = c = \text{const}. \tag{11}$$

Let us note that

$$a_i = \int_{t_0}^{t} S_i \, dt$$

represents the area swept out in the plane Π by the projection on it of the radius vector r_i of the point A_i from the time t_0 to t. Therefore by (11)

$$\Sigma m_i a_i = c(t - t_0). \tag{12}$$

Therefore: *the sum of the products of the masses and areas swept out by the projections of the radius vectors on the plane perpendicular to the axis is proportional to the time.*

Because of this the principle of conservation of angular momentum is also known as *the principle of conservation of areas.*

As we know (p. 187), the moment of the internal forces with respect to an arbitrary point is zero. Hence the moment of all the acting forces is reduced to the moment of the external forces. Therefore, if the moment of the external forces with respect to a certain fixed point A or the centre of mass is denoted by $\boldsymbol{M}^{(e)}$, then the equality (III) will assume the form

$$\boldsymbol{K}^{\bullet} = \boldsymbol{M}^{(e)}. \tag{III'}$$

In the theorems given previously we can therefore replace the moment of all the acting forces by the moment of the external forces.

If no external forces act on a system of points, then the principle of conservation of areas (of angular momentum) obviously holds, and hence the angular momentum with respect to each fixed point or centre of mass is then a constant vector. Since the angular momentum with respect to each fixed axis (or one passing through the centre of mass and not changing its direction) is then constant, equations (11) and (12) hold for motions which are executed by the projections of the points on an arbitrary fixed plane (or on one moving together with the centre of mass and not changing its direction).

Motion in a gravitational field. Let a system of material points $A_1, A_2, \dots$ of masses $m_1, m_2, \dots$ move in a gravitational field. If the only external forces are the weights of the points, then the total moment of the weights with respect to the centre of mass is zero (p. 195) and hence the angular momentum with respect to the centre of mass is constant.

Let us assume that a system of coordinates with its origin at the centre of gravity is moving with an advancing motion. In order to obtain the relative motion it is necessary to add to the acting forces the forces of transport (the force of Coriolis is zero because, by hypothesis, the system of coordinates is moving with an advancing motion).

Denote the gravitational acceleration vector by $\boldsymbol{g}$. Since the centre of gravity has an acceleration $\boldsymbol{g}$, the acceleration of transport is also $\boldsymbol{g}$. Therefore the force of transport for the individual points is $-m_1\boldsymbol{g}$,

$-m_2\mathbf{g}, \ldots$, respectively. We see then that the forces of transport are balanced by the weights of the points. Therefore the relative motion will be such as if the force of gravity were not acting. If there are no external forces besides the weights, then the angular momentum with respect to the centre of mass in relative motion will be a constant vector, and by (4), p. 200, will be equal to the angular momentum with respect to the centre of mass in absolute motion.

Example 1. Two material points A and B of masses m_1 and m_2 joined by a rigid massless rod are moving in a gravitational field. Therefore the forces $\mathbf{R}$ and $-\mathbf{R}$ of weight and reactions of the rod act on the points A and B. The reactions behave just like internal forces because they act along the line joining these points, are equal in magnitude, and have opposite senses. The centre of mass will therefore move (like a material point of mass $m_1 + m_2$ under the influence of gravity) with a vertical acceleration g along a straight line or along a parabola.

Let us denote the velocities of the points A and B by $\mathbf{v}_1$ and $\mathbf{v}_2$, and the centre of mass by S. The angular momentum with respect to the centre of mass is

$$\mathbf{K} = m_1\mathbf{v}_1 \times \overline{SA} + m_2\mathbf{v}_2 \times \overline{SB}. \tag{12}$$

Since

$$SA = \frac{m_2}{m_1 + m_2} AB, \qquad SB = \frac{m_1}{m_1 + m_2} AB,$$

(p. 156),

$$\overline{SA} = -\frac{m_2}{m_1 + m_2} \overline{AB}, \qquad \overline{SB} = \frac{m_1}{m_1 + m_2} \overline{AB},$$

from which we obtain after substituting in (12)

$$\mathbf{K} = \frac{m_1 m_2}{m_1 + m_2} (\mathbf{v}_2 - \mathbf{v}_1) \times \overline{AB}. \tag{13}$$

Let us assume that $\mathbf{K} \neq 0$; therefore $\mathbf{K} \perp \overline{AB}$. Since $\mathbf{K}$ is a constant vector, the segment AB (Fig. 138) is always parallel during motion to a certain plane Π perpendicular to the angular momentum $\mathbf{K}$.

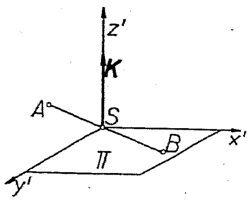

Fig. 138.

Let us choose the centre of mass as the origin of the coordinate system (x', y', z') moving with an advancing motion. Assume that the $x'y'$-plane is always parallel to Π. Therefore the z'-axis is parallel to $\mathbf{K}$. The rod AB therefore always remains in the $x'y'$-plane. It follows from this that the relative motion of the rod AB will be a rotation about the z'-axis (because the point S of the rod is motionless relative to the frame (x', y', z')). Since the angular momentum in relative motion is constant, the angular momentum with respect to the z'-axis will be, by formula (7), p. 201,

$$K_{z'} = I_{z'}\omega = \text{const}, \tag{14}$$

where the moment of inertia

$$I_{z'} = m_1 AS^2 + m_2 BS^2 = \frac{m_1 m_2}{m_1 + m_2} AB^2,$$

and ω denotes the angular velocity. Therefore $\omega = \text{const}$.

Hence the relative motion will be a rotation in the $x'y'$-plane about the centre of mass S with a constant angular velocity.

Rotation of a system about an axis. Let us assume that no external forces act on a system of material points U. Suppose that the system was at rest initially, and then some part of the system U_1 began to rotate about a certain fixed axis l under the influence of internal forces. Let us denote the moment of inertia of this part of the system with respect to l by I_1, and the angular velocity by ω_1. Then its angular momentum with respect to the axis of rotation is $K_1 = I_1\omega_1$.

Since the total angular momentum of a system must be zero, because the internal forces cannot change the angular momentum, the remaining part of the system U_2 must execute a motion whose angular momentum with respect to the l-axis is $K_2 = -K_1$, such that the sum of both angular momenta is zero (i. e. so that $K_1 + K_2 = 0$). Suppose that the motion of the other part is also a rotation about the l-axis (this case occurs if we assume e. g., that both parts can only rotate about l). If we denote the moment of inertia of the part U_2 with respect to l by I_2 and its angular velocity by ω_2, then $K_2 = I_2\omega_2$.

Since $K_1 + K_2 = 0$,

$$I_1\omega_1 + I_2\omega_2 = 0. \tag{15}$$

The preceding equation expresses the relation between the angular velocities of both parts of the system U. Since

$$\omega_1 / \omega_2 = -I_2 / I_1, \tag{16}$$

both parts of the system rotate in opposite directions and their angular velocities are in magnitude inversely proportional to the moments of inertia.

Denote by φ_1 and φ_2 the angles through which both parts U_1 and U_2 of the system U have turned in the time t. Since $\varphi_1^{\cdot} = \omega_1$, and $\varphi_2^{\cdot} = \omega_2$, it follows by (13), that $I_1\dot{\varphi}_1 + I_2\dot{\varphi}_2 = 0$, whence $I_1\varphi_1 + I_2\varphi_2 = c$. Assuming that $\varphi_1 = 0$ and $\varphi_2 = 0$ at the moment $t = 0$, we obtain $I_1\varphi_1 + I_2\varphi_2 = 0$, i. e.

$$\varphi_1 / \varphi_2 = -I_2 / I_1. \tag{17}$$

The angles of rotation are therefore in magnitude also inversely proportional to the moments of inertia of the two parts of the system.

If at a certain moment $\varphi_1 - \varphi_2 = 2k\pi$, where k is an arbitrary integer, then the position of the system is such as if it had turned through an angle φ_1.

We see then that the action of internal forces is sufficient for turning a system of points about an axis through an arbitrary angle φ. Such a rotation can occur for instance in the following manner: one part of the system turns through an angle φ, and the other part through an angle $2\pi - \varphi$ in the opposite direction.

This explains the fact that when a cat falls it can turn in the air in such a way as to fall on all fours.

If we start to turn a material wheel about a vertical axis with an angular velocity ω_1 on the deck of a boat, then the boat begins to turn in the opposite direction with an angular velocity ω_2; both velocities will satisfy relations (15) and (16), where I_1 and I_2 denote the moments of inertia of the wheel and boat respectively.

Example 2. A piece of paper rests on a smooth horizontal plane; the paper is pierced by a pin at the point O so that it can only turn about this point. An insect A crawls over the paper (Fig. 139).

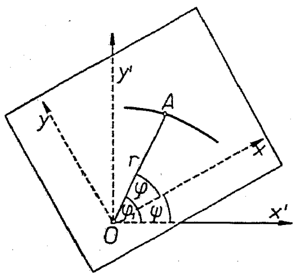

Fig. 139.

The external forces acting on the system consisting of the paper and insect are: the reaction of the pin with its origin at the point O, the weight of the paper and that of the insect as well as the reaction of the horizontal plane; these orces have a vertical direction. The moment of these forces with respect to the vertical z-axis passing through O is therefore equal to zero, and because of this the angular momentum with respect to the z-axis is constant.

Let us assume that the insect and the piece of paper were at rest at $t = 0$. The angular momentum with respect to the z-axis will therefore be constantly zero.

Select a fixed coordinate system (x', y') in the horizontal plane and

a moving system (x, y) on the piece of paper, both having a common origin at O. Denote the angle between OA and x' by φ_1, the angle between OA and x by φ, and the angle through which the paper has turned by ψ, i. e. the angle between x and x'. Finally, let m denote the mass of the insect, I the moment of inertia of the paper with respect to O and let $r = OA$. Then the angular momentum with respect to the z-axis is $mr^2\dot{\varphi}_1 + I\dot{\psi} = 0$, and since $\varphi_1 = \varphi + \psi$,

$$mr^2\dot{\varphi} + (mr^2 + I)\,\dot{\psi} = 0. \tag{18}$$

Let us suppose that the insect crawls along a curve whose equation is $r = f(\varphi)$. By (18), and from the fact that $\dot{\psi} / \dot{\varphi} = d\psi / d\varphi$, we obtain $d\psi / d\varphi = -mr^2 / (mr^2 + I)$. Therefore, integrating from φ_0 to φ, we get

$$\psi - \psi_0 = -\int_{\varphi_0}^{\varphi} \frac{mr^2}{mr^2 + I}\, d\varphi. \tag{19}$$

The difference $\psi - \psi_0$ represents the angle through which the paper has turned while the insect crawled along the curve $r = f(\varphi)$ from φ_0 to φ.

We see that the angle of rotation does not depend on the velocity of the insect, but only on the curve along which it crawls. In particular, if the insect crawls along the circle $r =$ const, then by (19)

$$\psi - \psi_0 = -\frac{mr^2}{mr^2 + I}(\varphi - \varphi_0). \tag{20}$$

Angular momentum in relative motion. Let the coordinate system $O'(x', y', z')$ move relative to the inertial system of coordinates $O(x, y, z)$. In order to determine the relative motion of the system of material points it is necessary to add the forces of transport and Coriolis to the acting forces. Denote by $\mathbf{K}_r$ the angular momentum of the relative motion with respect to the origin O', and by $\mathbf{M}$, $\mathbf{M}_t$ and $\mathbf{M}_C$ the moments of the acting forces, the force of transport, and the force of Coriolis, with respect to the point O'. Since Newton's laws apply to relative motion if we add the forces of transport and Coriolis to the acting forces (p. 135),

$$\dot{\mathbf{K}}_r = \mathbf{M} + \mathbf{M}_t + \mathbf{M}_C. \tag{21}$$

This formula becomes simpler in the case when O' coincides with the centre of mass of a system of material points, and the system of coordinates (x, y, z) moves with an advancing motion. This is so because as we have proved (p. 200), in this case $\mathbf{K}_r = \mathbf{K}$, where $\mathbf{K}$ denotes the angular momentum (with respect to the centre of mass) in absolute motion. Therefore $\dot{\mathbf{K}}_r = \dot{\mathbf{K}}$, and since $\dot{\mathbf{K}} = \mathbf{M}$, we obtain

$$\dot{\mathbf{K}}_r = \mathbf{M}. \tag{22}$$

Hence, if we are investigating motion relative to the centre of mass, then *the derivative of the angular momentum* (with respect to the centre of mass) *in relative motion is equal to the moment* (with respect to the centre of mass) *of the acting forces.*

§ 4. Work and potential of a system of points. Work. Let forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ act on the points of a system. The work of the force $\mathbf{P}_i$ is expressed (p. 94) by the formula

$$L_i = \int_{C_i} (P_{i_x}\,dx_i + P_{i_y}\,dy_i + P_{i_z}\,dz_i),$$

where C_i denotes the path of the i-th point.

The total work (or briefly the *work*) *of the forces* $\mathbf{P}_1, \mathbf{P}_2, \ldots$ is defined as the sum of the works of the separate forces.

Therefore the total work is

$$L = \Sigma \int_{C_i} (P_{i_x}\,dx_i + P_{i_y}\,dy_i + P_{i_z}\,dz_i). \tag{I}$$

The total work done by the forces from the time t_0 to t can be represented in the form (IV), p. 95,

$$L = \int_{t_0}^{t} \Sigma (P_{i_x} x_i^{\bullet} + P_{i_y} y_i^{\bullet} + P_{i_z} z_i^{\bullet})\,dt \tag{II}$$

or ((V), p. 95)

$$L = \int_{t_0}^{t} \Sigma (\mathbf{P}_i \mathbf{v}_i)\,dt, \tag{II'}$$

where $\mathbf{v}_i$ denote the velocities of the points, and $\mathbf{P}_i\mathbf{v}_i$ is a scalar product.

Work equal to zero. The cases in which the work of the forces acting on a system is zero are very important. We shall give several examples.

Example 1. For a rigid system of material points (p. 190) the following theorem holds:

The work of the internal forces in a rigid system is zero.

Proof. Let us first consider two points of the rigid system A_1 and A_2 (Fig. 140). Put:

$$\mathbf{r}_1 = \overline{OA_1}, \quad \mathbf{r}_2 = \overline{OA_2}, \quad \mathbf{r} = \overline{A_1A_2}, \tag{1}$$

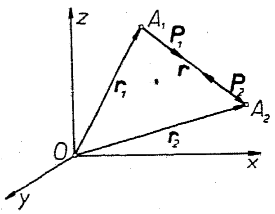

Fig. 140.

where O is the origin of the coordinate system. If $\mathbf{v}_1, \mathbf{v}_2$ are the velocities of the points A_1, A_2, then

$$\mathbf{v}_1 = \mathbf{r}_1^{\bullet}, \quad \mathbf{v}_2 = \mathbf{r}_2^{\bullet}. \tag{2}$$

By (1) we have $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, whence $\mathbf{r}^{\bullet} = \mathbf{r}_2^{\bullet} - \mathbf{r}_1^{\bullet}$, and therefore

$$\mathbf{r}^{\bullet} = \mathbf{v}_2 - \mathbf{v}_1. \tag{3}$$

Since $A_1A_2 =$ const, $\mathbf{r}^2 =$ const. Differentiating with respect to the time t, we obtain $2\mathbf{r}\mathbf{r}^{\bullet} = 0$. Hence by (3)

$$\mathbf{r}(\mathbf{v}_2 - \mathbf{v}_1) = 0. \tag{4}$$

Denote the force which the point A_2 (or A_1) exerts on the point A_1 (or A_2) by $\mathbf{P}_1$ (or $\mathbf{P}_2$). In virtue of the law of action and reaction

$$\mathbf{P}_1 = -\mathbf{P}_2. \tag{5}$$

Since the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ have the direction of the vector $\overline{A_1A_2} = \mathbf{r}$, we can assume that

$$\mathbf{P}_1 = \lambda\mathbf{r} \quad \text{and} \quad \mathbf{P}_2 = -\lambda\mathbf{r}, \tag{6}$$

where λ is a factor of proportionality depending on time (because the magnitude of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ can change in the course of time). By (II′) the work of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ is

$$L = \int_{t_0}^{t_1} (\mathbf{P}_1\mathbf{v}_1 + \mathbf{P}_2\mathbf{v}_2)\,dt. \tag{7}$$

From equations (6) we get $\mathbf{P}_1\mathbf{v}_1 + \mathbf{P}_2\mathbf{v}_2 = \lambda(\mathbf{r}\mathbf{v}_1 - \mathbf{r}\mathbf{v}_2) = \lambda\mathbf{r}(\mathbf{v}_1 - \mathbf{v}_2)$, and hence by (4) $\mathbf{P}_1\mathbf{v}_1 + \mathbf{P}_2\mathbf{v}_2 = 0$. It follows from this and (7) that

$$L = 0.$$

We have therefore proved that the work of the internal forces with which any two points of a rigid system react on each other is zero. The sum of the works of all the internal forces is hence also zero, q. e. d.

Example 2. Two material points A_1 and A_2 are connected by an inextensible (massless) string passing through a fixed point O (Fig. 141). Let us assume that there is no friction. We shall prove that the work of the forces exerted by the string on the points of the system is equal to zero.

Choose the point O as the origin of the system of coordinates. Let x_1, y_1, z_1 be the coordinates of the point A_1, and x_2, y_2, z_2 the coordinates of the point A_2. Put

$$r_1 = OA_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}, \quad \text{and} \quad r_2 = OA_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}. \tag{8}$$

Since the length of the string $l =$ const, and $r_2 + r_1 = l$,

$$r_1^{\bullet} + r_2^{\bullet} = 0. \tag{9}$$

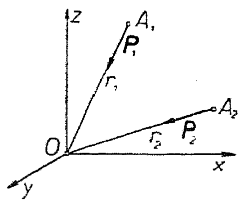

Fig. 141.

Denote the forces which the string exerts on the points A_1 and A_2 by $\mathbf{P}_1$ and $\mathbf{P}_2$. We have (p. 190)

$$|\mathbf{P}_1| = |\mathbf{P}_2|. \tag{10}$$

Putting $P = -|\mathbf{P}_1| = -|\mathbf{P}_2|$, we obtain:

$$P_{1_x} = P\frac{x_1}{r_1},\quad P_{1_y} = P\frac{y_1}{r_1},\quad \text{and}\quad P_{1_z} = P\frac{z_1}{r_1};$$

hence

$$P_{1_x}\dot{x}_1 + P_{1_y}\dot{y}_1 + P_{1_z}\dot{z}_1 = P(x_1\dot{x}_1 + y_1\dot{y}_1 + z_1\dot{z}_1)/r_1 = P\dot{r}_1,$$

and analogously

$$P_{2_x}\dot{x}_2 + P_{2_y}\dot{y}_2 + P_{2_z}\dot{z}_2 = P\dot{r}_2.$$

By (II), p. 208, the work of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ is then

$$L = \int_{t_0}^{t}[P\dot{r}_1 + P\dot{r}_2]\,dt = \int_{t_0}^{t}P[\dot{r}_1 + \dot{r}_2]\,dt.$$

From (9) we therefore get $L = 0$.

Example 3. Let us suppose that some body K moves in such a way that it constantly remains tangent to a certain surface Σ. Assuming that the forces of reaction of the surface have their points of application at the points of tangency, we see that the reactions change their points of application if the body comes in contact with the surface Σ at different points each time. Let us suppose that the work of the forces of reaction is in this case expressed by formula (II′), p. 208, where $\mathbf{v}_i$ denote the velocities of the points of tangency at which the reactions $\mathbf{P}_i$ have their points of application at a given time.

If the velocities of the points of tangency are always equal to zero, then we say that the body K *rolls* on the surface Σ.

Therefore: *when a body rolls the work of the forces of reaction is zero.*

Potential of a system. If the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ depend only on the position of the system of points, then the forces are said to form a *force field.*

Since the position of a system is defined by the coordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ of its points, the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ are functions of the variables $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ Consequently

$$P_{i_x} = F_i(x_1, y_1, z_1, \ldots),\quad P_{i_y} = \Phi_i(x_1, y_1, z_1, \ldots),\quad P_{i_z} = \Psi_i(x_1, y_1, z_1, \ldots).$$

If the total work in a force field does not depend on the path described by a system of points, but only on the initial and final positions of these points, then the force field is called a *conservative* or *potential field.*

Let us consider an arbitrary position S_0 of the system of points defined by the coordinates $x_1^0, y_1^0, z_1^0, x_2^0, y_2^0, z_2^0, \ldots$ Denote by V the work done by the forces in displacing the system from the position S_0 to the position S whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ If the field is a potential field, then the work V will not depend on the path described. Therefore V will be a function of the variables $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ The function V is known as the *potential* or the *force function.*

The following formulae can be proved by the same reasoning as in the case of one material point (p. 99):

$$P_{i_x} = \frac{\partial V}{\partial x_i},\quad P_{i_y} = \frac{\partial V}{\partial y_i},\quad P_{i_z} = \frac{\partial V}{\partial z_i}\qquad (i = 1, 2, \ldots).\quad \text{(III)}$$

Conversely, if there exists a function V satisfying equations (III), then the field is a potential field and V is a potential.

If a system is displaced from a position whose potential is V_1 to a position whose potential is V_2, then the work is

$$L = V_2 - V_1.$$

If each one of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ forms a potential field whose potentials are $\mathbf{V}_1, \mathbf{V}_2, \ldots$, then the field is a potential field whose potential is

$$V = V_1 + V_2 + \ldots \qquad (11)$$

Potential of the force of gravity. Let a system of points of masses $m_1, m_2, \ldots$ move in a gravitational field. If we assume that the z-axis is directed vertically upwards, then (p. 100)

$$V_1 = -m_1gz_1,\quad V_2 = -m_2gz_2,\quad \ldots$$

Therefore by (11) the field will be a potential field whose potential is

$$V = -(m_1z_1 + m_2z_2 + \ldots)\,g. \qquad (12)$$

Denoting the coordinate of the center of mass of the system by z_0, we shall have $m_1z_1 + m_2z_2 + \ldots = mz_0$, where m denotes the total mass of the system (p. 152). Therefore according to (12)

$$V = -mgz_0. \qquad (13)$$

Let us note that mg is the weight of the entire system (i. e. the sum of the weights of the separate points).

If in one position of the system the centre of gravity has coordinates $z_0^{(1)}$ and in the other $z_0^{(2)}$, then the work done by the force of gravity is $L = (-mgz_0^{(2)}) - (-mgz_0^{(1)})$, whence

$$L = mg(z_0^{(1)} - z_0^{(2)}). \qquad (14)$$

Hence: *the work in a gravitational field depends only on the difference of the levels of the centre of gravity of the system, and does not depend on the paths of its individual points.*

The work of the weights is therefore equal to the work that would be done by the total weight of the system whose point of application would be at the centre of its mass.

Potential of the internal forces. Let a system of points $A_1, A_2, \ldots, A_n$ of masses $m_1, m_2, \ldots, m_n$ be given. Assume that the internal forces with which two arbitrary points of the system react on each other depend in magnitude only on the distances between these points, i. e. if $\mathbf{P}^{ij}$ denotes the force with which the point $A_j(x_j, y_j, z_j)$ reacts on the point $A_i(x_i, y_i, z_i)$, and r_{ij} the distance between the points A_i and A_j (Fig. 142), then $|\mathbf{P}^{ij}|$ is a function of the distance r_{ij}, i. e.

$$|\mathbf{P}^{ij}| = f_{ij}(r_{ij}), \tag{15}$$

where

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}. \tag{16}$$

Let P_{ij} denote the projection of the force $\mathbf{P}^{ij}$ on the direction of $\overline{A_jA_i}$. Then:

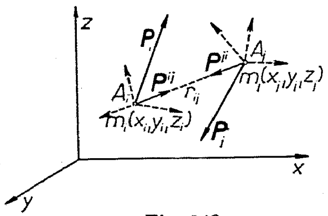

Fig. 142.

1° $|P_{ij}| = |\mathbf{P}^{ij}|$.

2° $P_{ij} \leq 0$, if the points A_i and A_j attract each other, and $P_{ij} \geq 0$ if the points A_i and A_j repel each other.

Since by the law of action and reaction $\mathbf{P}^{ij} = -\mathbf{P}^{ji}$, it follows that $P_{ij} = P_{ji}$, whence by (15) and condition 1°

$$P_{ij} = \pm f_{ij}(r_{ij}) \quad \text{or} \quad P_{ij} = F_{ij}(r_{ij}). \tag{17}$$

From the definition of the number P_{ij} it follows that the projections of the force $\mathbf{P}^{ij}$ on the axes of the system (x, y, z) can be written in the form:

$$P_x^{ij} = P_{ij}\frac{x_i - x_j}{r_{ij}}, \quad P_y^{ij} = P_{ij}\frac{y_i - y_j}{r_{ij}}, \quad P_z^{ij} = P_{ij}\frac{z_i - z_j}{r_{ij}}. \tag{18}$$

Let us put

$$V_{ij} = \int P_{ij}\,\mathrm{d}r_{ij} = \int F_{ij}(r_{ij})\,\mathrm{d}r_{ij}. \tag{19}$$

Since $P_{ij} = P_{ji}$ and $r_{ij} = r_{ji}$,

$$V_{ij} = V_{ji}. \tag{20}$$

We have

$$\frac{\partial V_{ij}}{\partial x_i} = \frac{\mathrm{d}V_{ij}}{\mathrm{d}r_{ij}} \cdot \frac{\partial r_{ij}}{\partial x_i} = P_{ij}\frac{x_i - x_j}{r_{ij}},$$

and therefore by (18):

$$\partial V_{ij} / \partial x_i = P_x^{ij}, \quad \partial V_{ij} / \partial y_i = P_y^{ij}, \quad \partial V_{ij} / \partial z_i = P_z^{ij}. \tag{21}$$

Let us set

$$V = \tfrac{1}{2}\Sigma V_{ij} = \tfrac{1}{2}\Sigma \int P_{ij}\,\mathrm{d}r_{ij}, \tag{22}$$

where the summation is extended over all number pairs i, j, such that $i \neq j$, $i \leq n$ and $j \leq n$.

Let $\mathbf{P}_i$ denote the sum of all the internal forces acting on the point A_i. Hence

$$\mathbf{P}_i = \sum_{j=1}^{n} \mathbf{P}^{ij} \quad \text{for} \quad j \neq i. \tag{23}$$

Let us calculate the partial derivative $\partial V / \partial x_i$. The variable x_i appears only in the functions V_{ij} and V_{ji}, where $j \neq i$. In virtue of this and (22)

$$\frac{\partial V}{\partial x_i} = \tfrac{1}{2}\sum_{j=1}^{n}\left[\frac{\partial V_{ij}}{\partial x_i} + \frac{\partial V_{ji}}{\partial x_i}\right] \qquad \text{for } i \neq j.$$

From equations (20) and (21) it therefore follows that

$$\frac{\partial V}{\partial x_i} = \tfrac{1}{2}\sum_{j=1}^{n} 2P_x^{ij} = \sum_{j=1}^{n} P_x^{ij} \qquad \text{for } i \neq j,$$

whence by (23):

$$\frac{\partial V}{\partial x_i} = P_{i_x}, \quad \frac{\partial V}{\partial y_i} = P_{i_y}, \quad \frac{\partial V}{\partial z_i} = P_{i_z}.$$

Hence V is a potential. We have therefore proved the following theorem:

If the internal forces with which the points of a system react on each other depend only on the distances of these points, then the internal forces form a potential field whose potential is given by formula (22).

Example 4. Let the points of a system attract each other according to Newton's law. Then $P_{ij} = -Km_im_j / r_{ij}^2$ for $i \neq j$, and hence according to (19) $V_{ij} = \int P_{ij}\,\mathrm{d}r_{ij} = Km_im_j / r_{ij}$, whence by (22)

$$V = \tfrac{1}{2}K\,\Sigma m_im_j / r_{ij}, \tag{24}$$

where the summation extends over every number pair i, j such that $i \neq j$, $i \leq n$ and $j \leq n$.

In particular, for two points we have

$$V = Km_1m_2 / r_{12}. \tag{25}$$

Example 5. Let the points of a system attract each other with forces proportional to the distances. Then $P_{ij} = -\lambda_{ij}^2 r_{ij}$ for $i \neq j$, where λ_{ij} is a factor of proportionality depending on the pair of points m_i, m_j. Therefore according to (19) $V_{ij} = \int P_{ij}\, dr_{ij} = -\frac{1}{2}\lambda_{ij}^2 r_{ij}^2$, from which by (22)

$$V = -\tfrac{1}{4}\Sigma\lambda_{ij}^2 r_{ij}^2. \tag{26}$$

§ 5. Kinetic energy of a system of points. Let there be given a system of points $m_1, m_2, \ldots$, having velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots$ at a certain moment t.

The *kinetic energy of a system* of points at the time t is defined as the sum of the kinetic energies of the separate points.

If we set $v_1 = |\mathbf{v}_1|$, $v_2 = |\mathbf{v}_2|, \ldots$, then the kinetic energy of the system will be

$$E = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \ldots = \tfrac{1}{2}\Sigma m_i v_i^2. \tag{I}$$

Kinetic energy of a system in an advancing motion. If a system moves with an advancing motion with a velocity $\mathbf{v}$ (i. e. if each of its points has this same velocity $\mathbf{v}$), then, putting $v = |\mathbf{v}|$, we have $E = \frac{1}{2}\Sigma m_i v^2 = \frac{1}{2}v^2\Sigma m_i$, or

$$E = \tfrac{1}{2}mv^2, \tag{II}$$

where m denotes the total mass of the system.

Kinetic energy in a rotating motion about an axis. Let a system of points rotate about an axis l with an angular velocity ω. Denoting the distances of the points of the system from the axis of rotation by $r_1, r_2, \ldots$, we have $v_i = r_i\omega$, and therefore $E = \frac{1}{2}\Sigma m_i v_i^2 = \frac{1}{2}\Sigma m_i r_i^2\omega^2 = \frac{1}{2}\omega^2\Sigma m_i r_i^2$. Since $\Sigma m_i r_i^2$ is the moment of inertia of the system with respect to the axis of rotation, setting $\Sigma m_i r_i^2 = I$, we obtain

$$E = \tfrac{1}{2}I\omega^2. \tag{III}$$

Theorem of König. Let an arbitrary system of coordinates with origin at O move with an advancing motion relative to a frame of reference. Denote the velocity of the point O by $\mathbf{u}$, the absolute velocities by $\mathbf{v}_i$, and the relative velocities of the material points m_i $(i = 1, 2, \ldots)$ by $\mathbf{w}_i$. Since $\mathbf{u}$ is the velocity of transport, it follows that

$$\mathbf{v}_i = \mathbf{u} + \mathbf{w}_i, \tag{1}$$

whence $\mathbf{v}_i^2 = (\mathbf{u} + \mathbf{w}_i)^2 = \mathbf{u}^2 + \mathbf{w}_i^2 + 2\mathbf{u}\mathbf{w}_i$. Putting $v_i = |\mathbf{v}_i|$, $w_i = |\mathbf{w}_i|$, and $u = |\mathbf{u}|$, we obtain

$$E = \tfrac{1}{2}\Sigma m_i v_i^2 = \tfrac{1}{2}\Sigma m_i u^2 + \tfrac{1}{2}\Sigma m_i w_i^2 + \Sigma m_i\mathbf{u}\mathbf{w}_i.$$

If we set $m = \Sigma m_i$, then we get

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}\Sigma m_i w_i^2 + \mathbf{u}\Sigma m_i\mathbf{w}_i. \tag{2}$$

Denote the absolute and relative velocities of the centre of mass by $\mathbf{v}_0$ and $\mathbf{w}_0$, respectively. By (1) we have $\mathbf{v}_0 = \mathbf{u} + \mathbf{w}_0$. Since $\Sigma m_i\mathbf{w}_i = m\mathbf{w}_0$, it follows from (2) that $E = \frac{1}{2}mu^2 + \frac{1}{2}\Sigma m_i w_i^2 + m\mathbf{u}\mathbf{w}_0$ or, writing $\mathbf{v}_0 - \mathbf{u}$ instead of $\mathbf{w}_0$,

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}\Sigma m_i\omega_i^2 + m\mathbf{u}(\mathbf{v}_0 - \mathbf{u}). \tag{IV}$$

The first term of this sum denotes the energy of the advancing motion of the system of points moving with a velocity $\mathbf{u}$. The second term denotes the kinetic energy of the relative motion, where the velocities $\mathbf{w}_i$ can be considered as the velocities of the points m_i relative to the point O which moves with a velocity $\mathbf{u}$.

Therefore, if we assume that the motion of the system consists of an advancing motion with a velocity of the arbitrary point O and a relative motion with respect to this point O, then *the kinetic energy of the system is equal to the sum of the kinetic energy of the advancing motion, the kinetic energy of the relative motion, and the product of the total mass of the system by the scalar product of the velocity of the point O and the relative velocity of the centre of mass.*

This theorem is known as the *theorem of König.*

In particular, if the centre of mass is chosen as the point O, then $\mathbf{u} = \mathbf{u}_0$, whence by (IV)

$$E = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}\Sigma m_i w_i^2. \tag{3}$$

Therefore, if the motion of a system is considered as a motion consisting of an advancing motion whose velocity is that of the centre of mass and a relative motion with respect to the centre of mass, then *the kinetic energy of the system is equal to the sum of the kinetic energy of the advancing motion and the kinetic energy of the relative motion.*

Principle of the equivalence of work and kinetic energy. Let the forces $\mathbf{P}_i$ act on the material points m_i. Denote the velocity of the point m_i at the time t by $\mathbf{v}_i$, its velocity at the time t_0 by $\mathbf{v}_i^{(0)}$, and the work of the force $\mathbf{P}_i$ during the time from t_0 to t by $L_{t_0t}^{(i)}$. Therefore ((3), p. 105) $\frac{1}{2}m_iv_i^2 - \frac{1}{2}m_i(v_i^{(0)})^2 = L_{t_0t}^{(i)}$, whence

$$\tfrac{1}{2}\Sigma m_iv_i^2 - \tfrac{1}{2}\Sigma m_i(v_i^{(0)})^2 = \Sigma L_{t_0t}^{(i)}.$$

Setting: $E = \frac{1}{2}\Sigma m_iv_i^2$, $E_0 = \frac{1}{2}\Sigma m_i(v_i^{(0)})^2$, $L_{t_0t} = \Sigma L_{t_0t}^{(i)}$, we obtain

$$E - E_0 = L_{t_0t}. \tag{V}$$

In this formula E denotes the kinetic energy of the system at the time t, and E_0 at the time t_0; the expression L_{t_0t} represents the sum of the works of the separate forces acting on the system, i. e. the total work which these forces did in the time from t_0 to t.

Therefore: *the increase in the kinetic energy of a system of material points is equal to the total work of the forces acting on the points of the system.*

This theorem is known as *the principle of the equivalence of work and kinetic energy.*

If the total work of the acting forces is zero, i. e. $L_{t_0t} = 0$, then by (V) $E - E_0 = 0$, or

$$E = E_0.$$

Therefore: *if the total work of the forces acting on the points of a system is constantly zero, then the kinetic energy of the system is constant.*

The above theorem is known as *the principle of conservation of kinetic energy.*

Let us assume that a system of acting forces possesses a potential. If we denote the potential at the time t by V, and the potential at the time t_0 by V_0, then $L_{t_0t} = V - V_0$, and hence by (V) $E - E_0 = V - V_0$ or

$$E - V = E_0 - V_0.$$

The magnitude $U = -V$ is called the *potential energy of the system.* Consequently

$$E + U = E_0 + U_0 = \text{const.} \tag{VI}$$

The sum $E + U$ is called the *total energy of the system.*

Therefore: *if a system of points moves in a potential field, then the total energy of the system is constant.*

These theorems are obviously generalizations of the corresponding theorems proved on p. 105 for one material point.

Kinetic energy in relative motion. Let the system of coordinates $O'(x', y', z')$ move relative to the inertial frame $O(x, y, z)$. Denote by E_r and $E_r^{(0)}$ the kinetic energy in relative motion at the times t and t_0, by L_{t_0t}, $L_{t_0t}^t$ and $L_{t_0t}^C$ the works in relative motion of the acting forces, the forces of transport and Coriolis during the time from t_0 to t. Since Newton's laws apply to relative motion if the forces of transport and Coriolis (p. 135) are added to the acting forces, it follows that

$$E_r - E_r^{(0)} = L_{t_0t} + L_{t_0t}^t + L_{t_0t}^C. \tag{4}$$

Since the acceleration of Coriolis is perpendicular to the relative velocity, the force of Coriolis is also perpendicular to this velocity.

Therefore the work of the forces of Coriolis in relative motion is zero, and because of this we can write

$$E_r - E_r^{(0)} = L_{t_0t} + L_{t_0t}^t. \tag{5}$$

Hence: *the increase in kinetic energy in relative motion is equal to the sum of the works in relative motion of the acting forces and of the forces of transport.*

In particular, let the point O' be situated at the centre of mass S of the system $O'(x', y', z')$ and let this system move with an advancing motion. Since the acceleration of transport is by this assumption equal to the acceleration $\mathbf{p}_0$ of the centre of mass, the forces of transport of the separate points of the system $m_1, m_2, \ldots$ are:

$$\mathbf{P}_{1t} = -m_1\mathbf{p}_0, \quad \mathbf{P}_{2t} = -m_2\mathbf{p}_0, \ \ldots \tag{6}$$

Denoting the relative velocities of the points of the system by $\mathbf{w}_1, \mathbf{w}_2, \ldots$, we obtain

$$L_{t_0t}^t = \int_{t_0}^{t} \mathbf{P}_{1t}\mathbf{w}_1\,dt + \int_{t_0}^{t} \mathbf{P}_{2t}\,\mathbf{w}_2 dt + \ldots,$$

whence according to (6)

$$L_{t_0t}^t = -\int_{t_0}^{t} \mathbf{p}_0(m_1\mathbf{w}_1 + m_2\mathbf{w}_2 + \ldots)\,dt.$$

The relative velocity $\mathbf{w}_0$ of the centre of mass is equal to zero because we have assumed that the centre S of the total mass m is always at the origin of the moving system $O'(x', y', z')$. Therefore $m_1\mathbf{w}_1 + m_2\mathbf{w}_2 + \ldots = m\mathbf{w}_0 = 0$, whence according to the last formula $L_{t_0t}^t = 0$. Hence by (5)

$$E_r - E_r^{(0)} = L_{t_0t}. \tag{7}$$

Hence: *the increase in kinetic energy in relative motion with respect to the centre of mass is equal to the work in relative motion of the acting forces.*

Example 1. A system of material points moves in a gravitational force field. Let the centre of mass S be the origin of the coordinate system (x', y', z') moving with an advancing motion. Assume that the z'-axis is vertical and has a downward sense.

Since the weights of the separate points have the direction of the z'-axis, then (just as in the absolute system, p. 212) the work of the weights in relative motion is equal to the work done by the total weight whose initial point is at the centre of mass. As the center of mass is at rest relative to the system (x', y', z'), for by hypothesis it is constantly at the origin of this system, the work of the weights in relative motion is zero. Therefore the increase in kinetic energy in relative motion is equal to the work of the remaining forces (excluding the weights) which act on the points of the system.

In particular, if the weights are the only forces acting on the points of the system, then the kinetic energy of this system in relative motion is constant.

Let us suppose, e. g., that at the moment $t = 0$ we have released freely a material point of mass m_1, and after T seconds, another point of mass m_2. After the time $t > T$ the velocities of the points m_1 and m_2 are:

$$v_1 = gt \text{ and } v_2 = g(t - T). \tag{8}$$

The velocity of the centre of mass v_0 is obtained from the equation

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v_0.$$

Therefore $v_0 = (m_1 v_1 + m_2 v_2) / (m_1 + m_2)$. The relative velocities of the points m_1 and m_2 are $w_1 = v_1 - v_0$ and $w_2 = v_2 - v_0$; hence $w_1 = m_2(v_1 - v_2) / (m_1 + m_2)$, and $w_2 = m_1(v_2 - v_1) / (m_1 + m_2)$, from which by (8) $w_1 = m_2 gT / (m_1 + m_2)$, and $w_2 = -m_1 gT / (m_1 + m_2)$.

The kinetic energy in relative motion is therefore

$$E_r = \tfrac{1}{2} m_1 w_1^2 = + \tfrac{1}{2} m_2 w_2^2 = m_1 m_2 g^2 T^2 / 2(m_1 + m_2) = \text{const.}$$

We see then that the kinetic energy in relative motion with respect to the centre of mass is constant.

Example 2. Two points A and B of masses m_1 and m_2, connected by a massless inextensible string, move in a vertical plane in such a way that the point A must remain constantly on the horizontal axis x and the point B on the vertical axis z.

Let us denote the length of the string by l and the coordinates of the points A, B by x, z (Fig. 143). The point A is acted upon by the reaction $\mathbf{R}_1$ perpendicular to the x-axis (we assume that there is no friction), the weight $\mathbf{Q}_1$, and the reaction $\mathbf{P}_1$ of the string; the point B is acted upon by the reaction $\mathbf{R}_2$ perpendicular to the z-axis, the weight $\mathbf{Q}_2$, and the reaction $\mathbf{P}_2$ of the string.

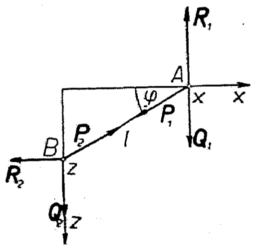

Fig. 143.

Of these forces, the following do no work: the reactions $\mathbf{R}_1$, $\mathbf{R}_2$, and the weight $\mathbf{Q}_1$, because these forces are perpendicular to the path. The forces $\mathbf{P}_1$ and $\mathbf{P}_2$ also do no work, since $\mathbf{P}_1 = -\mathbf{P}_2$ and furthermore the distance between

the points A, B is constant (p. 208). Only the weight $\mathbf{Q}_2$ therefore does work.

Suppose that at the time $t_0 = 0$ the points A, B had the coordinates x_0, z_0 and a zero velocity. The kinetic energy at the instant t is

$$E = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \dot{z}^2.$$

The work of the force $\mathbf{Q}_2$ is equal to $m_2 g(z - z_0)$ if the z-axis is given a downward sense. Therefore by the principle of equivalence of work and kinetic energy

$$\tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \dot{z}^2 = m_2 g(z - z_0). \tag{9}$$

Denote by φ the angle which the string makes with the x-axis at the time t, and by φ_0 the angle at the time $t_0 = 0$. We then have:

$$x = l \cos \varphi, \quad z = l \sin \varphi, \quad z_0 = l \sin \varphi_0,$$

whence

$$\dot{x} = -l\dot{\varphi} \sin \varphi, \quad \dot{z} = l\dot{\varphi} \cos \varphi. \tag{10}$$

Hence according to (9)

$$\tfrac{1}{2} l^2 \dot{\varphi}^2 (m_1 \sin^2\varphi + m_2 \cos^2\varphi) = m_2 g l(\sin \varphi - \sin \varphi_0). \tag{11}$$

From the above equation we can, knowing φ, calculate $\dot{\varphi}$, and then from equations (10) determine the velocities $\dot{x}$ and $\dot{y}$. Knowing φ, we can also determine the reactions $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{P}_1$, $\mathbf{P}_2$. Assume for simplicity's sake that $m_1 = m_2 = m$, and $\varphi_0 = 0$. We obtain from (11)

$$\tfrac{1}{2} l \dot{\varphi}^2 = g \sin \varphi. \tag{12}$$

Differentiating with respect to t, we get $l\dot{\varphi}\ddot{\varphi} = g\dot{\varphi} \cos \varphi$, whence

$$\ddot{\varphi} = g \cos \varphi / l. \tag{13}$$

Denoting the acceleration of the point A by $\mathbf{p}_1$, we obtain

$$m_1 \mathbf{p}_1 = \mathbf{R}_1 + \mathbf{Q}_1 + \mathbf{P}_1. \tag{14}$$

Forming the projection on the x-axis and putting $P = |\mathbf{P}_1| = |\mathbf{P}_2|$, we get

$$m\ddot{x} = -P \cos \varphi. \tag{15}$$

Since in virtue of (10)

$$\ddot{x} = -l\ddot{\varphi} \sin \varphi - l\dot{\varphi}^2 \cos \varphi, \tag{16}$$

from equations (15) and (16) we can obtain P, knowing φ, because $\dot{\varphi}$ and $\ddot{\varphi}$ can be calculated from equations (12) and (13). We get

$$P = 3mg \sin \varphi. \tag{17}$$

Forming the projection on the z-axis, we get from (14) $R_1 + mg + P \sin\varphi = 0$, or

$$R_1 = -mg - P \sin\varphi, \tag{18}$$

where R_1 denotes the projections of the force $\mathbf{R}_1$ on the z-axis.

Similarly, for the point B we have $m\mathbf{p}_2 = \mathbf{R}_2 + \mathbf{P}_2 + \mathbf{Q}_2$. Forming the projection on the x-axis and observing that $\mathbf{P}_2 = -\mathbf{P}_1$, we obtain the equation $R_2 + P\cos\varphi = 0$, whence

$$R_2 = -P\cos\varphi. \tag{19}$$

Formulae (17)—(19) determine the dependence of the reactions on the angle φ.

Example 3. Two material points A_1 and A_2 of masses m_1 and m_2, connected by a massless inextensible string passing through a fixed point O, move without friction in a horizontal plane passing through O. In this plane select a coordinate system (x, z) whose origin is at O (see Fig. 140). Since the directions of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, with which the string acts on the points A_1 and A_2, constantly pass through O, the points A_1 and A_2 will move with a central motion with centre at O.

Put $r_1 = OA_1$ and $r_2 = OA_2$; let φ_1, φ_2 denote the angles between the x-axis and the segments OA_1 and OA_2. The areal velocities of the points A_1 and A_2 are equal to $\frac{1}{2}r_1^2\dot{\varphi}_1$ and $\frac{1}{2}r_2^2\dot{\varphi}_2$, respectively. Since the areal velocities in a central motion are constant (p. 86), it follows that

$$r_1^2\dot{\varphi}_1 = c_1, \quad r_2^2\dot{\varphi}_2 = c_2, \tag{20}$$

where c_1 and c_2 are certain constants.

On p. 209 we proved that the total work of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ is zero. Therefore the kinetic energy of the system of points A_1 and A_2 has a constant value. Denoting the magnitudes of the velocities of the points A_1 and A_2 by v_1 and v_2, we obtain $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = c$, whence

$$m_1v_1^2 + m_2v_2^2 = h, \tag{21}$$

where c and $h = 2c$ are constants. But ((3), p. 47)

$$v_1^2 = \dot{r}_1^2 + r_1^2\dot{\varphi}_1^2 \text{ and } v_2^2 = \dot{r}_2^2 + r_2^2\dot{\varphi}_2^2;$$

hence by (20), expressing $\dot{\varphi}_1$ and $\dot{\varphi}_2$ in terms of r_1 and r_2, we obtain $v_1^2 = \dot{r}_1^2 + c_1^2/r_1^2$ and $v_2^2 = \dot{r}_2^2 + c_2^2/r_2^2$, from which by (21)

$$m_1\dot{r}_1^2 + m_2\dot{r}_2^2 + m_1c_1^2/r_1^2 + m_2c_2^2/r_2^2 = h. \tag{22}$$

Denote the length of the string by l. Then $r_1 + r_2 = l$ therefore $r_2 = l - r_1$, whence

$$\dot{r}_2 = -\dot{r}_1. \tag{23}$$

Substituting in (22), we obtain

$$(m_1 + m_2)\dot{r}_1^2 + m_1c_1^2/r_1^2 + m_2c_2^2/(l - r_1)^2 = h. \tag{24}$$

The differential equation (24) determines r_1 as a function of the time t. From equations (23) and (20) we obtain $r_2, \varphi_1, \varphi_2$. In order to determine the reactions $\mathbf{P}_1$, $\mathbf{P}_2$, let us note that if $\mathbf{p}_1$ denotes the acceleration of the point A_1, then $m_1\mathbf{p}_1 = \mathbf{P}_1$. Form the projection on the direction of $\overline{OA}_1$. Denoting the projections of $\mathbf{p}_1$ and $\mathbf{P}_1$ on $\overline{OA}_1$ by p_{1r} and P, we get

$$m_1p_{1r} = P. \tag{25}$$

By (II), p. 47, we have $p_{1r} = \ddot{r}_1 - r_1\dot{\varphi}_1^2$. Hence according to (20)

$$p_{1r} = \ddot{r}_1 - c_1^2/r_1^3. \tag{26}$$

In order to determine $\ddot{r}_1$, let us differentiate equation (24). We get

$$\dot{r}_1[(m_1 + m_2)\ddot{r}_1 - m_1c_1^2/r_1^3 + m_2c_2^2/(l - r_1)^3] = 0. \tag{27}$$

If $\dot{r}_1 \neq 0$, then in virtue of (25)—(27)

$$P = -\frac{m_1m_2}{m_1 + m_2}\left[\frac{c_2^2}{(l - r_1)^3} + \frac{c_1^2}{r_1^3}\right]. \tag{28}$$

From formula (28) we can obtain the reaction P knowing only r_1. Knowing P, we know $\mathbf{P}_1$ and $\mathbf{P}_2$ because $|\mathbf{P}_1| = |\mathbf{P}_2|$.

§ 6. Problem of two bodies. Let two material points of masses M and m attract each other according to Newton's law with a force of magnitude

$$P = KmM/r^2,$$

where r denotes the distance of these points. On p. 106 we examined the motion of the point m under the assumption that the point M is motionless. We proved that Kepler's law obtain in this case. Now we shall not assume that the point M is motionless, but that both points are unconstrained. Therefore under the influence of their mutual attraction, both points m and M will move. Obviously, their centre of mass will be at rest or in uniform straight line motion, because according to the law of action and reaction the sum of the forces acting on the points m and M is equal to zero. We can therefore place the origin of the inertial frame at the centre of gravity of both points.

Let x_1, y_1, z_1 be the coordinates of the point M, and x_2, y_2, z_2 those of the point m. Newton's equations of motion for the points M and m will have the form:

$$M\ddot{x}_1 = \frac{KmM}{r^2}\,\frac{x_2 - x_1}{r}, \quad M\ddot{y}_1 = \frac{KmM}{r^2}\,\frac{y_2 - y_1}{r}, \quad M\ddot{z}_1 = \frac{KmM}{r^2}\,\frac{z_2 - z_1}{r}, \tag{1}$$

$$m\ddot{x}_2 = -\frac{KmM}{r^2}\frac{x_2 - x_1}{r}, \quad m\ddot{y}_2 = -\frac{KmM}{r^2}\frac{y_2 - y_1}{r},$$
$$m\ddot{z}_2 = -\frac{KmM}{r^2}\frac{z_2 - z_1}{r}. \qquad (1')$$

Since the centre of gravity of the system M, m is at the origin of the coordinate system, $Mx_1 + mx_2 = 0$, $My_1 + my_2 = 0$, and $Mz_1 + mz_2 = 0$, whence $x_2 = -Mx_1/m$, $y_2 = -My_1/m$, and $z_2 = -Mz_1/m$. Therefore:

$$x_2 - x_1 = -\frac{M+m}{m}x_1, \quad y_2 - y_1 = -\frac{M+m}{m}y_1,$$
$$z_2 - z_1 = -\frac{M+m}{m}z_1. \qquad (2)$$

Hence

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \frac{M+m}{m}r_1, \qquad (3)$$

where $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ denotes the distance of the point M from the centre of gravity. Substituting in equations (1) the expressions from formulae (2) and (3), we obtain:

$$M\ddot{x}_1 = -\frac{Km^3M}{(M+m)^2 r_1^2}\cdot\frac{x_1}{r_1}, \quad M\ddot{y}_1 = -\frac{Km^3M}{(M+m)^2 r_1^2}\cdot\frac{y_1}{r_1},$$
$$M\ddot{z}_1 = -\frac{Km^3M}{(M+m)^2 r_1^2}\cdot\frac{z_1}{r_1}. \qquad (4)$$

Comparing these equations with equations (I), p. 106, we see that the motion of the point M is such as if this point were attracted by a motionless mass $m^3/(M+m)^2$ situated at the origin of the system.

Therefore: *if two points M and m attract each other according to Newton's law, then each one of them, for example M, moves relative to the centre of gravity (of both points) so as if a motionless mass $m^3/(M+m)^2$ were situated at the centre of gravity and attracted the point M according to Newton's law.*

Hence the investigation of a motion relative to the centre of mass of two points is reduced to the case considered on p. 106.

It follows from this that both points move along a conic at whose focus is found the centre of gravity of these points. The paths of both points are therefore plane paths.

Let us still examine the motion of the point m relative to the point M. Let us place at M the origin of the coordinate system (x', y', z') which

moves together with M with an advancing motion. Denoting the coordinates of m with respect to this coordinate system by ξ, η, ζ, we obtain

$$\xi = x_2 - x_1, \quad \eta = y_2 - y_1, \quad \zeta = z_2 - z_1. \qquad (5)$$

Multiplying equations (1) by m/M and subtracting from (1') we get in virtue of (5)

$$m\ddot{\xi} = -\frac{K(M+m)m}{r^2}\frac{\xi}{r}, \quad m\ddot{\eta} = -\frac{K(M+m)m}{r^2}\frac{\eta}{r},$$
$$m\ddot{\zeta} = -\frac{K(M+m)m}{r^2}\frac{\zeta}{r}. \qquad (6)$$

Comparing equations (6) with equations (I), p. 106, we see that the point m moves relative to the point M so as if M were motionless and its mass were increased by the mass of the point m.

Therefore: *if two points M and m attract each other according to Newton's law, then the relative motion of m with respect to M is such as if M were motionless and its mass were increased by the mass of the point m.*

In this case also, the investigation of the motion of one material point relative to another is therefore reduced to the case considered on p. 106.

Let us assume that the relative motion of the point m takes place along an ellipse of major axis $2a$, and let the time required to complete one revolution be T. By (10), p. 108, we obtain $a^3/T^2 = K(M+m)/4\pi^2$. We see, therefore, that in relative motion the ratio a^3/T^2 depends on the masses of both bodies. Since we are investigating the motions of the planets relative to the sun, assuming that M denotes the mass of the sun and m the mass of the planet, we see that *Kepler's third law* (p. 87), which refers to the relative motion of a planet with respect to the sun, *is not exact*. For another planet (using a corresponding notation)

$$a_1^3/T_1^2 = K(M+m_1)/4\pi^2, \qquad (7)$$

whence

$$\frac{a^3/T^2}{a_1^3/T_1^2} = \frac{M+m}{M+m_1} = \frac{1+m/M}{1+m_1/M}. \qquad (8)$$

In the solar system the ratio m/M is expressed in the thousandths and therefore the last fraction differs little from one. Accurate observations of planetary motions reveal these deviations from Kepler's third law.

Two celestial bodies which rotate about each other (far away from other bodies) are called *double stars*. Assuming that double stars attract each other according to Newton's law, we can apply to them the conclusions obtained in this §. Observations confirm these conclusions and at the same time the law of universal gravitation from which these conclusions were drawn.

§ 7. Problem of n bodies. Let n material points attract each other mutually with forces acting according to Newton's law of universal gravitation (p. 89). The so-called *problem of n bodies* is concerned with the investigation of motions in such a system of points.

This problem is important for astronomy. The sun and planets form such a system if we neglect the influence of the fixed stars which is very small because of their remoteness from the solar system.

The problem of two bodies with which we were concerned in § 6 is a particular case of the problem of n bodies.

The problem of n bodies is not solved in all generality. Even in the case of three bodies there are many questions still unanswered. By means of the *theory of perturbations*, however, we can determine the motions of the solar system with the desired accuracy.

In the problem of n bodies we have to deal only with internal forces. Therefore from the theorem on the centre of mass (p. 196), it follows that the centre of mass of a system is at rest or in uniform straight line motion. We can therefore choose the origin of the inertial system of coordinates at the centre of mass. With respect to such a chosen system of coordinates the momentum of the system of n points will be constantly zero (p. 195).

From the theorem concerning angular momentum (p. 202), it follows that the angular momentum of a system of points is constant. Hence the plane passing through the centre of mass and perpendicular to the angular momentum does not change its position.

In the case of the solar system the centre of mass lies in the sun (on account of the great mass of the sun as compared with the remaining planets).

The plane passing through the centre of mass of the solar system and perpendicular to the angular momentum was called the *invariable plane* by Laplace.

This plane does not change its position in space relative to the inertial system of coordinates whose origin is in the sun. According to calculations carried out by Laplace, the invariable plane forms an angle $\alpha = 1.7689°$ with the ecliptic.

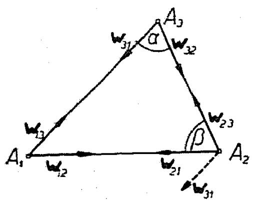

Fig. 144.

Problem of three bodies. Let there be given three material points A_1, A_2, A_3 of masses m_1, m_2, m_3 (Fig. 144). Denote the force with which m_j attracts m_i by $\mathbf{P}_{ij}$; let $\mathbf{w}_{ij}$ be the force with which a unit of mass situated at A_j attracts a unit of mass situated at A_i. According to Newton's law we therefore have

$$\mathbf{P}_{ij} = m_i m_j \mathbf{w}_{ij}. \tag{1}$$

From the law of action and reaction it follows that $\mathbf{P}_{ij} = -\mathbf{P}_{ji}$; hence

$$\mathbf{w}_{ij} = -\mathbf{w}_{ji}. \tag{2}$$

Denote the acceleration of the point m_i in an inertial system of coordinates by $\mathbf{p}_i$. Then $m_1\mathbf{p}_1 = \mathbf{P}_{12} + \mathbf{P}_{13} = m_1 m_2 \mathbf{w}_{12} + m_1 m_3 \mathbf{w}_{13}$, whence

$$\mathbf{p}_1 = m_2 \mathbf{w}_{12} + m_3 \mathbf{w}_{13}. \tag{3}$$

Similarly

$$\mathbf{p}_2 = m_1 \mathbf{w}_{21} + m_3 \mathbf{w}_{23}. \tag{4}$$

By (3) and (4), and because that in view of (2), $\mathbf{w}_{12} = -\mathbf{w}_{21}$, $\mathbf{w}_{13} = -\mathbf{w}_{31}$, we obtain:

$$\mathbf{p}_2 - \mathbf{p}_1 = (m_1 + m_2)\, \mathbf{w}_{21} + m_3(\mathbf{w}_{23} + \mathbf{w}_{31}). \tag{5}$$

The difference $\mathbf{p}_2 - \mathbf{p}_1$ represents the relative acceleration of the point A_2 with respect to A_1, i. e. the acceleration of A_2 relative to the coordinate system, whose origin is A_1, moving with an advancing motion. Put $\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1$. From equation (5) we get

$$m_2\mathbf{p} = m_2(m_1 + m_2)\, \mathbf{w}_{21} + m_2 m_3(\mathbf{w}_{23} + \mathbf{w}_{31}). \tag{6}$$

The right side of equation (6) represents the relative force of the point A_2 in motion relative to A_1.

The first one of its terms, i. e. $m_2(m_1 + m_2)\ \mathbf{w}_2$, represents the relative force which would act on the point A_2 if there were no third point A_3 (i. e. if there were $m_3 = 0$). This force would have (in agreement with the theorem given on p. 223) a direction towards A_1 and would be such as if the mass of A_1 were increased by the mass of A_2.

The second term of the sum, i. e. $m_2 m_3(\mathbf{w}_{23} + \mathbf{w}_{31})$, is called the *force of perturbation*; it is due to the action of the point A_3.

Example 1. Let m_1 denote the mass of the earth, m_2 the mass of the moon and m_3 the mass of the sun.

Approximately, the mass of the sun is $\frac{1}{3} \cdot 10^6$ times the mass of the earth, the distance of the earth from the sun is 400 times the distance of the earth from the moon, and finally the mass of the moon is $\frac{1}{80}$ of the mass of the earth. Hence:

$$m_3 = \tfrac{1}{3} \cdot 10^6 m_1, \quad m_2 = \tfrac{1}{80} m_1, \quad A_1A_3 = 400 A_1A_2. \tag{7}$$

From the triangle $A_1A_2A_3$ we obtain

$$399 A_1A_2 \leq A_2A_3 \leq 401 A_1A_2. \tag{8}$$

Because of the great distance of the sun from the earth and the moon (as compared with the distance of the moon from the earth), $\mathbf{w}_{23}$ and $\mathbf{w}_{31}$ will differ little from each other in magnitude and direction and they will have opposite senses. The absolute value of the sum $\mathbf{w}_{23} + \mathbf{w}_{31}$ is therefore small. Making use of (7) and (8), it can be shown that

$$\frac{m_2 m_3 |\mathbf{w}_{23} + \mathbf{w}_{31}|}{m_2(m_1 + m_2)|\mathbf{w}_{21}|} \leqq 1 \cdot 5 \cdot 10^{-2}. \tag{9}$$

We see by (6), therefore, that the force of perturbation due to the sun is small, and we can neglect it in the first approximation.

Hence: *In the first approximation the relative motion of the moon with respect to the earth is obtained by neglecting the attraction of the sun.*

An approximate investigation of the relative motion in the given case is therefore reduced to the problem of two bodies. This also refers to other planets having satellites.

Example 2. Let A_1 be the centre of the earth, A_2 a point on the surface of the earth and A_3 the centre of the moon. Let m_1, m_2, and m_3 denote the masses of the earth, the point A_2 and the moon, respectively. Assume that the points A_1, A_2, and A_3, are collinear.

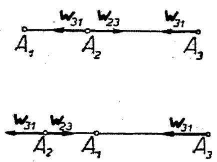

Fig. 145.

The vectors $\mathbf{w}_{23}$ and $\mathbf{w}_{31}$ have opposite senses. If A_2 lies between A_1 and A_3, then $|\mathbf{w}_{32}| > |\mathbf{w}_{31}|$ (Fig. 145a), and hence $\mathbf{w}_{23} + \mathbf{w}_{31}$ has the sense of $\mathbf{w}_{23}$. If A_1 lies between A_3 and A_2 (Fig. 145b), then $|\mathbf{w}_{23}| < |\mathbf{w}_{31}|$, and hence $\mathbf{w}_{23} + \mathbf{w}_{31}$ has the sense of $\mathbf{w}_{31}$. In both cases the force of perturbation of the moon $m_2 m_3(\mathbf{w}_{23} + \mathbf{w}_{31})$ is directed vertically upwards with respect to the earth. The action of this force explaines the tides.

Example 3. Let m_1 denote the mass of some planet, m_2 the mass of its satellite, a the mean distance of the planet from its satellite and T the time of one revolution of the satellite about the planet in relative motion (with respect to the planet). In virtue of (7), p. 223, we have

$$a^3 / T^2 = K(m_1 + m_2) / 4\pi^2. \tag{10}$$

If m_1', m_2', a', and T' denote the corresponding magnitudes for another planet and its satellite, then analogous to (10)

$$a'^3 / T'^2 = K(m_1' + m_2') / 4\pi^2. \tag{11}$$

By (10) and (11), $(m_1 + m_2) / (m_1' + m_2') = a^3 T'^2 / a'^3 T^2$. Neglecting

the masses of the satellites m_2 and m_2' because they are usually small as compared with the masses of the planets, we get

$$m_1 / m_1' = a^3 T'^2 / a'^3 T^2. \tag{12}$$

Therefore: *the ratio of the masses of two planets can be obtained from the observation of the motions of their satellites.*

Remark. We can also assume that m_1' denotes the mass of the sun, $m_2' = m_1$, a' the mean distance of the planet from the sun, and T' its period. Under these assumptions formula (12) represents the ratio of the mass of the given planet (possessing a satellite) to the mass of the sun.

§ 8. Motion of bodies of variable mass. Let us now investigate the motion of a body whose mass changes because particles leave the body (or new ones join it) during motion.

An example is that of a moving waggon into which sand is being poured (or from which sand is running out). A rocket is another example. When the fuel within a rocket burns, gases are expelled which propel the rocket. The mass of the rocket diminishes, therefore, by the mass of the escaping gases.

Let us assume that a body consists of a great number of small particles which can be considered as material points. Denote by m the mass of the body, by $\mathbf{v}$ the velocity of its centre of mass S at the time t, and by $m + \Delta m$ and $\mathbf{v} + \Delta\mathbf{v}$ the mass and velocity of the centre S at the time $t + \Delta t$. Finally, let $\mathbf{P}$ denote the sum of the forces acting on the body at the time t (Fig. 146).

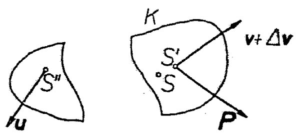

Fig. 146.

The mass of the particles leaving the body during the interval Δt is $(-\Delta m)$; let $\mathbf{u}$ and $\mathbf{u} + \Delta\mathbf{u}$ denote the velocities (at the times t and $t + \Delta t$) of the centre of mass S'' of the particles leaving the body.

Let us consider the system U of all the particles of which the body is composed at the time t. The momentum of this system at the time t is $\mathbf{H} = m\mathbf{v}$, and at the time $t + \Delta t$ it will be $\mathbf{H}' = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) + t + (-\Delta m)(\mathbf{u} + \Delta\mathbf{u})$. Hence the increase in the momentum is

$$\mathbf{H}' - \mathbf{H} = m\,\Delta\mathbf{v} - \Delta m(\mathbf{u} + \Delta\mathbf{u} - \mathbf{v}) + \Delta m\,\Delta\mathbf{v}.$$

Dividing by Δt and passing to the limit, we obtain

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - \frac{\mathrm{d}m}{\mathrm{d}t}(\mathbf{u} - \mathbf{v}). \tag{1}$$

Since the derivative of the momentum is equal to the sum of all the acting forces (p. 196), $d\mathbf{H} / dt = \mathbf{P}$. Therefore from (1)

$$m\mathbf{v}^{\cdot} - m^{\cdot}(\mathbf{u} - \mathbf{v}) = \mathbf{P}. \tag{2}$$

The above equation can be written in the form $m\mathbf{v}^{\cdot} + m^{\cdot}\mathbf{v} = m^{\cdot}\mathbf{u} + \mathbf{P}$, whence

$$d(m\mathbf{v}) / dt = m^{\cdot}\mathbf{u} + \mathbf{P}. \tag{3}$$

Formulae (2) and (3) apply equally to the case when new particles join the body. In equations (2) and (3) the vector $\mathbf{u}$ represents the velocity of the centre of mass S'' of the particles leaving or joining the body.

Substituting $\mathbf{u} - \mathbf{v} = \mathbf{w}$ in equation (2), we obtain

$$m\mathbf{v}^{\cdot} - m^{\cdot}\mathbf{w} = \mathbf{P}. \tag{4}$$

The vector $\mathbf{w}$ represents the relative velocity of the centre of mass of the particles leaving the body with respect to the centre of mass of the body.

***Example.* The motion of a rocket.** Denote the mass of a rocket by m, its velocity by $\mathbf{v}$, the relative velocity (with respect to the rocket) of the gases escaping from the rocket by $\mathbf{w}$ (Fig. 147), and the sum of the external forces acting on the body (such as gravity, the resistance of the air, etc.) by $\mathbf{P}$. With this notation formula (4) obtains.

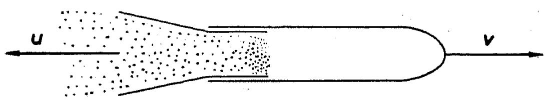

Fig. 147.

Let us suppose at first that the rocket moves in a horizontal plane along a straight line which we shall select as the x-axis, giving it a sense agreeing with the direction of the rocket. Put $v = |\mathbf{v}|$ and $w = |\mathbf{w}|$. Assume that $\mathbf{P} = 0$ (and hence that the force of gravity is balanced by the reaction of the plane; the resistance of the air and friction are neglected). Since $\mathbf{v}$ and $\mathbf{w}$ have opposite senses, by (4) $mv^{\cdot} + m^{\cdot}w = 0$, whence

$$v^{\cdot} = -\frac{m^{\cdot}}{m}w. \tag{5}$$

The relative velocity w of the escaping gases can be considered as constant. Integrating equation (5), we obtain

$$v = -w \ln m + c. \tag{6}$$

Assume that $m = m_0$ and $v = 0$ at $t = 0$. Then according to equation (6), $0 = -w \ln m_0 + c$, and hence $c = w \ln m_0$. Therefore by (6)

$$v = w \ln \frac{m_0}{m}, \tag{7}$$

whence $m_0 / m = e^{v/w}$ or

$$m = m_0 e^{-v/w}. \tag{8}$$

Let us suppose that the rocket attained a velocity $v = 100$ km/h = 27 m/sec. We can assume that $w = 1000$ m/sec is the velocity of the escaping gas. Therefore by (8) $m = m_0 e^{-0.027} = 0.973 m_0$, whence $m_0 - m = 0.03 m_0$.

Hence in order to realize a velocity of 100 km/h, it is necessary to burn an amount of fuel equal to 3% of the mass of the rocket.

Let the rocket now move vertically upwards. Assume that the z-axis is directed vertically upwards and let us retain the previous notation. We obtain from (4) (neglecting air resistance) $mv^{\cdot} + m^{\cdot}w = -mg$, whence

$$v^{\cdot} = -w\frac{m^{\cdot}}{m} - g. \tag{9}$$

Integrating (9) and assuming $v = 0$ and $m = m_0$ at $t = 0$ we obtain as previously

$$v = w \ln \frac{m_0}{m} - gt. \tag{10}$$

In order that the rocket may not fall back to earth and that it may penetrate interplanetary space it would be necessary to give it a velocity $v \geqq 12$ km/sec (p. 110). From equation (10) we obtain

$$v \leqq w \ln \frac{m_0}{m},$$

whence $e^{v/w} \leqq m_0 / m$ or $m e^{v/w} \leqq m_0$, and therefore

$$m_0 - m \geqq m(e^{v/w} - 1).$$

Putting $v = 12$ km/sec and $w = 1000$ m/sec = 1 km/sec, we get

$$m_0 - m \geqq 160000\, m.$$

In this inequality m denotes the mass of the rocket after attaining a velocity $v = 12$ km/sec, and $m_0 - m$ the mass of the propelling fuel burned. If we assume that $m = 1$ kg, then $m_0 - m \geqq 160000$ kg.

It is therefore necessary to burn 160000 kg of fuel in order that 1 kg of mass excape into space.

Hence in order to make an interplanetary journey in a rocket having together with its passengers a mass of one ton, it would be necessary to take along 160000 tons of fuel — which is obviously impossible. This shows that at the present state of technical sciences such a journey cannot be made. The matter would be pushed forward if w (the velocity of the escaping gases), which to-day is close to 2000 m/sec, could be markedly increased.

CHAPTER VI[1])

STATICS OF A RIGID BODY

I FREE BODY

§ 1. Rigid body. A material body which despite the action of forces does not sustain any deformations (i. e. in which the mutual distances of the points of the body do not undergo a change) is called a *rigid body*.

Rigid bodies are not found in nature, since every body becomes deformed more or less under the influence of the action of forces. However, if some body under the influence of forces experiences only small deformations not exceeding a certain limit, then we can take as a model of such a body a rigid body, and the conclusions that we shall draw will be approximately in agreement with experience (provided the forces are not large). From this arises the great importance of the theory of a rigid body for practical applications.

We shall consider in turn statics, kinematics and dynamics of a rigid body.

In the theory of a rigid body we shall meet, in addition to rigid material solids, rigid material surfaces and lines (p. 168) as models of bodies in which one or two dimensions are small in comparison with those remaining. Examples of such bodies are plates, rods, wires, etc.

Rigid systems of material points. It often proves useful to look upon a rigid body as a collection (system) of a large number of material points. We assume then, that the material points act on each other with certain forces which ensure that the system of points is rigid, i. e. that the mutual distances of its points do not undergo a change. These forces are called *internal forces*.

We assume that Newton's law of action and reaction (p. 173) applies to internal forces, i. e. that two points act on each other with

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[1]) For the understanding of this chapter the information included in chapters I and III (from p. 69 to 75) and the theorems on centre of gravity in chapter IV, §§ 1, 2, 6, 7 and 8, are sufficient.