tain fixed frame. Choose the ξ and η axes of the moving frame in the plane of the circle; let O be the origin and the line OA_2 the ξ-axis.

The angular velocity of the point A_1 relative to the chosen moving frame is called the (relative) *angular velocity of the point A_1 with respect to the point A_2*; we denote it by $\omega_{1,2}$.

It is easy to show that

$$\omega_{1,2} = \omega_1 - \omega_2. \tag{7}$$

Let us assume that the points A_1 and A_2 move with constant angular velocities. Denoting by T_1, T_2 the periods of revolutions of the corresponding points A_1, A_2 in the fixed frame, and by $T_{1,2}$ the period of revolution of the point A_1 relative to A_2 (i. e. the period of revolution of the point A_1 in the moving frame), we obtain: $T_1 = 2\pi / \omega_1$, $T_2 = 2\pi / \omega_2$, $T_{1,2} = 2\pi / \omega_{1,2}$. Hence by (7)

$$1 / T_{1,2} = 1 / T_1 - 1 / T_2. \tag{8}$$

The period of revolution of the minute hand of a clock is $T_1 = 1$ h, that of the hour hand $T_2 = 12$ h. From formula (8) we get: $1/T_{1,2} = 1 - \frac{1}{12}$, whence $T_{1,2} = \frac{12}{11}$ h $= 1$ h, 5 min, 27 sec. Therefore the hands coincide every 1 h, 5 min, 27 sec.

To a traveller circling the globe from west to east it seems that the journey lasted n mean solar days, because during the journey there were n days and n nights. However, returning to the place from which he started, he finds that the journey did not last n, but n' mean solar days. What is the relation between n and n'?

Let us denote by T_1 the time of the journey, and by T_2 the time taken by the sun to complete an apparent revolution about the earth. Consequently $T_1 = n'$, and $T_2 = -1$ (since the sun seemingly revolves about the earth from east to west, that is, in the direction opposite to that of the journey). The traveller assumed as the apparent mean solar day the period of time between two successive passings of the sun across the changing meridian on which he was. Since in the interval of n apparent days there were n' real days, the apparent mean solar day is equal to n' / n real days. Hence $T_{1,2} = n' / n$. Therefore by (8)

$$n / n' = 1 / n' + 1, \quad \text{or} \quad n' = n - 1.$$

Therefore the number of real days elapsed was one less than the number of apparent days.

If the traveller had gone from east to west, then (as is easily seen) the number of real days elapsed would be one greater than the number of apparent days.

CHAPTER III

DYNAMICS OF A MATERIAL POINT

I. DYNAMICS OF AN UNCONSTRAINED POINT

§ 1. Basic concepts of dynamics. The subject of dynamics is concerned with the investigations of the motion of bodies under the influence of forces which cause this motion.

In kinematics all frames of reference are, as we already know, equally valid; it is a matter of indifference how we measure time (i. e. what intervals of time we consider as equal). The laws of dynamics stated by Newton, however, are not valid for every frame of reference and every measure of time.

Inertial frame, absolute time. A frame of reference for which, along with a certain measure of time, the Newtonian laws of dynamics hold, is called an *inertial frame*, the corresponding measure of time — the measure of *absolute time*, and the motion of the body relative to the inertial frame — *absolute motion.*

Strictly speaking, we do not know at present of any example of either an inertial frame or of absolute time. Nevertheless, in a great number of problems we can select frames of reference and methods of measuring time in such a way, that the application of the laws of dynamics leads to results differing sufficiently little from experience, so that for all practical purposes the errors can be neglected.

For instance, if we are investigating the motion of small particles near the earth during a short interval of time, the results will be sufficiently accurate on the whole, if we take as an inertial frame the frame of reference attached to the earth, and if we base the measurement of absolute time on the assumption that the earth, relative to the fixed stars, revolves about its axis so as to make equal angles in equal times.

In other problems, however (such as Foucault's pendulum, the gyroscope, the motion of planets) the application of the laws of dynamics to a frame of reference attached to the earth does not lead to equally good results. Considerably better results are obtained here if we select for the inertial frame, a frame of reference whose origin is situated within the sun and whose axes point to the fixed stars.

In addition to the previously mentioned method of measuring time, which is based on the assumption that the earth revolves about its axis uniformly, there exist still other methods of measuring time which are given in astronomy.

In dynamics we assume that an *inertial frame and absolute time are given.*

Mass and force. Dynamics gives rise to new concepts such as mass and force. We assume that they are known to the reader from physics and we shall not enlarge upon their definition. We shall give only those of their properties which we assume about them in dynamics.

The mass of a body is expressed by a positive number which depends on the choice of the units of mass, i. e. on the choice of an arbitrary body whose mass is denoted by the number 1.

The ratio of two masses does not depend on the choice of the unit.

Therefore, if we denote by m_1 and m_2 the masses of two bodies expressed in terms of a certain unit, and by m_1' and m_2' the masses of these bodies expressed in terms of another unit, then

$$m_1 / m_2 = m_1' / m_2'.$$

For example, let the mass of a certain body A be m, if we take as the unit the mass of the body B. Let us assume, in addition, that the mass of the body A is m' and the mass of B is m'', if we select as the unit the mass of another body C. The ratio of the masses of the bodies A and B is $m / 1$, if the unit is the mass of the body B, and m' / m'', if the unit is the mass of the body C. Therefore, by hypothesis, $m / 1 = m' / m''$, whence

$$m' = mm''.$$

Hence, knowing the mass of a body in terms of a certain unit, we can compute it in terms of any other unit.

The mass of a body is independent of the time, i. e. the given body has the same mass at each moment.

We consider a *force* to be determined if its *magnitude* (absolute value), *direction, sense and point of application* are given. A force acting on a body can be thought of as a taut string or a stretched spring fastened to the body (see figure).

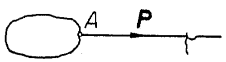

Fig. 69.

A force whose magnitude is expressed by the number 0 is called a *zero force*. We do not distinguish a direction or sense in connection with zero forces.

The magnitude of a non-zero force is expressed by a positive number which depends on the unit of force, i. e. on the choice of an arbitrary (non-zero) force whose magnitude we denote by the number 1.

The ratio of the magnitudes of two (non-zero) *forces does not depend on the choice of the unit.*

On the basis of this, from the magnitude of a force given in terms of a certain unit, we can determine its magnitude in terms of a different unit.

We represent the force acting on a body as a *vector*. With this in view, we select an arbitrary unit of length and an arbitrary unit of force. The given force is represented by a vector whose length is expressed by the same number as the magnitude of the force, while the origin, direction and sense are the same as the origin, direction and sense of the force. For example, a vector having five units of length represents a force having five units of force. A zero force represents a zero vector.

Operations on forces are defined as operations on vectors which represent these forces. For instance, if $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ represent certain forces, then the sum of these forces is the force which is represented by the vector $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$.

The moment of a force (represented by the vector $\mathbf{P}$) with respect to a point O is defined as the moment of the vector $\mathbf{P}$ with respect to O.

Material point. In dynamics we shall be concerned at first with the motion of points, and afterwards with the motion of bodies. As in kinematics, we shall sometimes regard a point as a model of the body (e. g. in the case when the dimensions of the body are small in comparison with the path).

The mass of a point is defined as the mass of the body which the given point represents; the point itself is then termed a *material point.*

If a force acts on a body whose image is the material point A, then this force is represented in the form of a vector whose initial point is at A.

The force acting on a material point can change with time in magnitude as well as in direction and sense.

§ 2. Newton's laws of danymics.

The laws of dynamics stated by Newton give the relations that obtain in absolute motion among the mass, acceleration and forces that act upon a material point.

Laws of motion. Let the frame of reference (x, y, z) be an inertial frame, and let t denote absolute time. Under these assumptions the laws of motion can be stated as follows:

I. *If m denotes the mass of a material point, $\mathbf{p}$ the acceleration at the moment t, $\mathbf{P}$ the sum of the forces acting on the material point at the moment t, then*

$$m\mathbf{p} = K\mathbf{P}, \tag{1}$$

where K denotes a certain number (positive), depending *only on the choice of units* of length, time, mass and force (and hence independent of the time, mass and force).

From equation (1) it follows that

$$m|\mathbf{p}| = K|\mathbf{P}|. \tag{2}$$

Therefore, if of the units mentioned, three are selected arbitrarily, the fourth can be so chosen that $K = 1$. For instance, select arbitrary units of time, length and mass, and for the unit of force select a force which gives a point of mass 1 an acceleration 1. For $m = 1$ and $|\mathbf{P}| = 1$ we have with these units $|\mathbf{p}| = 1$, and hence from formula (2) we obtain $K = 1$. Relation (1) then assumes the following form

$$m\mathbf{p} = \mathbf{P}. \tag{I}$$

Henceforth we shall always assume that the units are so chosen that $K = 1$. Newton's law will therefore always be taken in the form (I).

Forming the projections on the axes of the frame of reference, we obtain in virtue of (I):

$$mp_x = P_x, \; mp_y = P_y, \; mp_z = P_z. \tag{II}$$

Equations (I) and (II) are obviously equivalent.

Since $\mathbf{p} = \mathrm{d}\mathbf{v} / \mathrm{d}t$ and m is a constant, $m\mathbf{p} = \mathrm{d}(m\mathbf{v}) / \mathrm{d}t$. Hence relation (I) can also be written in the form

$$\mathrm{d}(m\mathbf{v}) / \mathrm{d}t = \mathbf{P}. \tag{III}$$

The vector $m\mathbf{v}$ is called the *momentum* (*quantity of motion*).

Therefore: *the derivative of the momentum* (with respect to time) *is equal to the sum of the forces acting on a material point.*

On a material point of mass m let there act forces whose sum $\mathbf{P}$ is constantly zero during a certain period of time. Then $m\mathbf{p} = 0$; hence the acceleration $\mathbf{p} = 0$, and consequently the point moves with uniform motion along a straight line (or is at rest). We therefore have the following law, known as *Newton's law of inertia*:

II. *If forces whose sum is zero act on a material point during a certain period of time, then the point is either at rest or in uniform motion along a straight line.*

Conversely, if a point is at rest or in uniform motion along a straight line, then the acceleration $\mathbf{p} = 0$, and since by (I) $m\mathbf{p} = \mathbf{P}$, it follows that the sum of the acting forces $\mathbf{P}$ is zero.

The forces acting on a material point usually arise from the action of other material points on the given point. For such forces Newton gave the following law, known as the *law of action and reaction*:

III. *If a material point A acts on a material point B with a certain force, then the point B also acts on the point A with a force equal in magnitude and direction, but opposite in sense; the forces which the points A and B exert on each other are always directed along the straight line AB joining these points.*

If the force which the point A exerts on the point B has a sense towards A (and hence the force which the point B exerts on A has a sense towards B), then we say that the points A and B *attract each other*; in the opposite case we say that the points *repel each other*.

Equilibrium of a point and forces. If a material point is at rest, then it is said to be in *equilibrium*. The forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ are said to be in *equilibrium*, or to *balance each other*, if their sum is zero, i. e. if $\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n = 0$.

Therefore, if a material point is in equilibrium, then the forces acting on this point are also in equilibrium. On the other hand, if the forces acting on a material point during a certain period of time are in equilibrium, then in this period of time the acceleration $\mathbf{p} = 0$, and hence the point is either in equilibrium or in uniform motion along a straight line.

Force of inertia. D'Alembert's principle. Law (I) can also be written in the form

$$\mathbf{P} + (-m\mathbf{p}) = 0. \tag{IV}$$

A vector $-m\mathbf{p}$ whose origin is at the point m is called a *force of inertia*.

One must not suppose that the vector $-m\mathbf{p}$ represents the force acting on the material point m. It is only for the sake of convenience that we call this vector a force (of inertia). Only forces whose sum is $\mathbf{P}$ act on the point m.

Relation (IV) can be stated as follows:

The forces acting on a material point are in equilibrium with the force of inertia.

The above formulation is equivalent to Newton's law I and is called *d'Alembert's principle*. This principle is very useful in connection with the investigation of the motion of the so-called constrained points.

§ 3. Systems of dynamical units. The fundamental units used in dynamics are the units of length, time and mass. By means of these units we define the unit of force. As a unit of force we select a force which gives to a mass 1 an acceleration 1.

C. g. s. system. In this system the unit of length is the *centimeter* (cm), the unit of mass the *gram* (g), the unit of time the *second* (sec) and that of force the *dyne*.

At first the meter (m = 100 cm) was to represent one forty millionth part of the earth's meridian. A small error, however, was made in the calculations. Today a meter is defined as the length of a certain bar preserved in Paris. Similarly, the unit of mass 1 g was at first to be the mass of 1 cm^3 of chemically pure water at 4° C under a pressure of 760 mm of mercury. At present, however, we take as 1 kilogram (kg = 1000 g) the mass of a certain block of platinum preserved in Paris.

The unit of time 1 sec is defined by means of the so-called mean solar day whose determination belongs to the subject of astronomy. The mean solar day = 24 hours (h), an hour = 60 minutes (min), a minute = = 60 sec.

The unit of force 1 dyne is the force which will impart an acceleration of 1 cm/sec^2 to a mass of 1 g.

The system of fundamental units (centimeter, gram, second) is called briefly the *c. g. s. system*.

Measurement of masses and forces. In the vicinity of the earth small freely falling bodies drop to the earth vertically with a uniformly accelerated motion (if air resistance is neglected). This acceleration (termed *gravitational*) is the same for all bodies at a given place on earth, but it changes with latitude. It is denoted by g. We shall take the gravitational acceleration to be in average $g = 981$ cm/sec^2.

Let m denote the mass of a small body. The force directed vertically downwards and of magnitude $Q = mg$ is called the *weight* of this body.

Weight is therefore proportional to the mass of the body; bodies having equal weights (at the same place on earth) have equal masses and conversely.

By means of an instrument called a *balance* (with whose principle we shall be acquainted in chapter VI) we can compare the weights of two bodies. Since the equality of masses follows from the equality of weights, the masses of bodies can be compared indirectly by means of a balance. Hence it follows that with the aid of a balance we can *measure*, i. e. we can determine the *masses* of bodies.

Forces are measured by *dynamical* or *statical* methods.

The dynamical method rests on Newton's first law ($\mathbf{P} = m\mathbf{p}$). From this formula we can determine the force $\mathbf{P}$ if we know the mass of the body m and the acceleration $\mathbf{p}$ which is imparted to it by $\mathbf{P}$.

The statical method is based on the fact that bodies change their shape (become deformed) when acted upon by forces. From a knowledge of the deformations we can infer in certain cases the magnitudes of the forces which cause these deformations. For instance, if a force directed vertically downwards acts at the lower end of a spring hanging vertically, then the spring becomes elongated. When the forces are small the elongation is proportional to the magnitude of the acting force. Instruments which serve to measure forces by statical methods are called *dynamometers*.

Metric gravitational system of units. In engineering the so-called *metric gravitational system of units* is generally used. In this system we assume as fundamental units the units of length, time and force. The unit of length is 1 m, of time 1 sec, and of force 1 kilogram (kg). This is the weight of 1 dcm^3 of water (under normal conditions) at a latitude of 45° north (where the gravitational acceleration $g = 981$ cm/sec^2 = = 9.81 m/sec^2).

If in the formula $|\mathbf{P}| = m|\mathbf{p}|$ we put $|\mathbf{P}| = 1$ and $|\mathbf{p}| = 1$, we obtain $m = 1$. Therefore the unit of mass will be a mass to which a force of 1 kg imparts an acceleration of 1 m/sec^2.

Let m be the mass of a body, Q its weight (at a latitude of 45° N) and let $g = 9.81$ m/sec^2. Then $Q = mg$, and therefore

$$m = Q \,/\, g = Q \,/\, 9.81. \tag{1}$$

From the above formula we can determine the mass of a body in terms of metric gravitational units when we know the weight of the body. Since the weight of 9.81 dcm^3 of water (at a latitude of 45° N) is 9.81 kg, the unit of mass in the metric gravitational system represents the mass of 9.81 dcm^3 of water.

In the c. g. s. system the mass of 9.81 dcm^3 of water is 9.81 kg (of mass) = 9810 g (of mass). Hence:

The unit of mass in the metric gravitational system = 9.81 kg (of mass) = 9810 g (of mass). (2)

In order to find the relation between the unit of force (kg) in the metric gravitational system and the unit of force (dyne) in the c. g. s. system, let us note that 1 dcm^3 of water (i. e. a mass of 1000 g) falls to the earth under the influence of its own weight of 1 kg with an acceleration of 981 cm/sec². Consequently 1 kg (of force) = $1000 \cdot 981$ dynes, whence

$$1 \text{ kg (of force)} = 981\,000 \text{ dynes.} \tag{3}$$

Dimensions of dynamical magnitudes. In dynamics there occur still other magnitudes (e. g. work, kinetic energy, etc.) whose units are defined in the same way as the unit of force by means of the fundamental units, i. e. length, mass and time. Similarly as for kinematic magnitudes (cf. Chap. II, § 11), we can introduce the notion of dimension for dynamical magnitudes. A knowledge of the dimension of a given magnitude enables one to determine easily the measure of this magnitude when the fundamental units are changed.

Suppose that we have chosen two systems of units of length, mass and time, which we denote respectively by L, M, T and L', M', T' and that these units are related as follows

$$L = \lambda L', \quad M = \mu M', \quad T = \tau T'. \tag{4}$$

Let the measure of some dynamical magnitude A be a in terms of the units L, M, T; and a' in terms of the units L', M', T'.

If it is possible to choose numbers α, β, γ such that for every two systems of units L, M, T and L', M', T' satisfying relations (4) there exists a relation of the form

$$a' = a\lambda^{\alpha}\mu^{\beta}\tau^{\gamma}, \tag{5}$$

then the *dimension* of the magnitude A is defined by the expression

$$L^{\alpha}M^{\beta}T^{\gamma}. \tag{6}$$

The dimension of the magnitude A is denoted by $[A]$, and the unit of the magnitude A in terms of the units L, M, T is represented by the symbol $L^{\alpha}M^{\beta}T^{\gamma}$.

The magnitude of A is therefore $aL^{\alpha}M^{\beta}T^{\gamma}$ in terms of the units L, M, T, and $a'L'^{\alpha}M'^{\gamma}T'^{\beta}$ in terms of the units L', M', T', whence

$$aL^{\alpha}M^{\beta}T^{\gamma} = a'L'^{\alpha}M'^{\beta}T'^{\gamma}. \tag{7}$$

Making use of formulae (4) and calculating formally, we obtain $aL^{\alpha}M^{\beta}T^{\gamma} = a(\lambda L')^{\alpha}(\mu M')^{\beta}(\tau T')^{\gamma}$, whence

$$MaL^{\alpha\beta}T^{\gamma} = (a\lambda^{\alpha}\mu^{\beta}\tau^{\gamma})\, L'^{\alpha}M'^{\beta}T'^{\gamma}. \tag{8}$$

By equating (7) and (8) we get (5). In this manner by means of formal reckoning we can, if we know the dimension of the magnitude A, obtain its measure when the units of length, mass and time are changed.

It is easy to generalize the theorem given on the page 51, which is very useful for the determination of the dimension.

Example 1. A force of magnitude (absolute value) P acting on a material point of mass m imparts to it an acceleration of magnitude p. Therefore $P = mp$, whence $[P] = [m] \cdot [p]$. Since $[m] = M$, and $[p] = LT^{-2}$,

$$[\text{force}] = LMT^{-2}. \tag{I}$$

The unit of force in the c. g. s. system is the dyne. Hence

$$\text{dyne} = \text{cm} \cdot \text{g} \cdot \text{sec}^{-2}. \tag{II}$$

Example 2. Represent a force of magnitude $6 \text{ m} \cdot \text{kg} \cdot \text{min}^{-2}$ in the c. g. s. system.

We have

$$6 \text{ m} \cdot \text{kg} \cdot \text{min}^{-2} = 6\,(100 \text{ cm})(1000 \text{ g})(60 \text{ sec})^{-2} = (6 \cdot 100 \cdot 1000 \cdot 60^{-2})(\text{cm} \cdot \text{g} \cdot \text{sec}^{-2}) = 166\tfrac{2}{3} \text{ dynes.}$$

§ 4. Equations of motion. One of the principal problems of dynamics is the determination of the motion of a material point when the mass m of this point and the force **P** acting on it are given. In the simplest case the force **P** can be given as a function of time, i. e. there are given functions:

$$P_x = F(t), \quad P_y = \Phi(t), \quad P_z = \Psi(t), \tag{1}$$

defining at each moment t (of a certain period of time) the projections of the force **P** on the coordinate axes.

We shall meet with cases, however, which are more complicated. It may happen that some region D has the property that a certain force **P** acts on a given material point situated anywhere within the region D.

If the force **P** depends only on the position of the point and does not depend on anything else (e. g. velocity), then the region D is called a *force field.*

An example of a force field is the earth's gravitational field: for on a given material point situated near the earth there acts the force of gravity which depends on the position of this point (and does not depend on the velocity).

In a force field the force **P** is therefore a function of the coordinates x, y, z of the given point. A field is defined if there are given functions:

$$P_x = F(x, y, z), \quad P_y = \Phi(x, y, z), \quad P_z = \Psi(x, y, z), \tag{2}$$

determining at each point of the field the projections of the force **P**. Therefore, if we are investigating the motion of a point in a force field, then we are dealing with a force which depends on the position of the point.

If a material point moves in a certain medium (e. g. in air), then, in addition to other forces acting on the material point, there is also the force of resistance which the medium offers in opposing the motion. This force depends, among other things, on the velocity of the material point. In this case we therefore have a force depending also on the velocity of the point.

In the most general case we shall assume that the force **P** depends on the time, position and velocity of the point. We shall therefore assume that the force **P** acting on a material point is defined by the functions:

$$P_x = F(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \quad P_y = \Phi(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \quad P_z = \Psi(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \tag{3}$$

whose values are the projections of this force which depend on the coordinates of the position of the point (x, y, z), its velocity $x^{\cdot}, y^{\cdot}, z^{\cdot}$, and on the time t.

Functions (3) *are usually assumed to be continuous and to have continuous partial derivatives in a certain region of the variables* $x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t$.

Obviously, in particular problems the force **P** does not have to depend on all the variables $x, y, \ldots, t$ it can be independent of some of them.

Theoretically, the force **P** can depend on higher derivatives (e. g. on the second, third, etc.) of the variables x, y, z. However, such cases are not encountered in practical problems and we shall not consider them here.

Let the motion of a material point be given by the functions:

$$x = f(t), \quad y = \varphi(t), \quad z = \psi(t). \tag{4}$$

In virtue of equations (II), p. 72, we obtain

$$mx^{\cdot\cdot} = P_x, \quad my^{\cdot\cdot} = P_y, \quad mz^{\cdot\cdot} = P_z. \tag{I}$$

If we assume that P_x, P_y, P_z are functions of the form (3), then equations (I) become

$$\begin{aligned} mx^{\cdot\cdot} &= F(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \\ my^{\cdot\cdot} &= \Phi(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \\ mz^{\cdot\cdot} &= \Psi(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t). \end{aligned} \tag{II}$$

These equations represent a system of differential equations of the second order, while the sought for functions are the functions (4).

Let us suppose that we are investigating a motion in the neighbourhood of a certain moment t_0. Assume that at the moment t_0 the point had coordinates x_0, y_0, z_0, and its velocity had projections $x^{\cdot}_0, y^{\cdot}_0, z^{\cdot}_0$. In addition, let us assume that the functions (3) are continuous, possessing continuous partial derivatives in the neighbourhood of the values $x_0, y_0, z_0, x^{\cdot}_0, y^{\cdot}_0, z^{\cdot}_0, t_0$.

From the theory of differential equations it is known that under the preceding assumptions there exists one and only one set of functions (4) continuous together with its first and second derivatives in the neighbourhood of the moment t_0, satisfying equations (II) as well as the relations:

$$f(t_0) = x_0, \quad \varphi(t_0) = y_0, \quad \psi(t_0) = z_0; \quad f'(t_0) = x^{\cdot}_0, \quad \varphi'(t_0) = y^{\cdot}_0, \quad \psi'(t_0) = z^{\cdot}_0. \tag{5}$$

This system, as the only system of functions (4) satisfying all the required conditions, therefore determines the motion of a material point having at the moment t_0 the coordinates x_0, y_0, z_0 and a velocity whose projections are $x^{\cdot}_0, y^{\cdot}_0, z^{\cdot}_0$.

We see from this that *the motion is completely determined when the mass of a point, the forces acting on it and the so-called initial conditions* (i. e. its position and velocity at the initial moment t_0) *are given.*

Equations (II) are called *Newton's laws of motion.*

Example. The force depends only on the time. Let the force **P** depend only on the time and be given by functions (1). The equations of motion (II) will therefore have the form:

$$mx^{\cdot\cdot} = F(t), \quad my^{\cdot\cdot} = \Phi(t), \quad mz^{\cdot\cdot} = \Psi(t).$$

Dividing by m and integrating both sides, we obtain for $t_0 = 0$:

$$x^{\cdot} = \frac{1}{m}\int_0^t F(t)\,dt + c_1, \quad y^{\cdot} = \frac{1}{m}\int_0^t \Phi(t)\,dt + c_2, \quad z^{\cdot} = \frac{1}{m}\int_0^t \Psi(t)\,dt + c_3.$$

Let us assume that at $t = 0$, $x^{\cdot} = x^{\cdot}_0$, $y^{\cdot} = y^{\cdot}_0$, $z^{\cdot} = z^{\cdot}_0$ (initial conditions). Substituting $t = 0$ in the above equations, we get $c_1 = x^{\cdot}_0$, $c_2 = y^{\cdot}_0$, $c_3 = z^{\cdot}_0$. Hence

$$x^{\cdot} = F_1(t) + x^{\cdot}_0, \quad y^{\cdot} = \Phi_1(t) + y^{\cdot}_0, \quad z^{\cdot} = \Psi_1(t) + z^{\cdot}_0, \tag{6}$$

where

$$F_1(t) = \frac{1}{m}\int_0^t F(t)\,dt, \text{ etc.}$$

Integrating equations (6), we obtain:

$$x = \int_0^t F_1(t)\,dt + x^{\cdot}_0 t + c_1', \quad y = \int_0^t \Phi_1(t)\,dt + y^{\cdot}_0 t + c_2', \quad z = \int_0^t \Psi_1(t)\,dt + z^{\cdot}_0 t + c_3'. \tag{7}$$

Let us now suppose that at $t = 0$, $x = x_0$, $y = y_0$, $z = z_0$. Therefore from equations (7), putting $t = 0$, we get $c_1' = x_0$, $c_2' = y_0$, $c_3' = z_0$. Hence

$$x = F_2(t) + x^{\cdot}_0 t + x_0,\quad y = \Phi_2(t) + y^{\cdot}_0 t + y_0,\quad z = \Psi_2(t) + z^{\cdot}_0 t + z_0, \tag{8}$$

where

$$F_2(t) = \int_0^t F_1(t)\,dt, \text{ etc.}$$

From equations (8) it follows that the motion will be defined if at the initial moment $t = 0$ the position of the point (i. e. x_0, y_0, z_0) and the initial velocity (i. e. $x^{\cdot}_0, y^{\cdot}_0, z^{\cdot}_0$) are given.

§ 5. Motion under the influence of the force of gravity. Let a force **P** of constant magnitude, direction and sence act on a material point of mass m.

We have to deal with this situation — when investigating the motion of small bodies near the earth and taking as the inertial frame a frame attached to the earth. If air resistance is neglected, then the only force acting on a projected body is the force of gravity which can be considered constant over a small region.

Let **P** denote the force of gravity. Then $|\mathbf{P}| = mg$ (g is the acceleration due to gravity). Let us select a frame (x, y, z) so that the sense of the z-axis is vertically upward. Then

$$P_x = 0,\quad P_y = 0,\quad P_z = -mg.$$

Newton's laws of motion (p. 72, formulae (II)) become: $mx^{\cdot\cdot} = 0$, $my^{\cdot\cdot} = 0$, $mz^{\cdot\cdot} = -mg$, or

$$x^{\cdot\cdot} = 0,\quad y^{\cdot\cdot} = 0,\quad z^{\cdot\cdot} = -g. \tag{1}$$

Integrating the above equations, we obtain:

$$x^{\cdot} = c_1,\quad y^{\cdot} = c_2,\quad z^{\cdot} = -gt + c_3. \tag{2}$$

Integrating once more, we get:

$$x = c_1 t + c_1',\quad y = c_2 t + c_2',\quad z = \tfrac{1}{2}gt^2 + c_3 t + c_3'. \tag{3}$$

The numbers $c_1, c_2, c_3, c_1', c_2', c_3'$ denote constants of integration which we shall determine from known initial conditions, i. e. the coordinates x_0, y_0, z_0 and the projections $x^{\cdot}_0, y^{\cdot}_0, z^{\cdot}_0$ of the velocity $\mathbf{v}_0$ of the moving point at the initial moment t_0. Without loss of generality we can assume that $t_0 = 0$; moreover (by selecting a system of coordinates suitably) we can assume that at $t_0 = 0$ the point was situated at the origin of the system and that the velocity $\mathbf{v}_0$ lay in the vertical plane zx.

We are therefore assuming that at $t = 0$, $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ and $y^{\cdot}_0 = 0$.

Substituting $t = 0$ in equations (2) and (3), we obtain

$$c_1 = x^{\cdot}_0,\ c_2 = y^{\cdot}_0 = 0,\quad c_3 = z^{\cdot}_0;\ c_1' = x_0 = 0,\quad c_2' = y_0 = 0,\quad c_3' = z_0 = 0.$$

Equations (2) and (3) hence take on the form:

$$x^{\cdot} = x^{\cdot}_0,\quad y^{\cdot} = 0,\quad z^{\cdot} = -gt + z^{\cdot}_0, \tag{2'}$$

$$x = x^{\cdot}_0 t,\quad y = 0,\quad z = -\tfrac{1}{2}gt^2 + z^{\cdot}_0 t. \tag{3'}$$

Since $y = 0$ constantly, the motion takes place in the vertical zx-plane.

We shall examine two cases: the so-called vertical projection and the oblique projection.

Vertical projection. Let us assume that at the moment $t = 0$ the velocity $\mathbf{v}_0$ was directed vertically (or was zero), and hence that $x^{\cdot}_0 = 0$. Putting $z^{\cdot} = v$ and $z^{\cdot}_0 = v_0$, we obtain from (2′) and (3′)

$$x^{\cdot} = 0,\ y^{\cdot} = 0,\ x = 0,\ y = 0, \tag{4}$$

$$v = -gt + v_0,\ z = -\tfrac{1}{2}gt^2 + v_0 t. \tag{5}$$

Since $x = 0$ and $y = 0$ constantly, the point moves along the z-axis, i. e. along a vertical. Moreover, we have $v^{\cdot} = p = -g$.

Therefore: *if the initial velocity has a vertical direction (or is zero), then a point moves with a uniformly accelerated motion along a vertical under the influence of the force of gravity.*

Assume that $v_0 > 0$, i. e. that at the initial moment the velocity had an upward sense (e. g. that we had projected the point vertically upwards with a velocity v_0). Let us denote the height of the projection by h, i. e. the maximum elevation the point will attain. In order to obtain h it is necessary to calculate the maximum of the function $z = -\tfrac{1}{2}gt^2 + v_0 t$. We get

$$h = v_0^2 / 2g \text{ at the moment } t = v_0 / g. \tag{6}$$

Oblique projection. Let us assume that the velocity $\mathbf{v}_0$ ($\neq 0$) makes an angle $\alpha \neq \pm\tfrac{1}{2}\pi$ with the x-axis. Setting $|\mathbf{v}_0| = v_0$, we get $x^{\cdot}_0 = v_0 \cos\alpha$, $z^{\cdot}_0 = v_0 \sin\alpha$. Therefore in virtue of (2′) and (3′)

$$x^{\cdot} = v_0\cos\alpha,\quad y^{\cdot} = 0,\quad z^{\cdot} = -gt + v_0\sin\alpha, \tag{7}$$

$$x = v_0 t\cos\alpha,\quad y = 0,\quad z = -\tfrac{1}{2}gt^2 + v_0 t\sin\alpha. \tag{8}$$

Since $\cos\alpha \neq 0$ and $v_0 \neq 0$, by the first of the equations (8) $t = v / v_0 \cos\alpha$, whence

$$z = -\frac{g}{2v_0^2\cos^2\alpha}x^2 + x\tan\alpha. \tag{9}$$

This equation is that of a parabola.

Hence: *a point projected obliquely moves along a parabola.*

The parabola cuts the x-axis in the points O and D. The length of the segment $OD = d$ is called the *range of the projection.*

In order to calculate d we substitute $z = 0$ in (9). We obtain

$$d = \frac{v_0^2}{g} \sin 2\alpha. \tag{10}$$

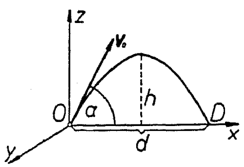

Fig. 70.

Therefore: *the maximum range of a projection with a given velocity v_0 occurs for the angle $\alpha = \frac{1}{4}\pi = 45°$.*

In order to obtain the height h it is necessary to calculate the maximum of the function (9). We get

$$h = v_0^2 \sin^2\alpha \,/\, 2g \quad \text{for} \quad x = v_0^2 \sin 2x \,/\, 2g. \tag{17}$$

§ 6. Motion in a resisting medium. A material point moving in a medium such as air, for instance, encounters resistance. Experiments show that air resistance can be expressed by a force depending only on the velocity of the point (for bodies the resistance also depends on the shape of the body). Resistance has the direction of the velocity, but an opposite sense. The magnitude of the resistance depends on the magnitude of the velocity, but not on its direction. Let us denote the magnitude of the resistance by Γ and the magnitude of the velocity by v. We can therefore write

$$\Gamma = f(v).$$

The function f is an increasing function with $f(0) = 0$. For velocities less than the velocity of sound (which is 333 m/sec in air) we can assume with great accuracy that

$$\Gamma = \lambda v^2, \tag{1}$$

where λ is a factor depending on the temperature and density of the air.

Vertical projection. Let us investigate the case of the falling point. Assume that the point falls along the z-axis to which we give a sense vertically downward. Consequently the component of the velocity $z^{\cdot} = v > 0$. The resistance is directed upwards and hence its projection on the z-axis is negative. Assuming that the magnitude of the resistance is expressed by formula (1), by putting $\lambda = km$ we obtain:

$$mz^{\cdot\cdot} = m\frac{dv}{dt} = mg - kmv^2. \tag{2}$$

From this

$$\frac{dv}{g - kv^2} = dt,$$

and hence

$$\int \frac{dv}{g - kv^2} = \int dt = t. \tag{3}$$

Setting

$$v_\infty = \sqrt{\frac{g}{k}},$$

we get

$$\int \frac{dv}{g - kv^2} = \frac{1}{k}\int \frac{dv}{v_\infty^2 - v^2} = \frac{1}{2kv_\infty}\ln\frac{v_\infty + v}{v_\infty - v} + c,$$

where c is the constant of integration. Hence by (3)

$$\frac{1}{2kv_\infty}\ln\frac{v_\infty + v}{v_\infty - v} + c = t. \tag{4}$$

Let us assume that at the initial moment $t = 0$ the velocity was $v = 0$. Substituting $v = 0$ and $t = 0$ in formula (4), we get $c = 0$. Consequently

$$v = \frac{e^{2kv_\infty t} - 1}{e^{2kv_\infty t} + 1} v_\infty = \left(1 - \frac{2}{e^{2kv_\infty t} + 1}\right) v_\infty. \tag{5}$$

From formula (5) it follows that v is always less than v_∞.

Hence: *the velocity of a point falling vertically in a resisting medium does not increase infinitely, but is always less than the limiting velocity v_∞.*

Oblique projection. Let us now assume that the point moves in the vertical plane xz. Let the magnitude of the resistance in all generality be given by $\Gamma = f(v)$. Denoting the resistance by Γ, we therefore obtain

$$\Gamma = -\frac{f(v)}{v}\mathbf{v}, \quad \text{whence} \quad \Gamma_x = -\frac{f(v)}{v}v_x \quad \text{and} \quad \Gamma_z = -\frac{f(v)}{v}v_z.$$

Therefore

$$\Gamma_x = -\frac{f(v)}{v}x^{\cdot}, \quad \Gamma_z = -\frac{f(v)}{v}z^{\cdot}.$$

The equations of motion will hence have the form

$$mx^{\cdot\cdot} = -\frac{f(v)}{v}x^{\cdot}, \quad mz^{\cdot\cdot} = mg - \frac{f(v)}{v}z^{\cdot}, \quad v = \sqrt{x^{\cdot 2} + z^{\cdot 2}}.$$

The science of exterior ballistics is concerned with the solution of the preceding equations. This problem is very difficult because the values of the function $f(v)$ are known only from measurements.

§ 7. Moment of momentum. Let a material point A of mass m move with a velocity $\mathbf{v}$. We have called the vector $m\mathbf{v}$ whose origin is at A the *momentum* or the *quantity of motion* (p. 72). If x, y, z are the coordinates of the point A, then the projections of the momentum on the coordinate axes are $m\dot{x}, m\dot{y}, m\dot{z}$, respectively.

Let $\mathbf{K}$ denote the moment of the momentum $m\mathbf{v}$ with respect to the origin of the coordinate system. Then (p. 18, formula (II)):

$$K_x = m(\dot{y}z - \dot{z}y), \quad K_y = m(\dot{z}x - \dot{x}z), \quad K_z = m(\dot{x}y - \dot{y}x). \tag{1}$$

Take the derivative (with respect to time) of the moment of momentum. We obtain:

$$\dot{K}_x = m(\ddot{y}z - \ddot{z}y), \quad \dot{K}_y = m(\ddot{z}x - \ddot{x}z), \quad \dot{K}_z = m(\ddot{x}y - \ddot{y}x). \tag{2}$$

Assume that the frame of reference is an inertial frame and that a force $\mathbf{P}$ acts on the point A. Then $m\ddot{x} = P_x$, $m\ddot{y} = P_y$, $m\ddot{z} = P_z$, whence by (2)

$$\dot{K}_x = P_y z - P_z y, \quad \dot{K}_y = P_z x - P_x z, \quad \dot{K}_z = P_x y - P_y x. \tag{3}$$

The expressions on the right hand sides of equations (3) represent the moments of the force $\mathbf{P}$ with respect to the axes of the frame. Therefore, denoting by $\mathbf{M}$ the moment of the force $\mathbf{P}$ with respect to the origin of the frame, we have by (3)

$$\dot{K}_x = M_x, \quad \dot{K}_y = M_y, \quad \dot{K}_z = M_z. \tag{4}$$

The above equations can be written as one vector equation:

$$\dot{\mathbf{K}} = \mathbf{M}. \tag{5}$$

The origin of the frame could have been chosen arbitrarily.

Hence: *the derivative of the moment of momentum with respect to an arbitrary fixed point is equal to the moment of the acting force with respect to this point.*

From equations (4) it also follows that *the derivative of the moment of momentum with respect to an arbitrary fixed axis is equal to the moment of the force with respect to this axis.*

Principle of conservation of areas. Let us suppose that the moment of the force $\mathbf{P}$ with respect to a certain axis l is constantly zero; therefore either the line on which the force $\mathbf{P}$ lies cuts the l-axis, or the force $\mathbf{P}$ is parallel to that axis. Let us choose the l-axis as the z-axis. Hence we have $M_z = 0$. In virtue of (4), $\dot{K}_z = 0$ or $K_z = \text{const}$. From this and (1) we obtain

$$m(\dot{x}y - \dot{y}x) = \text{const}, \quad \text{or} \quad \dot{x}y - \dot{y}x = \text{const}. \tag{6}$$

Let A' be the projection of the point A on the xy-plane. The point A' has the coordinates x, y. Therefore the areal velocity (p. 47) of A' is $-\frac{1}{2}(\dot{x}y - \dot{y}x)$. From formula (6) it follows that the areal velocity of the point A' is constant.

Therefore: *if the moment of a force with respect to a certain axis is constantly zero, then the moment of momentum of the motion with respect to this axis is constant, and the areal velocity of the projection of the motion on a plane perpendicular to this axis is constant.*

This theorem is called *the principle of conservation of moment of momentum* or *the principle of conservation of areas.*

§ 8. Central motion. If a material point moves in such a manner that its acceleration at each moment is directed along a line passing through a certain fixed point O, then the motion of the point is called a *central motion* and the point O the *centre of motion.*

For instance, the uniform motion of a point along the periphery of a circle is a central motion because the acceleration is constantly directed towards the centre of the circle which in this case is the centre of motion (p. 43).

Since the acceleration has the direction of the force acting on the material point, the line of action of the force in central motion passes through the centre of motion.

A force field in which the lines of action of the forces pass through a certain fixed point O is called a *central field* and the point O the *centre of the field.*

A point of mass M situated motionless at a fixed point O and attracting another point of mass m with a force depending only on mutual distance of these points forms a force field. This field is a central field because the force acting on the point m has — according to the law of action and reaction (p. 73, III) — a line of action passing through the point M.

The material point moves in a central field with central motion; the centre of motion obviously lies at the centre of the field.

Let us choose the origin of the coordinate system at the centre of the field. Since the line of action of the force constantly passes through the origin of the system, its moment with respect to each axis is zero. By the principle of conservation of areas the motion of the projection of the point on each of the coordinate planes has therefore a constant areal velocity, consequently:

$$\dot{y}z - \dot{z}y = a, \quad \dot{z}x - \dot{x}z = b, \quad \dot{x}y - \dot{y}x = c, \tag{1}$$

where a, b, c are certain constants. Multiplying the first equation by x, the second by y, the third by z, and adding, we obtain

$$ax + by + cz = 0. \tag{2}$$

Hence we see that the coordinates of the point constantly satisfy equation (2). This is the equation of a plane passing through the origin of the system (i. e. through the centre of the field). Therefore the point moves in a plane passing through the origin of the system.

If we select x and y axes in this plane, then from (1) it follows that the areal velocity in the plane of motion is constant; the radius vectors consequently sweep out equal areas in equal times.

Therefore: *the path in a central motion is a plane path lying in a plane passing through the centre; the radius vectors emanating from the centre sweep out equal areas in equal times.*

We shall now prove the converse theorem:

If the path of a point is a plane path and the radius vectors emanating from a certain fixed point O (lying in the plane of the path) sweep out equal areas in equal times, then the line of action of the force constantly passes through the point O.

Proof. Choose the origin of the system at O and the x and y axes in the plane of the motion. The point therefore moves in the xy-plane. Since the areal velocity is constant, $x^{\cdot}y - y^{\cdot}x = \text{const}$. Differentiating both sides we get $x^{\cdot\cdot}y - y^{\cdot\cdot}x = 0$; hence $(mx^{\cdot\cdot})\,y - (my^{\cdot\cdot})\,x = 0$, whence

$$P_x y - P_y x = 0. \tag{3}$$

Since $z^{\cdot\cdot} = 0$, $P_z = mz^{\cdot\cdot} = 0$. The force $\mathbf{P}$ therefore lies in the xy-plane; in virtue of (3) the moment of the force $\mathbf{P}$ with respect to O is zero; hence the line of action of the force $\mathbf{P}$ passes through O, q. e. d.

Remark. Let us assume that the areal velocity in a central motion is zero. Then $x^{\cdot}y - y^{\cdot}x = 0$, or (in polar coordinates) $r^2\varphi^{\cdot} = 0$. It follows from this that either $r = 0$ constantly, i. e. that the point is at rest, or $\varphi^{\cdot} = 0$ constantly, or $\varphi = \varphi_0 = \text{const}$, i. e. that the point moves along a line passing through the centre (and inclined at an angle φ_0 with the x-axis).

Hence: *if the areal velocity in a central motion is zero, then the point moves along a straight line passing through the centre.*

On the other hand, if we assume that the areal velocity is different from zero, then $r^2\varphi^{\cdot} \neq 0$, or $r \neq 0$.

Therefore: *if the areal velocity in a central motion is different from zero, then the point never passes through the centre.*

Binet's formula. A point A of mass m moves in a central force field in the xy-plane with an areal velocity different from zero. Let us introduce the polar coordinates r, φ and denote by P the projection of the force $\mathbf{P}$ on the radius vector $\overline{OA}$. Then $P_x = P\cos\varphi$ and $P_y = P\sin\varphi$; hence $P_x\cos\varphi + P_y\sin\varphi = P$, whence

$$P = m(x^{\cdot\cdot}\cos\varphi + y^{\cdot\cdot}\sin\varphi). \tag{4}$$

Since $x = r\cos\varphi$, $y = r\sin\varphi$ (cf. p. 47, formula (2)),

$$x^{\cdot\cdot}\cos\varphi + y^{\cdot\cdot}\sin\varphi = r^{\cdot\cdot} - r\varphi^{\cdot 2},$$

whence in virtue of (4)

$$P = m(r^{\cdot\cdot} - r\varphi^{\cdot 2}). \tag{5}$$

Let us denote the areal velocity by $\frac{1}{2}c$. By assumption $\frac{1}{2}c \neq 0$. Since the areal velocity in polar coordinates is $\frac{1}{2}r^2\varphi^{\cdot}$ (p. 47),

$$r^2\varphi^{\cdot} = c, \quad \text{or} \quad \varphi^{\cdot} = c\,/\,r^2. \tag{6}$$

Suppose that the path has the equation $r = f(\varphi)$. Then

$$r^{\cdot} = \frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{c}{r^2} = -c\frac{d(1/r)}{d\varphi}, \tag{7}$$

and hence

$$r^{\cdot\cdot} = \frac{dr^{\cdot}}{dt} = \frac{dr^{\cdot}}{d\varphi}\frac{d\varphi}{dt} = -c^2\frac{d^2(1/r)}{d\varphi^2}\frac{1}{r^2}. \tag{8}$$

From formula (5) in virtue of (6) and (8) we obtain:

$$P = m\left(-\frac{c^2}{r^2}\frac{d^2(1/r)}{d\varphi^2} - \frac{c^2}{r^3}\right),$$

and therefore

$$P = -\frac{mc^2}{r^2}\left(\frac{d^2(1/r)}{d\varphi^2} + \frac{1}{r}\right). \tag{I}$$

The above formula is called *Binet's formula.*

This formula enables one to determine the force acting in a central motion if one knows the equation of the path. Conversely, knowing the force P as a function of r and φ, we can determine the path.

§ 9. Planetary motions. Kepler's laws. On the basis of observations Kepler gave the following three laws relating to planetary motions:

1. The planets describe ellipses with the sun at one focus.
2. The radius vectors emanating from the sun sweep out equal areas in equal times.
3. The squares of the periods of two planets are proportional to the

third powers of their mean distances from the sun (where by the mean distance is meant the semi-major axis of the ellipse along which a planet moves).

The third law is not quite exact. The reason for this we shall know later. Let us further observe that Kepler's laws are strictly kinematic.

Corollaries from Kepler's laws. From Kepler's laws Newton deduced (by means of dynamics) the law defining the forces which cause the motion of the planets. From the first two of Kepler's laws it follows that the planets move along plane paths with a constant areal velocity. Therefore, in virtue of the converse theorem on p. 86, the forces acting on the planets are central forces whose lines of action pass through the sun.

Let us select axes x, y in the plane of motion of the planet, placing the origin of the frame in the sun as the focus of the ellipse along which the planet revolves. As the direction of the x-axis let us choose the direction of the major axis of the ellipse, and give to the x-axis a sense such that the centre of the ellipse will lie on the negative side of the x-axis. Denote the major axis of the ellipse by $2a$, the minor axis by $2b$, and the distance between the foci by $2e$. The equation of the ellipse in polar coordinates will then have the form

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi}, \tag{1}$$

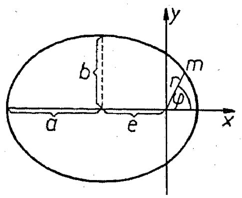

Fig. 71.

where

$$\varepsilon = \frac{e}{a} = \frac{\sqrt{a^2-b^2}}{a}. \tag{2}$$

From Binet's formula (p. 87, (I)) we can obtain the force acting on the planet. We have in virtue of (1)

$$\frac{1}{r} = \frac{1+\varepsilon\cos\varphi}{a(1-\varepsilon^2)}, \quad \text{whence} \quad \frac{d^2(1/r)}{d\varphi^2} = -\frac{\varepsilon\cos\varphi}{a(1-\varepsilon^2)},$$

and hence by (2) and Binet's formula

$$P = -\frac{mc^2a}{b^2r^2}. \tag{3}$$

The area of an ellipse is πab; denoting the period of the planet by T, and noting that $\frac{1}{2}c$ is the areal velocity, we obtain $\frac{1}{2}c = \pi ab / T$, or $c = 2\pi ab / T$. Therefore in virtue of (3) we get

$$P = -\frac{4\pi^2 ma^3}{r^2T^2}. \tag{4}$$

Since $P < 0$, the force is directed towards the sun.

By Kepler's third law we have for two planets $T^2 / T_1^2 = a^3 / a_1^3$, or $a^3 / T^2 = a_1^3 / T_1^2$. The ratio a^3 / T^2 therefore has a constant value for all planets. Putting

$$\mu = a^3 / T^2, \tag{5}$$

we obtain from (4)

$$P = -\frac{4\pi^2\mu m}{r^2}. \tag{6}$$

Hence: *the force under whose influence a planet moves is directed towards the sun, and is directly proportional (in magnitude) to the mass of the planet and inversely proportional to the square of the distance from the sun.*

Law of universal gravitation. The preceding result deduced from Kepler's laws suggested to Newton the supposition that the force acting on the planet is due to the mutual attraction of the planet and the sun. Newton generalized this reflection in the form of the law of *universal gravitation:*

Any two material points attract each other with forces whose magnitude is directly proportional to the product of the masses and inversely proportional to the square of the distance between them.

According to the law of action and reaction, the forces with which the material points attract each other are equal in magnitude, opposite in direction and act along the line joining these points. Denoting by m_1 and m_2 the masses of the points, by r the distance between them, and by P the magnitude of the force with which they attract each other, we therefore obtain

$$P = K\frac{m_1m_2}{r^2}, \tag{I}$$

where K is a certain constant, the so-called *gravitational constant* which depends only on the units of length, mass, and time.

From equation (I) we have $K = Pr^2 / m_1m_2$; consequently $[K] = [P][r]^2 / [m_1][m_2]$, and hence $[K] = L^3M^{-1}T^{-2}$. Measurements have shown that in the c. g. s. system:

$$K = 6.6 \cdot 10^{-8}\ \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{sec}^{-2}.$$

The gravitational constant can be measured by means of the so-called *Jolly's balance*. It is a balance having an upper and a lower pan on one side and a single pan on the other side. A body a of mass m is placed on the upper pan and balanced by a weight of mass m on the opposite pan.

Body a is next transferred to the lower pan; this will not disturb the equilibrium. However, if a body b of mass M is placed under the lower pan, then the balance will tilt. In order to restore equilibrium we must add a mass μ to the mass m.

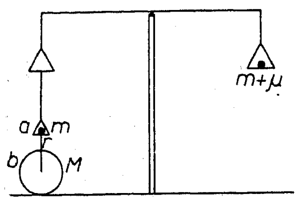

Fig. 72.

In the experiment the body b was a lead sphere. Since, as can be shown, a homogeneous sphere attracts an exterior point as if the entire mass of the sphere were concentrated at its centre, denoting by r the distance from the centre of the sphere to the body a, we have $KmM / r^2 = \mu g$, or

$$K = \mu g r^2 / mM.$$

Mass of the earth. It can be shown that a sphere composed of concentric layers of constant density attracts an exterior point as if the mass of the sphere were concentrated at its centre. Assuming that the earth satisfies the preceding condition, and denoting by M the mass of the earth, by R its radius and by Q the weight of a body of mass m (on the surface of the earth), we obtain $Q = KmM / R^2$. $Q = mg$, therefore $mg = KmM / R^2$, or

$$M = gR^2 / K. \tag{7}$$

Using $g = 9.81\ \text{m} \cdot \text{sec}^{-2}$, $R = 6300\ \text{km}$, $K = 6.6 \cdot 10^{-8}\ \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{sec}^{-2}$, we obtain (after changing m and km into cm)

$$M = 6 \cdot 10^{27}\ \text{g}.$$

The density of the earth is obtained from the formula

$$\varrho = M / \tfrac{4}{3}R^3\pi = 3g / 4KR\pi = 5.6\ \text{g/cm}^3.$$

Kepler's equation. We shall now determine the position of a planet at a given moment of time. Let us choose a system of coordinates in the plane of motion of the planet as on p. 88. In a rectangular coordinate system the ellipse along which a planet moves has the equation

$$(x + e)^2 / a^2 + y^2 / b^2 = 1.$$

Let us introduce an auxiliary angle u defined by the equations:

$$(x + e) / a = \cos u, \quad y / b = \sin u. \tag{II}$$

The angle u is called the *eccentric anomaly*.

Equations (II) define the angle u unambiguously. From (II) we get

$$x = a(\cos u - e / a), \quad y = b \sin u.$$

Substituting $\varepsilon = e / a$, $b = a\sqrt{1 - \varepsilon^2}$, we obtain

$$x = a(\cos u - \varepsilon), \quad y = a\sqrt{1 - \varepsilon^2} \sin u. \tag{8}$$

The radius vector r is obtained from the equation

$$r^2 = x^2 + y^2 = a^2(1 - \varepsilon \cos u)^2.$$

Therefore

$$r = a(1 - \varepsilon \cos u). \tag{III}$$

The angle φ which r makes with the x-axis is called the *true anomaly*.

From the equation of the ellipse in polar coordinates (p. 88, (1)) we get $r\varepsilon \cos \varphi = a(1 - \varepsilon^2) - r$; hence

$$r\varepsilon(1 + \cos \varphi) = (1 - \varepsilon)[a(1 + \varepsilon) - r],$$

whence by (III), $r(1 + \cos \varphi) = a(1 - \varepsilon)(1 + \cos u)$. Since $1 + \cos \varphi = 2 \cos^2 \frac{1}{2}\varphi$ and $1 + \cos u = 2 \cos^2 \frac{1}{2}u$,

$$\sqrt{r} \cos \tfrac{1}{2}\varphi = \sqrt{a(1 - \varepsilon)} \cos \tfrac{1}{2}u, \tag{IV}$$

and similarly

$$\sqrt{r} \sin \tfrac{1}{2}\varphi = \sqrt{a(1 + \varepsilon)} \sin \tfrac{1}{2}u;$$

whence

$$\tan \tfrac{1}{2}\varphi = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \tan \tfrac{1}{2}u. \tag{V}$$

Formulae (IV) and (V) determine φ unambiguously in terms of u.

Let us suppose that at the moment $t = 0$, $u = 0$, and hence $\varphi = 0$. The area of an ellipse is πab. If T denotes the period of revolution of a planet, then the areal velocity is $\pi ab / T$. Hence the radius vector sweeps out an area $\dfrac{\pi ab}{T} t$ during the time from 0 to t. This area can also be represented in the form of an integral

$$\frac{\pi ab}{T} t = \frac{1}{2} \int_0^{\varphi} r^2\, d\varphi. \tag{9}$$

Differentiating (V) we obtain

$$\frac{d\varphi}{\cos^2 \frac{1}{2}\varphi} = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \frac{du}{\cos^2 \frac{1}{2}u}.$$

Therefore by (IV)

$$d\varphi = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \cdot \frac{a(1 - \varepsilon)}{r} du = \frac{a\sqrt{1 - \varepsilon^2}}{r} du.$$

Substituting in (9) we obtain

$$\frac{\pi ab}{T} t = \frac{a\sqrt{1-\varepsilon^2}}{2} \int_0^u r\, du,$$

whence in virtue of (III),

$$\frac{\pi ab}{T} t = \frac{a^2\sqrt{1-\varepsilon^2}}{2} (u - \varepsilon \sin u).$$

From this and the fact that $a^2\sqrt{1-\varepsilon^2} = ab$, we get

$$u - \varepsilon \sin u = 2\pi t / T. \tag{VI}$$

The expression $2\pi t / T$ is called the *mean anomaly*.

Equation (VI) is called *Kepler's equation*.

By means of Kepler's equation we can determine u at each moment t, and then by equations (III), (IV), (V) the radius vector r and angle φ. Astronomy gives numerous methods for solving Kepler's equation.

In astronomy the eccentric anomaly u is usually denoted by the letter E, the true anomaly φ by v, and the mean anomaly $2\pi t / T$ by M.

§ 10. Work. Suppose that a material point was displaced from point A to B and that during this displacement a force **P** (there can be other forces besides) acted on it.

Constant force. Let us assume that the force **P** acting on a material point during its motion from A to B was constant in magnitude, direction, and sense (even though the motion could take place along a curve).

The work of the force **P** through the displacement AB is defined as the scalar product

$$\mathbf{P} \cdot \overline{AB}.$$

If work is denoted by L, then

$$L = \mathbf{P} \cdot \overline{AB}. \tag{I}$$

Let α be the angle between **P** and $\overline{AB}$. Consequently

$$L = |\mathbf{P}| \cdot |\overline{AB}| \cos\alpha. \tag{II}$$

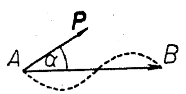

Fig. 73.

Work can be a positive or negative number, or zero. The work of a force **P** is zero if $\mathbf{P} = 0$ or $\overline{AB} = 0$ (i. e. when there is no displacement), or when $\alpha = \frac{1}{2}\pi$ (i. e. when the force is perpendicular to the displacement). If $P \neq 0$, $\overline{AB} \neq 0$, and $\cos\alpha \neq 0$, then work is a positive or negative number depending on whether α is an acute or obtuse angle.

If $\alpha = 0$ or $\alpha = \pi$ (i. e. if the force has the direction of the displacement), we have

$$L = \pm |\mathbf{P}| \cdot |\overline{AB}|,$$

where the sign depends on whether the force and the displacement have the same or opposite senses.

From theorems on a scalar product (Chapt. I, p. 7) it follows that the work of the force **P** through the displacement $\overline{AB}$ is equal to the product of the displacement and the projection of the force on the direction of the displacement, or the product of the force and the projection of the displacement on the direction of the force.

It should be noticed that — according to the definition — *work does not depend on the time it takes the material point to be displaced from A to B.*

Let us denote the projections of the displacement $\overline{AB}$ on the coordinate axes by Δx, Δy, Δz. Hence in virtue of (I)

$$L = P_x \Delta x + P_y \Delta y + P_z \Delta z. \tag{I}$$

If the point A has the coordinates x_0, y_0, z_0, and B x_1, y_1, z_1, then $\Delta x = x_1 - x_0$, etc. Consequently

$$L = P_x(x_1 - x_0) + P_y(y_1 - y_0) + P_z(z_1 - z_0). \tag{2}$$

Variable force. Let us now assume that the point moves along a curve C defined parametrically by the functions:

$$x = f(\sigma), \quad y = \varphi(\sigma), \quad z = \psi(\sigma), \qquad (\sigma' \leqq \sigma \leqq \sigma''). \tag{3}$$

Suppose along with this that if $\sigma_1 < \sigma_2$, then the position of the point corresponding to the value σ_1 occurs sooner than the position corresponding to the value σ_2.

Let us further assume that there acts on a material point a variable force **P** whose projections at an arbitrary point (x, y, z) of the path are given by the functions:

$$P_x = F(x, y, z), \quad P_y = \Phi(x, y, z), \quad P_z = \Psi(x, y, z). \tag{4}$$

Of course, we assume that the functions F, Φ, Ψ are defined at every point of the path. Let us form an arbitrary subdivision δ of the interval $\sigma'\sigma''$ by means of the points $\sigma' = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma''$. To these values of the parameter σ let there correspond the points

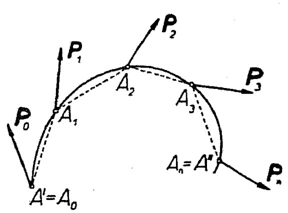

Fig. 74.

$$A' = A_0(x_0, y_0, z_0),\ A_1(x_1, y_1, z_1),\ \ldots,\ A_n(x_n, y_n, z_n) = A'',$$

on the curve C.

In virtue of (3)

$$x_i = f(\sigma_i),\quad y_i = \varphi(\sigma_i),\quad z_i = \psi(\sigma_i)\qquad \text{for } i = 0, 1, \ldots, n. \tag{5}$$

Let us put:

$$\Delta x_i = x_{i+1} - x_i,\quad \Delta y_i = y_{i+1} - y_i,\quad \Delta z_i = z_{i+1} - z_i,\quad (i = 0, 1, \ldots, n-1). \tag{6}$$

Finally, let us denote the forces acting at the points $A_0, A_1, \ldots, A_n$ by $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$. By (4)

$$P_{ix} = F(x_i, y_i, z_i),\quad P_{iy} = \Phi(x_i, y_i, z_i),\quad P_{iz} = \Psi(x_i, y_i, z_i). \tag{7}$$

If the force $\mathbf{P}$ acting through the displacements $\overline{A_0A_1}, \overline{A_1A_2}, \ldots$ were constant and equal to $\mathbf{P}_0, \mathbf{P}_1, \ldots$ respectively, then the work on these displacements would be expressed by the formulae:

$$L_0 = P_{0x}\,\Delta x_0 + P_{0y}\,\Delta y_0 + P_{0z}\,\Delta z_0,$$
$$L_1 = P_{1x}\,\Delta x_1 + P_{1y}\,\Delta y_1 + P_{1z}\,\Delta z_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Putting $L' = L_0 + L_1 + L_2 + \ldots$, we therefore obtain

$$L' = \sum_{i=0}^{n-1} (P_{ix}\,\Delta x_i + P_{iy}\,\Delta y_i + P_{iz}\,\Delta z_i). \tag{8}$$

The expression on the right side of the above equality obviously depends on the subdivision δ of the interval $\sigma'\sigma''$.

If L' tends to a certain limit for every normal sequence[1]) of subdivisions $\{\delta_k\}$ of the interval $\sigma'\sigma''$, then this limit is called *the work of the force* $\mathbf{P}$ *along the curve* C (or *along the length of the curve* C).

Expression (8) can be considered as the approximate value of the work L of the force $\mathbf{P}$.

The limit of expression (8) is the so-called line integral along the curve C

$$L = \int_C (P_x\,dx + P_y\,dy + P_z\,dz). \tag{III}$$

The line integral can be reduced to an ordinary definite integral by expressing the variables x, y, z as functions of the parameter σ. Making use of equations (3), we obtain

[1]) i. e. such that the length of the maximum interval of subdivision tends to zero.

$$L = \int_{\sigma'}^{\sigma''} [P_x f'(\sigma) + P_y \varphi'(\sigma) + P_z \psi'(\sigma)]\,d\sigma,$$

where $P_x = F(f(\sigma), \varphi(\sigma), \psi(\sigma))$, $P_y = \Phi(f(\sigma), \varphi(\sigma), \psi(\sigma))$, etc. In particular, if σ denotes the time, then $f'(\sigma) = x^{\cdot}$, $\varphi'(\sigma) = y^{\cdot}$, $\psi'(\sigma) = z^{\cdot}$. Consequently

$$L = \int_{t'}^{t''} [P_x x^{\cdot} + P_y y^{\cdot} + P_z z^{\cdot}]\,dt. \tag{IV}$$

Since $x^{\cdot}, y^{\cdot}, z^{\cdot}$ are the projections of the velocity $\mathbf{v}$, $P_x x^{\cdot} + P_y y^{\cdot} + P_z z^{\cdot} = \mathbf{P}\cdot\mathbf{v}$. Hence

$$L = \int_{t'}^{t''} (\mathbf{P}\cdot\mathbf{v})\,dt. \tag{V}$$

Remark. Formula (III) is correct when the positions of the moving point on the curve follow each other in the order which corresponds to an increase of the parameter σ. However, if the contrary is true, i. e. if $\sigma_1 > \sigma_2$, then the position corresponding to σ_1 occurs later than that corresponding to σ_2, and hence it is necessary to substitute in formula (III) $-dx, -dy, -dz$ for dx, dy, dz. We obtain then

$$L = -\int_C (P_x\,dx + P_y\,dy + P_z\,dz).$$

Therefore, if a material point has moved along curve C from A' to A'' and the force $\mathbf{P}$ has done work L, then, if the point moves along the curve C from A'' to A' (in this case the positions will follow each other in an order opposite to that before), the same force $\mathbf{P}$ is going to do work $-L$.

Work of a sum of forces. Let us suppose that a material point moving along a curve C was acted upon by two forces $\mathbf{P}$ and $\mathbf{Q}$. Put $\mathbf{R} = \mathbf{P} + \mathbf{Q}$. Denote by L the work of the force $\mathbf{R}$, by L' the work of the force $\mathbf{P}$, and by L'' the work of the force $\mathbf{Q}$. Then

$$L = \int_C (R_x\,dx + R_y\,dy + R_z\,dz) =$$
$$= \int_C [(P_x + Q_x)\,dx + (P_y + Q_y)\,dy + (P_z + Q_z)\,dz] =$$
$$= \int_C (P_x\,dx + P_y\,dy + P_z\,dz) + \int_C (Q_x\,dx + Q_y\,dy + Q_z\,dz) =$$
$$= L' + L'',$$

and hence

$$L = L' + L''. \tag{VI}$$

We can therefore say that *the work done by a sum of two (or more) forces along a certain curve is equal to the sum of the works done by the separate forces along this curve.*

Dimension and units of work. By (II), p. 92, we have [work] = = [force] · [distance] = $LMT^{-2}L$, and hence

$$[\text{work}] = L^2MT^{-2}.$$

The unit of work in the c. g. s. system is the *erg*. It is the work done by a force of 1 dyne acting through a distance of 1 cm. Consequently

$$\text{erg} = \text{cm}^2 \cdot \text{g} \cdot \text{sec}^{-2}.$$

A greater unit is the *Joule* (J) = 10^7 ergs. In the metric gravitational system the unit of work is the *kilogram-meter* (kgm). It is the work done by a force of 1 kg acting through a distance of 1 m. Since 1 kg (of force) = = 981 000 dynes, and 1 m = 100 cm,

$$\text{kgm} = 9.81 \cdot 10^7 \text{ ergs} = 9.81 \text{ J}.$$

§ 11. Potential force field. We called the region D (p. 77) a *force field* if on a material point, situated anywhere in the region D, there acts a force depending only on the position of that point.

The force field is defined by the given functions:

$$P_x = F(x, y, z), \quad P_y = \Phi(x, y, z), \quad P_z = \Psi(x, y, z), \tag{1}$$

which determine the projections of the acting force **P** at the point with coordinates x, y, z.

Stress field. It can happen that a force **P** is proportional to the mass m of a material point. Then the force acting on a unit mass (i. e. the force $\mathbf{P}/m$) at a certain point of the field is called a *stress field* at this point.

An example of such a field is the earth's gravitational field. The weight of a body is proportional to the mass of the body. On the earth's surface the stress field is equal in magnitude to g (gravitational acceleration).

Lines of force. Certain curves in a force field called *lines of force* deserve special consideration. These are curves having the property that a tangent at an arbitrary point has the same direction as the force acting at that point. For instance, in the earth's gravitational field the lines of force are vertical lines. Lines of force are defined by the system of differential equations:

$$dx / P_x = dy / P_y = dz / P_z. \tag{2}$$

Definition of a potential field. If a material point in a force field moves from a point A to a point B along some arc AB, then the work done by the acting force **P** (p. 94, (III)) is

$$L = \int_{AB} (P_x\,dx + P_y\,dy + P_z\,dz). \tag{I}$$

The work will in general depend not only on the points A and B, but also on the path described, i. e. on the arc AB. Fields in which the work depends only on the points A and B, and not on the arc AB, play an important role in mechanics. Therefore, if a material point moves from A to B along various paths in such a field, then the force **P** always does the same work. Such fields are called *potential* or *conservative fields*.

Therefore: *a potential field is a force field in which the work does not depend on the path, but only on its origin and end points.*

If a point in a potential field has traversed a closed path (or has left the point A and returned to A), then the work done throughout the length of the path is zero. The work in a potential field depends only on the origin and end points, hence, if they coincide, the work is such as if the point had not moved at all.

Conversely, if a force field has the property that the work along every closed path is zero, then the field is a potential field. Let us choose two arbitrary points A, B and arc AMB, ANB. Denote by L' the work along arc AMB and by L'' along arc ANB. By hypothesis, the work along the closed curve $AMBNA$ is zero. This work can be represented as the sum of the works: from A to B along the arc AMB and from B to A along the arc BNA. Since the work along the arc BNA is equal to $-L''$, $L' + + (-L'') = 0$, whence $L' = L''$. Therefore the work along both arcs is the same.

We can therefore say that *for a force field to be a potential field, it is necessary and sufficient that the work along every closed curve in the field be zero.*

Potential. Let us select an arbitrary coordinate system (x, y, z) and a point A in a potential field. If we look upon the point A as fixed, then the work L_{AB}, where B is an arbitrary point of the field, will depend only on the coordinates x, y, z of the point B. Therefore the work L_{AB} will be a function of the coordinates x, y, z. Denoting this function by $V(x, y, z)$, we obtain

$$L_{AB} = V(x, y, z). \tag{3}$$

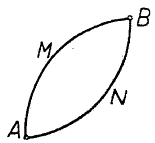

Fig. 75.

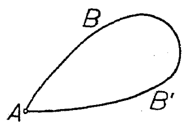

Fig. 76.

The function $V(x, y, z)$ is called a *force function* or a *potential*.

Let us consider some point B' with coordinates x', y', z'. The work along an arbitrary curve $ABB'A$ is zero. Therefore $L_{AB} + L_{BB'} + L_{B'A} = 0$.

But $L_{B'A} = -L_{AB'} = -V(x', y', z')$. Hence by (3) $V(x, y, z) + L_{BB'} - V(x', y', z') = 0$, whence

$$L_{BB'} = V(x', y', z') - V(x, y, z). \tag{II}$$

Formula (II) can be stated as follows:

In the passage from one point to another the work is equal to the difference of potentials between these points.

We have defined the potential as a function depending on the choice of the point A. Had we chosen another point $A'(x', y', z')$, the potential would have been expressed by another function $V'(x, y, z)$.

Since by the definition of a potential we have for an arbitrary point $B(x, y, z)$

$$V'(x, y, z) = L_{A'B} = V(x, y, z) - V(x', y', z'),$$

hence

$$V(x, y, z) - V'(x, y, z) = V(x', y', z') = \text{const}.$$

Therefore the difference of both functions V and V' is constant. We see from this that in a potential force field the function is defined to within a certain constant (as in an indefinite integral). As formula (II) shows, this constant does not play any role, since for the magnitude of the work there enters only the difference of the potentials at the two points.

Dimension of the potential. Since by definition the potential is equal to work, the dimension of the potential is the same as the dimension of work. Therefore

$$[\text{potential}] = L^2MT^{-2}.$$

The units of work are equally units of potential.

Relation between force and potential. Let us move a material point from the point $A(x_0, y, z)$ to the point $B(x, y, z)$ along a line parallel to the x-axis. By (I) and (II) the work is:

$$L_{AB} = V(x, y, z) - V(x_0, y, z), \quad \text{or} \quad L_{AB} = \int_{AB} (P_x\,dx + P_y\,dy + P_z\,dz).$$

Since the point was translated along a parallel to the x-axis, $dy = 0$ and $dz = 0$. Therefore

$$L_{AB} = \int_{AB} P_x\,dx = \int_{x_0}^{x} P_x\,dx,$$

whence

$$V(x, y, z) - V(x_0, y, z) = \int_{x_0}^{x} P_x\,dx.$$

Taking the partial derivative with respect to x we get $\partial V / \partial x = P_x$; we obtain analogous formulae for the remaining partial derivatives.

Hence: *the partial derivatives of the potential are equal to the corresponding projections of the force on the coordinate axes*, i. e.

$$\frac{\partial V}{\partial x} = P_x, \quad \frac{\partial V}{\partial y} = P_y, \quad \frac{\partial V}{\partial z} = P_z. \tag{III}$$

Conversely, if we assume that in a given force field there exists a function V satisfying relations (III), then the field is a potential field. For let a given function V satisfy relations (III). Then the work from the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$ along an arbitrary arc AB is:

$$L_{AB} = \int_{AB} (P_x\,dx + P_y\,dy + P_z\,dz) = \int_{AB} \left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz\right).$$

Since the expression in the parenthesis of the last integral is the total differential dV, we obtain the formula

$$L_{AB} = \int_{AB} dV = V(x_2, y_2, z_2) - V(x_1, y_1, z_1), \tag{4}$$

expressing the fact that the work does not depend on the path, but only on the end points. In virtue of (4) the function V is therefore a potential.

Hence: *if for a force field there exists a function V satisfying relations* (III), *then the force field is a potential field and the function V is a potential.*

Potential surfaces. If c is an arbitrary constant, the surfaces defined by the equation

$$V(x, y, z) = c \tag{5}$$

are called *potential surfaces*.

Therefore: *a potential surface is one along which the potential has a constant value.*

In differential geometry it is proved that the direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ of the normal to the surface (5) at the point (x, y, z) satisfy the conditions:

$$\cos\alpha : \cos\beta : \cos\gamma = \frac{\partial V}{\partial x} : \frac{\partial V}{\partial y} : \frac{\partial V}{\partial z},$$

whence, by (III),

$$\cos\alpha : \cos\beta : \cos\gamma = P_x : P_y : P_z.$$

Since the direction cosines $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ of the force $\mathbf{P}$ satisfy similar conditions: $\cos\alpha' : \cos\beta' : \cos\gamma' = P_x : P_y : P_z$, the force $\mathbf{P}$ is normal to the potential surface.

Therefore: *at every point of a potential surface the acting force is perpendicular to this surface.* It follows from this that *the lines of force are perpendicular to potential surfaces.*

Let us select two neighbouring potential surfaces S and S' having potentials c and c', where $c' > c$. From an arbitrary point A of the surface S let us draw a normal to this surface intersecting the surface S' at the point A'.

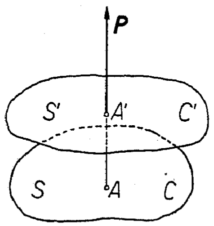

Fig. 77.

The work along the displacement from A to A' is $L_{AA'} = c' - c > 0$. Since the work is positive, the force $\mathbf{P}$ has the sense of the displacement $\overline{AA'}$.

Hence: *with respect to a potential surface the force points in the direction of increasing potential.*

Approximately we have $L_{AA'} = |\mathbf{P}|\, AA' = c' - c$, or $|\mathbf{P}| = (c' - c) / AA'$.

Hence: *on one and the same potential surface the force is approximately inversely proportional to the segment of the normal enclosed between this surface and a neighbouring potential surface.*

§ 12. Examples of potential fields. Let us now look at several kinds of potential fields which are frequently dealt with in practice.

Constant field. If a force $\mathbf{P}$ in a certain field is constant in magnitude, direction and sense, then the field is called a *constant field.*

The earth's gravitational field in a small neighbourhood of a given point on the earth-surface is a constant field.

Let us select a coordinate system (x, y, z), giving to the z-axis the direction of the force $\mathbf{P}$, but an opposite sense. Putting $|\mathbf{P}| = mg$, we obtain

$$P_x = 0, \quad P_y = 0, \quad P_z = -mg.$$

It is easy to verify that the function

$$V = -mgz$$

is a potential because we have

$$\partial V / \partial x = 0 = P_x, \quad \partial V / \partial y = 0 = P_y, \quad \partial V / \partial z = -mg = P_z.$$

Hence: *a constant field is a potential field.*

The work from the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$ along an arbitrary path is $L_{AB} = -mgz_2 - (-mgz_1)$; hence

$$L_{AB} = mg(z_1 - z_2). \tag{1}$$

By hypothesis, the force $\mathbf{P}$ is the force of gravity, and it is clear that $z_1 - z_2 = h$ is the difference between the levels at which the points A and B are situated. Therefore, putting $|\mathbf{P}| = Q = mg$, we obtain

$$L_{AB} = Qh.$$

The potential surface has the equation $V = \text{const.}$; hence $-mgz = \text{const.}$, or $z = \text{const.}$ Therefore, potential surfaces are level surfaces (i. e. perpendicular to the direction of the force). Since the lines of force are perpendicular to the potential surfaces, the lines of force are straight lines parallel to the z-axis, i. e. vertical lines.

Central fields. If the direction of a force in a force field always passes through a certain fixed point O, then the field is called a *central field* and the point O the *centre of the field* (p. 85).

Let us assume that in a given central field the magnitude of the force at an arbitrary point A depends only on the distance r of the point A from the centre O. Denote by P the projection of the force $\mathbf{P}$ acting at A on the direction of $\overline{OA}$. Therefore P is a function of r. Set

$$P = f(r).$$

Let the origin of the coordinate system be at O. Denoting by x, y, z the coordinates of the point A and by α the angle which $\overline{OA}$ makes with the x-axis, we obtain $\cos\alpha = x / r$. Therefore

$$P_x = P\cos\alpha = P\frac{x}{r} \text{ and similarly } P_y = P\frac{y}{r}, \quad P_z = P\frac{z}{r}. \tag{2}$$

Put $V = \int P\,dr = \int f(r)\,dr$. Since $r = \sqrt{x^2 + y^2 + z^2}$, $\partial r / \partial x = x / r$, $\partial r / \partial y = y / r$, and $\partial r / \partial z = z / r$. Therefore

$$\frac{\partial V}{\partial x} = \frac{dV}{dr}\cdot\frac{\partial r}{\partial x} = f(r)\frac{x}{r} = P\frac{x}{r} = P_x,$$

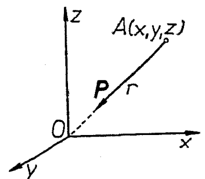

Fig. 78.

and analogously

$$\frac{\partial V}{\partial y} = P_y, \quad \frac{\partial V}{\partial z} = P_z.$$

Our field is hence a potential field and the function V is a potential.

Therefore: *a central field in which the force depends only on the distance of the point from the centre is a potential field, and the potential is expressed by the formula*

$$V = \int P\,dr. \tag{3}$$

Since the potential at the point A is a function of the distance r of the point A from the centre O, the potential has a constant value on spheres with centre at O. Hence the potential surfaces in this case will be spheres with centre at O. The lines of force are obviously straight lines passing through the point O.

Newtonian gravitational field. Let us suppose that a point of mass m is attracted with a force $\mathbf{P}$ by a fixed point of mass M according to Newton's law of gravitation (p. 89, (I)), i. e. that

$$|\mathbf{P}| = KmM / r^2.$$

Since the force is directed towards the point M, the field is a central field whose centre is the point M. Therefore, according to the definition of the number P, $P = -KmM / r^2$.

We have

$$V = \int P \, dr = -\int KmM \, dr / r^2.$$

Consequently

$$V = KmM / r. \tag{4}$$

Hence the work along an arbitrary arc $A'A$ is

$$L_{A'A} = KmM\left(\frac{1}{r} - \frac{1}{r'}\right),$$

where r and r' denote the distances of the points A and A' from the centre. If, in particular, we select the point A' at infinity i. e. if we put $r' = \infty$, then we shall obtain

$$L_{\infty A} = KmM / r = V. \tag{5}$$

Therefore: *in a Newtonian gravitational field the potential at a point A is equal to the work a force would do in bringing a material point from infinity to A.*

Axial field. A force field having the property that at every point of the field the line of action of the force cuts a certain fixed line l at right angles is called an *axial field*, and the line l is called the *axis of the field*.

Let us assume that the magnitude of the force $\mathbf{P}$ acting at an arbitrary point A depends only on the distance r of the point from the axis of the field. Put $P = -|\mathbf{P}|$ or $P = |\mathbf{P}|$ depending on whether the force $\mathbf{P}$ is pointed towards or away from the axis l. Since the magnitude of the force $\mathbf{P}$ is a function of r (i. e. the distance of the point A from the axis l), we can write

$$P = f(r).$$

Let us select a system of coordinates in which the axis of the field is the z-axis. It is easy to see that the projections of the force $\mathbf{P}$ acting at the

point $A(x, y, z)$ are $P_x = Px / r$, $P_y = Py / r$, and $P_z = 0$, where $r = \sqrt{x^2 + y^2}$. Put

$$V = \int P dr = \int f(r) \, dr.$$

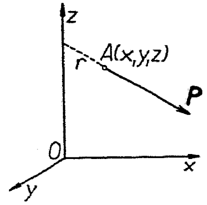

Fig. 79.

Therefore

$$\frac{\partial V}{\partial x} = \frac{dV}{dr} \cdot \frac{\partial r}{\partial x} = P\frac{x}{r} = P_x.$$

Similarly

$$\frac{\partial V}{\partial y} = P_y.$$

Since V does not depend on z (because r does not depend on z),

$$\frac{\partial V}{\partial z} = 0 = P_z.$$

It follows from this that the given field is a potential field and V is the potential.

Hence: *an axial field in which the magnitude of the force depends only on the distance of the point from the axis is a potential field and the potential is*

$$V = \int P \, dr. \tag{6}$$

It is easy to see that in this case the potential surfaces are cylinders whose common axis is the axis of the field. The lines of force are straight lines cutting the axis at right angles.

For instance, if $P = m\omega^2 r$ (ω constant), then $V = \int P \, dr = \int m\omega^2 r \, dr$, and hence $V = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2(x^2 + y^2)$. The potential surfaces are obtained by setting $V = \text{const.}$ Therefore $\frac{1}{2}m\omega^2(x^2 + y^2) = \text{const}$, whence $x^2 + y^2 = \text{const}$; this is the equation of a cylinder whose axis is the z-axis.

Sum of potential fields. Let there be given several force fields $\mathbf{P}_1, \mathbf{P}_2, \ldots$ in a certain region D. A force field $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots$ in the region D is called the *sum of the force fields* $\mathbf{P}_1, \mathbf{P}_2, \ldots$

If the force fields $\mathbf{P}_1, \mathbf{P}_2, \ldots$ are potential fields, then — as is easily shown — *the sum of the fields is also a potential field whose potential V is equal to the sum of the potentials $V_1, V_2, \ldots$ of the separate fields.*

For let us put $V = V_1 + V_2 + \ldots$ We have

$$\frac{\partial V}{\partial x} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial x} + \ldots = P_{1x} + P_{2x} + \ldots = P_x,$$

and analogously

$$\frac{\partial V}{\partial y} = P_y, \quad \frac{\partial V}{\partial z} = P_z.$$

Therefore V is the potential of the sum of the given fields.

Let us suppose, for example, that a point of mass m is attracted according to Newton's law by two fixed points of masses m_1 and m_2 with forces $\mathbf{P}_1$ and $\mathbf{P}_2$. The resultant force will therefore be $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. On p. 103 we have shown that the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ have potentials. Hence according to (4), p. 102, denoting the distances of m from m_1 and m_2 by r_1 and r_2, we obtain:

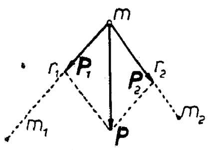

Fig. 80.

$$V_1 = K\frac{mm_1}{r_1} \quad \text{and} \quad V_2 = K\frac{mm_2}{r_2}.$$

The force $\mathbf{P}$ therefore has the potential $V = V_1 + V_2$. Consequently, $V = Km(m_1 / r_1 + m_2 / r_2)$. Similarly, if a point of mass m is attracted by n fixed points of masses $m_1, m_2, \ldots, m_n$ according to Newton's law, then

$$V = Km\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + \ldots + \frac{m_n}{r_n}\right), \tag{7}$$

where $r_1, r_2, \ldots, r_n$ denote the distances of the point m from the points $m_1, m_2, \ldots, m_n$, respectively.

§ 13. Kinetic and potential energy. Let a force $\mathbf{P}$ act on a material point $A(x, y, z)$ of mass m. Then (p. 78, (I)):

$$mx^{\cdot\cdot} = P_x, \quad my^{\cdot\cdot} = P_y, \quad mz^{\cdot\cdot} = P_z.$$

Multiply both sides of the first equation by $x^{\cdot}$, of the second by $y^{\cdot}$, of the third by $z^{\cdot}$, and add. We obtain

$$m(x^{\cdot}x^{\cdot\cdot} + y^{\cdot}y^{\cdot\cdot} + z^{\cdot}z^{\cdot\cdot}) = P_x x^{\cdot} + P_y y^{\cdot} + P_z z^{\cdot}. \tag{1}$$

Let us denote the absolute value of the velocity of the point A by v. Then $v^2 = x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}$, whence $\mathrm{d}(v^2) / \mathrm{d}t = 2(x^{\cdot}x^{\cdot\cdot} + y^{\cdot}y^{\cdot\cdot} + z^{\cdot}z^{\cdot\cdot})$, and hence $\mathrm{d}(\frac{1}{2}mv^2) / \mathrm{d}t = m(x^{\cdot}x^{\cdot\cdot} + y^{\cdot}y^{\cdot\cdot} + z^{\cdot}z^{\cdot\cdot})$. Substituting this equation in (1), we obtain

$$\mathrm{d}(\tfrac{1}{2}mv^2) / \mathrm{d}t = P_x x^{\cdot} + P_y y^{\cdot} + P_z z^{\cdot}.$$

Integrating both sides (with respect to t) from the initial time t_0 to t, we get

$$\int_{t_0}^{t} \frac{\mathrm{d}(\frac{1}{2}mv^2)}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_0}^{t} [P_x x^{\cdot} + P_y y^{\cdot} + P_z z^{\cdot}]\,\mathrm{d}t. \tag{2}$$

Let v_0 be the absolute value of the velocity at the initial moment t_0; the left hand side of (2) becomes $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$, and the right hand side equals (p. 95, (IV)) the work that the force $\mathbf{P}$ did during the time from t_0 to t. Let us denote this work by $L_{t_0 t}$. Equation (2) can therefore be written in the form

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = L_{t_0 t}. \tag{3}$$

The expression $\frac{1}{2}mv^2$ is called the *kinetic energy* of the point.

Putting

$$E = \tfrac{1}{2}mv^2, \quad E_0 = \tfrac{1}{2}mv_0^2 \tag{4}$$

we obtain:

$$E - E_0 = L_{t_0 t}. \tag{I}$$

Hence: *the increase in kinetic energy in a certain time is equal to the work of the acting force in this time.*

This theorem is called the *equivalence of work and kinetic energy.*

In particular, if the work of the force $\mathbf{P}$ is constantly zero, then $E - E_0 = 0$, i. e. $E = E_0$, and hence by (4) $v = v_0$. Therefore the point has a velocity which is constant in magnitude. Hence, if the force is e. g. constantly perpendicular to the path, then the point moves with a uniform motion. An example is the uniform motion along a circle of a point under the influence of a force constant in magnitude and directed towards the centre of the circle.

Let the point now move in a potential field. Denote by V and V_0 the potentials the point possesses at the moments t and t_0, respectively. Then $L_{t_0 t} = V - V_0$, whence by (I) $E - E_0 = V - V_0$, i. e.

$$E - V = E_0 - V_0. \tag{5}$$

The expression $-V$ is called the *potential energy.*

Setting $-V = U$, and $-V_0 = U_0$, we obtain

$$E + U = E_0 + U_0 = \text{const.} \tag{II}$$

The sum of the kinetic and potential energies, i. e. the expression $E + U$, is called the *total energy.*

Hence: *if a point moves in a potential field, then its total energy is constant.*

The preceding theorem is called the *principle of conservation of total energy.*

Dimension of kinetic and potential energies. By (4) $[E] = [m][v^2]$; hence

$$[\text{kinetic energy}] = L^2MT^{-2}.$$

Therefore kinetic energy has the dimension of work. The units of work are consequently also units of kinetic energy.

By definition, potential energy has the dimension of a potential and therefore also has the dimension of work (p. 98).

§ 14. Motion of a point attracted by a fixed mass. Motion along a curve of the second degree. Let a material point A of mass m be attracted by a fixed point of mass M with a force $\mathbf{P}$ acting according to Newton's law. Let us place the origin O of the coordinate system at the point M. Denoting (as on p. 101) by P the projection of the force on the direction of $\overline{OA}$, we obtain

$$P = -K\frac{mM}{r^2}. \tag{1}$$

Denoting the coordinates of the point A by x, y, z, we obtain (p. 101):

$$P_x = P\frac{x}{r} = -K\frac{mM}{r^2}\cdot\frac{x}{r}, \text{ etc.}$$

Therefore the equations of motion of the point A are:

$$mx^{\cdot\cdot} = -K\frac{mM}{r^2}\frac{x}{r},\quad my^{\cdot\cdot} = -K\frac{mM}{r^2}\frac{y}{r},\quad mz^{\cdot\cdot} = -K\frac{mM}{r^2}\frac{z}{r}. \tag{I}$$

In our case the force $\mathbf{P}$ has the potential $V = KmM / r$ (p. 102), and therefore the potential energy $U = -KmM / r$. By the principle of conservation of total energy $\frac{1}{2}mv^2 - KmM / r = \text{const}$, whence putting $\mu = KM$,

$$v^2 = 2\mu / r + h, \quad \text{where } h = \text{const}. \tag{2}$$

Since the motion in the problem under consideration is central (p. 85), the path is a plane curve. Let us assume, therefore, that the motion takes place in the xy-plane and that the areal velocity is different from zero.

From Binet's formula (p. 87, (I)) we obtain by (1)

$$-\frac{KmM}{r^2} = -\frac{mc^2}{r^2}\left[\frac{d^2(1/r)}{d\varphi^2} + \frac{1}{r}\right], \tag{3}$$

and since $KM = \mu$,

$$\frac{d^2(1/r)}{d\varphi^2} + \frac{1}{r} = \frac{\mu}{c^2}. \tag{4}$$

Let us set $1 / r = u$. We obtain

$$\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{c^2}. \tag{5}$$

A particular solution of the above equation is $u = \mu / c^2$. The general solution of the homogeneous equation

$$\frac{d^2u}{d\varphi^2} + u = 0$$

is — as is easily verified — of the form $u = a\cos\varphi + b\sin\varphi$. The general solution of (5) will therefore be

$$u = \mu / c^2 + a\cos\varphi + b\sin\varphi,$$

where a and b are arbitrary constants. Setting $a = \varrho\cos\varphi_0$, $b = \varrho\sin\varphi_0$ (where ϱ and φ_0 are arbitrary constants) and substituting $1 / r = u$ back again, we obtain the general solution of (4)

$$1 / r = \mu / c^2 + \varrho\cos(\varphi - \varphi_0). \tag{6}$$

Now the general equation of a conic section, if the pole is at a focus, has the form

$$\frac{1}{r} = \frac{1}{p} + \frac{\varepsilon}{p}\cos(\varphi - \varphi_0),$$

where p is a parameter, ε the distance between the foci, and φ_0 the angle the axis of the curve makes with the axis of the coordinate system. Equation (6) is therefore the equation of a conic section. By comparing them we get

$$p = c^2 / \mu \quad \text{and} \quad \varepsilon = \varrho c^2 / \mu. \tag{7}$$

Such a curve is an ellipse, hyperbola or parabola, depending on whether $\varepsilon < 1$, $\varepsilon > 1$ or $\varepsilon = 1$. In order to recognize the type of conic section, we must calculate the constant ϱ. We shall determine it from formula (2).

We have $v^2 = r^{\cdot 2} + r^2\varphi^{\cdot 2}$. Since $\frac{1}{2}c$ is the areal velocity, $\frac{1}{2}c = \frac{1}{2}r^2\varphi^{\cdot}$ i. e. $\varphi^{\cdot} = c / r^2$; therefore (p. 87, formula (7))

$$v^2 = c^2\left(\frac{d(1/r)}{d\varphi}\right)^2 + \frac{c^2}{r^2}.$$

By (6) we obtain

$$v^2 = \mu^2 / c^2 + 2\mu\varrho\cos(\varphi - \varphi_0) + c^2\varrho^2. \tag{8}$$

Determining r in terms of φ from formula (6) and substituting in formula (2), we obtain $v^2 = 2\mu^2 / c^2 + 2\mu\varrho\cos(\varphi - \varphi_0) + h$, whence by comparing with formula (8)

$$\varrho = \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}}, \tag{9}$$

and hence by (7)

$$\varepsilon = \sqrt{1 + \frac{hc^2}{\mu^2}}.$$

Therefore $\varepsilon \lesseqqgtr 1$ depending on whether $h \lesseqqgtr 0$. Setting $t = t_0$, $v = v_0$, $r = r_0$, we obtain from formula (2) $h = v_0^2 - 2\mu / r_0$; consequently:

$h \lesseqqgtr 0$ *depending on whether* $v_0^2 \lesseqqgtr 2\mu / r_0$.

It follows from this that *the type of conic section does not depend on the direction of the velocity, but only on its magnitude.*

We can therefore determine the type of conic if we know one position of a point and its speed at that position.

Comets, for instance, move within the limits of the solar system under the influence of the sun's attraction and hence move (with respect to the sun) along conics.

Let us now assume that a point moves along an ellipse whose equation is $1/r = 1/p + (\varepsilon/p)\cos(\varphi - \varphi_0)$. From Binet's formula we get

$$-\frac{KMm}{r^2} = -\frac{c^2}{p}\frac{m}{r^2}, \quad \text{whence } KM = c^2/p.$$

Let a and b be the axes of the ellipse and T the period. The areal velocity will then be $\frac{1}{2}c = ab\pi / T$. Since $p = b^2/a$, $KM = 4\pi^2 a^3 / T^2$, whence

$$a^3 / T^2 = KM / 4\pi^2. \tag{10}$$

It follows from this that *the ratio a^3/T^2 depends only on the mass of the attracting body and not on the mass of the moving point.*

If the sun were at rest, then the ratio a^3/T^2 would be a constant for the planets (such as is required by Kepler's third law). The sun, however, is not at rest, since it is attracted by the planets. This fact accounts for the deviations from Kepler's law.

We shall consider this matter later in connection with the *problem of two bodies* (chap. V).

Motion along a straight line. Let us examine, in addition, the particular case when the areal velocity is zero. The motion in this case takes place along a straight line passing through the centre of the field, i. e. through the point M (p. 86). Since v denotes the absolute value of the velocity,

$$v = |r^{\cdot}|. \tag{11}$$

Let us suppose that at $t = 0$, $r = r_0$ and $v = v_0$. From equation (2), p. 106, it follows that

$$v^2 = 2\mu / r + h, \tag{12}$$

whence

$$h = v_0^2 - 2\mu / r_0. \tag{13}$$

Let us assume that at $t = 0$ the velocity vector of the moving point was directed away from the point M, that is, that the point was receding from M. Therefore at $t = 0$, $r^{\cdot} > 0$.

Let us consider the two cases depending on whether $h \geqq 0$, or $h < 0$.

1° $h \geqq 0$. By (12) $v^2 \geqq 2\mu / r$ constantly; hence $v^2 > 0$; therefore $v > 0$ constantly. It follows from this that the point will never stop, but will always move away from M. Hence $r^{\cdot} > 0$ constantly, whence by (11) $v = r^{\cdot}$ during the entire time of the motion. From (12) we obtain

$$r^{\cdot} = v = \sqrt{2\mu / r + h}, \quad \text{whence} \quad \mathrm{d}r / \sqrt{2\mu / r + h} = \mathrm{d}t. \tag{14}$$

Consequently

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{2\mu r^{-1} + h}} = t. \tag{15}$$

From the above equation it follows that when t tends to ∞, r also tends to ∞, and hence the point recedes to infinity.

2° $h < 0$. In this case there exists an $r = r_1$ for which $v = 0$. We obtain the value of r_1 from (12) by putting $v = 0$ and $r = r_1$. We get

$$r_1 = -2\mu : h. \tag{16}$$

It is easy to show that $r_1 > r_0$. For we have $2\mu > 2\mu - r_0 v_0^2 = r_0(2\mu / r_0 - v_0^2) = r_0(-h)$. Since $h < 0$, $-2\mu / h < r_0$, and therefore by (16) $r_1 > r_0$.

At the beginning of the motion, so long as $r > r_1$, the point will move away from M. During this period $r^{\cdot} > 0$ constantly; therefore by (11) $r^{\cdot} = v$ and as a consequence of this, formulae (14) and (15) will hold.

Substituting r_1 for the upper limit of integration in formula (15), we obtain the time t_1 for which $r = r_1$. Therefore

$$\int_{r_0}^{r_1} \frac{\mathrm{d}r}{\sqrt{2\mu r^{-1} + h}} = t_1.$$

At $t = t_1$ we shall have $v = 0$, and for $t > t_1$ the point returns and will come closer to M.

Assuming that the earth is a sphere composed of concentric homogeneous layers (i. e. of constant density), it can be shown that the earth attracts an exterior material point as though the entire mass of the earth were concentrated at its centre O. The results obtained can therefore be applied to the

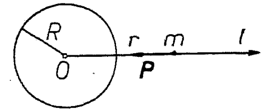

Fig. 81.

motion of bodies attracted by the earth, denoting the mass of the earth (concentrated at its centre O) by M, and assuming that the origin of the coordinate system is at the point O, while the moving point is above the surface of the earth, i. e. that $r \geqq R$, where R is the radius of the earth (Fig. 81).

Example. Let us assume that a material point was thrown from the surface of the earth vertically upwards with a velocity v_0. Therefore $r_0 = R$ and by (13) $h = v_0^2 - 2\mu / R$. From formula (7), p. 90, it follows that $\mu = KM = gR^2$, where g denotes the gravitational acceleration. Hence

$$h = v_0^2 - 2gR.$$

If $v_0 < \sqrt{2gR}$, then $h < 0$, and hence the point will return to the earth again. On the other hand, if $v_0 \geqq \sqrt{2gR}$, then $h \geqq 0$, and hence the point will never return to the earth again.

Assuming that $R = 6300$ km, $g = 9.81$ m/sec², we obtain $\sqrt{2gR} = 12$ km/sec. Therefore, if the body is thrown upwards with a velocity $v_0 \geqq 12$ km/sec, then it will never return to the earth again. This result does not take into account the resistance of the air.

§ 15. Harmonic motion.

Simple harmonic motion. On a material point of mass m in a central field let there act a force $\mathbf{P}$ which is always directed towards the centre O, and whose magnitude is proportional to the distance of the point from O.

The force $\mathbf{P}$ is called an *elastic force.*

Let us assume for the present that the point moves along the x-axis whose origin is O. Denoting the coordinate of the point m by x, and the component of the force by P, we shall therefore have

$$P = -\lambda^2 x, \tag{1}$$

where λ^2 is the constant of proportionality. Hence $m\ddot{x} = -\lambda^2 x$. Putting $k^2 = \lambda^2 / m$, we obtain $\ddot{x} = -k^2 x$, whence

$$\ddot{x} + k^2 x = 0. \tag{2}$$

From equation (2) it follows that the magnitude of the acceleration of the point is proportional to the distance of the point from O and always directed towards O.

A motion having this property is called a *simple harmonic* (or *oscillatory*) motion.

The differential equation (2) is a linear equation of the second order with constant coefficients. The roots of the characteristic equation

$r^2 + k^2 = 0$ are $r_{1,2} = \pm ki$. The general solution of equation (2) is therefore

$$x = c_1 \sin kt + c_2 \cos kt. \tag{3}$$

Writing constants c_1, c_2 in the form $c_1 = a \cos kt_0$, $c_2 = -a \sin kt_0$, (where a and t_0 are arbitrary constants and $a \geqq 0$), we obtain

$$x = a \sin k(t - t_0), \tag{4}$$

whence, starting the calculation of time from the moment t_0,

$$x = a \sin kt. \tag{I}$$

The constant a is called the *amplitude.*

Since $|\sin kt| \leqq 1$, the amplitude a represents the greatest deviation of the point from O. For $t = \pm \pi / 2k$ we get $x = \pm a$. The path of the point is therefore the line segment from $-a$ to a. Let us put

$$T = 2\pi / k. \tag{5}$$

Then $a \sin k(t + T) = a \sin(kt + 2\pi) = a \sin kt$. Therefore by (I) the point occupies the same position at the times t and $t + T$. The motion is therefore *periodic* of *period* T.

Substituting in (I) for k the value determined from (5), we obtain

$$x = a \sin \frac{2\pi}{T} t. \tag{II}$$

If n denotes the number of periods in 1 second, then $n = 1 / T$. Hence in virtue of (II)

$$x = a \sin 2n\pi t. \tag{III}$$

Differentiating (II), we obtain:

$$\dot{x} = v = \frac{2a\pi}{T} \cos \frac{2\pi}{T} t, \quad \ddot{x} = p = -\frac{4a\pi^2}{T^2} \sin \frac{2\pi}{T} t. \tag{6}$$

By (II) and (6) we can form the following table giving the position, velocity, and acceleration of the point at $t = 0, \frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T$, and T:

t	0	$\frac{1}{4}T$	$\frac{1}{2}T$	$\frac{3}{4}T$	T
x	0	a	0	$-a$	0
v	$2a\pi / T$	0	$-2a\pi / T$	0	$2a\pi / T$
p	0	$-4a\pi^2 / T^2$	0	$4a\pi^2 / T^2$	0

From the table we see that during the period T the point moves from the origin of the coordinate system to the point $x = a$, then returns through the point O and arrives at the point $x = -a$, then returns to O etc. The maximum velocity is at O, whereas at the end points of the path (i. e. at the points $x = \pm a$) the velocity is zero. The acceleration, on the other hand, is greatest at the end points, i. e. for $x = \pm a$; at O the acceleration is zero.

Example. A sphere of mass m is attached at the lower end of a spring hanging vertically (Fig. 82). Let O denote the point at which the mass is at rest (in equilibrium). If the sphere is depressed along the vertical from its position of equilibrium, then the sphere will begin to oscillate vertically. If the mass of the spring is small, then we can assume as an approximation that the spring acts on the sphere with a force $\mathbf{P}$ proportional to the extension (or contraction), and is directed constantly towards the point A_0 which was the position of the end of the unstretched spring before the sphere was attached to it.

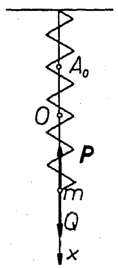

Fig. 82.

Let O be the origin of the x-axis directed vertically downwards. Putting $A_0O = d$, we obtain

$$P = -\lambda^2(x + d),$$

where λ is a constant depending on the spring. Since the sphere is in equilibrium at O, and $P = -\lambda^2 d$ (because $x = 0$), it follows that $-\lambda^2 d + mg = 0$, whence $\lambda^2 = mg / d$. During the motion $m\ddot{x} = P + mg = -\lambda^2(x + d) + mg$; hence $m\ddot{x} + \lambda^2 x = 0$; therefore $\ddot{x} + k^2 x = 0$, where

$$k^2 = \lambda^2 / m = g / d.$$

By (I), p. 111, the solution of the above equation is $x = a \sin kt$; therefore

$$x = a \sin \sqrt{\frac{g}{d}}\, t.$$

The sphere will therefore execute a simple harmonic motion about the point O. By (5) the period of the motion is

$$T = 2\pi / k = 2\pi\sqrt{d / g} = 2\pi\sqrt{m} / \lambda.$$

The period of the motion therefore depends on the mass of the point.

Plane harmonic motion. Let a point move in a central force field in which the force $\mathbf{P}$ is directed towards the centre of the field and is (in magnitude) proportional to the distance of the point from the centre.

Let us select the centre of the field as the origin O of the coordinate

system. Since a central motion is a plane motion, we can assume that it takes place in the xy-plane.

According to Newton's law $m\mathbf{p} = \mathbf{P}$, where $\mathbf{p}$ denotes the acceleration. The acceleration is therefore directed towards the centre of the field and is (in magnitude) proportional to the distance of the point from the centre.

A motion having this property is called a *plane harmonic* motion and the force $\mathbf{P}$ is called an *elastic* force (cf. p. 110).

By hypothesis, we have

$$P_x = -\lambda^2 x, \quad \text{and} \quad P_y = -\lambda^2 y,$$

where λ is a constant of proportionality. The equations of motion will have the form

$$m\ddot{x} = -\lambda^2 x, \quad m\ddot{y} = -\lambda^2 y.$$

As before, putting $k^2 = \lambda^2 / m$, we obtain

$$\ddot{x} = -k^2 x, \quad \ddot{y} = -k^2 y. \tag{7}$$

On p. 111, formula (4), we showed that the solution of the above equations is:

$$x = a' \sin k(t - t_0'), \quad y = a'' \sin k(t - t_0''), \tag{8}$$

where a', a'', t_0', t_0'' are arbitrary constants.

As is easily shown, this motion is also periodic of period $T = 2\pi / k$. From equations (8) we obtain:

$$a''x \cos kt_0'' - a'y \cos kt_0' = a'a'' \cos kt \sin k(t_0'' - t_0'),$$
$$a''x \sin kt_0'' - a'y \sin kt_0' = a'a'' \sin kt \sin k(t_0'' - t_0').$$

Squaring each of the equations and adding, we obtain

$$a''^2x^2 + a'^2y^2 - 2a'a''xy \cos k(t_0'' - t_0') = [a'a'' \sin k(t'' - t')]^2. \tag{9}$$

If $a' = 0$, or $a'' = 0$, or $t_0'' - t_0' = n\pi / k$ (where n is an integer), then equation (9) is the equation of a straight line. In the remaining cases (9) is the equation of an ellipse whose centre is at the origin of the coordinate system.

Hence: *a plane harmonic motion takes place along a straight line passing through the centre of the field, or along an ellipse whose centre is the centre of the field.*

A plane harmonic motion along a line is obviously a simple harmonic motion.

Damped harmonic motion. On a material point moving along the x-axis, let there act in addition to an elastic force $\mathbf{P}$ (i. e. a force which is proportional to the distance from the centre and directed towards the centre), another force **Q** (*damping* or retarding the motion) which is in magnitude proportional to the velocity, but directed opposite to it.

The motion which the point will then execute is called a *damped harmonic* motion.

Denoting the components of the forces **P** and **Q** by P and Q, we can write:

$$P = -\lambda^2 x, \quad \text{and} \quad Q = -2\mu x^{\cdot}, \tag{10}$$

where λ^2 and $\mu > 0$ are constants of proportionality. Therefore $mx^{\cdot\cdot} = -\lambda^2 x - 2\mu x^{\cdot}$. Putting

$$\lambda^2 / m = k^2, \quad \text{and} \quad \mu / m = \varepsilon, \tag{11}$$

we therefore obtain

$$x^{\cdot\cdot} + 2\varepsilon x^{\cdot} + k^2 x = 0. \tag{IV}$$

Equation (IV) is a linear differential equation of the second order with constant coefficients. Its characteristic equation is

$$r^2 + 2\varepsilon r + k^2 = 0; \tag{12}$$

hence

$$r_{1,2} = -\varepsilon \pm \sqrt{\varepsilon^2 - k^2}. \tag{13}$$

We shall consider three cases here, depending on whether the discriminant $\varepsilon^2 - k^2$ is negative, positive, or zero.

1° $\varepsilon^2 - k^2 < 0$. This case arises when ε is small, i. e. when the damping force **Q** is small. Let us set

$$\sqrt{k^2 - \varepsilon^2} = k_1. \tag{14}$$

Therefore, by (13) $r_{1,2} = -\varepsilon \pm ik_1 t$. Hence the general solution of equation (IV) in this case is

$$x = e^{-\varepsilon t}(c_1 \sin k_1 t + c_2 \cos k_2 t).$$

Writing constants c_1, c_2 in the form $c_1 = A \cos k_1 t_0$ and $c_2 = -A \, . \sin k_1 t_0$, where $A > 0$ and t_0 are arbitrary constants, we obtain

$$x = Ae^{-\varepsilon t} \sin k_1 (t - t_0). \tag{15}$$

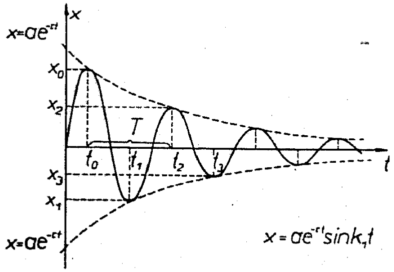

Fig. 83.

Let us select as a new initial time, the time t_0; therefore let us substitute $t - t_0 = t'$. We get $x = Ae^{-\varepsilon(t'+t_0)} \sin k_1 t'$; writing t again instead of t' and putting $Ae^{-\varepsilon t_0} = a$, we obtain

$$x = ae^{-\varepsilon t} \sin k_1 t, \qquad \text{where } a > 0. \tag{V}$$

The graph of the above function is shown in the Fig. 83. In order to determine the extrema of this function, it is necessary to determine the places where the derivative $x^{\cdot} = ae^{-\varepsilon t}(k_1 \cos k_1 t - \varepsilon \sin k_1 t)$ is zero. Hence $x^{\cdot} = 0$ for those values of t, for which

$$\tan k_1 t = k_1 / \varepsilon. \tag{16}$$

If t_0 is the smallest positive root of equation (16), then the remaining roots have the form

$$t_n = t_0 + n\pi / k_1, \tag{17}$$

where n is an arbitrary integer. Examining the sign of the second derivative, we establish that a maximum occurs for an even n, and a minimum for an odd n. It follows from this that at the times t_n the derivative $x^{\cdot}$ changes its sign and therefore the velocity changes its sense.

The times t_n are called *times of return*, while the corresponding positions of the moving point — *points of return.*

The points of return occur periodically every π / k_1 seconds, successively, once to the right and once to the left of the origin O.

The time $T_1 = 2\pi / k_1$ is called, as before, the *period of the motion.*

The time $\frac{1}{2}T_1 = \pi / k_1$ between two times of return is called a *period of oscillation.*

Hence by (17) we have

$$t_n = t_0 + \tfrac{1}{2}nT_1. \tag{18}$$

Let us take under consideration two successive points of return x_n, x_{n+1}, which correspond to the times t_n, t_{n+1}. By (V) and (18) we have

$$|x_n| = ae^{-\varepsilon(t_0 + \frac{1}{2}nT_1)}|\sin k_1 t_0|, \quad |x_{n+1}| = ae^{-\varepsilon(t_0 + \frac{1}{2}(n+1)T_1)}|\sin k_1 t_0|,$$

whence

$$|x_{n+1}| / |x_n| = e^{-\frac{1}{2}\varepsilon T_1}.$$

It follows from this that the coordinates x_n (in absolute value) decrease to zero in geometric progression.

Hence: *if the damping force is small, then the maximum displacements of the points follow each other in equal intervals of time* (period of oscillation) *and decrease to zero in geometric progression.*

2° $\varepsilon^2 - k^2 > 0$. This case arises when the damping force is large. It is easy to verify that the roots of the characteristic equation (13) are in this case negative. Denoting them by $-\varrho_1$ and $-\varrho_2$, we obtain the general solution of the equation (IV) in the form

$$x = Ae^{-\varrho_1 t} + Be^{-\varrho_2 t}, \tag{VI}$$

where A and B are arbitrary constants with $\varrho_1 > 0$ and $\varrho_2 > 0$.

When the time t increases, then x tends to zero rapidly. It is not difficult to verify that there exists at most one point of return. The velocity is therefore zero at most once.

3° $\varepsilon^2 - k^2 = 0$. With this assumption the characteristic equation (13) has a double root $-\varepsilon$. The general solution of (IV) has the form

$$x = e^{-\varepsilon t}(At + B), \tag{VII}$$

where A and B are arbitrary constants.

When the time increases, x tends to zero rapidly. As before, there exists at most one point of return, and therefore the velocity becomes zero at most only once.

Forced harmonic motion. On a material point moving along the x-axis let there act, in addition to an elastic force $\mathbf{P}$ and a damping force $\mathbf{Q}$, a force $\mathbf{R}$ directed along the x-axis and depending only on time.

The component of the force $\mathbf{R}$ will therefore be

$$R = mw\, f(t),$$

where w is a constant.

Let us suppose that the force $\mathbf{R}$ is periodic, e. g. that

$$R = mw \sin(\alpha t + \beta), \tag{19}$$

where α and β are constants. The equation of motion has the form (cf. (IV), p. 114):

$$x^{\cdot\cdot} + 2\varepsilon x^{\cdot} + k^2 x = w \sin(\alpha t + \beta), \tag{20}$$

where the meaning of the constants ε and k is the same as before. In order to obtain the general solution of equation (20), we determine one particular solution of the form

$$x = b \sin(\alpha t + \gamma). \tag{21}$$

Having the determination of b and γ in mind, let us substitute (21) in (20). We get

$$(k^2 - \alpha^2)\, b \sin(\alpha t + \gamma) + 2\alpha\varepsilon b \cos(\alpha t + \gamma) = w \sin(\alpha t + \beta). \tag{22}$$

Setting $\alpha t + \gamma = 0$ the first time, and $\alpha t + \gamma = \frac{1}{2}\pi$ the second time, we get

$$2\alpha\varepsilon b = w \sin(\beta - \gamma), \quad (k^2 - \alpha^2)\, b = w \cos(\beta - \gamma), \tag{23}$$

whence

$$b^2 = \frac{w^2}{(k^2 - \alpha^2)^2 + 4\alpha^2\varepsilon^2}, \quad \tan(\beta - \gamma) = \frac{2\alpha\varepsilon}{k^2 - \alpha^2}, \tag{24}$$

and from these equations we determine b and γ.

On the basis of (24) it is easy to verify that (21) satisfies (20) identically for every t.

Let us consider the case $\varepsilon^2 - k^2 < 0$. The general solution of the homogeneous equation $x^{\cdot\cdot} + 2\varepsilon x^{\cdot} + k^2 x = 0$ is given by formula (15), p. 114. Therefore the general solution of equation (20) is

$$x = Ae^{-\varepsilon t} \sin k_1(t - t_0) + b \sin(\alpha t + \gamma), \text{ where } k_1 = \sqrt{k^2 - \varepsilon^2}. \tag{25}$$

As t increases, the first term tends to zero rapidly, and the motion becomes approximately harmonic with the equation

$$x = b \sin(\alpha t + \gamma).$$

The amplitude of this motion is b. The force $\mathbf{R}$ is periodic with period $T' = 2\pi / \alpha$. The period of the damped harmonic motion is $T_1 = 2\pi / k_1$. Let us suppose that the periods T' and T_1 differ little from each other, so that α differs little from k_1. If the damping force is small, then ε is small; hence $k_1 = \sqrt{k^2 - \varepsilon^2}$ differs little from k. Therefore k also will differ little from α. By (24), b can therefore be large even when w is small (i. e. when the force $\mathbf{R}$ is small).

We see from this that *a small periodic force with a period near that of the motion can cause large displacements of the point from the centre if the damping force is small.*

A company of soldiers marching across a bridge will cause it to vibrate. If the periods of the steps and the vibration of the bridge differ little from each other, the displacements of the bridge can become large so rapidly that the bridge will collapse. Similarly, when an automobile experiences bumps on a bad road, even small bumps, but ones whose period is near the natural period of the car springs, then the vibrations can become so large that the car springs will break.

Lissajous' curves. On a material point let there act a force $\mathbf{P}$ whose projections on the coordinate axes are (in magnitude) proportional to the coordinates of the point and directed towards the origin of the system. We can therefore assume that:

$$P_x = -\lambda_1^2 x, \quad P_y = -\lambda_2^2 y, \quad P_z = -\lambda_3^2 z,$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants. The equations of motion have the form:

$$mx^{\cdot\cdot} = -\lambda_1^2 x, \quad my^{\cdot\cdot} = -\lambda_2^2 y, \quad mz^{\cdot\cdot} = -\lambda_3^2 z.$$

Putting $\lambda_1^2 / m = k_1^2$, $\lambda_2^2 / m = k_2^2$ and $\lambda_3^2 / m = k_3^2$, we obtain:

$$x^{\cdot\cdot} = -k_1^2 x, \quad {}^{\cdot\cdot} = -y\, k_2^2 y, \quad z^{\cdot\cdot} = -k_3^2 z. \tag{26}$$

The solutions of the above equations (cf. p. 111, formula (4)) are the functions:

$$x = a_1 \sin k_1(t - t_0'), \quad y = a_2 \sin k_2(t - t_0''), \quad z = a_3 \sin k_3(t - t_0'''). \quad (27)$$

The periods of these functions are (p. 111, formula (5)):

$$T_1 = 2\pi / k_1, \quad T_2 = 2\pi / k_2; \quad T_3 = 2\pi / k_3. \quad (28)$$

If the motion is periodic of period T, then the ratios $T : T_1$, $T : T_2$, $T : T_3$ must be integers. Therefore the ratios $T_1 : T_2$, $T_1 : T_3$, and $T_2 : T_3$ (or because of (28) the ratios $k_2 : k_1$, $k_3 : k_1$, and $k_3 : k_2$) must be rational numbers. Therefore, if not all of these ratios are rational numbers, then the motion is not a periodic motion.

In the case of motion in the plane, the paths of the motion defined by equations (27) are called *Lissajous' curves*; they play an important role in acoustics.

Example. The motion takes place in the xy-plane. Let $k_2 : k_1 = 2$, and $t_0' = t_0'' = 0$.

Putting $k_1 = k$ and $k_2 = 2k$, we obtain by (27):

$$x = a_1 \sin kt, \quad \text{and} \quad y = a_2 \sin 2kt.$$

Since $y = 2a_2 \sin kt \cos kt$, it follows that $\sin kt = x / a_1$ and $\cos kt = a_1 y / 2a_2 x$, whence $(x / a_1)^2 + (a_1 y / 2a_2 x)^2 = 1$; therefore

$$4a_2 x^4 - 4a_1^2 a_2^2 x^2 + a_1^4 y^2 = 0.$$

The path will therefore be a curve of the fourth degree.

§ 16. Conditions for equilibrium in a force field. If a material point in a certain force field is in equilibrium at the point A, then obviously the force $\boldsymbol{P}$ acting at A is equal to zero. Conversely, if at a certain point $A(x_0, y_0, z_0)$ of the field the force $\boldsymbol{P} = 0$, then the material point situated at A at the time $t = t_0$ without initial velocity (i. e. $\boldsymbol{v}_0 = 0$) will remain constantly at rest, i. e. in equilibrium. This follows from the fact that the initial conditions determine the motion unambiguously, and rest (i. e. motion defined by the equations $x = x_0, y = y_0, z = z_0$) satisfies the initial conditions and the equation $m\boldsymbol{p} = \boldsymbol{P}$; for, we have constantly $\boldsymbol{p} = 0$ and $\boldsymbol{P} = 0$.

In a potential field the partial derivatives of the potential V are equal, as we know, to the projections of the force on the axes of the coordinate system (§ 11, p. 99). Therefore, if the point A is a position of equilibrium, then at the point A:

$$\partial V / \partial x = 0, \ \partial V / \partial y = 0, \ \partial V / \partial z = 0. \quad (1)$$

The above equations hold in particular at those points for which the maxima or minima of the potential occur.

Hence: *the points at which the extrema of a potential occur are the positions of equilibrium.*

The positions of equilibrium can also arise, however, at such points for which the potential does not have an extremum; for, equations (1) represent only the necessary conditions for the existence of an extremum.

Stable equilibrium. Let a material point be in equilibrium at the point A in a force field.

Equilibrium is said to be *stable* if a material point, after being displaced slightly from the point A and after receiving initially a small amount of kinetic energy, will constantly move at a small distance from A and possess constantly a small amount of kinetic energy. Strictly speaking, equilibrium at A is stable, if for every two numbers $R > 0$ and $\varepsilon > 0$, we can choose numbers $R_0 > 0$ and $\varepsilon_0 > 0$, such that a material point situated anywhere at a distance less than R_0 from A, after receiving initially kinetic energy in amount less than ε_0, will move at distance from A constantly less than R and possess kinetic energy constantly less than ε.

If the equilibrium at the point A is not stable, then this point is said to be in an *unstable equilibrium.*

Dirichlet's theorem. *In a potential field a point at which the potential attains a proper maximum is the position of stable equilibrium.*

Proof. In a certain potential field let the potential V attain a proper maximum at the point A (a function is said to attain a *proper maximum* at the point A if, in a certain region about this point, it assumens its greatest value only at the point A).

Let us assume that the potential has the value zero at A; this we can always obtain by adding a suitable constant, since a potential is defined only to within a certain constant (p. 98).

Let us take arbitrary $R > 0$ and $\varepsilon > 0$. Without any loss of generality of proof we can also choose an R so small that in a sphere K with centre at A and radius R, the potential is negative everywhere outside of A. Let us denote the maximum potential on the surface of the sphere K by L; therefore $L < 0$.

Now let ε_0 be an arbitrary number satisfying the inequalities:

$$\varepsilon_0 > 0, \quad \varepsilon_0 < -\tfrac{1}{2}L, \quad \varepsilon_0 < \tfrac{1}{2}\varepsilon. \quad (2)$$

Since the potential is zero at A, there exists a sphere K_0 with centre at A and radius $R_0 < R$, such that

$$-\varepsilon_0 < V \leqq 0 \text{ in sphere } K_0. \tag{3}$$

Let us place the material point anywhere at a distance $< R_0$ from A (i. e. in sphere K_0) and give it an initial kinetic energy

$$E_0 < \varepsilon_0. \tag{4}$$

By (5), p. 105,

$$E - V = E_0 - V_0 \tag{5}$$

constantly during the motion.

Since by (3) $-\varepsilon_0 < V_0$, we have on account of (5) and (4)

$$E - V < 2\varepsilon_0. \tag{6}$$

As $E \geqq 0$, we obtain $-V \leqq 2\varepsilon_0$, whence by (2) $-V < -L$, so that $V > L$. Therefore the material point never goes outside the surface of the sphere K (because the potential on it is $\leqq L$); its motion will hence take place inside the sphere K, i. e. at a distance from A less than R. In addition, within the sphere K, $V \leqq 0$ constantly, i. e. $-V \geqq 0$; therefore by (6) $E < 2\varepsilon_0$, whence by (1) $E < \varepsilon$. Hence the equilibrium at A is stable, q. e. d.

Example. Let us consider a force field in which $P_x = -k^2x$, $P_y = -k^2y$, $P_z = -k^2z$. The field is hence a potential field with a potential $V = -\frac{1}{2}k^2r^2$, where $r^2 = x^2 + y^2 + z^2$.

The point $A(0, 0, 0)$ is the position of stable equilibrium because at this point the potential attains the largest value zero, and beyond it is negative.

We shall prove now directly the stability of equilibrium at A.

Let $R > 0$ and $\varepsilon > 0$ be given arbitrary numbers. Let us place a material point at a distance r_0 from A and give it a kinetic energy E. Therefore $E + \frac{1}{2}k^2r^2 = E_0 + \frac{1}{2}k^2r_0^2$, whence

$$E \leqq E_0 + \tfrac{1}{2}k^2r_0^2. \tag{7}$$

In addition $\frac{1}{2}k^2r^2 \leqq E_0 + \frac{1}{2}k^2r_0^2$, whence

$$r \leqq \sqrt{\frac{2}{k^2}E_0 + r_0^2}. \tag{8}$$

If we therefore choose ε_0 and R_0 such that

$$\varepsilon_0 + \tfrac{1}{2}k^2R_0^2 < \varepsilon \quad \text{and} \quad \sqrt{\frac{2}{k^2}\varepsilon_0 + R_0^2} < R$$

simultaneously, then we obtain for every $E_0 < \varepsilon_0$ and $r_0 < R_0$ by (7) and (8) $E < \varepsilon$ and $r < R$. Thus we have proved that the equilibrium at A is stable.

II. DYNAMICS OF A CONSTRAINED POINT

§ 17. Equations of motion. So far we have examined the motion of an *unconstrained* material point, i. e. one which could execute arbitrary motions when acted upon by suitable forces. However, we shall also encounter problems in which the motions of the point are subject to certain restraints, e. g., that the point must always remain on a certain line, surface, etc.

Example. Let us imagine that a small sphere is strung on a stiff wire (e. g. in the form of a circle). Whatever the forces acting on the sphere, it can execute only those motions during which it will always remain on the wire. Therefore, the problem in this case is that of investigating the motion of a material point which must always remain on a certain curve.

Such a point is said to be *constrained*, and the restraining conditions which the motions of the constrained point must satisfy are called *constraints*.

Reaction. When inquiring into the motion of constrained points, we shall assume that there acts on the constrained point (besides the given forces) a certain additional force which causes the point to maintain constraints. This additional force is called the *reaction*.

We attribute this reaction to the action on a material point by the bodies causing the constraints. The reaction of the wire is therefore e. g. the force with which the wire resists its being left by the sphere strung on it.

Let a material point A be constrained to remain on the curve C (Fig. 84). Let the reaction at a certain position of the point A be **R**. The component **N** of the reaction, perpendicular to the tangent, is called the *normal reaction*, the tangential component **T** is called the *tangential reaction* or *friction*.

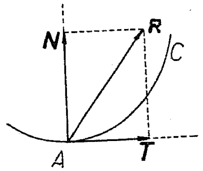

Fig. 84.

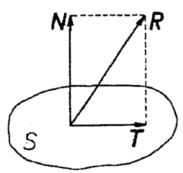

Fig. 85.

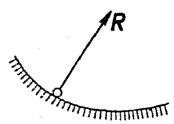

Fig. 86.

Similarly, if a point is constrained to remain on a certain surface S (Fig. 85), then the vector component of the reaction perpendicular to the surface S is called the *normal reaction*, whereas the tangential vector component is called the *tangential reaction* or *friction*.

Therefore, every time we assume that there is no friction, we are assuming equivalently that the reaction is perpendicular to the curve (surface). If there is no friction, the curve (surface) is said to be *smooth*.

If we only assume that a point A lying on a certain side of a surface cannot pass to the other side (even though it can leave this surface), then the reaction is regarded as being directed towards that side of the surface on which the point lies (Fig. 86).

For instance, if a small ball lies on a table, then the reaction of the table is directed upwards.

Equations of motion. We have defined the reaction as an additional force which causes the constrained point to maintain constraints. Therefore, if we add the reaction **R** to the acting force **P**, then we can regard the material point as an unconstrained point. Denoting the mass by m, and the acceleration of the point by **p**, we therefore obtain

$$m\mathbf{p} = \mathbf{P} + \mathbf{R}. \tag{I}$$

In this manner the investigation of the motion of a constrained point is reduced to the investigation of an unconstrained point. If we assume in addition, that the reaction satisfies certain special conditions, e. g. that there is no friction, then (as we shall show later) equation (I) is sufficient to determine the motion.

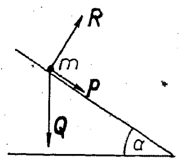

Fig. 87.

Example. Let a point of mass m slide down a plane, inclined at an angle α with the horizontal, under the influence of its weight $Q = mg$.

Let us assume that there is no friction. The reaction **R** is therefore perpendicular to the plane (Fig. 87).

Denoting the acceleration of the point by **p**, we have by (I) $m\mathbf{p} = \mathbf{Q} + \mathbf{R}$. Forming the projections on the inclined plane and putting $p = |\mathbf{p}|$, we obtain $mp = mg \sin\alpha$, whence

$$p = g \sin\alpha.$$

Kinetic energy. The increase in the kinetic energy of a constrained point is equal to the sum of the works of the acting force **P** and the reaction **R**. Under the assumption that there is no friction, the reaction is perpendicular to the path, and therefore the work of the reaction is zero.

It follows from this that, if there is no friction, the increase in kinetic energy is equal only to the work done by the force **P**.

In particular, *if there is no friction, then the sum of the kinetic and potential energies of a point moving in a potential field is constant.*

§ 18. Motion of a constrained point along a curve. Motion along a plane curve. Let us assume that a point A of mass m is to remain on a plane curve C, and that the force **P** acting on the point lies in the plane of the curve C. Let us suppose that there is no friction, i. e. that the reaction **R** is perpendicular to the curve.

Denoting the acceleration of the point by **p**, we have (cf. formula (I), p. 122)

$$m\mathbf{p} = \mathbf{P} + \mathbf{R}. \tag{1}$$

Let us give the tangent t a sense agreeing with that of the curve, and the normal n a sense towards the centre of curvature (Fig. 88). Let p_t, p_n, P_t, P_n be the projections of the acceleration **p** and the force **P** on the tangent and the normal, and let R be the projection of the reaction **R** on the normal. Forming the projections on the tangent and normal, we obtain from equation (1)

$$mp_t = P_t, \quad mp_n = P_n + R. \tag{2}$$

Let v denote the projection of the velocity on the tangent, and ϱ the radius of curvature. Then (p. 41):

$$p_t = v^{\cdot}, \quad p_n = v^2 / \varrho,$$

whence by (2):

$$mv^{\cdot} = P_t, \quad mv^2 / \varrho = P_n + R. \tag{I}$$

The first of the equations (I) enables one to determine the motion if one knows the force **P** or its projection P_t. This equation can also be written in another form, namely:

$$ms^{\cdot\cdot} = P_t, \tag{3}$$

where s denotes the arc coordinate on the curve C.

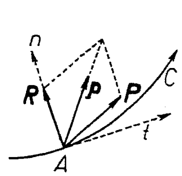

Fig. 88.

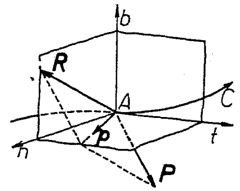

Fig. 89.

The second of the equations (I) enables one to calculate the reaction **R** if one knows the velocity v.

Motion along a space curve. Let us assume that the path is a space curve C and that there is no friction.

Let us give the tangent t a sense agreeing with that of the curve, the principal normal n a sense towards the centre of curvature, and finally the binormal b a sense such that the system (t, n, b) has a sense agreeing with that of the coordinate system (Fig. 89).

Let us form projections on the tangent, the principal normal, and the binormal. Since the projection of the acceleration on the binormal (p. 42) and the projection **R** on the tangent are zero, we obtain from the equation $m\mathbf{p} = \mathbf{P} + \mathbf{R}$:

$$mp_t = P_t, \quad mp_n = R_n + P_n, \quad 0 = P_b + R_b,$$

whence

$$mv^{\cdot} = P_t, \quad mv^2 / \varrho = P_n + R_n, \quad P_b + R_b = 0. \tag{II}$$

The first of the equations (II) enables one to determine the motion; from the remaining two equations in (II) one can calculate the components R_n and R_b, and hence the reaction **R**.

Motion of a heavy constrained point. Let the force of gravity act on a constrained material point of mass m. Let us assume that there is no friction. The potential of the gravitational force is $V = -mgz$ (the z-axis being directed vertically upwards). By the principle of conservation of total energy we therefore obtain $\frac{1}{2}mv^2 + mgz = \text{const}$, or after simplifying

$$v^2 + 2gz = h. \tag{III}$$

Knowing the velocity v_0 and the coordinate z_0 at a certain moment t_0, we can determine the constant h. We get

$$h = v_0^2 + 2gz_0, \quad \text{whence} \quad v^2 + 2gz = v_0^2 + 2gz_0. \tag{4}$$

From (III) it follows that $2gz \leq h$, and hence $z \leq h / 2g$. Hence the maximum height to which the point can rise is

$$z_{\max} = \left(\frac{1}{2g}\right) h = \left(\frac{1}{2g}\right) v_0^2 + z_0. \tag{5}$$

If a point is situated several times at the same level $z = z'$ during the motion, then by (III) we have $v^2 = h - 2gz'$.

Hence: *on one and the same level a point has one and the same velocity.*

Example 1. A curve C along which a point falls is situated in the vertical xz-plane. The equation of the curve C is $z = f(x)$. Let us assume that at the time $t = 0$ the point is at $A(x_0, z_0)$ and has a velocity $\mathbf{v}_0 = 0$.

Denoting the arc coordinate by s and noting that $v = s^{\cdot}$, we get by (4):

$$s^{\cdot 2} + 2gz = 2gz_0, \quad \text{whence} \quad s^{\cdot 2} = 2g(z_0 - z).$$

Let us select a sense on the curve C which agrees with the initial motion of the point (i. e. a downward sense). Up to the time when the material point arrives at the point B, situated at the same height as the point A (Fig. 90), we have $s^{\cdot} = \sqrt{2g(z_0 - z)}$, and hence $\mathrm{d}s / \sqrt{2g(z_0 - z)} = \mathrm{d}t$. Since $\mathrm{d}s = \sqrt{1 + f'^2(x)}\,\mathrm{d}x$,

$$\int_{x_0}^{\xi} \sqrt{\frac{1 + f'^2(x)}{2g[f(x_0) - f(x)]}}\,\mathrm{d}x = t. \tag{6}$$

The above formula gives the time at which a material point arrives at the point D having coordinates $x = \xi$, $y = f(\xi)$. If x_1 denotes the abscissa of the point B, then for $x_0 \leq \xi < x_1$ integral (6) has a finite value, and hence the time t is finite. For $\xi = x_1$ the integrand becomes infinite because by hypothesis $z_0 = f(x_0) = f(x_1)$. In this case the value of the integral can be finite or infinite. It follows from this that the material point may arrive at the point B or not: this will depend on the shape of the

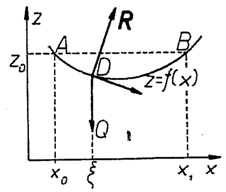

Fig. 90.

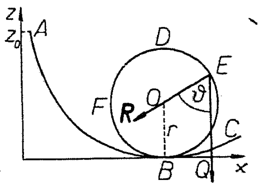

Fig. 91.

curve C. It is easy to show that if the tangent at the point B is not horizontal (i. e. if $f'(x_1) \neq 0$), then the value of (6) is finite, and hence the material point will arrive at the point B.

Example 2. Let a point slide in a vertical plane along a curve C, a portion of which, namely, $BEDF$ is a circle with centre at O and radius r. Let us assume there is no friction. Let us also assume that the point need not always remain on the curve C, just so that it does not go over to the other side; the reaction will therefore be directed towards the side on which the point is situated (Fig. 91).

Let us ask from what height z_0 should a point be released, without initial velocity, in order to traverse the periphery of the circle $BEDF$.

Let us select an arbitrary point E on the circle. Denote by v the velocity of the point at E and by ϑ the angle which the radius OE makes with the vertical. By (I), p. 123, $mv^2 / r = mg \cos\vartheta + R$, whence

$$R = \frac{m}{r}(v^2 - gr\cos\vartheta).$$

Since $R \geqq 0$ (because the reaction must be directed towards the side of the point, i. e. towards the centre of the circle),

$$v^2 - gr\cos\vartheta \geqq 0. \tag{7}$$

Since the point was released from a height z_0 without initial velocity, denoting the ordinate of the point E by z, we shall have $v^2 + 2gz = 2gz_0$. Determining v^2 from this equation and substituting in (7), we obtain $2gz_0 - 2gz - gr\cos\vartheta \geqq 0$, whence

$$z_0 \geqq z + \tfrac{1}{2}r\cos\vartheta. \tag{8}$$

The inequality (8) is the necessary and sufficient condition which must be satisfied by the height z_0 in order that the point traverse the periphery $BDEF$. The right side of this inequality attains its maximum value at the highest point on the circle, at which $z = 2r$ and $\vartheta = 0$. Substituting these values in (8), we obtain

$$z_0 \geqq 5r / 2.$$

Hence, if a material point is released from a height $z_0 \geqq 5r / 2$, then the point will go completely around the circle.

If, on the other hand, $z_0 < 5r / 2$, then at a certain point of the circle, namely, at that point at which $z_0 = z + \frac{1}{2}r\cos\vartheta$ our material point will leave the circle. This is so, because were the point to move farther along the circle, then, as is easily verified, we should have $R < 0$, which is impossible, since this would mean that the point is pressed to the curve. After leaving the circle the point will obviously fall only under the influence of its weight.

Example 3. A point of mass m moves under the action of the force of gravity along a helix

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = k\varphi. \tag{9}$$

We have

$$x^{\cdot} = -r\varphi^{\cdot}\sin\varphi, \quad y^{\cdot} = r\varphi^{\cdot}\cos\varphi \quad \text{and} \quad z^{\cdot} = k\varphi^{\cdot},$$

hence $v^2 = x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2} = (r^2 + k^2)\varphi^{\cdot 2}$, whence by (III), p. 124, we obtain $(r^2 + k^2)\varphi^{\cdot 2} + 2gk\varphi = h$, and therefore

$$\mathrm{d}t / \mathrm{d}\varphi = \pm\sqrt{(r^2 + k^2)} / \sqrt{(h - 2gk\varphi)},$$

and finally

$$t = \pm\frac{1}{gk}\sqrt{r^2 + k^2}\sqrt{h - 2gk\varphi} + c.$$

The sign on the right hand side and the constant c depend on the initial conditions. Expressing φ in terms of t and substituting in (9), we obtain the equations of motion.

§ 19. Motion of a constrained point along a surface. Let a force **P** act on a material point of mass m. Let us assume that there is no friction and that the point is to remain constantly on the surface S whose equation is

$$F(x, y, z) = 0. \tag{1}$$

The reaction **R** is therefore perpendicular to S. From differential geometry it is known that the direction numbers of the normal to the surface are proportional to the partial derivatives $\partial F / \partial x$, $\partial F / \partial y$, $\partial F / \partial z$. Since the reaction **R** has the direction of the normal

$$R_x = \lambda\,\partial F / \partial x, \quad R_y = \lambda\,\partial F / \partial y, \quad R_z = \lambda\,\partial F / \partial z, \tag{2}$$

where λ is a factor of proportionality depending on time. Therefore $\lambda = \lambda(t)$.

From the equation $m\mathbf{p} = \mathbf{P} + \mathbf{R}$ we obtain by (2):

$$mx^{\cdot\cdot} = P_x + \lambda\frac{\partial F}{\partial x}, \quad my^{\cdot\cdot} = P_y + \lambda\frac{\partial F}{\partial y}, \quad mz^{\cdot\cdot} = P_z + \lambda\frac{\partial F}{\partial z}. \tag{I}$$

Equations (1) and (I) taken together determine the unknown functions of time $x = f(t)$, $y = \varphi(t)$, $z = \psi(t)$ and $\lambda = \lambda(t)$. After determining these functions we can calculate the reaction **R** from equations (2).

Example I. A heavy point of mass m moves over the surface of a right circular cylinder (the z-axis being directed vertically upwards)

$$x^2 + y^2 = r^2.$$

We have here $F(x, y, z) \equiv x^2 + y^2 - r^2 = 0$, $P_x = 0$, $P_y = 0$, and $P_z = -mg$; hence by (I):

$$mx^{\cdot\cdot} = 2\lambda x, \quad my^{\cdot\cdot} = 2\lambda y, \quad mz^{\cdot\cdot} = -mg. \tag{3}$$

The third of the equations (3) gives after integrating

$$z = -\tfrac{1}{2}gt^2 + at + b, \tag{4}$$

where a and b are constants. Let the initial conditions for $t = 0$ be:

$$x_0 = r, \quad y_0 = 0, \quad z_0 = 0, \quad x_0^{\cdot} = 0, \quad y_0^{\cdot} = u, \quad z_0^{\cdot} = w, \tag{5}$$

where u and w denote certain constants ($x_0^{\cdot} = 0$, because at the time

$t = 0$ the velocity $\mathbf{v}_0$ is tangent to the cylinder, and hence perpendicular to the x-axis). By (4) and (5) we get $b = 0$, and $a = w$; therefore

$$z = -\tfrac{1}{2}gt^2 + wt. \tag{6}$$

Since $v^2 + 2gz = v_0^2 + 2gz_0$, $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2gz = u^2 + w^2$, whence by (6) $\dot{x}^2 + \dot{y}^2 + (-gt + w)^2 + 2wgt - g^2t^2 = u^2 + w^2$, and therefore

$$\dot{x}^2 + \dot{y}^2 = u^2. \tag{7}$$

Hence the projection of the point on the horizontal plane moves along the circle $x^2 + y^2 = r^2$ with a constant velocity u; the angular velocity is therefore $\omega = u / r$. From this $x = r\cos(ut / r + \varphi_0)$, and $y = r\sin(ut / r + \varphi_0)$. Since at $t = 0$, according to (5), $x_0 = r$ and $y_0 = 0$, we can take $\varphi_0 = 0$. We therefore get:

$$x = r\cos\frac{u}{r}t, \quad y = r\sin\frac{u}{r}t. \tag{8}$$

Equations (6) and (8) define the motion of the point. We obtain the factor λ from equations (3) by substituting for $\ddot{x}$ and x the values obtained from (8). We get $\lambda = -mu^2 / 2r^2$, whence by (2):

$$R_x = -\frac{mu^2}{r^2}x, \quad R_y = -\frac{mu^2}{r^2}y, \quad R_z = 0,$$

and finally

$$R = \sqrt{R_x^2 + R_y^2} = \frac{mu^2}{r^2}\sqrt{x^2 + y^2} = \frac{mu^2}{r}.$$

Hence: the reaction is constant in magnitude and always perpendicular to the axis of the cylinder.

Example 2. A point of mass m, under the influence of gravity, moves on a sphere (the z-axis being directed vertically upwards)

$$x^2 + y^2 + z^2 - r^2 = 0. \tag{9}$$

In virtue of (I), p. 127:

$$m\ddot{x} = 2\lambda x, \quad m\ddot{y} = 2\lambda y, \quad m\ddot{z} = 2\lambda z - mg. \tag{10}$$

Equations (10) cannot be solved by means of elementary functions. Nevertheless, we can deduce certain consequences without solving these equations.

Let us note that the reaction $\mathbf{R}$ is constantly directed towards the center of the sphere, and hence that its projection $\mathbf{R}'$ on the horizontal plane is constantly directed towards the origin of the coordinate system. Consequently, $\mathbf{R}'$ is a central force.

Since the projection of the force of gravity on the horizontal plane is zero, denoting by $\mathbf{p}'$ the projection of the acceleration of the point on the horizontal plane, we obtain $m\mathbf{p}' = \mathbf{R}'$.

It follows from this (p. 86) that the motion of the projection will be a central motion. The path of projection will therefore be either a straight line l passing through the origin O, or a curve C which will never pass through the origin (p. 86).

In the first case the motion of the point itself will take place in a vertical plane whose trace is l; hence the point will move along a meridian. This case will occur if the point is given an initial velocity tangent to the meridian, because then the projection of the velocity (on the xy-plane) will be directed towards the origin O, the areal velocity of the projection will be zero, and the path of the projection will be a straight line passing through the centre O.

In the second case, when the path of the projection is the curve C never passing through O, we will have, denoting by r_0 and r_1 the smallest and the largest distance of the projection from O, $r_0^2 \leqq x^2 + y^2 \leqq r_1^2$.

By (9) $z^2 = r^2 - (x^2 + y^2)$; hence $r^2 - r_1^2 \leqq z^2 \leqq r^2 - r_0^2$, whence

$$\sqrt{r^2 - r_1^2} \leqq |z| \leqq \sqrt{r^2 - r_0^2}.$$

It follows from this that the point goes around the sphere between two horizontal planes. This case will occur if the initial velocity $\mathbf{v}_0$ of the point is not tangent to the meridian, because then the projection of the velocity $\mathbf{v}_0$ on the xy-plane will not be directed towards O and the areal velocity of the projection will be different from zero.

§ 20. Mathematical pendulum. A *mathematical pendulum* is a material point m suspended in a gravitational field by a weightless and inextensible string fixed at one end at the point S.

The string acts on the material point only when it is in tension; the reaction $\mathbf{R}$ is directed along the string towards the point S. The distance of the point m from S is constantly not greater than the length l of the string. The point can therefore move within and on the surface of a sphere K with centre at S and radius l.

Let the string be in tension and make an angle $< \frac{1}{2}\pi$ with the vertical SO. If we release the point m freely (i. e. without an initial velocity), then the point will move in a vertical plane passing through S along a circle with centre at S and radius l.

Taking an arbitrary sense on the circle, let us denote the position of the point A (lying on the lower half of the circle) by means of the arc coordinate s, calculated from the lowest point of the circle O. Let us denote by φ the angle between SO and SA, and let the sign of the angle φ agree with the sense of the arc OA. Therefore

$$s = l\varphi. \tag{1}$$

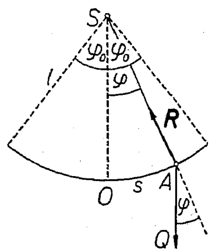

Fig. 92.

Forming the projections of the force of gravity **Q** and the reaction **R** on the tangent at the point A, we obtain $ms^{\cdot\cdot} = -mg\sin\varphi$, and since by (1) $s^{\cdot\cdot} = l\varphi^{\cdot\cdot}$, it follows that $ml\varphi^{\cdot\cdot} = -mg\sin\varphi$, and hence

$$\varphi^{\cdot\cdot} = -\frac{g}{l}\sin\varphi. \tag{I}$$

Suppose that at $t = 0$, we had $\varphi = \varphi_0 > 0$. During the entire motion obviously $-\varphi_0 \leqq \varphi \leqq \varphi_0$, since the point cannot rise to position higher than the initial position.

If φ_0 is sufficiently small, then we can assume with a good approximation that $\sin\varphi = \varphi$. Therefore by (1) we obtain

$$\varphi^{\cdot\cdot} + \frac{g}{l}\varphi = 0,$$

and since according to (1) $\varphi = s/l$,

$$s^{\cdot\cdot} + \frac{g}{l}s = 0. \tag{2}$$

Comparing equation (2) with the equation of harmonic motion (p. 110) we see that the point will move with a harmonic motion. In our case $k = \sqrt{g/l}$, so that the period of motion (by (5), p. 111) is

$$T = 2\pi\sqrt{l/g}. \tag{3}$$

Formula (3) is an approximate formula derived on the assumption that the angle φ_0 is small. It is interesting to note that the *period T does not depend on the angle of the displacement.*

Let us now discard the assumption that angle φ_0 is small. Let us multiply both sides of equation (I) by $\varphi^{\cdot}$ and integrate. We obtain:

$$\tfrac{1}{2}\varphi^{\cdot 2} = \frac{g}{l}\cos\varphi + c. \tag{4}$$

$\varphi = \varphi_0$ and $s^{\cdot} = 0$ for $t = 0$; therefore by (1) $\varphi^{\cdot} = 0$. From equation (4) for $t = 0$ we get $0 = g\cos\varphi_0/l + c$, whence $c = -g\cos\varphi_0/l$, and hence $\tfrac{1}{2}\varphi^{\cdot 2} = g(\cos\varphi - \cos\varphi_0)/l$; therefore

$$\varphi^{\cdot} = \pm\sqrt{2g/l}\sqrt{\cos\varphi - \cos\varphi_0}. \tag{5}$$

Let us suppose that we are investigating the motion of the point from the time $t = 0$ to the time when the point reaches the same elevation on the opposite side of the line OS. Therefore $\varphi^{\cdot} \leqq 0$, and (5) will be

$$\varphi^{\cdot} = -\sqrt{\frac{2g}{l}}\sqrt{\cos\varphi - \cos\varphi_0}, \quad\text{whence}\quad -\sqrt{\frac{l}{2g}}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = dt.$$

Denoting the period of oscillation by T, we obtain

$$-\sqrt{\frac{l}{2g}}\int_{\varphi_0}^{-\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = \tfrac{1}{2}T;$$

therefore

$$T = \sqrt{\frac{2l}{g}}\int_{-\varphi_0}^{\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = 2\sqrt{\frac{2l}{g}}\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}}. \tag{6}$$

Let us introduce a new variable u by means of the equation $\sin\tfrac{1}{2}\varphi = \sin u \sin\tfrac{1}{2}\varphi_0$. Since $\cos\varphi - \cos\varphi_0 = 2(\sin^2\tfrac{1}{2}\varphi_0 - \sin^2\tfrac{1}{2}\varphi)$, we obtain

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi}\frac{du}{\sqrt{1 - \sin^2 u \sin^2\tfrac{1}{2}\varphi_0}}. \tag{7}$$

Evaluating the integral by means of a series expansion, we obtain:

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + (\tfrac{1}{2})^2\sin^2(\tfrac{1}{2}\varphi_0) + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\sin^4(\tfrac{1}{2}\varphi_0) + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\sin^6(\tfrac{1}{2}\varphi_0) + \ldots\right].$$

For small φ_0 we obtain formula (3) by omitting the terms of the series beginning with the second term.

§ 21. Equilibrium of a constrained point. If a constrained point is in equilibrium it means that the acting force **P** balances the reaction **R**. Therefore

$$\mathbf{P} + \mathbf{R} = 0. \tag{I}$$

The above equation represents the *necessary condition for equilibrium.*

If there is no friction and the point is constrained to remain on the surface, then — as we know — the reaction is perpendicular to the surface. In the case of equilibrium, therefore, the acting force **P** must also be perpendicular to the surface.

Conversely, if at a certain time t the force **P** is perpendicular to the surface S, and the point has a velocity $\mathbf{v} = 0$, then $\mathbf{P} + \mathbf{R} = 0$, so that the point will remain at rest. For, suppose that $\mathbf{P} + \mathbf{R} \neq 0$; then the point would move along a certain curve C lying on a surface S. Let us note that at the time t the normal acceleration is $p_n = v^2 / \varrho = 0$. From the equation $m\mathbf{p} = \mathbf{P} + \mathbf{R}$, after forming the projections on the tangent to C, we obtain $mp_t = 0$, because $\mathbf{P}$ and $\mathbf{R}$ are perpendicular to the tangent. Since $p_n = 0$ and $p_t = 0$, it follows that $\mathbf{p} = 0$. Therefore we would have $\mathbf{P} + \mathbf{R} = m\mathbf{p} = 0$, which is contrary to hypothesis.

Hence: *the necessary and sufficient condition for equilibrium of a constrained point having to remain (without friction) on a certain surface is that the acting force be perpendicular to the surface.*

A similar theorem holds for a curve.

Stable equilibrium. We define the stable equilibrium of a constrained point in a manner similar to that for an unconstrained point (p. 119), with this difference, that the displacement from the position of equilibrium has to be consistent with the constraints. A point will therefore be in *stable equilibrium* if after a small displacement (consistent with the constraints) from the position of equilibrium, and after receiving initially a small amount of kinetic energy, it will move constantly in the vicinity of the position of equilibrium and possess constantly a small amount of kinetic energy.

Equilibrium in a potential field. Let a material point in a potential field be constrained to remain on a certain surface whose equation is $F(x, y, z) = 0$. Let us assume that there is no friction.

If at a certain point $A(x, y, z)$ of the surface S the potential V attains an extremum with respect to the points on that surface, then the point A is the position of equilibrium.

For, by hypothesis, the point A is an extremum of the function V with the subsidiary condition $F(x, y, z) = 0$. Therefore by a theorem from the theory of maxima and minima there exists a constant λ such that:

$$\frac{\partial V}{\partial x} + \lambda\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial V}{\partial y} + \lambda\frac{\partial F}{\partial y} = 0, \qquad \frac{\partial V}{\partial z} + \lambda\frac{\partial F}{\partial z} = 0.$$

Therefore

$$P_x + \lambda\frac{\partial F}{\partial x} = 0, \qquad P_y + \lambda\frac{\partial F}{\partial y} = 0, \qquad P_z + \lambda\frac{\partial F}{\partial z} = 0.$$

Since $\partial F / \partial x$, $\partial F / \partial y$, $\partial F / \partial z$ are proportional to the direction

cosines of the normal at A, the force $\mathbf{P}$ has the direction of the normal, i. e. the point A is actually the position of equilibrium.

If a point A is a proper maximum of a potential with respect to the points of a surface S, then the point A is the position of stable equilibrium.

The proof is similar to that on p. 119.

The above remarks apply equally to the case when the material point is constrained to remain on a curve.

Let a point in a gravitational field be constrained to remain on a surface S whose equation is $z = f(x, y)$ (the z-axis being directed vertically upwards). The positions of equilibrium are those points at which the force of gravity is perpendicular to the surface, i. e. at which the tangent plane is horizontal. These points can be the highest or lowest points (relative to the surrounding ones) or so-called saddle points. The proper maximum of the potential $V = -mgz$ occurs at those points for which the function $z = f(x, y)$ attains a proper minimum. Stable equilibrium therefore occurs at the lowest points. The points A, B are then positions of stable equilibrium; whereas C is a position of unstable equilibrium (see Fig. 93.).

If we displace the point from the position A, e. g. to A' and impart to it a small velocity, then it will move in the depression around the point A with a small velocity. If, on the other hand, we displace the point (even ever so slightly) from the position C to the position C', then obviously it will move away from C under the influence of its weight.

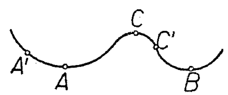

Fig. 93.

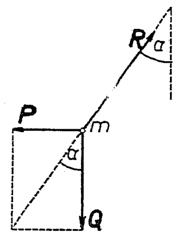

Fig. 94.

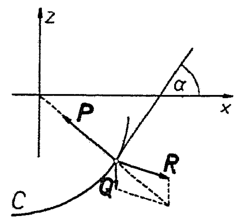

Fig. 95.

Example 1. A heavy material point hanging on a string making an angle α with the vertical is in equilibrium under the influence of a horizontal force $\mathbf{P}$ (Fig. 94). The point is acted upon by the reaction $\mathbf{R}$ of the string directed along the string (towards the point of suspension), the weight $\mathbf{Q}$, and the force $\mathbf{P}$. Therefore

$$\mathbf{R} + \mathbf{Q} + \mathbf{P} = 0.$$

Putting $|\mathbf{R}| = R$, $|\mathbf{P}| = P$ and $|\mathbf{Q}| = mg$, we obtain from the triangle formed by the forces **R**, **Q** and **P**

$$P = mg \tan\alpha, \qquad R = mg / \cos\alpha.$$

***Example* 2.** The equation of a curve C lying in the xz-plane is $z = f(x)$. A heavy point on the curve C is attracted towards the origin O of the coordinate system by a force **P** whose magnitude is proportional to the distance of the point from O. In what position will the point be in equilibrium if we assume that there is no friction?

In a position of equilibrium the force **P**, the weight **Q**, and the reaction **R** balance each other (Fig. 95, p. 133); hence

$$\mathbf{P} + \mathbf{Q} + \mathbf{R} = 0. \tag{1}$$

The projections of the force **P** on the axes of the coordinate system are:

$$P_x = -\lambda^2 x, \quad P_z = -\lambda^2 z, \tag{2}$$

where λ is a constant of proportionality. Let α denote the angle which the tangent at the position of equilibrium makes with the x-axis. Projecting on the tangent, we obtain from (1) and (2) $-\lambda^2 x\cos\alpha - \lambda^2 z \sin\alpha - mg\cdot\sin\alpha = 0$. Dividing by $\cos\alpha$ and noting that $\tan\alpha = z'$, we get

$$\lambda^2 x + \lambda^2 zz' + mgz' = 0. \tag{3}$$

Knowing the function $z = f(x)$, we can determine the x coordinate of the position of equilibrium from equation (3).

For example, if the curve C is the parabola $z = x^2 - a$, then by (3) we have $\lambda^2 x + 2\lambda^2(x^2 - a)\,x + 2mgx = 0$, whence

$$x_1 = 0, \text{ and } x_{2,3} = \pm\sqrt{\frac{\lambda^2(2a-1) - 2mg}{2\lambda^2}}.$$

The solutions $x_{2,3}$ exist provided that the expression under the radical is positive.

Let us ask now: *what is the curve on which a point is everywhere in equilibrium?*

For such a curve equation (3) must be satisfied identically. Integrating it, we obtain $\frac{1}{2}\lambda^2 x^2 + \frac{1}{2}\lambda^2 z^2 + mgz = \text{const.}$, whence

$$x^2 + \left(z + \frac{mg}{\lambda^2}\right) = \text{const.}$$

Such a curve is therefore an arbitrary circle with centre at the point $(0, -mg/\lambda^2)$.

III. DYNAMICS OF RELATIVE MOTION

§ 22. Laws of motion. Let us suppose that we are investigating the motion of a material point in a frame (x, y, z) moving relative to the inertial frame. Considering the inertial frame (*vide* p. 69) as fixed and the frame (x, y, z) as moving, we obtain (p. 60):

$$\mathbf{p}_a = \mathbf{p}_r + \mathbf{p}_t + \mathbf{p}_C \quad \text{or} \quad \mathbf{p}_r = \mathbf{p}_a - \mathbf{p}_t - \mathbf{p}_C, \tag{1}$$

where $\mathbf{p}_a, \mathbf{p}_r, \mathbf{p}_t, \mathbf{p}_C$ denote the accelerations: absolute, relative, transport and Coriolis. Multiplying (1) on both sides by the mass m of the given point, we get

$$m\mathbf{p}_r = m\mathbf{p}_a - m\mathbf{p}_t - m\mathbf{p}_C. \tag{2}$$

Let us put:

$$\mathbf{P}_a = m\mathbf{p}_a, \quad \mathbf{P}_t = -m\mathbf{p}_t, \quad \mathbf{P}_C = -m\mathbf{p}_C. \tag{I}$$

Since $\mathbf{p}_a$ is the acceleration of a point relative to the inertial frame, $\mathbf{P}_a$ is according to Newton's law the force acting on the given material point; it is called the *absolute force*. The vector $\mathbf{P}_t$ is called the *force of transport* or the *centrifugal force*, and the vector $\mathbf{P}_C$ the *force of Coriolis* or the *compound centrifugal force*.

It should be noted that the vectors $-m\mathbf{p}_t$ and $-m\mathbf{p}_C$ do not represent any forces; we have called them forces of transport and of Coriolis only for practical reasons.

By (2) and (I)

$$m\mathbf{p}_r = \mathbf{P}_a + \mathbf{P}_t + \mathbf{P}_C. \tag{II}$$

According to Newton's law we have $m\mathbf{p} = \mathbf{P}$ in an inertial frame; we see that equation (II) has a similar form.

Hence: *the laws of motion in a moving frame of reference are such as if the frame were an inertial frame, subject to the condition, however, that to the acting forces we add the force of transport and the force of Coriolis.*

The sum of the forces: absolute, transport, and Coriolis, is called the *relative force* and we denote it by $\mathbf{P}_r$.

Therefore

$$\mathbf{P}_r = \mathbf{P}_a + \mathbf{P}_t + \mathbf{P}_C. \tag{3}$$

Equation (II) can therefore be written in the form

$$m\mathbf{p}_r = \mathbf{P}_r. \tag{III}$$

An observer, being at rest relative to a moving frame and taking it as the inertial frame, will judge that the force acting on the material point is just the relative force $\mathbf{P}_r$. If the frame began its motion at a certain time t_0, then it will seem to the observer that in addition to the force $\boldsymbol{P}_a$ acting previously, a new force $\boldsymbol{P}_t + \boldsymbol{P}_C$ began to act from the time t_0. For instance, a person riding on a merry-go-round judges that in addition to the force of gravity, there acts on him still another force directed from the centre of motion and trying to throw him off the merry-go-round (the centrifugal force). However, to an observer at rest relative to the inertial frame, the forces of transport and Coriolis obviously do not exist.

If a moving frame moves with an advancing motion with a constant velocity relative to a certain inertial frame, then $\boldsymbol{p}_t = 0$ and $\boldsymbol{p}_C = 0$ (*vide* p. 61); because of this $\boldsymbol{P}_t = 0$ and $\boldsymbol{P}_C = 0$, and by (II)

$$m\boldsymbol{p}_r = \boldsymbol{P}_a.$$

Therefore for such a moving frame hold the Newton's laws.

Hence: *every coordinate system which moves with an advancing motion with a constant velocity relative to an inertial frame is also an inertial frame.*

We see from this that the laws of mechanics will never enable us to decide whether a given inertial frame is at rest or not.

If we are investigating the motion of a material point in a certain frame of reference (x, y, z), then we can obtain the relative force $\boldsymbol{P}_r$ from equation (III). If we know in addition the absolute force $\boldsymbol{P}_a$ from another source and if we observe that $\boldsymbol{P}_a \neq \boldsymbol{P}_r$, then we shall be able to establish that the frame (x, y, z) is not an inertial frame and hence that it moves relative to every inertial frame.

§ 23. Examples of motion. Advancing motion of a frame. If a frame moves with an advancing motion, the acceleration of Coriolis $\boldsymbol{p}_C = 0$ (p. 61), and hence the force of Coriolis $\boldsymbol{P}_C = 0$. The acceleration of transport is constant for all points and is equal to the acceleration of the origin of the frame (relative to the inertial frame). Therefore the force of transport is constant. It follows from this that the force of transport forms a potential field (p. 100). By (II), p. 135, we then have

$$m\boldsymbol{p}_r = \boldsymbol{P}_a + \boldsymbol{P}_t. \tag{I}$$

Example 1. An inclined plane moves with a constant horizontal acceleration $\boldsymbol{a}$. A heavy point of mass m is situated on the inclined plane. Friction is not considered. What acceleration will the point m have with respect to the inclined plane?

The absolute forces are: the weight $\boldsymbol{Q}$ and the reaction $\boldsymbol{R}$ perpendicular to the inclined plane. The force of transport is $-m\boldsymbol{a}$. Let us select as the x-axis the intersection of the inclined plane with the vertical plane passing through m and give to it a downward sense (Fig. 96). Denot-

ing by α the angle which the inclined plane makes with the horizontal and forming the projections on the x-axis, we obtain from (I)

$$p = g \sin\alpha - a\cos\alpha, \tag{1}$$

where $p = p_{r_x}$, and $a = |\boldsymbol{a}|$. We see from this that $p > 0$ or $p < 0$, depending on whether $a < g\tan\alpha$ or $a > g\tan\alpha$.

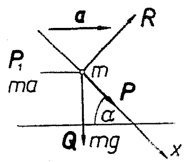

Fig. 96.

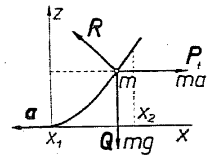

Fig. 97.

Example 2. A frame (x, y, z) moves with an advancing motion with a constant horizontal acceleration $\boldsymbol{a}$ in a gravitational force field. Let us assume that the z-axis is directed vertically upwards and that the x-axis has the direction of the acceleration $\boldsymbol{a}$, but an opposite sense.

The force of transport is $\boldsymbol{P}_t = -m\boldsymbol{a}$; putting $a = |\boldsymbol{a}|$ we obtain $P_{t_x} = ma$, $P_{t_y} = 0$, and $P_{t_z} = 0$. It is easy to see that the force of transport forms a potential field having the potential $V_t = max$; the potential of the force of gravity is $V_g = -mgz$. The relative force therefore forms a field having the potential

$$V = max - mgz. \tag{2}$$

If only the force of gravity $\boldsymbol{Q}$ acts on the material point, then applying the theorem on the conservation of total energy and setting $v = |\boldsymbol{v}_r|$, we obtain by (2) $\frac{1}{2}mv^2 - V = \text{const}$, whence

$$v^2 - 2ax + 2gz = h, \tag{3}$$

where h is a certain constant.

Let us suppose now that we are investigating the motion of a constrained point which is to remain on a curve $z = x^2$ lying in the xz-plane (Fig. 97).

Let us assume that at $t = 0$, $x = 0$ and $v = 0$. If friction is neglected, then the reaction is perpendicular to the path and does no work. Hence equation (3) applies to the motion. From the initial conditions it follows that $h = 0$; hence $v^2 - 2ax + 2gx^2 = 0$, whence

$$v^2 = 2x(a - gx). \tag{4}$$

Since $v^2 \geqq 0$, $2x(a - gx) \geqq 0$; it follows from this that $0 \leqq x \leqq a/g$. The motion will therefore take place along the arc closed between the abscissae $x_1 = 0$ and $x_2 = a/g$. Since $v = ds/dt = (ds/dx)\cdot(dx/dt) = = x^{\cdot}\sqrt{1 + (dz/dx)^2} = x^{\cdot}\sqrt{1 + 4x^2}$, it follows in virtue of (4) that $x^{\cdot 2}(1 + 4x^2) = 2x(a - gx)$, whence

$$\sqrt{\frac{1 + 4x^2}{2x(a - gx)}}\, dx = dt,$$

and hence

$$\int_0^x \sqrt{\frac{1 + 4x^2}{2x(a - gx)}}\, dx = t.$$

The above formula is valid from the momet $t = 0$ until the time when the point reaches the abscissa $x_2 = a/g$. For $x_2 = a/g$ in virtue of (4) we have $v = 0$. After that the return motion will take place until the time when the point reaches the abscissa $x = 0$, etc.

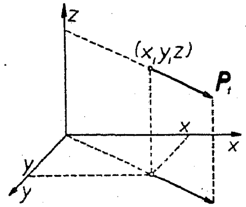

Fig. 98.

Rotary motion of a frame. Let a frame (x, y, z) rotate about the z-axis with a constant angular velocity ω (Fig. 98). The acceleration of transport has the projections: $p_{t_x} = -x\omega^2$, $p_{t_y} = -y\omega^2$ and $p_{t_z} = 0$. Therefore for the force of transport we have:

$$P_{t_x} = mx\omega^2, \quad P_{t_y} = my\omega^2, \quad P_{t_z} = 0.$$

It is easy to see that the force of transport forms a field having the potential

$$V = \tfrac{1}{2}m\omega^2(x^2 + y^2). \tag{5}$$

The acceleration of Coriolis will be $\mathbf{p}_C = 2\mathbf{v}_r \times \omega$ (p. 62). The projections of the relative velocity on the x, y, z axes are $x^{\cdot}, y^{\cdot}, z^{\cdot}$, whereas $\omega_x = 0$, $\omega_y = 0$, and $\omega_z = \omega$. Therefore $p_{C_x} = 2y^{\cdot}\omega$, $p_{C_y} = -2x^{\cdot}\omega$ and $p_{C_z} = 0$, whence

$$P_{C_x} = -2my^{\cdot}\omega, \quad P_{C_y} = 2mx^{\cdot}\omega, \quad P_{C_z} = 0.$$

The equations of motion will therefore have the form (p. 135, formula (II)):

$$mx^{\cdot\cdot} = P_{a_x} + mx\omega^2 - 2my^{\cdot}\omega, \quad my^{\cdot\cdot} = P_{a_y} + my\omega^2 + 2mx^{\cdot}\omega, \quad mz^{\cdot\cdot} = P_{a_z}. \tag{6}$$

The work of a force in relative motion is called *relative work*.

Since $\mathbf{p}_C$ is perpendicular to $\mathbf{v}_t$, $\mathbf{P}_C$ is perpendicular to $\mathbf{v}_r$; therefore the force of Coriolis does no relative work. The relative work of a relative force is hence reduced to the work of the absolute force and the force of transport.

If the absolute force is the force of gravity, then taking the z-axis as directed vertically upwards, we obtain $V_g = -mgz$ as the potential of the force of gravity. The force of gravity together with the force of transport forms a potential field having the potential

$$V = -mgz + \tfrac{1}{2}m\omega^2(x^2 + y^2).$$

Therefore, if we set $v = |\mathbf{v}_r|$, then by the principle of equivalence of work and kinetic energy (p. 105) we get $\frac{1}{2}mv^2 - V = \text{const}$; hence

$$\tfrac{1}{2}mv^2 + mgz - \tfrac{1}{2}m\omega^2(x^2 + y^2) = \text{const}.$$

Therefore

$$v^2 + 2gz - \omega^2(x^2 + y^2) = h, \tag{7}$$

where h is a constant.

If we are investigating the motion of a constrained point along a curve (or surface) motionless relative to a frame (x, y, z), then under the assumption that there is no friction, the reaction does no relative work; hence formula (7) also applies in this case.

Example 3. A plane curve C revolves with a constant angular velocity ω about a vertical axis lying in its plane. Determine the motion of a constrained point moving along a curve C under the influence of the force of gravity.

Let us choose the z-axis directed vertically upwards as the axis of revolution, and the xz-plane as the plane of the curve C. Let the equation of the curve C be $z = f(x)$. Because $y = 0$, we get by (7): $v^2 + 2gz - \omega^2x^2 = h$. Assuming that at $t = 0$, $x = x_0$, $z = z_0 = f(x_0)$, and $v = 0$, we obtain $h = 2gz_0 - \omega^2x_0^2$; $ds = dx\sqrt{1 + f'^2(x)}$, whence $v = s^{\cdot} = = x^{\cdot}\sqrt{1 + f'^2(x)}$; therefore

$$x^{\cdot 2}(1 + f'^2(x)) + 2g\,f(x) - \omega^2x^2 = h. \tag{8}$$

From this equation we can determine x as a function of the time t.

Example 4. In particular, let the curve C (in above example) be the straight line l passing through the origin O of the frame and inclined at an angle φ with the z-axis (Fig. 99).

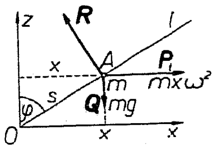

Fig. 99.

In order to determine the motion of a point along the line l, one could apply formula (8) by substituting $z = f(x) = x \cot \varphi$. However, we shall derive the equations of motion directly.

The force of transport is perpendicular to the z-axis and equal to $mx\omega^2$. Let s denote the length of the segment OA. Since the force of Coriolis and the reaction are perpendicular to the line l, the projection of the relative force on l is equal to $-mg \cos \varphi + mx\omega^2 \sin \varphi$. As $x = s \sin \varphi$, we obtain

$$ms^{\cdot\cdot} = -mg \cos \varphi + ms\omega^2 \sin^2\varphi,$$

whence

$$s^{\cdot\cdot} - s\omega^2 \sin^2\varphi = -g \cos \varphi. \tag{9}$$

The homogeneous equation $s^{\cdot\cdot} - s\omega^2 \sin^2\varphi = 0$ has a general solution of the form $s = ae^{\omega t \sin\varphi} + be^{-\omega t \sin\varphi}$. Since a particular solution of equation (9) is

$$s = \frac{g \cos \varphi}{\omega^2 \sin^2\varphi},$$

the general solution of this equation will be

$$s = ae^{\omega t \sin\varphi} + be^{-\omega t \sin\varphi} + \frac{g \cos \varphi}{\omega^2 \sin^2\varphi}. \tag{10}$$

The constants a and b are determined from initial conditions. In particular, if $\varphi = \frac{1}{2}\pi$, i. e. the line l is the x-axis, then

$$s = ae^{\omega t} + be^{-\omega t}. \tag{11}$$

§ 24. Relative equilibrium. If a material point is in equilibrium (i. e. at rest) relative to a moving frame, then the relative acceleration $\boldsymbol{p}_r = 0$, and the relative velocity $\boldsymbol{v}_r = 0$. It follows from this that the acceleration of Coriolis $\boldsymbol{p}_C$ is also equal to zero, and hence the force of Coriolis $\boldsymbol{P}_C = 0$. From equation (II), p. 135, we therefore obtain

$$\boldsymbol{P}_a + \boldsymbol{P}_t = 0. \tag{I}$$

Hence: *when a point is in relative equilibrium, the absolute force is in equilibrium with the force of transport.*

Relative equilibrium in a frame moving with an advancing motion. If a frame moves with an advancing motion, the acceleration of transport has a constant value for all points; hence the force of transport must also be the same at every point.

If, in particular, the moving frame moves with an advancing motion with a constant velocity, then $\boldsymbol{p}_t = 0$, whence $\boldsymbol{P}_t = 0$, and equation (I) expressing the condition for equilibrium reduces to the form $\boldsymbol{P}_a = 0$.

Example 1. A heavy point of mass m is hanging on an inextensible string in an elevator moving with an acceleration $\boldsymbol{p}$. Let the string have a vertical direction and let the point be in equilibrium relative to the elevator (i. e. to the frame attached to the elevator). The acting forces, namely the weight $\boldsymbol{Q}$ and the tension in the string $\boldsymbol{T}$, are therefore balanced by the force of transport (Fig. 100). Let us put $p = |\boldsymbol{p}|$ and $T = |\boldsymbol{T}|$.

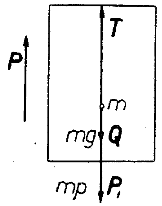

Fig. 100.

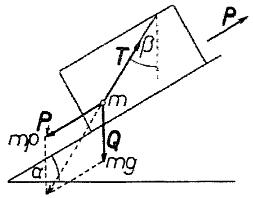

Fig. 101.

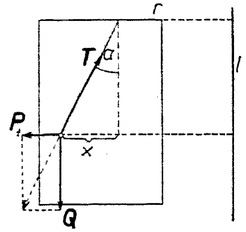

Fig. 102.

The acceleration of transport is $\boldsymbol{p}$. Let us assume that it is directed upwards. Therefore the force of transport is directed downwards and is in magnitude equal to mp. Forming the projections of the forces on the axis directed vertically upwards, we obtain $T - mg - mp = 0$, whence $T = mg + mp$. The tension in the string is therefore greater than the weight. If one held the body in one's hand, one would feel an increase in its weight.

Conversely, if the acceleration is directed downwards, then $T = mg - mp$; the tension in the string is smaller than the weight and the body seems lighter in this case.

Finally, if $p = 0$, then $T = mg$. Hence the tension in the string is equal to the weight during the uniform motion of the elevator.

Example 2. A carriage of a cog-wheel railway moves with an acceleration $\boldsymbol{p}$ along a path inclined at an angle α with the horizontal. A material point hanging on an inextensible string is in equilibrium relative to the carriage. Let β denote the angular deviation of the string from the vertical.

The weight $\boldsymbol{Q}$ of the point and the tension $\boldsymbol{T}$ of the string are in equilibrium with the force of transport $\boldsymbol{P}_t$ (Fig. 101); therefore

$$\boldsymbol{T} + \boldsymbol{Q} + \boldsymbol{P}_t = 0. \tag{1}$$

The acceleration of transport is $\boldsymbol{p}$. Let us assume that it is directed upwards. $\boldsymbol{P}_t$ is hence directed downwards, and $|\boldsymbol{P}_t| = m|\boldsymbol{p}|$. Forming the projections on the horizontal and vertical axes, we obtain from (1):

$$T \sin \beta - mp \cos \alpha = 0, \quad T \cos \beta - mp \sin \alpha - mg = 0, \tag{2}$$

where $T = |\mathbf{T}|$, and $p = |\mathbf{p}|$. From equations (2) we obtain:

$$T = m\sqrt{p^2 + g^2 + 2pg \sin\alpha}, \quad \tan\beta = \frac{p\cos\alpha}{g + p\sin\alpha}.$$

In particular, when $\alpha = 0$, i. e. when the path is horizontal, $T = m\sqrt{p^2 + g^2}$, and $\tan\beta = p / g$. Hence in the railway carriage we can determine the acceleration of the carriage from the angular deviation of the string from the vertical: because we have $p = g \tan\beta$.

Example 3. A railway carriage moves along a horizontal curved path with a constant velocity v. We may suppose that the carriage turns about a certain vertical line l. Let a heavy point of mass m hanging on an inextensible string be in equilibrium relative to the carriage (Fig. 102).

Let us denote by α the angle made by the string with the vertical, and by r the distance from the point of suspension of the string to the line l. The distance of the point m from the l-axis is therefore $r' = r + x = r + d \sin\alpha$ (where d is the length of the string). The acceleration of transport $\mathbf{p}_t$ of the point m is perpendicular to l and directed towards l, while $|\mathbf{p}_t| = v^2 / (r + x)$. The force of transport, having an opposite sense, is $|\mathbf{P}_t| = mv^2 / (r + x)$. Since the weight $\mathbf{Q}$ and the tension $\mathbf{T}$ in the string are in equilibrium with the force of transport, we obtain from the triangle of forces

$$\tan\alpha = |\mathbf{P}_t| / |\mathbf{Q}| = v^2 / g(r + x).$$

When x is small in comparison with r, then $\tan\alpha = v^2 / gr$.

Example 4. A heavy point of mass m is constrained to remain on a curve C revolving about a fixed vertical line l with an angular velocity ω. Friction is not considered. In what position will the point be in equilibrium relative to the curve C?

Let us choose a moving frame (x, y, z) revolving together with the curve C about the l-axis with an angular velocity ω, taking l as the z-axis directed upwards. Let the curve C which is at rest relative to the frame (x, y, z) be given parametrically by means of the functions:

$$x = f(\sigma), \quad y = \varphi(\sigma), \quad z = \psi(\sigma). \tag{3}$$

In a position of relative equilibrium the weight $\mathbf{Q}$, the reaction $\mathbf{R}$, and the force of transport $\mathbf{P}_t$ balance each other (Fig. 103). Therefore the sum $\mathbf{Q} + \mathbf{P}_t$ is perpendicular to the curve C. Denoting the coordinates of the point in relative equilibrium by x, y, z, we obtain $p_{t_x} = -x\omega^2$, $p_{t_y} = -y\omega^2$, and $p_{t_z} = 0$, from which $P_{t_x} = mx\omega^2$, $P_{t_y} = my\omega^2$, and $P_{t_z} = 0$. Therefore the sum $\mathbf{P}_t + \mathbf{Q}$ has the projections: $mx\omega^2$, $my\omega^2$, and $-mg$.

The direction numbers of the tangent are proportional to the derivatives $x' = f'(\sigma)$, $y' = \varphi'(\sigma)$, $z' = \psi'(\sigma)$. From the condition that $\mathbf{P}_t + \mathbf{Q}$ is perpendicular to the tangent it therefore follows (after dividing by m) that

$$xx'\omega^2 + yy'\omega^2 - gz' = 0. \tag{4}$$

From this equation we can determine the value of the parameter σ corresponding to the position of relative equilibrium.

In particular, if the curve C having the equation $z = \psi(x)$ is a plane curve lying in the xz-plane, then for the position of equilibrium we obtain from equation (4) (putting $x = \sigma$, $y = 0$, and $z = \psi(\sigma)$) or directly from the figure:

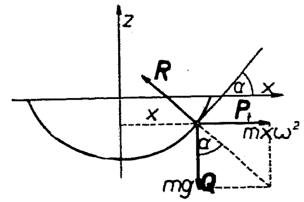

Fig. 103.

$$\tan\alpha = \psi'(x) = x\omega^2 / g. \tag{5}$$

For example, if the equation of the curve C is $z = -\sqrt{r^2 - x^2}$ (i. e. the lower portion of the circle $x^2 + z^2 = r^2$), then from (5) we get $x / \sqrt{r^2 - x^2} = x\omega^2 / g$, whence $x_1 = 0$, and $x_{2,3} = \pm\sqrt{r^2 - g^2 / \omega^4}$. The solutions $x_{2,3}$ exist only when $r^2 - g^2 / \omega^4 \geqq 0$, i. e. when $\omega \geqq \sqrt{g / r}$.

We ask now: *what are the curves on which a point is at every position in relative equilibrium?*

For such curves equation (4) must be satisfied identically, i. e. for every value of the parameter σ. Therefore we obtain from (4)

$$\frac{1}{2}\frac{\mathrm{d}(x^2 + y^2)}{\mathrm{d}\sigma}\omega^2 - g\frac{\mathrm{d}z}{\mathrm{d}\sigma} = 0.$$

Integrating, we get $\frac{1}{2}(x^2 + y^2)\omega^2 - gz = \text{const}$, whence

$$z = \frac{\omega^2}{2g}(x^2 + y^2) + c, \tag{6}$$

where $c = \text{const}$.

Equation (6) represents a system of paraboloids of revolution generated by revolving the parabola $z = \frac{\omega^2}{2g}x^2 + c$ about the z-axis. By (6) the curve lying on any one of these paraboloids satisfies equation (4) identically. The plane curves satisfying equation (4) are obtained by forming section of the paraboloid with an arbitrary plane. We get ellipses and parabolas as sections. In particular, the section with the vertical plane $y = 0$ will be the parabola $z = \frac{\omega^2}{2g}x^2 + c$.

§ 25. Motion relative to the earth. Force of gravity. Let us take as a frame of reference an inertial frame whose origin is in the sun and whose axes are directed towards the fixed stars. The earth is not at rest relative to such a frame. When investigating the motion of a point during a short interval of time, we can confine ourselves to the rotary motion of the earth only about a certain axis.

Let a point hung on a string in a certain place on the earth's surface be at rest relative to the earth. The absolute forces are: the attraction of earth $\boldsymbol{A}$ and the tension $\boldsymbol{T}$ in the string (equal in magnitude and direction to the weight, but opposite in sense).

The earth's force of attraction is not equal to the weight because, in the contrary case, it would be in equilibrium with the tension in the string, and the point would be at rest or in uniform motion along a straight line. However, this is not the case because the point rotates together with the earth about its axis.

Applying the conditions of relative equilibrium (§ 24, p. 140) to the frame attached to the earth, we can say that the attraction $\boldsymbol{A}$ of the earth and the tension $\boldsymbol{T}$ in the string are in equilibrium with the force of transport $\boldsymbol{P}_t$. Hence

$$\boldsymbol{A}+\boldsymbol{T}+\boldsymbol{P}_t=0.$$

Since the weight of the body $\boldsymbol{Q}=-\boldsymbol{T}$, it follows that $\boldsymbol{A}-\boldsymbol{Q}+\boldsymbol{P}_t=0$, whence

$$\boldsymbol{Q}=\boldsymbol{A}+\boldsymbol{P}_t. \tag{I}$$

Hence: *the weight of a body is the resultant of the centrifugal force* (force of transport) *and the earth's force of attraction.*

Magnitude and direction of the earth's attraction. Let us suppose that the earth has the form of a solid of revolution whose axis is the earth's axis of revolution. In addition, let us suppose that the density of the earth is distributed symmetrically with respect to the centre of mass. Then it can be proved that the force of attraction is directed constantly towards the earth's centre of mass.

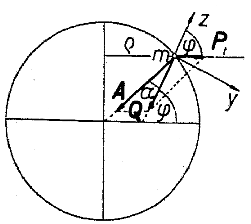

Fig. 104.

Let α be the angle which the force $\boldsymbol{A}$ makes with the vertical, i. e. with the weight $\boldsymbol{Q}$ (Fig. 104). Let us denote the radius of the parallel of latitude on which the material point lies by ϱ, the latitude of this point by φ (i. e. the angle made by the vertical passing through the point and the equatorial plane), and finally the angular velocity of the earth rotating by ω.

The force of transport lies in the plane of the parallel of latitude and is directed away from the axis of revolution, while $|\boldsymbol{P}_t|=m\varrho\omega^2$. Let us form the projections of the force $\boldsymbol{A}$, $\boldsymbol{Q}$, and $\boldsymbol{P}_t$ on the z-axis directed vertically as well as on the horizontal y-axis (i. e. perpendicular to the vertical) lying in the plane of the meridian (i. e. in the plane of the forces) and directed southwards. Setting $A=|\boldsymbol{A}|$, and $Q=|\boldsymbol{Q}|$, we obtain by (I)

$$-\boldsymbol{Q}=-A\cos\alpha+m\varrho\omega^2\cos\varphi,\quad -A\sin\alpha+m\varrho\omega^2\sin\varphi=0.$$

Therefore:

$$A\cos\alpha=Q+m\varrho\omega^2\cos\varphi,\quad A\sin\alpha=m\varrho\omega^2\sin\varphi. \tag{1}$$

Hence, knowing Q, ϱ, ω, and φ, we can calculate A and α. On the equator $\varphi=0$; therefore by (1) we get $\alpha=0$. Denoting by A_0, Q_0, and ϱ_0 the corresponding values on the equator, we obtain

$$A_0=Q_0+m\varrho_0\omega^2. \tag{2}$$

Knowing Q_0 and ϱ_0, we can calculate A_0. Knowing A_0, we obtain

$$m\varrho_0\omega^2\,/\,A_0=\tfrac{1}{289}=(\tfrac{1}{17})^2. \tag{3}$$

If the velocity of the earth were $\omega_1=17\omega$, then $m\varrho_0\omega_1^2\,/\,A_0=1$, whence $A_0=m\varrho_0\omega_1^2$. From this and (2) we obtain $Q_0=A_0-m\varrho_0\omega_1^2=0$; hence if the earth were to turn 17 times faster, then bodies on the equator would be deprived of their weight.

Let us now assume that the earth is a sphere composed of concentric layers of constant density. Then, as can be demonstrated, A must be constant on the earth's surface. Therefore $A=A_0$. Denoting the radius of the earth by R, we obtain

$$\varrho=R\cos(\varphi-\alpha). \tag{4}$$

By (1)

$$\sin\alpha=\frac{mR\omega^2}{A_0}\cos(\varphi-\alpha)\sin\varphi,$$

and since $\varrho_0=R$, by (3)

$$\sin\alpha=\tfrac{1}{289}\cos(\varphi-\alpha)\sin\varphi.$$

Angle α is very small; hence taking as an approximation $\sin\alpha=\alpha$, and $\cos(\varphi-\alpha)=\cos\varphi$, we get

$$\alpha=\frac{1}{2\cdot289}\sin2\varphi.$$

We see from this that α has the greatest value for $\varphi=45°$. Putting $\alpha=0$, we get by (1) and (4)

$$Q = A_0 - mR\omega^2 \cos^2\varphi = A_0 \left(1 - \frac{mR\omega^2}{A_0} \cos^2\varphi\right),$$

whence by (3), as $\varrho_0 = R$, we obtain

$$Q = A_0(1 - \cos^2\varphi \,/\, 289).$$

Force of Coriolis. When investigating the motion of a point relative to the earth, it is necessary to add the forces of transport and Coriolis to the absolute forces. Let us assume that in addition to the force of attraction $\mathbf{A}$, a force $\mathbf{P}$ acts on a material point. Denoting the acceleration relative to the earth by $\mathbf{p}$, we obtain $m\mathbf{p} = \mathbf{P} + \mathbf{A} + \mathbf{P}_t + \mathbf{P}_C$, and since $\mathbf{A} + \mathbf{P}_t$ is equal to the weight $\mathbf{Q}$,

$$m\mathbf{p} = \mathbf{P} + \mathbf{Q} + \mathbf{P}_C. \tag{5}$$

Therefore: *when inquiring into the motion of a point relative to the earth it is necessary to add the force of Coriolis* $\mathbf{P}_C$ *to the force* $\mathbf{P}$ *and the weight* $\mathbf{Q}$.

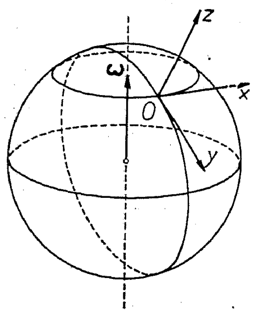

Fig. 105.

At a given place on the earth let us select a z-axis directed vertically upwards, a horizontal x-axis directed towards the east and a horizontal y-axis directed towards the south (Fig. 105). The axis of revolution will lie in the yz-plane and make an angle of $90° - \varphi$ with the z-axis (cf. Fig. 104). Since the earth revolves from west to east, the vector of angular velocity $\boldsymbol{\omega}$ lying on the axis of revolution is directed from the south pole to the north pole. Putting $|\boldsymbol{\omega}| = \omega$, we obtain

$$\omega_x = 0, \quad \omega_y = -\omega \cos\varphi, \quad \omega_z = \omega \sin\varphi \tag{N}$$

on the northern hemisphere, and

$$\omega_x = 0, \quad \omega_y = -\omega \cos\varphi, \quad \omega_z = -\omega \sin\varphi \tag{S}$$

on the southern hemisphere.

Denoting by $\mathbf{v}$ the velocity of the point relative to the earth, we obtain $\mathbf{p}_C = 2\mathbf{v} \times \boldsymbol{\omega}$; therefore $\mathbf{P}_C = -m\mathbf{p}_C = -2m\mathbf{v} \times \boldsymbol{\omega}$, whence

$$\mathbf{P}_C = 2m\boldsymbol{\omega} \times \mathbf{v}, \tag{6}$$

whence by (N), for the northern hemisphere:

$$P_{C_x} = -2m\omega(v_y \sin\varphi + v_z \cos\varphi), \quad P_{C_y} = 2m\omega v_x \sin\varphi, \quad P_{C_z} = 2m\omega v_x \cos\varphi. \tag{II}$$

If the point has only a vertical velocity, i. e. if $v_x = 0$ and $v_y = 0$, we obtain $P_{C_x} = -2m\omega v_z \cos\varphi$, $P_{C_y} = 0$, and $P_{C_z} = 0$. When the point rises, $v_z > 0$; hence $P_{C_x} < 0$ and $\mathbf{P}_C$ is directed horizontally towards the west; whereas when the point falls, $v_z < 0$, $P_{C_x} > 0$, and therefore $\mathbf{P}_C$ is directed horizontally towards the east.

It follows from this that *a falling body is deflected towards the east under the influence of the force of Coriolis.*

If a point moves constantly in a horizontal plane, i. e. when $v_z = 0$, then $P_{C_x} = -2m\omega v_y \sin\varphi$, and $P_{C_y} = 2m\omega v_x \sin\varphi$. The horizontal component of the force of Coriolis is therefore perpendicular to the velocity and has with respect to it a sense to the right.

Therefore: *a point moving in a horizontal plane in the northern hemisphere tends to be deflected* (under the influence of the force of Coriolis) *to the right of the direction of the velocity.*

It is for this reason, for example, that the right rail is pressed down more than the left rail by moving trains.

The effects of the force of Coriolis are small because the force is small. For in virtue of (6) we have $|\mathbf{P}_C| = 2m\omega|\mathbf{v}| \sin\varepsilon$, where ε denotes the angle between $\mathbf{v}$ and the axis of revolution. Since

$$\omega = \frac{2\pi}{T}\,\text{sec}^{-1} = \frac{2\pi}{24 \cdot 60 \cdot 60}\,\text{sec}^{-1} = 0.00007\ \text{sec}^{-1},$$

where T denotes the period of one revolution of the earth about its axis, thus $\mathbf{P}_C$ is small.

Let us form the projections of (5) on the x, y and z axes. By (II) we obtain:

$$m\ddot{x} = P_x - 2m\omega(\dot{y} \sin\varphi + \dot{z} \cos\varphi), \quad m\ddot{y} = P_y + 2m\omega\dot{x} \sin\varphi, \quad m\ddot{z} = P_z - mg + 2m\omega\dot{x} \cos\varphi. \tag{III}$$

Deviation to the east of a falling body. We shall concern ourselves with the determination of the deviation from the vertical of a freely falling material point.

Let us assume that for $t = 0$ we have

$$x = 0, \quad y = 0, \quad z = 0, \quad \mathbf{v} = 0. \tag{7}$$

By (III), under the assumption that $\mathbf{P} = 0$, we obtain:

$$\ddot{x} = -2\omega(\dot{y} \sin\varphi + \dot{z} \cos\varphi), \quad \ddot{y} = 2\omega\dot{x} \sin\varphi, \quad \ddot{z} = -g + 2\omega\dot{x} \cos\varphi. \tag{8}$$

Integrating and making use of the initial conditions (7), we get

$$x^{\cdot} = -2\omega(y \sin\varphi + z \cos\varphi), \quad y^{\cdot} = 2\omega x \sin\varphi,$$
$$z^{\cdot} = -gt + 2\omega x \cos\varphi. \tag{9}$$

Substituting the values $y^{\cdot}$ and $z^{\cdot}$ in (8), we obtain

$$x^{\cdot\cdot} = -4\omega^2 x + 2\omega g t \cos\varphi.$$

The above equation could be integrated and the result substituted in (9) from which y and z could be determined. We shall obtain an approximate solution by neglecting the term $-4\omega^2 x$, which is very small in comparison with $2\omega g t \cos\varphi$. We get $x^{\cdot\cdot} = 2\omega g t \cos\varphi$, whence

$$x = \tfrac{1}{3}\omega g t^3 \cos\varphi. \tag{10}$$

Omitting the term $2\omega x \cos\varphi$ in the third of the equations (9), as being small compared with $-gt$, we obtain $z^{\cdot} = -gt$, whence $z = -\frac{1}{2}gt^2$. When the point reaches the level $z = -h$, then $-h = -\frac{1}{2}gt^2$; therefore $t = \sqrt{2h/g}$. Hence by (10)

$$x = \tfrac{2}{3}\omega h \cos\varphi \sqrt{2h/g}. \tag{11}$$

This formula represents the deviation to the east (because $x > 0$) of a body falling from a height h.

At Harvard University experiments were performed with $h = 23$ m and $\varphi = 42°$. From about a thousand experiments deviation between 1.3 mm and 1.7 mm was obtained. From the approximation formula (11) one gets instead 1.8 mm. The difference is therefore not great.

Foucault's pendulum. Let us investigate the influence of the force of Coriolis on the motion of a pendulum. Let us place the origin of the coordinate system (x, y, z) at the point of suspension of an inextensible string at whose end a heavy point of mass m is fastened. Let l be the length of the pendulum (i. e. of the string). Since the reaction $\boldsymbol{P}$ of the string acts on the point along the string, denoting the coordinates of the point m by x, y, z, we obtain

$$P_x = \lambda m x, \quad P_y = \lambda m y, \quad P_z = \lambda m z,$$

where λ is a factor of proportionality depending on time.

By (III), p. 147, after dividing by m, we obtain

$$x^{\cdot\cdot} = \lambda x - 2\omega(y^{\cdot} \sin\varphi + z^{\cdot} \cos\varphi), \quad y^{\cdot\cdot} = \lambda y + 2\omega x^{\cdot} \sin\varphi, \tag{12}$$
$$z^{\cdot\cdot} = \lambda z - g + 2\omega x^{\cdot} \cos\varphi. \tag{13}$$

We shall concern ourselves only with an approximate solution of

equations (12) and (13). Let us assume that the angle of oscillation is sufficiently small so that we can take as an approximation

$$z = -l, \; z^{\cdot} = 0, \; z^{\cdot\cdot} = 0. \tag{14}$$

From equation (13) we then obtain $0 = -\lambda l - g + 2\omega x^{\cdot} \cos\varphi$, whence $\lambda = (-g + 2\omega x^{\cdot} \cos\varphi)/l$. Omitting the second term in the numerator as being small compared with the first one, we get

$$\lambda = -g/l. \tag{15}$$

The factor λ can therefore be considered approximately as a constant. By (14) and (15) the equations (12) take the form

$$x^{\cdot\cdot} = -\frac{g}{l}x - 2\omega y^{\cdot} \sin\varphi, \quad y^{\cdot\cdot} = -\frac{g}{l}y + 2\omega x^{\cdot} \sin\varphi. \tag{16}$$

Multiplying the first of the equations (16) by $x^{\cdot}$, the second by $y^{\cdot}$, and adding, we obtain

$$x^{\cdot\cdot}x^{\cdot} + y^{\cdot\cdot}y^{\cdot} = -\frac{g}{l}(xx^{\cdot} + yy^{\cdot}),$$

whence after integrating

$$x^{\cdot 2} + y^{\cdot 2} = -\frac{g}{l}(x^2 + y^2) + a, \tag{17}$$

where a is a certain constant. Multiplying the first of the equations (16) by y, the second by x, and substracting, we get

$$yx^{\cdot\cdot} - xy^{\cdot\cdot} = -2\omega(yy^{\cdot} + xx^{\cdot}) \sin\varphi,$$

whence after integrating

$$yx^{\cdot} - xy^{\cdot} = -\omega(x^2 + y^2) \sin\varphi + b, \tag{18}$$

where b is a certain constant. Let us introduce the polar coordinates:

$$x = r \cos\psi, \quad y = r \sin\psi.$$

From (17) and (18) we obtain

$$r^{\cdot 2} + r^2\psi^{\cdot 2} = -\frac{g}{l}r^2 + a, \tag{19}$$

$$r^2\psi^{\cdot} = r^2\omega \sin\varphi - b. \tag{20}$$

Let us introduce a new coordinate system (x_1, y_1, z_1) having with the preceding system (x, y, z) a common origin as well as a common z-axis, and revolving about the z-axis with an angular velocity $\omega \sin\psi$ in the direction from east to south, i. e. from x to y. For the polar coordinates r_1, ψ_1 we obtain in the new system the formulae:

$$r = r_1, \quad \psi = \psi_1 + \omega t \sin\varphi. \tag{21}$$

By substituting (21) in (20) we obtain in terms of the new coordinates r_1, ψ_1 the equation $r_1^{\cdot 2}(\psi_1^{\cdot} + \omega \sin\varphi) = r_1^2\omega \sin\varphi - b$, whence

$$r_1^2\psi_1^{\cdot} = -b, \tag{22}$$

and by substituting (21) in equation (19) we get

$$r_1^{\cdot 2} + r_1^2\psi_1^{\cdot 2} + 2r_1^2\psi_1^{\cdot}\omega \sin\varphi + r_1^2\omega^2 \sin^2\varphi = -gr_1^2 / l + a,$$

from which after neglecting the term $r_1^2\omega^2 \sin^2\varphi$ as being very small and applying equation (22), we obtain

$$r_1^{\cdot 2} + r_1^2\psi_1^{\cdot 2} = -gr_1^2 / l + a_1, \tag{23}$$

where $a_1 = a + 2b\omega \sin\varphi = \text{const.}$

It is easy to verify that (22) and (23) are the equations of the motion whose equations in terms of the coordinates x_1, y_1, z_1 are:

$$x_1^{\cdot\cdot} = -gx_1 / l, \quad y_1^{\cdot\cdot} = -gy_1 / l. \tag{24}$$

Indeed, this is the form which equations (16) assume for $\omega = 0$. Hence, introducing polar coordinates, we obtain, as is seen from equations (19) and (20) for $\omega = 0$, equations (22) and (23).

Equations (24) represent the motion of a point under the influence of a force $\boldsymbol{P}$ whose projections are:

$$P_{x_1} = -gmx_1 / l, \quad P_{y_1} = -gmy_1 / l. \tag{25}$$

This is an elastic force, i. e. one directed constantly towards the origin of the coordinate system and directly proportional to the distance of the point from the origin of the system. On p. 112 we showed that motion under the influence of an elastic force takes place along an ellipse. Hence a material point will execute a motion in the system (x_1, y_1, z_1) along an ellipse. Because this system also revolves about the z-axis with an angular velocity $\omega \sin\varphi$, the axis of this ellipse will revolve with an angular velocity $\omega \sin\varphi$ from east to south. The period of revolution is

$$T = 2\pi / \omega \sin\varphi.$$

Since one revolution of the earth lasts 24 hours it follows that, $2\pi / \omega = 24$ h, whence $T = 24 / \sin\varphi$ h. For $\varphi = 45°$ we get $T = 34$ h.

This phenomenon was first confirmed experimentally by L. Foucault; it constitutes a proof of the earth's rotation about its axis.

CHAPTER IV

GEOMETRY OF MASSES

I. SYSTEMS OF POINTS

§ 1. Statical moments. Statical moment of a point. Let us consider an arbitrary plane Π. It divides space into two parts; we can consider one of these parts as positive, and the other as negative. Let A denote a certain material point and d its distance from the plane Π. We shall write $\sigma = +d$ or $\sigma = -d$, depending on whether A lies in the positive or negative part of space.

Denoting the mass of the point A by m, we shall call the expression

$$M_\Pi = m\sigma$$

the *statical moment* of the material point A with respect to the plane Π.

The statical moment of a point can therefore be a positive or negative number or zero (it is zero for every point A lying in the plane Π).

If we choose one of the coordinate planes xy, yz, zx, as the plane Π, then we shall consider as the positive part of space that part in which is found the positive part of the axis perpendicular to the chosen coordinate plane. If the point A of mass m has the coordinates x, y, z, then by the preceding convention we have:

$$M_{xy} = mz, \quad M_{yz} = mx, \quad M_{zx} = my,$$

where M_{xy}, M_{yz}, M_{zx} denote the corresponding statical moments of the point A with respect to the xy, yz and zx planes.

Statical moment of a system of points. A collection of material points is called a *system of points*, and the sum of the statical moments of its separate points is called the *(total) statical moment of the system of points.*

If the statical moments with respect to the plane Π of the material