CHAPTER II

KINEMATICS OF A POINT

I. MOTION RELATIVE TO A FRAME OF REFERENCE

§ 1. Time. In kinematics, in addition to known geometric concepts, there arises the concept of time. For purposes of theoretical kinematics it is sufficient to assume that to each moment there is assigned a certain number t, and that there are assigned smaller numbers for moments before t than for moments after t. Conversely, to each ordering of numbers t there should correspond a certain moment: to a larger number a later moment.

In theoretical kinematics it is entirely immaterial in what way the above ordering of time was defined. In any concrete problem we proceed in the following manner. We select an arbitrary *unit of time*, e. g. a second, and an arbitrary moment which we call the *initial moment*. To the initial moment we assign the number 0. Every other moment is represented by a number t whose absolute value is the number of seconds that elapsed between the initial and given moments. The number t is positive for moments after, and negative for moments before the initial moment.

§ 2. Frame of reference. In kinematics we assume that a certain system of coordinates, called a *frame of reference*, is given.

A body moves relative to a frame of reference if the coordinates of the points of the body change. The problem of kinematics is to describe the motion of the body relative to a frame of reference when the coordinates of the points of that body are given at each moment of time.

In kinematics it is a matter of indifference how a frame of reference was selected. In concrete problems we select a frame of reference attached to certain bodies like the earth, the sun, the fixed stars, etc.

The motion of a body depends on the frame of reference. Relative to one frame a body may be at rest, but relative to another frame it may

be in motion. A passenger sitting in a car of a moving train is at rest relative to a frame attached to the car, but in motion relative to the earth.

In concrete problems a question arises as to whether it is not possible to examine the motion of a body independent of other bodies. Such a motion would be the so-called absolute motion. It appears, however, by observing that points in space are indistinguishable, that by measuring distances and appealing to theorems of geometry, it is not possible to prove in any way whether a body examined at two different moments has, or has not, altered its position. The concept of absolute motion is therefore useless. Hence we must confine ourselves to the study of relative motion, i. e. to the motion of a body relative to other bodies.

§ 3. Motion of a point. We shall concern ourselves at first with the motion of one point because the description of the motion of a body is reduced to the description of the motion of its points. Moreover, in many cases the discription of the motion of a body is reduced in practice to that of the motion of one of its points, e. g. if the dimensions of the body are small in comparison with the path traversed (the motion of the earth around the sun, the motion of a bullet) or if the motion of one point determines the motion of the entire body (e. g. the motion of a car).

Let us denote the coordinates of a moving point M relative to a certain frame of reference by x, y, z. The coordinates x, y, z depend on the time and are thus functions of the variable t:

$$x = f(t), \quad y = \varphi(t), \quad z = \psi(t).$$

These functions give us a description of the motion of the point M relative to the chosen frame of reference. Knowing them, we can obtain the coordinates x, y, z of the point M at any time t.

We assume that the functions f, φ and ψ are continuous together with their first and second derivatives in the interval $[t_0, t_1]$ during which the motion is being examined.

The motion of the point can be characterized by means of one vector function. Let us put $\mathbf{r} = \overline{OM}$ (O being the origin of the reference frame). Hence

$$\mathbf{r} = \mathbf{F}(t).$$

The above vector function describes the motion in its entirety, giving at each moment a vector $\mathbf{r}$ and consequently the position of the point M.

Let us note that the functions f, φ and ψ give the components of the vector $\mathbf{r}$.

The curve that is described by the point during its motion is called a *path* or a *trajectory*.

Suppose that the path of the point is the arc L. Let us give this arc a certain sense and select on it an arbitrary point O, which we shall call the *initial point* (Fig. 43). The position of the point M on the arc L will be determined by giving a number s whose absolute value is equal to the length of the arc OM, and which is positive or negative depending on whether the sense of OM agrees, or does not agree, with the sense originally selected. The number s is called the *arc coordinate* of the point M on the arc L.

The motion of the point M along the arc L will also be determined by the function

$$s = f(t),$$

which gives the arc coordinate s of the point M at each moment t.

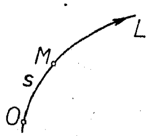

Fig. 43.

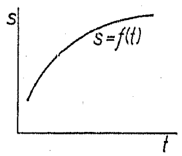

Fig. 44.

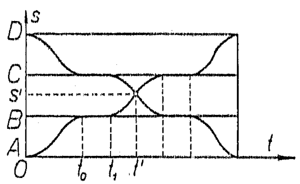

Fig. 45.

§ 4. Graph of a motion. Let the motion of a point along a curve L be defined by the function $s = f(t)$. Select two perpendicular axes s and t.

The graph of the function $s = f(t)$ is called the *graph* or the *diagram* of the motion (Fig. 44).

In Fig. 45 we have a graph of the motion of two trains, one running from station A to station D (through stations B, C), and the other running from D to A. From the diagram we read, for instance, that at $t = 0$ the train departed from station A and arrived at B at $t = t_0$. It left the station B at $t = t_1$, etc. The coordinates (t', s') of the point of intersection of both graphs represent the time and place at which both trains meet.

§ 5. Velocity. Suppose that a point moving along a curve L is at the point A at the time t, and at the point B at the time $t + \Delta t$.

The vector $\overline{AB}$ is called the *displacement* of the moving point during the time Δt. The quotient

$$\overline{AB} \,/\, \Delta t = \overline{AC} \tag{1}$$

represents the displacement per unit of time. The above quotient is also called the *average velocity vector*, or briefly, the *average velocity* during the time Δt.

Let us assume that the limit of the ratio (1) exists as $\Delta t \to 0$. Denote this limit by $\mathbf{v}$. Then

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\overline{AB}}{\Delta t}. \tag{I}$$

The vector $\mathbf{v}$ is called the *velocity vector*, or briefly, the *velocity* at the time t.

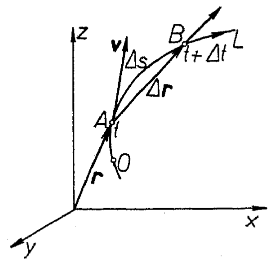

Fig. 46.

As $\Delta t \to 0$ the secant AB tends to the tangent. Therefore: *the velocity vector is tangent to the path.*

Let the motion be defined by the functions $x = f(t)$, $y = \varphi(t)$, $z = \psi(t)$. Denote the coordinates of the point A by x, y, z and those of the point B by $x + \Delta x$, $y + \Delta y$, $z + \Delta z$. The projections of $\overline{AB}$ are $\Delta x, \Delta y, \Delta z$. The projections of the quotient $\overline{AB} \,/\, \Delta t$ will therefore be the ratios $\Delta x : \Delta t$, $\Delta y : \Delta t$, $\Delta z : \Delta t$.

It follows from this that the projections of the velocity $\mathbf{v}$ are expressed by the formulae:

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = f'(t), \quad v_y = \frac{dy}{dt} = \varphi'(t), \quad v_z = \frac{dz}{dt} = \psi'(t).$$

In mechanics the derivate with respect to time is denoted by a dot above, after the dependent variable. Thus

$$v_x = x^{\cdot}, \quad v_y = y^{\cdot}, \quad v_z = z^{\cdot}. \tag{II}$$

Hence: *the projections of the velocity vector on the coordinate axes are equal to the derivatives (with respect to time) of the coordinates of the moving point.*

Let the motion be defined now by the vector function $\mathbf{r} = \mathbf{F}(t)$. Setting $\overline{AB} = \Delta \mathbf{r}$, we obtain

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\overline{AB}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{r}^{\cdot}.$$

Therefore

$$\mathbf{v} = \mathbf{r}^{\cdot}. \tag{III}$$

Velocity as a derivative of the path. Finally, let the motion of the point along the path L be defined by the function $s = f(t)$, where s denotes the arc coordinate. Since the velocity is tangent to the path, it is sufficient to give its magnitude and sense in order to determine it at the point A. From the definition of velocity it follows that

$$|\mathbf{v}| = \lim_{\Delta t\to 0}\left|\frac{\overline{AB}}{\Delta t}\right| = \lim_{\Delta t\to 0}\left|\frac{\overline{AB}}{\Delta s}\right|\cdot\left|\frac{\Delta s}{\Delta t}\right|,$$

where $|\Delta s|$ denotes the length of the arc AB. Since $\lim\limits_{\Delta s\to 0}\left|\dfrac{\overline{AB}}{\Delta s}\right| = 1$,

$$|\mathbf{v}| = \lim_{\Delta t\to 0}\left|\frac{\Delta s}{\Delta t}\right| = \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| = |s^{\cdot}|.$$

Let us draw a tangent at the point A and give to it a sense agreeing with that chosen for the curve L. If $s^{\cdot} > 0$, then $\Delta s > 0$ for small $\Delta t > 0$; therefore the point moves along the path in the positive direction, and hence $\mathbf{v}$ has a sense agreeing with that of the tangent. Similarly, if $s^{\cdot} < 0$, then $\mathbf{v}$ has a sense opposite to that of the tangent. Therefore, if v denotes the component of the velocity vector along the tangent to which we have assigned a sense agreeing with that of the path, then

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = s^{\cdot}. \tag{IV}$$

§ 6. Acceleration. Suppose that at the time t a point was at A and had a velocity $\mathbf{v}$, while at the time $t+\Delta t$ it was at B and had a velocity $\mathbf{v}'$. Put $\Delta\mathbf{v} = \mathbf{v}' - \mathbf{v}$.

The limit $\lim\limits_{\Delta t\to 0}\dfrac{\Delta\mathbf{v}}{\Delta t} = \mathbf{p}$ is called the *acceleration vector*, or briefly, the *acceleration* at the time t.

Let the motion be defined by the functions $x = f(t)$, $y = \varphi(t)$, $z = \psi(t)$. We have $p_x = \lim\limits_{\Delta t\to 0}\dfrac{\Delta v_x}{\Delta t}$. Since $v_x = f'(t)$ and $v_x' = f'(t+\Delta t)$, it follows that $\Delta v_x = v_x' - v_x = f'(t+\Delta t) - f'(t)$. Therefore

$$p_x = \lim_{\Delta t\to 0}\frac{f'(t+\Delta t) - f'(t)}{\Delta t} = f''(t);$$

similarly $p_y = \varphi''(t)$ and $p_z = \psi''(t)$.

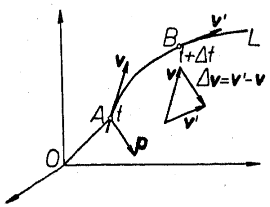

Fig. 47.

The derivatives

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2},\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2},\ \frac{\mathrm{d}^2z}{\mathrm{d}t^2}$$

are denoted by $x^{\cdot\cdot}$, $y^{\cdot\cdot}$, $z^{\cdot\cdot}$. Hence

$$p_x = x^{\cdot\cdot},\qquad p_y = y^{\cdot\cdot},\qquad p_z = z^{\cdot\cdot}. \tag{I}$$

Therefore: *the projections of the acceleration on the coordinate axes are equal to the second derivatives of the coordinates of the moving point.*

If the motion is defined by the vector function

$$\mathbf{r} = \mathbf{F}(t),$$

then — as follows from the definition of the second derivative — we have

$$\mathbf{p} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{r}^{\cdot\cdot}. \tag{II}$$

Example 1. A point moves in a plane in such a way that its coordinates at the time t are expressed by the following equations:

$$x = a\cos kt,\qquad y = b\sin kt\qquad (a>0,\ b>0,\ k>0). \tag{1}$$

Determine the velocity, acceleration and the path.

We have

$$x^{\cdot} = -ak\sin kt,\qquad y^{\cdot} = bk\cos kt,$$
$$x^{\cdot\cdot} = -ak^2\cos kt,\qquad y^{\cdot\cdot} = -bk^2\sin kt,$$

and hence the absolute value of the velocity will be

$$|\mathbf{v}| = k\sqrt{a^2\sin^2 kt + b^2\cos^2 kt},$$

and the absolute value of the acceleration

$$|\mathbf{p}| = k^2\sqrt{a^2\cos^2 kt + b^2\sin^2 kt}.$$

In order to determine the path it is necessary to find a relation between x and y, i. e. it is necessary to eliminate t. Dividing the equations (1) by a and b respectively, squaring and adding, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence the path is an ellipse with axes $2a$ and $2b$. The velocity vector is obviously tangent to the ellipse. If we make use of equations (1), then the components of the acceleration vector can be written in the form:

$$x^{\cdot\cdot} = -k^2x,\qquad y^{\cdot\cdot} = -k^2y,$$

whence

$$|\mathbf{p}| = k^2\sqrt{x^2+y^2},\qquad y^{\cdot\cdot}/x^{\cdot\cdot} = y/x.$$

The acceleration vector is therefore proportional to the distance from the origin of the coordinate system and always directed toward it.

Example 2. Determine the velocity, the acceleration and the path of a point whose motion is defined by the equations:

$$x = \tfrac{1}{2}a(1+\cos t),\qquad y = \tfrac{1}{2}a\sin t,\qquad z = a\sin\tfrac{1}{2}t\qquad (a>0). \tag{2}$$

Differentiating, we obtain

$$x^{\cdot} = -\tfrac{1}{2}a\sin t, \quad y^{\cdot} = \tfrac{1}{2}a\cos t, \quad z^{\cdot} = \tfrac{1}{2}a\cos\tfrac{1}{2}t,$$
$$x^{\cdot\cdot} = -\tfrac{1}{2}a\cos t, \quad y^{\cdot\cdot} = -\tfrac{1}{2}a\sin t, \quad z^{\cdot\cdot} = -\tfrac{1}{4}a\sin\tfrac{1}{2}t,$$

whence

$$|\mathbf{v}| = \tfrac{1}{2}a\sqrt{1+\cos^2\tfrac{1}{2}t}, \quad \text{and} \quad |\mathbf{p}| = \tfrac{1}{2}a\sqrt{1+\tfrac{1}{4}\sin^2\tfrac{1}{2}t}.$$

In order to determine the path of the point it is necessary to eliminate t from the equations (2), which will give us two relations between x, y, z defining the space curve along which the point moves. This elimination in general presents great computational difficulties; in this example, however, it is easy to accomplish. Squaring the equations (2) and adding, we obtain

$$x^2+y^2+z^2 = a^2.$$

Similarly, from the first two equations (2) it follows that

$$(x-\tfrac{1}{2}a)^2+y^2 = (\tfrac{1}{2}a)^2.$$

From the equations obtained we see that the path of the point is the curve of intersection of a sphere and a circular cylinder.

***Example* 3.** Uniform straight line motion. A point moves in such a way that the acceleration is always zero. We are assuming then that $\mathbf{p} = 0$. Hence

$$x^{\cdot\cdot} = 0, \quad y^{\cdot\cdot} = 0, \quad z^{\cdot\cdot} = 0.$$

After integrating, we get

$$x^{\cdot} = c_1, \quad y^{\cdot} = c_2, \quad z^{\cdot} = c_3, \tag{3}$$

where c_1, c_2, c_3 are certain constants. Integrating once more, we obtain

$$x = c_1t+d_1, \quad y = c_2t+d_2, \quad z = c_3t+d_3. \tag{4}$$

Here d_1, d_2, d_3 also denote certain constants. The equations (4) represent the parametric equations of a straight line. From (3) it follows that the velocity vector is constant. Therefore, the motion takes place along a straight line with a constant velocity. Such a motion is called a *uniform straight line motion*.

Let us note that setting $\mathbf{p} = 0$ is equivalent to the assumption that the velocity vector is constant. For we have $\mathbf{p} = \mathbf{v}^{\cdot}$. Therefore, if $\mathbf{v} =$ const, then $\mathbf{p} = 0$ and conversely, if $\mathbf{p} = 0$, then $\mathbf{v}^{\cdot} = 0$ and hence $\mathbf{v} =$ const.

Hodograph. Let a point move along the curve L (Fig. 48). Choose an arbitrary point O. At each moment t draw from O a velocity vector which

the moving point has at the given moment (Fig. 49). The end points of these velocity vectors describe a curve H which is called a *hodograph*.

On the hodograph we denoted the end points of the velocity vectors which a moving point along the curve L has at A, B, C by A', B', C'. If the point moves along the path L, then the end point of the corresponding velocity vector moves along the hodograph.

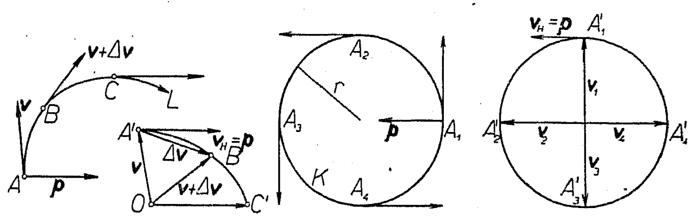

Fig. 48. Fig. 49. Fig. 50. Fig. 51.

Denote by $\mathbf{v}_H$ the velocity of the point on the hodograph at A'. From the definition of velocity we have $\mathbf{v}_H = \lim\limits_{\Delta t\to 0}\dfrac{\overline{A'B'}}{\Delta t}$. But $\overline{A'B'} = \Delta\mathbf{v}$; hence

$$\mathbf{v}_H = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t} = \mathbf{p}.$$

Therefore: *the acceleration of a moving point is equal to the velocity of the corresponding point on the hodograph.*

***Example* 4.** If a point moves along a straight line, then the direction of the velocity is constant. Therefore the hodograph is also a straight line.

***Example* 5.** Let us assume that a point moves along a circle K with a velocity which is constant in magnitude (Fig. 50).

The hodograph will be a circle (Fig. 51). The point will move along the hodograph with a velocity which is constant in magnitude.

Since the velocity of the point A_1' on the hodograph is tangent to the hodograph, it is perpendicular to $\mathbf{v}_1$. It follows from this that the acceleration of a point moving along the circle K is directed towards the centre of the circle and is constant in magnitude.

§ 7. Resolution of the acceleration along a tangent and a normal. Motion along a plane curve. Let the motion of the point A along the path L be defined by the function $s = f(t)$, where s denotes the arc coordinate. Draw a tangent t and a normal n at the point A. Give the tangent a sense agreeing with that of the curve L, and the normal — a sense towards the centre of curvature.

The projection of the acceleration $\mathbf{p}$ on the tangent is called the *tangential acceleration* $\mathbf{p}_t$; the projection of the acceleration on the normal is called the *normal acceleration* $\mathbf{p}_n$. Obviously

$$\mathbf{p} = \mathbf{p}_t + \mathbf{p}_n. \tag{I}$$

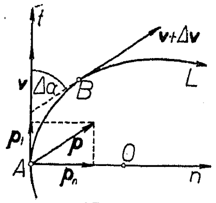

Fig. 52.

Having been given the tangential and normal accelerations, we can therefore determine the acceleration $\mathbf{p}$. The tangential acceleration will be defined by giving the component p_t along the tangent. Similarly, the component p_n along the normal defines the normal acceleration.

Let us denote by $\Delta\alpha$ the acute angle between the tangents at the point A whose arc coordinate is s, and the neighbouring point B whose arc coordinate is $s + \Delta s$. Assume that $\Delta s > 0$. We then have

$$\lim_{\Delta t\to 0}\frac{\Delta\alpha}{\Delta s} = \frac{1}{\varrho},$$

where ϱ denotes the radius of curvature. As we know,

$$\mathbf{p} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t};$$

therefore

$$p_t = \lim_{\Delta t\to 0}\frac{\mathrm{Proj}_t\Delta\mathbf{v}}{\Delta t},$$

where $\mathrm{Proj}_t\Delta\mathbf{v}$ denotes the component of $\Delta\mathbf{v}$ along the tangent t. But $\mathrm{Proj}_t\Delta\mathbf{v} = \mathrm{Proj}_t(\mathbf{v} + \Delta\mathbf{v}) - \mathrm{Proj}_t\mathbf{v}$. Hence $\mathrm{Proj}_t\Delta\mathbf{v} = (v + \Delta v)\cos\Delta\alpha - v$. Therefore

$$p_t = \lim_{\Delta t\to 0}\frac{(v+\Delta v)\cos\Delta\alpha - v}{\Delta t} = v\lim_{\Delta t\to 0}\frac{\cos\Delta\alpha - 1}{\Delta t} + \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}\cos\Delta\alpha. \tag{1}$$

But

$$\lim_{\Delta t\to 0}\frac{\cos\Delta\alpha - 1}{\Delta t} = \lim_{\Delta t\to 0}\frac{\cos\Delta\alpha - 1}{\Delta\alpha}\cdot\frac{\Delta\alpha}{\Delta s}\cdot\frac{\Delta s}{\Delta t}. \tag{2}$$

From known rule for evaluating indeterminate forms we obtain

$$\lim_{\Delta t\to 0}\frac{\cos\Delta\alpha - 1}{\Delta\alpha} = \lim_{\Delta\alpha\to 0}\frac{\cos\Delta\alpha - 1}{\Delta\alpha} = \lim_{\Delta\alpha\to 0}\frac{-\sin\Delta\alpha}{1} = 0.$$

Therefore in virtue of (2)

$$\lim_{\Delta t\to 0}\frac{\cos\Delta\alpha - 1}{\Delta t} = 0\cdot\frac{1}{\varrho}\cdot v = 0,$$

whence by (1):

$$p_t = \mathrm{d}v/\mathrm{d}t = s^{\cdot\cdot}.$$

Let us now evaluate p_n. We have

$$p_n = \lim_{\Delta t\to 0}\frac{\mathrm{Proj}_n\Delta\mathbf{v}}{\Delta t}.$$

But $\mathrm{Proj}_n\Delta\mathbf{v} = \mathrm{Proj}_n(\mathbf{v} + \Delta\mathbf{v}) - \mathrm{Proj}_n\mathbf{v} = (v + \Delta v)\sin\Delta\alpha$. Hence

$$p_n = \lim_{\Delta t\to 0}(v + \Delta v)\frac{\sin\Delta\alpha}{\Delta t}.$$

Since

$$\lim_{\Delta t\to 0}\frac{\sin\Delta\alpha}{\Delta t} = \lim_{\Delta t\to 0}\frac{\sin\Delta\alpha}{\Delta\alpha}\cdot\frac{\Delta\alpha}{\Delta s}\cdot\frac{\Delta s}{\Delta t} = 1\cdot\frac{1}{\varrho}\cdot v,$$

it follows that

$$p_n = v^2/\varrho.$$

Therefore the tangential acceleration p_t and the normal acceleration p_n are expressed by the formulae

$$p_t = \mathrm{d}v/\mathrm{d}t = s^{\cdot\cdot}, \qquad p_n = v^2/\varrho, \tag{II}$$

where ϱ is the radius of curvature.

Since the tangential acceleration is perpendicular to the normal acceleration, it follows that

$$|\mathbf{p}| = \sqrt{p_t^2 + p_n^2}. \tag{III}$$

From formula (1) we see that $p_n \geqq 0$. Therefore the normal acceleration is always directed towards the centre of curvature.

Let us note that the tangential acceleration depends only on the change of the absolute value of the velocity and not on the change of its direction. The tangential acceleration is constantly zero if and only if $v = \mathrm{const.}$

The normal acceleration depends on the radius of curvature ϱ and therefore on the change of the direction of the velocity. The normal acceleration is constantly zero if constantly $v = 0$ or $1/\varrho = 0$. In the first case the point is at rest, and in the second case the motion is along a straight line.

Example. A point moves along a circle of radius r with a velocity v whose absolute value is constant. Therefore by (I)

$$p_t = \mathrm{d}v/\mathrm{d}t = 0, \quad p_n = v^2/r = \mathrm{const.}$$

Hence the acceleration is constantly directed towards the centre of the circle and its absolute value is constant (cf. p. 39, example 5).

Motion along a space curve. Let a point move along a space curve L. Denote the velocity of the point at A by $\mathbf{v}$ and its velocity at B by $\mathbf{v} + \Delta\mathbf{v}$.

Pass a plane Π through the tangent at A parallel to the tangent at B. The vectors $\mathbf{v}$ and $\mathbf{v} + \Delta\mathbf{v}$ drawn from the point A lie in this plane. Therefore the vector $\Delta\mathbf{v}/\Delta t$ lies in the plane Π. As the point B tends to the point A, the plane Π tends to the so-called *osculating plane* at A. It therefore follows that the acceleration $\mathbf{p} = \lim_{\Delta t \to 0} \Delta\mathbf{v} / \Delta t$ lies in the osculating plane.

Hence: *the acceleration vector lies in the osculating plane.*

The tangent lies in the osculating plane. The line perpendicular to the tangent and lying in the osculating plane is called the *principal normal*. The centre of curvature lies on the principal normal. Forming the projections of the acceleration on the tangent and the normal, we obtain

$$\mathbf{p} = \mathbf{p}_t + \mathbf{p}_n,$$

which is analogous to formula (I) obtained in connection with plane motion.

Giving the tangent a sense agreeing with that of the curve, and the normal a sense towards the centre of curvature and proceeding as before, we obtain:

$$p_t = \mathrm{d}v/\mathrm{d}t = s^{\cdot\cdot} \quad \text{and} \quad p_n = v^2/\varrho,$$

where ϱ denotes the radius of curvature.

The above relations are identical with those of (II) in the case of plane motion.

Example 1. Uniform motion. Let a point A move along a curve L on which an initial point O and a sense have been chosen. Assume that the velocity of the point A is constant in magnitude (i. e. in absolute value). Such a motion is called a *uniform motion* along the curve L.

We are assuming, therefore, that $v = s^{\cdot} = \text{const}$. Integrating, we obtain

$$s = vt + s_0. \tag{3}$$

Substituting $t = 0$, we get $s = s_0$. Hence the constant s_0 denotes the arc coordinate of the point A at the time $t = 0$.

Uniform motion is defined by a function of the first degree with respect to t. Conversely, an arbitrary function of the first degree $s = at + b$ defines a uniform motion with the velocity $v = s^{\cdot} = a$. In addition $s_0 = b$.

By (3) we have

$$p_t = v^{\cdot} = s^{\cdot\cdot} = 0, \quad p_n = v^2/\varrho.$$

Since the tangential acceleration is zero, the acceleration is constantly directed towards the centre of curvature. The magnitude of the acceleration is p_n. Therefore the magnitude of the acceleration is inversely proportional to the radius of curvature ϱ (or directly proportional to the curvature $K = 1/\varrho$).

In particular, if a point moves uniformly along a circle of radius r, then $\varrho = r$, and hence

$$p_n = v^2/r.$$

Therefore: *if a point moves along a circle uniformly, then the acceleration is of constant magnitude and it is directed towards the centre of the circle.*

Example 2. Uniformly accelerated motion. A point moving along a curve L has a constant tangential acceleration. Such a motion is called a *uniformly accelerated motion* along the curve L.

Assuming that $p_t = p = \text{const}$, we get $s^{\cdot\cdot} = p_t = p$; hence

$$s^{\cdot} = v = pt + c_1, \quad s = \tfrac{1}{2}pt^2 + c_1 t + c_2. \tag{4}$$

Uniformly accelerated motion is defined by the function $s = f(t)$ of the second degree in t. Conversely, an arbitrary function of the second degree $s = at^2 + bt + c$ defines a uniformly accelerated motion because after differentiating twice:

$$p_t = s^{\cdot\cdot} = 2a = \text{const}.$$

Let us assume that in a uniformly accelerated motion defined by function (4), a point had a velocity $v = v_0$ and an arc coordinate $s = s_0$ at $t = 0$.

Putting $t = 0$, we obtain from (4) $v_0 = c_1$, $s_0 = c_2$. Substituting in equations (4), we obtain

$$v = pt + v_0, \quad s = \tfrac{1}{2}pt^2 + v_0 t + s_0.$$

In particular, if $v_0 = 0$ and $s_0 = 0$, we get

$$v = pt \quad \text{and} \quad s = \tfrac{1}{2}pt^2.$$

Example 3. Motion along a cycloid. Let a circle of radius r roll along a straight line. Examine the motion of an arbitrary point on the circumference of the circle.

Assume that at the beginning a given point P on the periphery of the circle is a point of tangency of the circle and the straight line. Let us select this point as the origin of the coordinate system with the line as the x-axis. If the circle turns through an angle φ, the point will occupy a new position $P'(x, y)$. In order to express the coordinates x, y as functions of the angle φ, let us note that the new position of the point can be obtained

Fig. 53.

by first turning the circle clockwise about the centre through the angle φ and then translating it along the x-axis through the segment PQ which is equal to the arc $P'Q$ subtending the angle φ. Since the length of the arc $P'Q$ is $r\varphi$, it follows that $x = -r\sin\varphi + r\varphi$, $y = r - r\cos\varphi$, or:

$$x = r(\varphi - \sin\varphi), \qquad y = r(1 - \cos\varphi).$$

After one revolution of the circle, i. e. for $\varphi = 2\pi$, the point P is again a point of tangency, after which the motion repeats itself. The resulting curve consisting of congruent arcs is called the *cycloid*.

Assume that the circle revolves uniformly, i. e. that the angle φ is proportional to the time: $\varphi = \omega t$ (where ω is a constant). The equations of the motion of the point P are then

$$x = r(\omega t - \sin\omega t), \quad y = r(1 - \cos\omega t).$$

Differentiating them twice with respect to time, we obtain the components of velocity and acceleration

$$x^{\cdot} = r\omega(1 - \cos\omega t), \quad y^{\cdot} = r\omega\sin\omega t,$$
$$x^{\cdot\cdot} = r\omega^2\sin\omega t, \qquad y^{\cdot\cdot} = r\omega^2\cos\omega t.$$

Hence

$$|\mathbf{v}| = \sqrt{r^2\omega^2(2 - 2\cos\omega t)} = 2r\omega|\sin\tfrac{1}{2}\omega t|, \qquad |\mathbf{p}| = r\omega^2. \tag{5}$$

Because of the fact that the motion repeats itself after one complete revolution, we can confine ourselves to the time interval $0 \leq t \leq 2\pi/\omega$. From the above equations we see that the magnitude of the acceleration of the point P is constant, but the magnitude of the velocity changes. At $t = 0$ or $t = 2\pi/\omega$, i. e. when the point is on the line, the velocity is zero, whereas at $t = \pi/\omega$, i. e. when the point attains its highest position, the velocity is greatest and equal to $2r\omega$.

Let us give the cycloid a sense agreeing with that of the motion of the point. During the time $0 \leq t \leq 2\pi/\omega$ we have $v = |\mathbf{v}|$ and hence by (5)

$$v = 2r\omega\sin\tfrac{1}{2}\omega t.$$

The tangential acceleration will hence be

$$p_t = v^{\cdot} = r\omega^2\cos\tfrac{1}{2}\omega t.$$

In order to determine the normal acceleration, we calculate the curvature from a well-known formula

$$\frac{1}{\varrho} = \frac{|x^{\cdot}y^{\cdot\cdot} - y^{\cdot}x^{\cdot\cdot}|}{(x^{\cdot 2} + y^{\cdot 2})^{\frac{3}{2}}} = \frac{r^2\omega^3(1 - \cos\omega t)}{8r^3\omega^3\sin^3\omega t/2} = \frac{1}{4r\sin\omega t/2}.$$

Hence

$$p_n = v^2/\varrho = r\omega^2\sin\tfrac{1}{2}\omega t.$$

Since $ds = v\,dt$, the distance covered by the point up to the time t is

$$s = \int_0^t ds = \int_0^t v\,dt = \int_0^t 2\omega r\sin\tfrac{1}{2}\omega t\,dt = 4r(1 - \cos\tfrac{1}{2}\omega t).$$

In particular, for $t = 2\pi/\omega$ we obtain the length of the path of the cycloid for one revolution of the circle. The length turns out to be $8r$.

§ 8. Angular velocity and acceleration. Let a point A move along a circle of radius r and centre M. Let us select a sense on the circle and an initial point O (Fig. 54). Denote by φ the angle between the radii MA and MO assumed to agree with the sense selected. We have $s = r\varphi$; therefore

$$s^{\cdot} = r\varphi^{\cdot} \quad \text{and} \quad s^{\cdot\cdot} = r\varphi^{\cdot\cdot}. \tag{1}$$

The derivative $\varphi^{\cdot}$, which is usually denoted by ω, is called the *angular velocity*.

The derivative $\varphi^{\cdot\cdot}$ is called the *angular acceleration* and it is denoted by ε. We obviously have $\varphi^{\cdot} = \omega$ and $\omega^{\cdot} = \varepsilon$. Therefore by (1)

$$v = r\omega \quad \text{and} \quad p_t = r\varepsilon, \tag{I}$$

and since $p_n = v^2/r$, it follows that

$$p_n = r\omega^2. \tag{II}$$

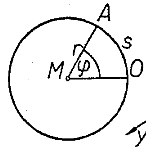

Fig. 54.

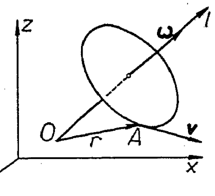

Fig. 55.

Angular velocity vector. Let a point rotate about a certain axis l. This means that it moves along a circle lying in a plane perpendicular to l, whose centre is the point of intersection of this plane and the axis l (Fig. 55).

Let the point have an angular velocity ω at the time t. Denote the vector lying on l and having a length $|\boldsymbol{\omega}|$ by $\boldsymbol{\omega}$. Select the sense of the vector $\boldsymbol{\omega}$ in such a way that a person, having his head at the terminal point and his feet at the initial point of the vector, sees the motion proceeding from his right hand to his left hand.

The vector $\boldsymbol{\omega}$ is called the *angular velocity vector*.

It is easy to verify that the velocity $\mathbf{v}$ of the point A is equal to the moment of $\boldsymbol{\omega}$ with respect to A:

$$\mathbf{v} = \mathrm{Mom}_A \boldsymbol{\omega}. \tag{2}$$

If the vector $\overline{OA}$ is denoted by r, where O is an arbitrary point on the line l, then $\mathbf{v} = \boldsymbol{\omega} \times \overline{OA}$ and hence

$$\mathbf{v} = \mathbf{r} \times \boldsymbol{\omega}. \tag{III}$$

Denoting the projections of the vector $\boldsymbol{\omega}$ on the coordinate axes by $\omega_x, \omega_y, \omega_z$, the coordinates of the point A by x, y, z and the coordinates of the point O by x_0, y_0, z_0, we obtain

$$v_x = \omega_z(y - y_0) - \omega_y(z - z_0), \quad v_y = \omega_x(z - z_0) - \omega_z(x - x_0), \quad v_z = \omega_y(x - x_0) - \omega_x(y - y_0). \tag{IV}$$

If, in particular, l passes through the origin of the system of coordinates, then, setting $x_0 = y_0 = z_0 = 0$, we obtain

$$v_x = \omega_z y - \omega_y z, \quad v_y = \omega_x z - \omega_z x, \quad v_z = \omega_y x - \omega_x y. \tag{V}$$

§ 9. Plane motion in a polar coordinate system. If a point moves in the xy-plane, then its position is completely determined by the length of the segment $OA = r$, called the *radius vector*, and the angle φ which OA makes with the x-axis. The motion of the point will therefore be defined by two functions

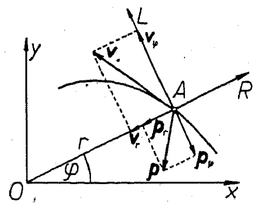

Fig. 56.

$$r = F(t), \quad \varphi = f(t).$$

Since $x = r\cos\varphi$, $y = r\sin\varphi$, taking derivatives with respect to t, we obtain:

$$x^{\cdot} = r^{\cdot}\cos\varphi - r\varphi^{\cdot}\sin\varphi, \quad y^{\cdot} = r^{\cdot}\sin\varphi - r\varphi^{\cdot}\cos\varphi, \tag{1}$$

$$x^{\cdot\cdot} = r^{\cdot\cdot}\cos\varphi - 2r^{\cdot}\varphi^{\cdot}\sin\varphi - r\varphi^{\cdot 2}\cos\varphi - r\varphi^{\cdot\cdot}\sin\varphi, \quad y^{\cdot\cdot} = r^{\cdot\cdot}\sin\varphi + 2r^{\cdot}\varphi^{\cdot}\cos\varphi - r\varphi^{\cdot 2}\sin\varphi + r\varphi^{\cdot\cdot}\cos\varphi. \tag{2}$$

From equations (1) and (2) we can determine $x^{\cdot}, x^{\cdot\cdot}, y^{\cdot}, y^{\cdot\cdot}$ if we know $r^{\cdot}, r^{\cdot\cdot}, \varphi^{\cdot}, \varphi^{\cdot\cdot}$, and conversely. From (1) we obtain

$$v^2 = x^{\cdot 2} + y^{\cdot 2} = r^{\cdot 2} + r^2\varphi^{\cdot 2}, \tag{3}$$

$$r^{\cdot} = \frac{xx^{\cdot} + yy^{\cdot}}{\sqrt{x^2 + y^2}}, \quad \varphi^{\cdot} = \frac{xy^{\cdot} - yx^{\cdot}}{x^2 + y^2}. \tag{4}$$

It is often convenient to resolve the velocity and acceleration not in the directions of the coordinate axes, but in the direction of the radius vector and a direction perpendicular to it, while the positive sense along these directions is chosen as in the figure. These components are called the *radial* and *transverse* components, respectively.

If a_x, a_y are the components of an arbitrary vector $\mathbf{a}$ beginning at the point $A(r, \varphi)$, then projecting this vector on the axis AR as well as on AL, we obtain for the radial component a_r and the transverse component a_φ:

$$a_r = a_x\cos\varphi + a_y\sin\varphi, \quad a_\varphi = -a_x\sin\varphi + a_y\cos\varphi.$$

Applying these formulae to the velocity and acceleration vectors, we obtain by equations (1) and (2):

$$v_r = r^{\cdot}, \quad v_\varphi = r\varphi^{\cdot}, \tag{I}$$

$$p_r = r^{\cdot\cdot} - r\varphi^{\cdot 2}, \quad p_\varphi = r\varphi^{\cdot\cdot} + 2r^{\cdot}\varphi^{\cdot} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\varphi^{\cdot}). \tag{II}$$

Example. A point moves along the spiral $r = a + b\varphi$ in such a way that the angle φ is proportional to the time t. Hence $\varphi = \omega t$, where ω is the factor of proportionality. Then $\varphi^{\cdot} = \omega$, $\varphi^{\cdot\cdot} = 0$, $r^{\cdot} = b\varphi^{\cdot} = b\omega$ and $r^{\cdot\cdot} = 0$, whence

$$v_r = b\omega, \quad p_r = -(a + b\omega t)\,\omega^2, \quad v_\varphi = (a + b\omega t)\,\omega, \quad p_\varphi = 2b\omega^2.$$

§ 10. Areal velocity. Let a motion take place in the xy-plane. Denote the area of the region swept out by the radius vector during the time from t to $t + \Delta t$ by ΔS. From the formula for calculating areas in polar coordinates we obtain

$$\Delta S = \tfrac{1}{2}\int_{\varphi}^{\varphi+\Delta\varphi} r^2\,\mathrm{d}\varphi,$$

whence, on the basis of the mean value theorem,

$$\Delta S = \tfrac{1}{2} r_s^2\,\Delta\varphi, \tag{1}$$

where r_s denotes the mean value between the maximum and minimum of the radius r during the time from t to $t + \Delta t$.

The $\lim\limits_{\Delta t \to 0} \dfrac{\Delta S}{\Delta t}$ is called the *areal velocity* and it is denoted by A. Therefore by (1)

$$A = \tfrac{1}{2} r^2 \varphi^{\cdot}. \tag{I}$$

From formulae (4) on p. 47 we obtain $\varphi^{\cdot} = \frac{1}{r^2}(xy^{\cdot} - yx^{\cdot})$. Hence by (I)

$$A = \tfrac{1}{2}(xy^{\cdot} - yx^{\cdot}). \tag{II}$$

Example. Determine the velocity and acceleration of a point moving along the circle $x^2 - 2ax + y^2 = 0$ with a constant areal velocity h.

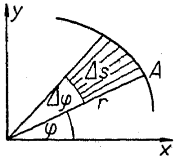

Fig. 57.

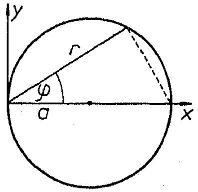

Fig. 58.

The equation of the given circle in polar coordinates is $r = 2a\cos\varphi$. The condition that the areal velocity be constant is expressed by

$$\tfrac{1}{2}r^2\varphi^{\cdot} = h, \quad \text{or} \quad \varphi^{\cdot} = 2h/r^2. \tag{2}$$

Differentiating the equation of the circle, we obtain

$$r^{\cdot} = -2a\varphi^{\cdot}\sin\varphi = -\frac{4ah}{r^2}\sin\varphi.$$

Therefore $$v_r = -\frac{4ah}{r^2}\sin\varphi, \quad v_\varphi = \frac{2h}{r}.$$

As the areal velocity is constant, we have from formula (II), p. 47,

$$p_\varphi = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\varphi^{\cdot}),$$

it follows that $p_\varphi = 0$. Therefore the acceleration always has the direction of the radius vector.

Differentiating the first of the relations (2), we obtain $2rr^{\cdot}\varphi^{\cdot} + r^2\varphi^{\cdot\cdot} = 0$ and hence

$$\varphi^{\cdot\cdot} = -\frac{2r^{\cdot}\varphi^{\cdot}}{r} = \frac{16ah^2}{r^5}\sin\varphi;$$

differentiating $r^{\cdot}$, we get

$$r^{\cdot\cdot} = -2a\varphi^{\cdot\cdot}\sin\varphi - 2a\varphi^{\cdot 2}\cos\varphi = -2a\sin\varphi\frac{16ah^2}{r^5}\sin\varphi -$$

$$-2a\cos\varphi\frac{4h^2}{r^4} = -\frac{32a^2h^2}{r^5}\sin^2\varphi - 4a^2\cos^2\varphi\frac{4h^2}{r^5} = -\frac{16a^2h^2}{r^5}(1 + \sin^2\varphi).$$

Since

$$r\varphi^{\cdot 2} = r\frac{4h^2}{r^4} = r^2\frac{4h^2}{r^5} = 4a^2\cos^2\varphi\frac{4h^2}{r^5} = \frac{16a^2h^2}{r^5}\cos^2\varphi,$$

from the formula $p_r = r^{\cdot\cdot} - r\varphi^{\cdot 2}$ we obtain

$$p_r = -32a^2h^2/r^5.$$

§ 11. Dimensions of kinematic magnitudes. The measure of velocity, acceleration, etc. depends on the units of length and time. We shall concern ourselves with the question of how these measures change when the units of length and time undergo changes.

Let us select an arbitrary unit of length L and a unit of time T. Assume that a point in uniform motion traversed a path of length sL in the time tT (e. g. if L denotes cm, and T sec, then $sL = s$ cm, $tT = t$ sec). The measure of the velocity v using units L and T is

$$v = s/t.$$

Let us now select new units of length and time L', T'. Assume that the new units and the preceding are connected by the relations

$$L = \lambda L', \qquad T = \tau T', \tag{1}$$

where λ and τ indicate how many new units are contained in the old. Denoting the measures of length, time and velocity in the new units by s', t' and v', we obtain

$$v' = s'/t'.$$

Since $sL = s\lambda L'$ and $tT = t\tau T'$, it follows that $s' = s\lambda$, $t' = t\tau$, whence

$$v' = \frac{s\lambda}{t\tau} = \frac{s}{t}\cdot\frac{\lambda}{\tau}$$

and hence

$$v' = v\frac{\lambda}{\tau} \quad (\text{or } v' = v\lambda\tau^{-1}).$$

The unit of velocity (i. e. the velocity whose measure is 1) in units L and T is denoted by

$$\frac{L}{T} \text{ or } LT^{-1}.$$

The velocity whose measure is v we denote by

$$v\frac{L}{T} \quad \text{or} \quad vLT^{-1}.$$

For example, $5\ \mathrm{cm}\cdot\mathrm{sec}^{-1}$ denotes a velocity whose measure is 5, if the unit of length is the cm and the unit of time the sec.

Let us now suppose that we have chosen new units of length and time L' and T' which are connected with the old units by equations (1). Substitute in vLT^{-1} for L and T, $\lambda L'$ and $\tau T'$ respectively, and then transform the resulting expression as if the letters L' and T' denoted numbers. We obtain

$$vLT^{-1} = v\lambda L'\tau^{-1}T'^{-1} = v\lambda\tau^{-1}L'T'^{-1}.$$

Substitute $v' = v\lambda\tau^{-1}$. Then

$$vLT^{-1} = v'L'T'^{-1}.$$

Let us note that, according to the definition, v' is the measure of velocity in units L' and T'; consequently, $v'L'T'^{-1}$ represents the velocity in units of length and time L' and T'.

We see, therefore, that the symbol LT^{-1} enables us to determine the measure of the velocity in changed units by formal calculation.

Example. Determine the measure of the velocity $12\ \text{cm}\cdot\text{sec}^{-1}$ in units m and min.

We have $\text{cm} = 0.01\ \text{m}$, $\text{sec} = \frac{1}{60}\ \text{min}$. Calculating formally, we obtain $12\ \text{cm}\cdot\text{sec}^{-1} = 12\cdot 0.01\ \text{m}\cdot(\frac{1}{60}\ \text{min})^{-1} = 12\cdot 0.01\cdot 60\ \text{m}\cdot\text{min}^{-1} = 7.2\ \text{m}\cdot\text{min}^{-1}$.

Therefore 7.2 is the measure of the given velocity in units m and min.

The expression LT^{-1}, in which L and T do not denote particular units, but are symbols representing arbitrary units of length and time, is called the *dimension of velocity*.

General definition of dimension. The notion of dimension given above for velocity can be generalized to other magnitudes such as acceleration, angular velocity and acceleration, etc.

We shall call the *dimension* of any magnitude A the expression

$$L^\alpha T^\beta,$$

where the exponents α, β are numbers satisfying the condition: if a is the measure of the magnitude A in units of length and time L, T, and a' its measure in units L', T' connected with L and T by equations (1), then

$$a' = a\lambda^\alpha\tau^\beta. \tag{2}$$

The dimension of the magnitude A is denoted by $[A]$. Hence

$$[A] = L^\alpha T^\beta.$$

The unit of the magnitude A in units of length and time L, T is denoted by $L^\alpha T^\beta$. Therefore, if a is the measure of the magnitude A in terms of the unit $L^\alpha T^\beta$, then this magnitude is denoted by

$$aL^\alpha T^\beta.$$

Suppose now that we have introduced new units of length and time L', T' connected with the preceding by equations (1), p. 49. Calculating formally, we obtain $aL^\alpha T^\beta = a(\lambda L')^\alpha(\tau T')^\beta = a\lambda^\alpha L'^\alpha\tau^\beta T'^\beta$, and hence $aL^\alpha T^\beta = (a\lambda^\alpha\tau^\beta)\,L'^\alpha T'^\beta$. Therefore, denoting the measure of the given magnitude in units L' and T' by a', we obtain from (2)

$$aL^\alpha T^\beta = a'L'^\alpha T'^\beta. \tag{3}$$

Thus formal calculation permits us to determine the measure in terms of the changed units.

Example. The dimension of acceleration is LT^{-2} (as can be verified by employing the same method as used in connection with the velocity). Represent an acceleration of $5\ \text{cm}\cdot/\text{sec}^{-2}$ in units m and min.

Since $1\ \text{cm} = 0.01\ \text{m}$, $1\ \text{sec} = \frac{1}{60}\ \text{min}$, it follows that $\lambda = 0.01$, $\tau = \frac{1}{60}$, $\alpha = 1$, $\beta = -2$, and hence by (3), $5\ \text{cm}\cdot\text{sec}^{-2} = 5(0.01\ \text{m})(\frac{1}{60}\ \text{min})^{-2} = 5\cdot 0.01\cdot 60^2\ \text{m}\cdot\text{min}^{-2}$, or $5\ \text{cm}\cdot\text{sec} = 180\ \text{m}\cdot\text{min}^{-2}$.

Determination of dimension. The following theorem is useful for the determination of dimensions:

Let there be given the magnitudes A and B for which

$$[A] = L^\alpha T^\beta, \quad [B] = L^\gamma T^\delta, \tag{4}$$

as well as a third magnitude C depending on A and B in such a manner that if we denote by a, b, c the measures of the magnitudes A, B, C expressed in arbitrary units L and T, we always have

$$c = ga^pb^q, \tag{5}$$

where the numbers g, p, q do not depend on the units L and T.

From these assumptions follows

$$[C] = L^{p\alpha+q\gamma}T^{p\beta+q\delta}. \tag{I}$$

This formula can be written in still another way. Dimension $[C]$ can be obtained if we reckon formally as follows:

$$[C] = (L^\alpha T^\beta)^p(L^\gamma T^\delta)^q = L^{p\alpha}T^{p\beta}L^{q\gamma}T^{q\delta} = L^{p\alpha+q\gamma}T^{p\beta+q\delta};$$

we can therefore give formula (I) in the form

$$[C] = [A]^p[B]^q.$$

Proof. Let us select new units L' and T' connected with the preceding by equations (1), p. 49. Denote the measures of the magnitudes A, B, C in terms of the new units by a', b', c'. According to assumption (5) we have $c' = ga'^pb'^q$. By (4), $a' = \lambda^\alpha\tau^\beta a$ and $b' = \lambda^\gamma\tau^\delta b$ hence $c' = g(\lambda^\alpha\tau^\beta a)^p(\lambda^\gamma\tau^\delta b)^q = \lambda^{p\alpha+q\gamma}\tau^{p\beta+q\delta}c$, and by the definition of dimension follows formula (I), q. e. d.

From the above theorem we obtain the following corollaries:

Corollary I. *If formula* (5) *has the form*

$$c = ab, \quad \text{then} \quad [C] = [A][B] = L^{\alpha+\gamma}T^{\beta+\delta},$$
$$c = a/b, \quad \text{then} \quad [C] = [A]/[B] = L^{\alpha-\gamma}T^{\beta-\delta},$$
$$c = a^p, \quad \text{then} \quad [C] = [A]^p = L^{\alpha p}T^{\beta p}.$$

Examples: 1. Velocity is expressed by the formula $v = s/t$. Therefore

$$[v] = [s]/[t] = L/T = LT^{-1}.$$

2. Acceleration in uniformly accelerated motion is given by the formula $p = v/t$. Hence

$$[p] = [v]/[t] = LT^{-1}/T = LT^{-2}.$$

A similar result is obtained if we use formula $s = \frac{1}{2}pt^2$. We calculate from it $p = 2s/t^2$. Therefore

$$[p] = [s]/[t]^2 = L/T^2 = LT^{-2}.$$

3. Angular velocity is given by $\omega = \mathrm{d}\varphi/\mathrm{d}t$. The dimension of the angle φ is L^0T^0 because the measure of φ is independent of the units of length. Hence

$$[\omega] = 1/T = T^{-1}.$$

4. Angular acceleration is given by $\varepsilon = \mathrm{d}^2\varphi/\mathrm{d}t^2$. Therefore

$$[\varepsilon] = 1/T^2 = T^{-2}.$$

Corollary 2. Certain constants can depend on the choice of the units of length and time. We can therefore speak of the *dimension* of these *constants*.

Example. In a certain motion the absolute value of the acceleration is proportional to the square of the velocity. Denoting the constant of proportionality by k, we have

$$p = kv^2.$$

In units of cm and sec, $k = 2$. Calculate k in units of m and min. We have $k = p/v^2$, from which $[k] = [p]/[v]^2 = LT^{-2}/(LT^{-1})^2$; hence $[k] = L^{-1}$ and

$$2\ \mathrm{cm}^{-1} = 2(\tfrac{1}{100}\ \mathrm{m})^{-1} = 200\ \mathrm{m}^{-1}.$$

Therefore $k = 200$ in a system of units m and min.

II. CHANGE OF FRAME OF REFERENCE

§ 12. Relation among coordinates. The velocity and acceleration of a point depend on the frame of reference relative to which the motion of the point is being examined. The motion of one and the same point will therefore be described differently by two observers moving relative to each other.

If we are travelling in a train, for instance, then the passengers travelling with us are at rest relative to us. To an observer standing near the tracks

the passengers move with the velocity of the train. We can express this in the following way: relative to a frame attached to the train the passengers are at rest, and relative to a frame attached to the earth the passengers move with the velocity of the train.

The motions of the planets and sun relative to a frame of reference attached to the earth are very complicated. Copernicus discovered that the motions of the planets are represented much more simply if we choose as a frame of reference a frame attached to the sun.

Let there be given a frame of reference $O(x, y, z)$ and a second frame $M(\xi, \eta, \zeta)$ moving relative to the former (Fig. 59). In order to differentiate one frame from the other, we shall call (x, y, z) a fixed frame and (ξ, η, ζ) a moving frame. The motion of one and the same point will be represented differently in both frames.

We shall be concerned with the problem of representing the motion of the point A relative to one frame when this motion is known relative to another frame.

This problem is very important and we shall meet it in many situations.

Denote by x_0, y_0, z_0 the coordinates of the origin M in the frame $O(x, y, z)$, and by ξ_0, η_0, ζ_0 the coordinates of the origin O in the frame $M(\xi, \eta, \zeta)$. Let $\alpha_1, \alpha_2, \ldots, \gamma_3$ be the angles between the axes of both frames as indicated in the table:

axes	ξ	η	ζ
x	α_1	α_2	α_3
y	β_1	β_2	β_3
z	γ_1	γ_2	γ_3

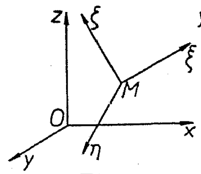

Fig. 59.

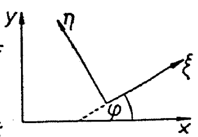

Fig. 60.

If x, y, z and ξ, η, ζ are the coordinates of the point A in the first and second frames, respectively, then, as it is known from analytic geometry,

$$\begin{aligned} x &= x_0 + \xi\cos\alpha_1 + \eta\cos\alpha_2 + \zeta\cos\alpha_3, \\ y &= y_0 + \xi\cos\beta_1 + \eta\cos\beta_2 + \zeta\cos\beta_3, \\ z &= z_0 + \xi\cos\gamma_1 + \eta\cos\gamma_2 + \zeta\cos\gamma_3, \end{aligned} \qquad \text{(I)}$$

$$\begin{aligned} \xi &= \xi_0 + x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1, \\ \eta &= \eta_0 + x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2, \\ \zeta &= \zeta_0 + x\cos\alpha_3 + y\cos\beta_3 + z\cos\gamma_3. \end{aligned} \qquad \text{(I')}$$

The motion of the frame $M(\xi, \eta, \zeta)$ relative to $O(x, y, z)$ will be known if the coordinates x_0, y_0, z_0 and the angles $\alpha_1, \alpha_2, \ldots, \gamma_3$ are given for each moment t. Therefore x_0, y_0, z_0 and $\alpha_1, \alpha_2, \ldots, \gamma_3$ are functions of the time t. If the motion of A relative to $M(\xi, \eta, \zeta)$ is defined by the functions $\xi = f(t)$, $\eta = \varphi(t)$, $\zeta = \psi(t)$, then the motion relative to $O(x, y, z)$ is obtained from formulae (I)

$$x = x_0 + f(t) \cos \alpha_1 + \varphi(t) \cos \alpha_2 + \psi(t) \cos \alpha_3$$

and similarly for y and z.

If the motion of A takes place in the plane Π, then selecting axes x, y and ξ, η in this plane and denoting by φ the angle between the axes x and ξ (Fig. 60), we obtain

$$x = x_0 + \xi \cos\varphi - \eta \sin\varphi, \qquad y = y_0 + \xi \sin\varphi + \eta \cos\varphi, \qquad \text{(II)}$$

$$\xi = \xi_0 + x \cos\varphi + y \sin\varphi, \qquad \eta = \eta_0 - x \sin\varphi + y \cos\varphi. \qquad \text{(II')}$$

If the directions of the axes of the moving frame $M(\xi, \eta, \zeta)$ do not change, then this frame is said to move with an *advancing* motion.

In this case the angles $\alpha_1, \alpha_2, \ldots, \gamma_3$ are constant.

We say that a moving frame revolves about the axis l with an angular velocity ω, if the points lying on the axes ξ, η, ζ revolve about the axis l with an angular velocity ω.

Let the frame $M(\xi, \eta, \zeta)$ revolve about the ζ-axis with an angular velocity ω. Let us assume that the fixed frame $O(x, y, z)$ coincided with the moving frame $M(\xi, \eta, \zeta)$ at the time $t = 0$. We then have $x_0 = y_0 = z_0 = \xi_0 = \eta_0 = \zeta_0 = 0$. Denote the angle between the x and ξ axes by φ. Obviously $\varphi = \omega t$. Since the axes x, y and ξ, η constantly lie in one plane, by (II) and (II'):

$$x = \xi \cos \omega t - \eta \sin \omega t, \qquad y = \xi \sin \omega t + \eta \cos \omega t, \qquad z = \zeta, \qquad \text{(III)}$$

$$\xi = x \cos \omega t + y \sin \omega t, \qquad \eta = -x \sin \omega t + y \cos \omega t, \qquad \zeta = z. \qquad \text{(III')}$$

Example 1. The motion of a point relative to a fixed frame is defined by the equations

$$x = a \cos \omega t, \quad y = b \sin \omega t, \qquad (1)$$

and hence the path of the point is an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

How is the motion of this point represented in a moving frame with the same origin, if this frame revolves in a positive direction with an angular velocity ω?

We assume that both frames are coincident at the moment $t = 0$.

Denoting by ξ, η the coordinates of an arbitrary point relative to the moving frame, we have

$$\xi = x \cos \omega t + y \sin \omega t, \qquad \eta = -x \sin \omega t + y \cos \omega t,$$

for by hypothesis the angle φ between the x and ξ axes is equal to ωt. Substituting on the right side of these formulae the expressions (1), we obtain the equations of the path described by the point in the moving frame

$$\xi = a \cos^2\omega t + b \sin^2\omega t, \qquad \eta = (b - a) \sin \omega t \cos \omega t,$$

or, making use of the identities $\cos^2\omega t = \frac{1}{2}(1 + \cos 2\omega t)$, $\sin^2\omega t = \frac{1}{2}(1 - \cos 2\omega t)$ and $\sin \omega t \cos \omega t = \frac{1}{2} \sin 2\omega t$:

$$\xi = \frac{a+b}{2} + \frac{a-b}{2} \cos 2\omega t, \qquad \eta = \frac{b-a}{2} \sin 2\omega t,$$

whence

$$\left[\xi - \frac{a+b}{2}\right]^2 + \eta^2 = \left(\frac{a-b}{2}\right)^2.$$

Hence: *relative to the moving frame the point describes a circle when $a \neq b$, and it remains at rest when $a = b$.*

Example 2. Motion along a helix. An important example of the motion of a point along a space curve is *helical motion* which arises in the following manner. The moving frame (ξ, η, ζ) revolves with a constant angular velocity ω about the ζ-axis, while relative to the moving frame, the point moves uniformly with a velocity c along the line $\xi = r$, $\eta = 0$ (i. e. along the line parallel to the ζ-axis and cutting the ξ-axis in the point $\xi = r$).

Such a motion arises, for example, when a circular cylinder rotates about its axis with an angular velocity ω, while a point moves along the generatrix of the cylinder with uniform motion.

At $t = 0$ let the fixed frame (x, y, z) be coincident with the moving frame (ξ, η, ζ), while the moving point has the coordinates $(r, 0, 0)$. By (III)

$$x = r \cos \omega t, \qquad y = r \sin \omega t, \qquad z = ct. \qquad (2)$$

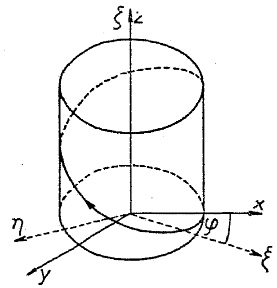

Fig. 61.

The above equations can also be obtained directly from the figure. They represent the parametric equations of a helix. The motion of the point therefore takes place along a helix. Differentiating (2) we obtain

$$x^{\cdot} = -r\omega\,\sin\omega t,\quad y^{\cdot} = r\omega\,\cos\omega t,\quad z^{\cdot} = c,$$
$$x^{\cdot\cdot} = -r\omega^2\cos\omega t,\quad y^{\cdot\cdot} = -r\omega^2\sin\omega t,\quad z^{\cdot\cdot} = 0.$$

Consequently

$$|\mathbf{v}| = \sqrt{r^2\omega^2 + c^2},\quad |\mathbf{p}| = r\omega^2.$$

Hence: *in motion along a helix the absolute values of the velocity and acceleration are constant.*

§ 13. Relation among velocities. The velocity of a point A relative to a fixed frame (x, y, z) is called *absolute velocity*; we denote it by $\mathbf{v}_a$.

The velocity of a point relative to a moving frame is called *relative velocity*; we denote it by $\mathbf{v}_r$.

Let us imagine that the point A, whose motion we are investigating, is attached *rigidly* to the moving frame (ξ, η, ζ), i. e. that its coordinates ξ, η, ζ do not change. Under this assumption the point A connected with the moving frame would possess a certain velocity relative to the fixed frame. This velocity is called the *velocity of transport* and we denote it by $\mathbf{v}_t$.

We can also say that the velocity of transport of the point A at a given moment is the velocity of a point attached to the moving frame and coinciding at the given moment with the point A.

Let us suppose, for example, that a passenger is running along the aisle of a train. As the fixed frame let us take the frame attached to the earth, as the moving frame — the frame attached to the train.

A person standing near the track will observe the motion of the passenger relative to the fixed frame, and a person sitting in the car — relative to the moving frame.

The velocity of the passenger, as observed by the person near the track, will be *absolute velocity*. The velocity, as observed by the passenger sitting in the car, will be *relative velocity*. The *velocity of transport* will be the velocity of that point on the floor of the aisle which is touched at the given moment by the passenger running along the aisle.

The velocity of transport in this case will therefore be the velocity of the train. The absolute velocity will be greater or smaller than the velocity of transport depending on whether the passenger runs in the same or opposite direction of the motion of the train.

We shall now concern ourselves with the relations that obtain among the absolute, relative and transport velocities.

The point A has the coordinates x, y, z relative to the fixed frame.

Consequently the projections of the absolute velocity on the axes of the fixed frame will be:

$$v_{a_x} = x^{\cdot},\quad v_{a_y} = y^{\cdot},\quad v_{a_z} = z^{\cdot}. \tag{1}$$

Similarly, the projections of the relative velocity on the axes of the moving frame will be:

$$v_{r_\xi} = \xi^{\cdot},\quad v_{r_\eta} = \eta^{\cdot},\quad v_{r_\zeta} = \zeta^{\cdot}. \tag{2}$$

In order to equate the absolute velocity with the relative velocity, let us form the projections of the relative velocity on the axes of the fixed frame. We obtain

$$v_{r_x} = \xi^{\cdot}\cos\alpha_1 + \eta^{\cdot}\cos\alpha_2 + \zeta^{\cdot}\cos\alpha_3,\qquad \text{etc.} \tag{3}$$

By (I), p. 53, the coordinates x, y, z relative to the fixed frame are

$$x = x_0 + \xi\cos\alpha_1 + \eta\cos\alpha_2 + \zeta\cos\alpha_3,\qquad \text{etc.}$$

Differentiating the above expression, we get from (1)

$$v_{a_x} = x^{\cdot} = x_0^{\cdot} + \xi\frac{\mathrm{d}\cos\alpha_1}{\mathrm{d}t} + \eta\frac{\mathrm{d}\cos\alpha_2}{\mathrm{d}t} + \zeta\frac{\mathrm{d}\cos\alpha_3}{\mathrm{d}t} + \xi^{\cdot}\cos\alpha_1 + $$
$$+ \eta^{\cdot}\cos\alpha_2 + \zeta^{\cdot}\cos\alpha_3. \tag{4}$$

The velocity of transport is obtained by supposing that the point A is attached rigidly to the moving frame, i. e. that the coordinates ξ, η, ζ are constant, or that $\xi^{\cdot} = 0$, $\eta^{\cdot} = 0$, $\zeta^{\cdot} = 0$. Therefore in virtue of (4) the projections of the velocity of transport on the axes of the frame (x, y, z) are

$$v_{t_x} = x_0^{\cdot} + \xi\frac{\mathrm{d}\cos\alpha_1}{\mathrm{d}t} + \eta\frac{\mathrm{d}\cos\alpha_2}{\mathrm{d}t} + \zeta\frac{\mathrm{d}\cos\alpha_3}{\mathrm{d}t},\qquad \text{etc.} \tag{5}$$

By (3) and (5) we obtain from (4) $v_{a_x} = v_{t_x} + v_{r_x}$ and similarly $v_{a_y} = v_{t_y} + v_{r_y}$, $v_{a_z} = v_{t_z} + v_{r_z}$, or

$$\mathbf{v}_a = \mathbf{v}_t + \mathbf{v}_r. \tag{I}$$

We have thus proved that *the absolute velocity is equal to the sum of the velocity of transport and the relative velocity.*

When the moving frame moves with an advancing motion the angles $\alpha_1, \alpha_2, \ldots, \gamma_3$ are constant; hence the derivatives

$$\frac{\mathrm{d}\cos\alpha_1}{\mathrm{d}t},\ \frac{\mathrm{d}\cos\alpha_2}{\mathrm{d}t},\ \ldots,\ \frac{\mathrm{d}\cos\gamma_3}{\mathrm{d}t}$$

are zero. Therefore from formula (5) we obtain

$$v_{t_x} = x_0^{\cdot},\quad v_{t_y} = y_0^{\cdot},\quad v_{t_z} = z_0^{\cdot}.$$

Hence, if $\mathbf{v}_0$ is the velocity of the origin of the moving frame, then

$$\mathbf{v}_t = \mathbf{v}_0.$$

Therefore: *if a frame moves with an advancing motion, then the velocity of transport is the same for all points and equal to the velocity of the origin of the frame.*

Remark. We say that the point A executes two motions simultaneously: one with relative velocity, the other with velocity of transport. The motion relative to a fixed frame is termed a *compound motion* of both component motions or their *resultant* motion.

The velocity of the resultant motion is therefore the sum of the velocities of the component motions. In order that the velocity of the resultant motion be defined, it is sufficient to give the velocities of the component motions; it is not necessary to say, in addition to this, which of the velocities is relative and which is the velocity of transport.

Example 1. A train is moving with a velocity $\mathbf{u}$; along the floor of a car a point A rolls with a velocity $\mathbf{v}$ relative to the floor. What is the velocity of the point A relative to the earth?

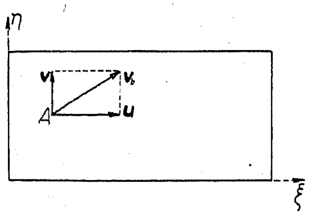

Fig. 62.

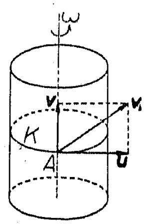

Fig. 63.

Let us assume that the axes of the frame (x, y, z) are attached to the earth and the axes ξ, η, ζ to the car (Fig. 62). The velocity of transport of the point A is therefore $\mathbf{u}$, because this is the velocity A would have relative to the earth were it at rest relative to the car. Let the relative velocity of A be $\mathbf{v}$. Hence its absolute velocity $\mathbf{v}_a$ (i. e. the velocity relative to the earth) is

$$\mathbf{v}_a = \mathbf{u} + \mathbf{v}.$$

Example 2. A cylinder revolves about its axis with an angular velocity ω. A point A moves along the generatrix with a velocity $\mathbf{v}$ (relative to the generatrix). What is the absolute velocity of the point A?

The relative velocity is $\mathbf{v}$. In order to determine the velocity of transport, let us note that if the point A were attached to the cylinder, then it would move along the circle K with an angular velocity ω (Fig. 63). Therefore, if r denotes the radius of the base of the cylinder, then the velocity of transport $\mathbf{u}$ is tangent to the circle K and $|\mathbf{u}| = r\omega$. The absolute velocity is then $\mathbf{v}_a = \mathbf{v} + \mathbf{u}$, and since $\mathbf{v} \perp \mathbf{u}$,

$$|\mathbf{v}_a| = \sqrt{v^2 + u^2} = \sqrt{v^2 + r^2\omega^2} \qquad (v = |\mathbf{v}|, \quad u = |\mathbf{u}|).$$

§ 14. Relations among accelerations. We shall consider now the relations that obtain among the accelerations of a point relative to various frames. Let us assume that we have two frames: a fixed (x, y, z) and a moving (ξ, η, ζ).

The acceleration of the point A relative to a fixed frame is called the *absolute acceleration* $\mathbf{p}_a$.

The acceleration of a point relative to a moving frame is called the *relative acceleration* $\mathbf{p}_r$.

The acceleration which a point A would possess (relative to a fixed frame), were it attached rigidly to a moving frame, is called the *acceleration of transport* $\mathbf{p}_t$.

We can also say that the acceleration of transport is the acceleration of that point attached to a moving frame which coincides with the point A at a given moment.

For example, suppose that a passenger runs along the aisle of a car. If we select as the fixed frame a frame attached to the earth, and as the moving frame a frame attached to the car, then: the absolute acceleration will be the acceleration observed by a person standing near the track, the relative acceleration will be the acceleration observed by the passenger travelling in this car, finally, the acceleration of transport will be the acceleration relative to the earth of that point on the floor which the running passenger touches at a given moment.

Denote by x, y, z the coordinates of the point A relative to the fixed frame, and by ξ, η, ζ those relative to the moving frame.

The projections of the absolute acceleration $\mathbf{p}_a$ on the x, y, z axes are:

$$p_{a_x} = x^{\cdot\cdot}, \quad p_{a_y} = y^{\cdot\cdot}, \quad p_{a_z} = z^{\cdot\cdot}. \tag{1}$$

The projections of the relative acceleration $\mathbf{p}_r$ on the ξ, η, ζ axes are:

$$p_{r_\xi} = \xi^{\cdot\cdot}, \quad p_{r_\eta} = \eta^{\cdot\cdot}, \quad p_{r_\zeta} = \zeta^{\cdot\cdot}. \tag{2}$$

Let us form the projections of the vector $\mathbf{p}_r$ on the axes of the fixed frame. We obtain

$$p_{r_x} = \xi^{\cdot\cdot} \cos\alpha_1 + \eta^{\cdot\cdot} \cos\alpha_2 + \zeta^{\cdot\cdot} \cos\alpha_3, \qquad \text{etc.} \tag{3}$$

By (I), p. 53, we have

$$x = x_0 + \xi \cos \alpha_1 + \eta \cos \alpha_2 + \zeta \cos \alpha_3, \qquad \text{etc.} \qquad (4)$$

We obtain the acceleration of transport by assuming that the point A is rigidly attached to the moving frame, or that ξ, η, ζ are constants, and therefore that the derivatives $\xi^{\cdot}, \eta^{\cdot}, \zeta^{\cdot}, \xi^{\cdot\cdot}, \eta^{\cdot\cdot}, \zeta^{\cdot\cdot}$ are equal to zero.

The projections of the vector $\boldsymbol{p}_t$ on the x, y, z axes are obtained by differentiating (4) twice under the assumption that ξ, η, ζ are constants:

$$p_{t_x} = x_0^{\cdot\cdot} + \xi \frac{d^2 \cos \alpha_1}{dt^2} + \eta \frac{d^2 \cos \alpha_2}{dt^2} + \zeta \frac{d^2 \cos \alpha_3}{dt^2}, \qquad \text{etc.} \qquad (5)$$

If $\alpha_1, \alpha_2, \ldots$ are constants, then $p_{t_x} = x_0^{\cdot\cdot}$.

Therefore: *if a frame moves with an advancing motion, then the acceleration of transport is for all points equal to the acceleration of the origin of the frame.*

Let us differentiate (4) twice. We obtain

$$\begin{aligned} x^{\cdot\cdot} = x_0^{\cdot\cdot} &+ \xi \frac{d^2 \cos \alpha_1}{dt^2} + \eta \frac{d^2 \cos \alpha_2}{dt^2} + \zeta \frac{d^2 \cos \alpha_3}{dt^2} + \\ &+ \xi^{\cdot\cdot} \cos \alpha_1 + \eta^{\cdot\cdot} \cos \alpha_2 + \zeta^{\cdot\cdot} \cos \alpha_3 + \qquad (6) \\ &+ 2\left(\xi^{\cdot} \frac{d \cos \alpha_1}{dt} + \eta^{\cdot} \frac{d \cos \alpha_2}{dt} + \zeta^{\cdot} \frac{d \cos \alpha_3}{dt}\right), \qquad \text{etc.} \end{aligned}$$

According to (3) and (5) the expressions in the first and second lines denote the projections of $\boldsymbol{p}_t$ and $\boldsymbol{p}_r$ on the x, y, z axes.

Denote by $\boldsymbol{p}_C$ the vector whose projections on the x, y and z axes are expressed by the formulae

$$p_{C_x} = 2\left(\xi^{\cdot} \frac{d \cos\alpha_1}{dt} + \eta^{\cdot} \frac{d \cos\alpha_2}{dt} + \zeta^{\cdot} \frac{d \cos\alpha_3}{dt}\right), \qquad \text{etc.} \qquad (7)$$

The vector $\boldsymbol{p}_C$ is called the *acceleration of Coriolis.*

In virtue of (1), (3), (5) and (7), formula (6) can be written in the form $p_{a_x} = p_{t_x} + p_{r_x} + p_{C_x}$. Similarly, we obtain $p_{a_y} = p_{t_y} + p_{r_y} + p_{C_y}$ and $p_{a_z} = p_{t_z} + p_{r_z} + p_{C_z}$. We may therefore write

$$\boldsymbol{p}_a = \boldsymbol{p}_t + \boldsymbol{p}_r + \boldsymbol{p}_C. \qquad (I)$$

Hence: *the absolute acceleration is equal to the sum of the accelerations: transport, relative, and Coriolis.*

Acceleration of Coriolis. In order to understand the meaning of the acceleration of Coriolis, draw from the origin M of the moving frame the vector of relative velocity $\overline{MB} = \boldsymbol{v}_r$. The coordinates of the point B

relative to the frame (ξ, η, ζ) are v_{r_ξ}, v_{r_η}, v_{r_ζ} or $\xi^{\cdot}, \eta^{\cdot}, \zeta^{\cdot}$. Let us imagine that the point B is rigidly attached to the frame (ξ, η, ζ). The velocity $\boldsymbol{u}$ of the point B relative to the fixed frame (x, y, z) is therefore equal to its velocity of transport (because its relative velocity is zero). Writing $\xi^{\cdot}, \eta^{\cdot}, \zeta^{\cdot}$ instead of ξ, η, ζ, we obtain by (5), p. 57:

$$u_x = x_0^{\cdot} + \xi^{\cdot} \frac{d \cos \alpha_1}{dt} + \eta^{\cdot} \frac{d \cos \alpha_2}{dt} + \zeta^{\cdot} \frac{d \cos \alpha_3}{dt}, \qquad \text{etc.}$$

Equating with formula (7), we obtain:

$$u_x = x_0^{\cdot} + \tfrac{1}{2} p_{C_x}, \quad u_y = y_0^{\cdot} + \tfrac{1}{2} p_{C_y}, \quad u_z = z_0^{\cdot} + \tfrac{1}{2} p_{C_z}.$$

Therefore, if we denote by $\boldsymbol{v}_0$ the velocity of the origin of the moving frame, then $\boldsymbol{u} = \boldsymbol{v}_0 + \frac{1}{2}\boldsymbol{p}_C$, whence

$$\boldsymbol{p}_C = 2(\boldsymbol{u} - \boldsymbol{v}_0). \qquad (II)$$

The difference $\boldsymbol{u} - \boldsymbol{v}_0$ is the velocity of the point B relative to the origin M of the moving frame (ξ, η, ζ) (cf. § 15, p. 65, (I)).

Therefore: *in order to obtain the acceleration of Coriolis, draw from the origin of the moving frame the vector of relative velocity $\boldsymbol{v}_r$ and imagine that this vector is attached rigidly to the moving frame.*

The acceleration of Coriolis is equal to twice the velocity of the end point of the vector $\boldsymbol{v}_r$ relative to its initial point.

It follows from this that *the acceleration of Coriolis is zero if the relative velocity is zero, or if the moving frame moves with an advancing motion.* For in these cases the origin and the terminus of the vector $\boldsymbol{v}_r$ have the same velocity. This can also be easily deduced from formula (7) by putting $\xi^{\cdot} = 0$, $\eta^{\cdot} = 0$, $\zeta^{\cdot} = 0$ or $\alpha_1 = \text{const}$, $\alpha_2 = \text{const}$, $\alpha_3 = \text{const}$, etc.

Let the frame (ξ, η, ζ) revolve about a certain axis l with an angular velocity $\boldsymbol{\omega}$ (Fig. 64). Maintaining the previous notation and selecting an arbitrary point O on the axis l, we obtain

$$\boldsymbol{u} = \overline{OB} \times \boldsymbol{\omega}, \quad \boldsymbol{v}_0 = \overline{OM} \times \boldsymbol{\omega}.$$

Then $\boldsymbol{p}_C = 2(\boldsymbol{u} - \boldsymbol{v}_0) = 2(\overline{OB} - \overline{OM}) \times \boldsymbol{\omega}$. But $\overline{OB} - \overline{OM} = \boldsymbol{v}_r$. Hence

$$\boldsymbol{p}_C = 2\boldsymbol{v}_r \times \boldsymbol{\omega}. \qquad (III)$$

Hence we see that *the acceleration of Coriolis is equal to twice the vector product of the relative velocity vector and the angular velocity vector.*

The acceleration of Coriolis is therefore perpendicular to the axis of revolution and the relative velocity, and its magnitude is

$$|\boldsymbol{p}_C| = 2|\boldsymbol{\omega}||\boldsymbol{v}_r| \sin \alpha,$$

where α is the angle between the axis of revolution and the relative velocity vector. Hence it follows that the acceleration of Coriolis is zero (in addition to the case when $\boldsymbol{v}_r = 0$) when $\alpha = 0$, i. e. when $\boldsymbol{v}_r$ is parallel to $\boldsymbol{\omega}$.

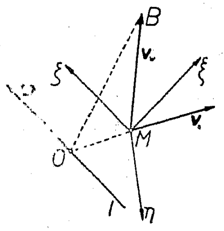

Fig. 64.

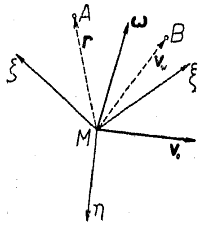

Fig. 65.

We shall show later (in chapter VII) that at every moment the velocities of the points attached rigidly to the moving frame (ξ, η, ζ) are such, as if the frame were executing two motions simultaneously: the first, an advancing motion with a velocity equal to that of the origin of the frame, and the second, a rotation about a certain axis, which passes through the origin of the frame, with an angular velocity $\boldsymbol{\omega}$.

This axis of rotation is called the *instantaneous axis of rotation*; $\boldsymbol{\omega}$ is called the *instantaneous angular velocity*.

At each moment of time there may be a different instantaneous axis of rotation and different $\boldsymbol{\omega}$'s. Therefore, denoting by $\boldsymbol{v}_0$ the velocity of the origin M of the frame, we obtain for the velocity of transport $\boldsymbol{v}_t$ of the point A the formula

$$\boldsymbol{v}_t = \boldsymbol{v}_0 + \overline{MA} \times \boldsymbol{\omega}.$$

Hence, if we denote by B the end of the relative velocity vector drawn from the origin of the frame, and by $\boldsymbol{u}$ (as before) the velocity of transport of the point B, then $\boldsymbol{u} = \boldsymbol{v}_0 + \overline{MB} \times \boldsymbol{\omega} = \boldsymbol{v}_0 + v_r \times \boldsymbol{\omega}$, whence by (II), p. 61

$$\boldsymbol{p}_C = 2\boldsymbol{v}_r \times \boldsymbol{\omega}. \qquad \text{(IV)}$$

Formula (IV) represents the acceleration of Coriolis in the general case.

The acceleration of Coriolis is therefore zero: 1° *when* $\boldsymbol{\omega} = 0$ (i. e. when the frame moves with an advancing motion), 2° *when* $\boldsymbol{v}_r = 0$, 3° *when* $\boldsymbol{\omega} \parallel \boldsymbol{v}_r$.

Example 1. A train moves along a straight track with an acceleration $\boldsymbol{p}$. A point moves along the floor of the carriage with an acceleration $\boldsymbol{a}$ relative to the carriage. Determine the acceleration of the point relative to the earth.

The relative acceleration is $\boldsymbol{p}_r = \boldsymbol{a}$ and the acceleration of transport is $\boldsymbol{p}_t = \boldsymbol{p}$. Since the frame of reference attached to the carriage moves with an advancing motion, the acceleration of Coriolis $\boldsymbol{p}_C = 0$. Therefore the absolute acceleration (i. e. the acceleration relative to the earth) is

$$\boldsymbol{p}_a = \boldsymbol{a} + \boldsymbol{p}.$$

Example 2. A cylinder of radius r rotates about an axis with an angular velocity ω. A point A moves along the generatrix of the cylinder with a constant velocity $\boldsymbol{v}$ relative to the generatrix. Determine the acceleration of the point A relative to the fixed frame.

Taking the axis of the cylinder as the ζ-axis, let us select a moving frame attached to the cylinder. If the point A were attached rigidly to the frame (ξ, η, ζ), then it would move along the circle K with an angular velocity ω. The acceleration of transport $\boldsymbol{p}_t$ is therefore directed towards the centre of the circle K and $|\boldsymbol{p}_t| = r\omega^2$. By hypothesis, the relative acceleration $\boldsymbol{p}_r = 0$. Since the relative velocity $\boldsymbol{v}_r = \boldsymbol{v}$ is parallel to the axis of rotation, the acceleration of Coriolis $\boldsymbol{p}_C = 0$. Hence

$$\boldsymbol{p}_a = \boldsymbol{p}_t.$$

Example 3. A horizontal plane revolves about a vertical axis with an angular velocity ω. A point A moves along the plane and at a certain moment has a velocity $\boldsymbol{v}_r$ and an acceleration $\boldsymbol{p}_r$ relative to the plane. Determine the acceleration relative to the fixed frame at that moment.

Let O be the point of intersection of the axis of revolution with the moving plane. If the point A were attached to the moving plane, then it would move along a circle with centre at O and radius $OA = r$ with an angular velocity ω. Hence the acceleration of transport $\boldsymbol{p}_t$ is directed towards O and $|\boldsymbol{p}_t| = r\omega^2$.

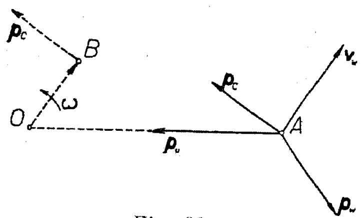

Fig. 66.

In order to determine the acceleration of Coriolis $\boldsymbol{p}_C$, select a point O as the origin of the moving frame attached to the moving plane and draw from O the vector $\overline{OB} = \boldsymbol{v}_r$. Since the velocity of the point O is zero, $\frac{1}{2}\boldsymbol{p}_C$ is equal to the velocity of the point B. Therefore $\boldsymbol{p}_C \perp \boldsymbol{v}_r$ and $|\boldsymbol{p}_C| = 2\,OB\,\omega = 2|\boldsymbol{v}_r|\omega$.

We should obviously have obtained the same result if we had used the formula $\mathbf{p}_C = 2\mathbf{v}_r \times \boldsymbol{\omega}$, and the fact that the angular velocity vector $\boldsymbol{\omega}$ has the direction of the axis of revolution and is therefore perpendicular to the moving plane.

The absolute acceleration is obtained by adding together vectors $\mathbf{p}_r$, $\mathbf{p}_t$ and $\mathbf{p}_C$.

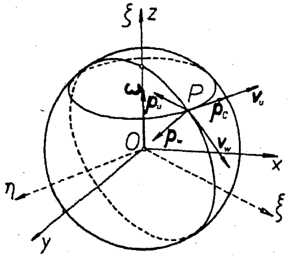

Fig. 67.

Example 4. A sphere of radius r revolves about a fixed axis with a constant angular velocity $\boldsymbol{\omega}$. A point P moves with a constant velocity $\mathbf{c}$ along a great circle passing through the axis of revolution. Determine the velocity and acceleration of this point relative to the fixed frame (x, y, z).

Let (ξ, η, ζ) denote the moving frame whose ζ-axis is the same as that of the fixed frame, while the plane $\xi\zeta$ is the plane of the meridian along which the point P moves. This frame revolves along with the sphere about the z-axis with a constant angular velocity $\boldsymbol{\omega}$ (which in the figure is represented by the vector drawn on the z-axis). The velocity of the point P relative to the moving frame, i. e. the relative velocity $\mathbf{v}_r$, is a vector of length c tangent to the meridian (along which the point P moves). The velocity of transport $\mathbf{v}_t$ is equal to the velocity of a point of the parallel of latitude passing through P. Since this point moves along a circle of radius $\varrho = r\cos\varphi$, where φ denotes the latitude of this parallel of latitude (i. e. the angle between OP and the equatorial plane), the velocity $\mathbf{v}_t$ is tangent to the parallel of latitude and is equal to $\varrho\omega = r\omega\cos\varphi$. The velocities $\mathbf{v}_r$ and $\mathbf{v}_t$ are perpendicular to each other; hence $|\mathbf{v}_a| = \sqrt{c^2 + r^2\omega^2\cos^2\varphi}$. The absolute velocity $\mathbf{v}_a$ forms with the meridians an angle Θ defined by the formula $\tan\Theta = |\mathbf{v}_t| / |\mathbf{v}_r| = (r\omega / c)\cos\varphi$. Since the point P moves along the meridian with a constant velocity c, its relative acceleration $\mathbf{p}_r$ is directed towards the centre of the sphere and $|\mathbf{p}_r| = c^2/r$. Similarly, the acceleration of transport $\mathbf{p}_t$ is directed towards the centre of the parallel of latitude (passing through P) and $|\mathbf{p}_t| = \varrho\omega^2 = r\omega^2\cos\varphi$. The acceleration of Coriolis $\mathbf{p}_C = 2\mathbf{v}_r \times \boldsymbol{\omega}$ is perpendicular to $\boldsymbol{\omega}$ and $\mathbf{v}_r$ and hence perpendicular to the equatorial plane and has the same sense as $\mathbf{v}_t$. Since, as is easily seen, the angle between $\boldsymbol{\omega}$ and $\mathbf{v}_r$ is $\pi - \varphi$, $|\mathbf{p}_C| = 2|\mathbf{v}_r||\boldsymbol{\omega}|\sin\varphi = 2c\,\omega\sin\varphi$. Adding together the vectors $\mathbf{p}_r$, $\mathbf{p}_t$ and $\mathbf{p}_C$, we obtain the absolute acceleration $\mathbf{p}_a$.

§ 15. Determination of relative motion. Up to this time we have been concerned with the determination of the motion relative to a fixed frame, having been given the motion relative to a moving frame. We frequently encounter the converse problem, i. e. we have to find the motion relative to a moving frame, knowing this motion relative to a fixed frame.

From formulae (I), pp. 60 and 57, we obtain for the relative velocity and relative acceleration:

$$\mathbf{v}_r = \mathbf{v}_a - \mathbf{v}_t, \quad \mathbf{p}_r = \mathbf{p}_a - \mathbf{p}_t - \mathbf{p}_C. \tag{I}$$

Therefore: *the relative velocity is obtained by adding to the absolute velocity the velocity of transport with an opposite sense. The relative acceleration is obtained by adding to the absolute acceleration the acceleration of transport and Coriolis with opposite senses.*

Motion relative to a point. Let the points A_1 and A_2 move relative to a certain fixed frame (x, y, z) with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Let us place the origin of the moving frame (ξ, η, ζ) at A_2, and let us assume that this frame moves with an advancing motion.

The motion of the point A_1 relative to the frame (ξ, η, ζ) will be called *relative motion with respect to the point A_2*.

Such a motion would be observed by a person moving with an advancing motion together with the point A_2.

Let us denote by $\mathbf{v}_{1,2}$ the velocity of the point A_1 relative to the point A_2, i. e. relative to the frame (ξ, η, ζ). Since the velocity of transport is equal to the velocity $\mathbf{v}_2$ of the point A_2, and the absolute velocity is equal to $\mathbf{v}_1$, it follows $\mathbf{v}_1 = \mathbf{v}_{1,2} + \mathbf{v}_2$, whence $\mathbf{v}_{1,2} = \mathbf{v}_1 - \mathbf{v}_2$, or

$$\mathbf{v}_{1,2} = \mathbf{v}_1 + (-\mathbf{v}_2). \tag{II}$$

Denoting by $\mathbf{p}_{1,2}$ the acceleration of the point A_1 relative to A_2, i. e. relative to the frame (ξ, η, ζ), and observing that the acceleration of Coriolis is zero, we obtain $\mathbf{p}_{1,2} = \mathbf{p}_1 - \mathbf{p}_2$, or

$$\mathbf{p}_{1,2} = \mathbf{p}_1 + (-\mathbf{p}_2). \tag{III}$$

Therefore: *the velocity (acceleration) of a point A_1 relative to A_2 is obtained by adding to the velocity (acceleration) of A_1 the velocity (acceleration) of the point A_2 with an opposite sense*

Example 1. The points A_1 and A_2 move uniformly along the x and y axes, respectively, with the velocities c_1 and c_2. Determine the velocity of the point A_1 relative to A_2.

The sought for velocity $\mathbf{v}_{1,2}$ is the difference between the velocities of A_1 and A_2. Hence $\mathbf{v}_{1,2} = \mathbf{v}_1 - \mathbf{v}_2$. The projections of $\mathbf{v}_{1,2}$ on the x and y axes are c_1 and $-c_2$. Therefore

$$|\mathbf{v}_{1,2}| = \sqrt{c_1^2 + c_2^2}.$$

Example 2. A point A_1 moves along a circle of radius r with uniform motion, while the point A_2 moves in such way that it is always at the other end of the diameter passing through A_1. Determine the velocity and acceleration of the point A_1 relative to A_2.

Obviously, the velocity and acceleration of both points are equal in absolute value and have opposite senses. Denoting by $\mathbf{v}$ and $\mathbf{p}$ the velocity and acceleration of the point A_1, we obtain

$$\mathbf{v}_{1,2} = \mathbf{v} - (-\mathbf{v}) = 2\mathbf{v}, \quad \text{and} \quad \mathbf{p}_{1,2} = \mathbf{p} - (-\mathbf{p}) = 2\mathbf{p}.$$

Since $|\mathbf{v}_{1,2}| = \text{const}$, the tangential acceleration of the rolative motion is zero; consequently $\mathbf{p}_{1,2}$ is the normal acceleration. Hence $|\mathbf{p}_{1,2}| = |\mathbf{v}_{1,2}|^2 / \varrho = 4|\mathbf{v}|^2 / \varrho$, and since $|\mathbf{p}_{1,2}| = 2|\mathbf{p}| = 2|\mathbf{v}|^2 / r$,

$$4|\mathbf{v}|^2 / \varrho = 2|\mathbf{v}|^2 / r, \quad \text{or} \quad \varrho = 2r.$$

Therefore the motion of the point A_1 relative to A_2 takes place along a circle with centre at A_2 and radius $2r$, with a velocity twice as large as the velocity of the point A_2.

Example 3. A body A moves along the x-axis with a constant velocity u, and emits every T seconds small particles which move with uniform motion along the x-axis with a velocity c. Let ν denote the frequency of emission (i. e. the number of particles emitted per second), and λ the distance between two successively emitted particles. We obviously have $\nu = 1 / T$.

Since the relative velocity of an emitted particle with respect to A is $c - u$, the distance of the particle from A after the time T is $\lambda = (c - u)\, T$. Consequently

$$\nu = (c - u) / \lambda. \tag{1}$$

Let us suppose now that an observer B moves along the x-axis with a constant velocity v. Let us denote by ν' the relative frequency of emission (i. e. the number of particles per second met by an observer), and by T' the time between the meetings of two successive particles. Since the velocity of the particles relative to B is $c - v$, $\lambda = (c - v)\, T'$. Hence

$$\nu' = (c - v) / \lambda. \tag{2}$$

In virtue of (1) and (2) we obtain

$$\nu' = \nu(c - v) / (c - u) = \nu(1 - v / c) / (1 - u / c). \tag{3}$$

Let us assume that the velocity c is large as compared with u and v. Since for small x's we have $1 / (1 - x) = 1 + x$, from (3)

$$\nu' = \nu\left(1 - \frac{v}{c}\right)\left(1 + \frac{u}{c}\right) = \nu\left[1 - \frac{v - u}{c} - \frac{vu}{c^2}\right].$$

Neglecting the last term enclosed by the brackets as very small compared with what remains, we finally have

$$\nu' = \nu[1 - (v - u) / c]. \tag{4}$$

In particular, if $u = 0$, we get

$$\nu' = \nu(1 - v / c). \tag{5}$$

Example 4. A swarm of small particles is moving through space with a constant velocity $\mathbf{v}$. A body A moves through this swarm with a velocity $\mathbf{u}$. The relative velocity of the particles with respect to A is therefore $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Let us denote by w, v and u the absolute values of these velocities, by φ the angle between $\mathbf{w}$ and $\mathbf{v}$, and by α the angle between $\mathbf{w}$ and $\mathbf{u}$. From the triangle with sides $\mathbf{v}$, $-\mathbf{u}$, $\mathbf{w}$ we obtain

$$\sin\varphi = \frac{u}{v}\sin\alpha. \tag{6}$$

We shall give an application of the above formula.

As the fixed frame let us choose a frame attached to the sun and the fixed stars. A certain star G sends to the earth rays of light travelling with a velocity $\mathbf{v}$ ($v = 300\,000$ km/sec). The earth moves with a velocity $\mathbf{u}$ ($u = 30$ km/sec). Therefore the rays of light have a relative velocity $\mathbf{w}$ with respect to the earth. An observer on the earth who wants to see the star G must set his telescope in the direction of the relative velocity $\mathbf{w}$. He will therefore see G seemingly at the place G'. The angle φ, denoting the deviation from the true direction, can be calculated from (6).

Since v is large as compared with u, the angle φ is very small. From formula (6) we obtain

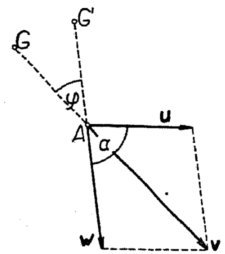

Fig. 68.

$$\sin\varphi = \frac{\sin\alpha}{10\,000}.$$

For $\alpha = \frac{1}{2}\pi$ (i. e. $\sin\alpha = 1$) we obtain for the angle φ the maximum value $\varphi = 22''$.

Example 5. On the periphery of a circle with centre O the points A_1 and A_2 are moving with the angular velocities ω_1 and ω_2 relative to a certain fixed frame. Choose the ξ and η axes of the moving frame in the plane of the circle; let O be the origin and the line OA_2 the ξ-axis.

The angular velocity of the point A_1 relative to the chosen moving frame is called the (relative) *angular velocity of the point A_1 with respect to the point A_2*; we denote it by $\omega_{1,2}$.

It is easy to show that

$$\omega_{1,2} = \omega_1 - \omega_2. \tag{7}$$

Let us assume that the points A_1 and A_2 move with constant angular velocities. Denoting by T_1, T_2 the periods of revolutions of the corresponding points A_1, A_2 in the fixed frame, and by $T_{1,2}$ the period of revolution of the point A_1 relative to A_2 (i. e. the period of revolution of the point A_1 in the moving frame), we obtain: $T_1 = 2\pi / \omega_1$, $T_2 = 2\pi / \omega_2$, $T_{1,2} = 2\pi / \omega_{1,2}$. Hence by (7)

$$1 / T_{1,2} = 1 / T_1 - 1 / T_2. \tag{8}$$

The period of revolution of the minute hand of a clock is $T_1 = 1$ h, that of the hour hand $T_2 = 12$ h. From formula (8) we get: $1/T_{1,2} = 1 - \frac{1}{12}$, whence $T_{1,2} = \frac{12}{11}$ h $= 1$ h, 5 min, 27 sec. Therefore the hands coincide every 1 h, 5 min, 27 sec.

To a traveller circling the globe from west to east it seems that the journey lasted n mean solar days, because during the journey there were n days and n nights. However, returning to the place from which he started, he finds that the journey did not last n, but n' mean solar days. What is the relation between n and n'?

Let us denote by T_1 the time of the journey, and by T_2 the time taken by the sun to complete an apparent revolution about the earth. Consequently $T_1 = n'$, and $T_2 = -1$ (since the sun seemingly revolves about the earth from east to west, that is, in the direction opposite to that of the journey). The traveller assumed as the apparent mean solar day the period of time between two successive passings of the sun across the changing meridian on which he was. Since in the interval of n apparent days there were n' real days, the apparent mean solar day is equal to n' / n real days. Hence $T_{1,2} = n' / n$. Therefore by (8)

$$n / n' = 1 / n' + 1, \quad \text{or} \quad n' = n - 1.$$

Therefore the number of real days elapsed was one less than the number of apparent days.

If the traveller had gone from east to west, then (as is easily seen) the number of real days elapsed would be one greater than the number of apparent days.

CHAPTER III

DYNAMICS OF A MATERIAL POINT

I. DYNAMICS OF AN UNCONSTRAINED POINT

§ 1. Basic concepts of dynamics. The subject of dynamics is concerned with the investigations of the motion of bodies under the influence of forces which cause this motion.

In kinematics all frames of reference are, as we already know, equally valid; it is a matter of indifference how we measure time (i. e. what intervals of time we consider as equal). The laws of dynamics stated by Newton, however, are not valid for every frame of reference and every measure of time.

Inertial frame, absolute time. A frame of reference for which, along with a certain measure of time, the Newtonian laws of dynamics hold, is called an *inertial frame*, the corresponding measure of time — the measure of *absolute time*, and the motion of the body relative to the inertial frame — *absolute motion.*

Strictly speaking, we do not know at present of any example of either an inertial frame or of absolute time. Nevertheless, in a great number of problems we can select frames of reference and methods of measuring time in such a way, that the application of the laws of dynamics leads to results differing sufficiently little from experience, so that for all practical purposes the errors can be neglected.

For instance, if we are investigating the motion of small particles near the earth during a short interval of time, the results will be sufficiently accurate on the whole, if we take as an inertial frame the frame of reference attached to the earth, and if we base the measurement of absolute time on the assumption that the earth, relative to the fixed stars, revolves about its axis so as to make equal angles in equal times.

In other problems, however (such as Foucault's pendulum, the gyroscope, the motion of planets) the application of the laws of dynamics to a frame of reference attached to the earth does not lead to equally good results. Considerably better results are obtained here if we select for the inertial frame, a frame of reference whose origin is situated within the sun and whose axes point to the fixed stars.