CHAPTER I

THEORY OF VECTORS

I. OPERATIONS ON VECTORS

§ 1. Preliminary definitions. Magnitudes which can be characterized by means of one real number are called *scalars*. Examples of scalars are: mass, work, kinetic energy, etc.

A *vector* is a line segment in which the initial point is distinguished from the terminal point. Points are classified as *zero vectors*.

Magnitudes such as velocity, acceleration and force can be represented by means of vectors. A vector will be denoted by bold face type, for example $\mathbf{a}$; a vector whose origin is A and terminus is B will be denoted by $\overline{AB}$ (Fig. 1). In a drawing an arrow serves to mark the terminus of a vector. The origin of a vector is also called *a point of application*.

By the *length* or *absolute value* of the vector $\overline{AB}$ is meant the length of the line segment AB and it is denoted by $|\overline{AB}|$.

Two vectors having the same direction (i. e. parallel vectors) can have the same or opposite *senses* (Fig. 2).

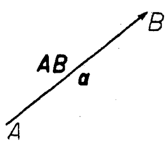

Fig. 1.

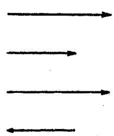

Fig. 2.

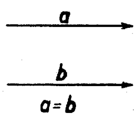

Fig. 3.

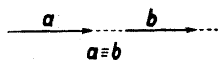

Fig. 4.

The vectors $\mathbf{a}$ and $\mathbf{b}$ having equal lengths, directions and senses are said to be *equal* (Fig. 3) and we write

$$\mathbf{a} = \mathbf{b}.$$

Two vectors having equal lengths and directions but opposite senses are called *opposite* vectors. The vector opposite to $\mathbf{a}$ is denoted by $-\mathbf{a}$ (Fig. 11).

The straight line on which a vector lies is called the *position* of the vector.

Equal vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ having the same position (i. e. lying on the same line) are termed *equipollent* (Fig. 4):

$$\boldsymbol{a} \equiv \boldsymbol{b}.$$

Two zero vectors are considered to be equal and equipollent.

Equal vectors will often be denoted by one and the same letter (whenever there is no likelihood of committing an error).

The *projection* of the vector $\boldsymbol{a}$ on a line (or plane) is the vector whose initial and terminal points are the projections of the corresponding points of $\boldsymbol{a}$.

Suppose that there is given in space a coordinate system $O(x, y, z)$ which is either rectangular or oblique. Rotate the x-axis about O in the xy-plane through an angle $< \pi$ so that the positive side of the x-axis falls on the positive side of the y-axis. If to an observer situated on the same side of the xy-plane as the positive side of the z-axis, the rotation is clockwise, then the coordinate system $O(x, y, z)$ is said to be *left-handed* and in the contrary case *right-handed*.

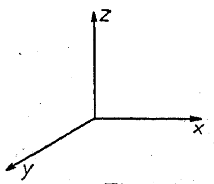

Fig. 5.

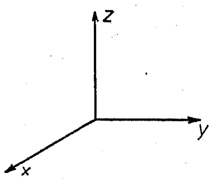

Fig. 6.

In this book we shall consistently use a left-handed rectangular system (i. e. as in Fig. 5, and not as in Fig. 6).

We say that *the system of vectors* ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$), not parallel to the same plane, has a *left (*or *right) sense*, if upon passing the x, y and z axes through an arbitrary point O parallel to and in the same direction as the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, we obtain a left-handed (or right-handed) system.

§ 2. Components of a vector. Let $\boldsymbol{a}$ represent an arbitrary vector and $\boldsymbol{a}'$ its projection on a given x-axis.

The component of the vector $\boldsymbol{a}$ *with respect to the x-axis*, which we shall denote by a_x, is a number defined in the following way: $a_x = |\boldsymbol{a}'|$ if $\boldsymbol{a}'$ has the same direction as the x-axis, but $a_x = -|\boldsymbol{a}'|$ in the contrary case.

We obviously have

$$a_x = |\boldsymbol{a}| \cos \alpha, \tag{1}$$

where α denotes the angle between the vector $\boldsymbol{a}$ and the x-axis (Fig. 8).

Suppose that a rectangular coordinate system (x, y, z) is given. Denoting the components of $\boldsymbol{a}$ with respect to the coordinate axes by a_x, a_y, a_z, and the angles which the vector makes with the axes by α, β, γ (Fig. 7), we obtain by (1):

$$a_x = |\boldsymbol{a}| \cos \alpha, \quad a_y = |\boldsymbol{a}| \cos \beta, \quad a_z = |\boldsymbol{a}| \cos \gamma. \tag{I}$$

Therefore: *equal vectors have equal components* with respect to the coordinate axes.

By the identity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ well-known from analytic geometry and by (I)

$$|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{II}$$

$$\cos \alpha = a_x / |\boldsymbol{a}|, \quad \cos \beta = a_y / |\boldsymbol{a}|, \quad \cos \gamma = a_z / |\boldsymbol{a}|. \tag{III}$$

From equations (II) and (III) it follows that the components of a vector define its length, direction and sense.

Hence, two *vectors* $\boldsymbol{a}$ and $\boldsymbol{b}$ *having correspondingly equal components* with respect to a rectangular coordinate system (i. e. for which $a_x = b_x$, $a_y = b_y$, $a_z = b_z$) *are equal*.

If the vector $\boldsymbol{a}$ lies in the xy-plane (Fig. 8), then

$$a_x = |\boldsymbol{a}| \cos \alpha, \quad a_y = |\boldsymbol{a}| \sin \alpha, \tag{IV}$$

$$|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2}, \quad \cos \alpha = a_x / |\boldsymbol{a}|, \quad \sin \alpha = a_y / |\boldsymbol{a}|. \tag{V}$$

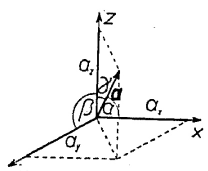

Fig. 7.

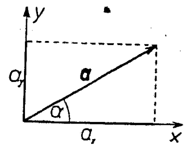

Fig. 8.

Often (when an error is precluded) the projections of the vector $\boldsymbol{a}$ on the coordinate axes are also termed the components a_x, a_y, a.

It is easy to show that if the points A and A' have coordinates x, y, z and x', y', z' respectively, then the vector $\boldsymbol{a} = \overline{AA'}$ has components: $a_x = x' - x$, $a_y = y' - y$, and $a_z = z' - z$.

§ 3. Sum and difference of vectors. Every vector which can be obtained in the following manner is said to be the *sum* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

From an arbitrary point O we draw a vector equal to $\boldsymbol{a}$ and from the terminal point of this vector a second vector equal to $\boldsymbol{b}$; the vector whose initial point is O and whose terminal point is the terminal point of the second vector we call *the sum of the vectors* $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 9) and we denote it by

$$\boldsymbol{a} + \boldsymbol{b}.$$

For opposite vectors (Fig. 11) we therefore obtain in particular

$$\boldsymbol{a} + (-\boldsymbol{a}) = 0.$$

The sum of several vectors, for example $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$, is obtained by forming the sum $\boldsymbol{b} + \boldsymbol{c}$ and then adding the result to the vector $\boldsymbol{a}$ (Fig. 10).

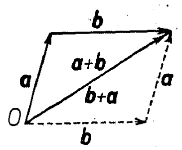

Fig. 9.

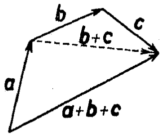

Fig. 10.

Vectors obey the commutative and associative laws of addition. Hence:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a},$$

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}).$$

From these laws it follows that the sum of any number of vectors remains unaltered if the order of the terms is changed, or if several are enclosed by a parenthesis. For example:

$$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d} + \boldsymbol{e} = \boldsymbol{a} + \boldsymbol{c} + \boldsymbol{e} + \boldsymbol{b} + \boldsymbol{d} =$$
$$= (\boldsymbol{a} + \boldsymbol{c}) + \boldsymbol{e} + (\boldsymbol{b} + \boldsymbol{d}).$$

The *difference* $\boldsymbol{a} - \boldsymbol{b}$ is defined as the sum $\boldsymbol{a} + (-\boldsymbol{b})$. Therefore from the definition

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}).$$

Figures 12 and 13 show how to determine the difference.

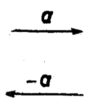

Fig. 11.

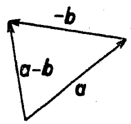

Fig. 12.

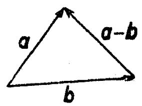

Fig. 13.

Since

$$(\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a},$$

it follows that *the difference added to the subtrahend gives as a result the minuend.*

§ 4. Product of a vector by a number. *The product of a vector* $\boldsymbol{a}$ *by a number* m is defined as the vector which has the same direction as $\boldsymbol{a}$, a length $|m|$ times that of $\boldsymbol{a}$, and a sense agreeing with or opposite to that

of $\boldsymbol{a}$, depending on whether $m > 0$ or $m < 0$. The product of $\boldsymbol{a}$ by m is denoted by

$$m\boldsymbol{a}.$$

If $m = 0$ or $\boldsymbol{a} = 0$, then $m\boldsymbol{a} = 0$.

We evidently have (Fig. 14 for $m = 2$):

$$(-m)\,\boldsymbol{a} = -m\boldsymbol{a}.$$

Hence it follows that

$$(-1)\,\boldsymbol{a} = -\boldsymbol{a}.$$

For a product it is easy to demonstrate the distributive law for multiplication with respect to addition and the associative law:

$$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}, \quad (m + p)\,\boldsymbol{a} = m\boldsymbol{a} + p\boldsymbol{a}, \quad m(p\boldsymbol{a}) = (mp)\,\boldsymbol{a},$$

where m and p denote numbers (Figs 15 and 16).

From the above laws follow the usual algebraic rules of addition and multiplication.

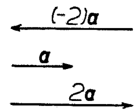

Fig. 14.

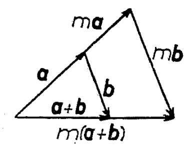

Fig. 15.

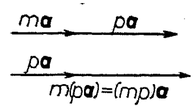

Fig. 16.

Division of a vector by a number (different from zero) is defined as multiplication by the reciprocal of that number. Therefore:

$$\frac{\boldsymbol{a}}{m} = \frac{1}{m}\boldsymbol{a}.$$

§ 5. Components of a sum and product. It is easy to show that the *projection* (on a line or plane) of a *sum* of vectors is equal to the *sum of the projections* of these vectors (Fig. 17). Hence:

$$\text{Proj}\,(\boldsymbol{a} + \boldsymbol{b}) = \text{Proj}\,\boldsymbol{a} + \text{Proj}\,\boldsymbol{b}.$$

Similarly, *the projection of a product* of a vector by a number is equal to *the product of the projection* of the vector by this number (Fig. 18). Therefore:

$$\text{Proj}\,(m\boldsymbol{a}) = m\,\text{Proj}\,\boldsymbol{a}.$$

If the vector $\boldsymbol{a}$ has components a_x, a_y, a_z, and the vector $\boldsymbol{b}$ components b_x, b_y, b_z, then the vector $\boldsymbol{s} = \boldsymbol{a} + \boldsymbol{b}$ has components $s_x = a_x + b_x$, $s_y = a_y + b_y$, $s_z = a_z + b_z$.

This follows from the theorem on the projection of a sum of vectors.

Similarly, from the theorem on the projection of the product of a vector by a number, it follows that the vector $\boldsymbol{c} = m\boldsymbol{a}$ has components

$$c_x = ma_x, \quad c_y = ma_y, \quad c_z = ma_z.$$

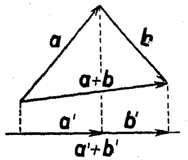

Fig. 17.

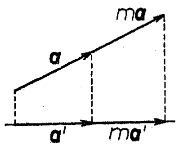

Fig. 18.

For example, if $\boldsymbol{d} = 5\boldsymbol{a} - 3\boldsymbol{b} - 2\boldsymbol{c}$, then:

$$d_x = 5a_x - 3b_x - 2c_x,$$
$$d_y = 5a_y - 3b_y - 2c_y,$$
$$d_z = 5a_z - 3b_z - 2c_z.$$

§ 6. Resolution of a vector. The sum of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ having a common origin, but not lying on the same line, represents the diagonal of a parallelogram with these vectors as sides. Similarly, the sum of the three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ having a common origin, but not lying in the same plane, represents the diagonal of a parallelepiped having these vectors as edges.

The resolution of a given vector into the sum of two or three vectors having given directions is based on the preceding theorems.

Let us suppose that a vector $\boldsymbol{s}$ and two non-parallel lines l and m lying in a certain plane parallel to $\boldsymbol{s}$ are given. If we want to represent the vector $\boldsymbol{s}$ as a sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ parallel to l and m, then let us form a parallelogram whose sides are parallel to l and m, and whose diagonal is $\boldsymbol{s}$. For this purpose we draw lines from the initial and terminal points of the vector $\boldsymbol{s}$ parallel to l and m. The sides of the parallelogram obtained will determine the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 19).

It is easy to see that such a resolution is possible in only one way.

Similarly, if a vector $\boldsymbol{s}$ and three lines l, m, n not parallel to the same plane are given and we want to represent $\boldsymbol{s}$ as the sum of the three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ parallel to l, m, n, then we form a parallelepiped with edges parallel to l, m, n whose diagonal is $\boldsymbol{s}$. We therefore draw lines l', m', n' from the initial point O of the vector $\boldsymbol{s}$ parallel to l, m, n; then from the terminal point of $\boldsymbol{s}$ we draw a line parallel to n to the point of intersection G of this line with the plane formed by l', m'; finally from the point G we draw parallels to l and m. The points of intersection of these lines with l' and m' are the end points of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ whose initial point is O. Vector $\boldsymbol{c}$ is equal to the vector joining point G with the end of vector $\boldsymbol{s}$ (Fig. 20).

Only one such resolution is possible, since there exists only one parallelepiped having edges parallel to l, m, n, and a diagonal $\boldsymbol{s}$.

A particular case of such a resolution is the representation of a vector by means of unit vectors. We denote the projections of the vector $\boldsymbol{a}$ on the axes of the system (x, y, z) by $\boldsymbol{a}', \boldsymbol{a}'', \boldsymbol{a}'''$. We obviously have (Fig. 21):

$$\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{a}'' + \boldsymbol{a}'''.$$

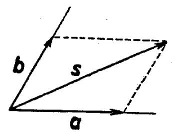

Fig. 19.

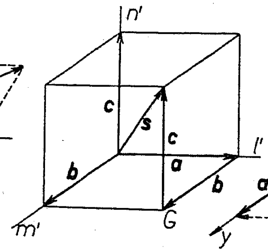

Fig. 20.

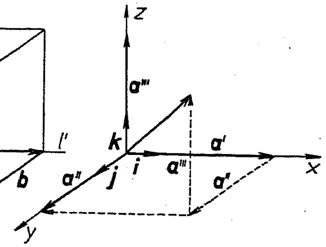

Fig. 21.

On the coordinate axes let us select vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of unit length agreeing in direction with the corresponding axes. From the definition of a_x, a_y, a_z (§ 2, p. 2) it follows that

$$\boldsymbol{a}' = a_x\boldsymbol{i}, \quad \boldsymbol{a}'' = a_y\boldsymbol{j}, \quad \boldsymbol{a}''' = a_z\boldsymbol{k}.$$

Therefore

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}. \qquad \text{(I)}$$

The vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are called *unit* vectors. Formula (I) expresses the vector $\boldsymbol{a}$ in terms of components and unit vectors.

§ 7. Scalar product. The *scalar product* of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ forming an angle φ (Fig. 22) is defined as the number $|\boldsymbol{a}||\boldsymbol{b}| \cos \varphi$.

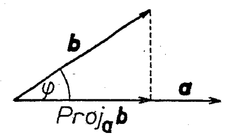

Fig. 22.

We denote the scalar product by $\boldsymbol{a} \cdot \boldsymbol{b}$ or $\boldsymbol{ab}$. Therefore

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \varphi. \qquad \text{(I)}$$

The scalar product is zero not only when $\boldsymbol{a} = 0$ or $\boldsymbol{b} = 0$, but also when $\boldsymbol{a} \perp \boldsymbol{b}$, because then $\varphi = \pi/2$ and hence $\cos \varphi = 0$. However, if $\boldsymbol{a} \neq 0$ and $\boldsymbol{b} \neq 0$, then the scalar product can be positive or negative depending on whether φ is acute or obtuse.

The *scalar product is commutative* because we have

$$\boldsymbol{b} \cdot \boldsymbol{a} = |\boldsymbol{b}||\boldsymbol{a}| \cos \varphi = |\boldsymbol{a}||\boldsymbol{b}| \cos \varphi = \boldsymbol{a} \cdot \boldsymbol{b}.$$

The expression $|\boldsymbol{b}| \cos \varphi$ represents the projection of the vector $\boldsymbol{b}$ on the axis determined by the vector $\boldsymbol{a}$ and agreeing with it in direction. This projection is called the *projection of* $\boldsymbol{b}$ *on the direction of* $\boldsymbol{a}$ and is denoted by $\mathrm{Proj}_{a}\boldsymbol{b}$. Therefore

$$\mathrm{Proj}_{a}\boldsymbol{b} = |\boldsymbol{b}| \cos\varphi, \quad \mathrm{Proj}_{b}\boldsymbol{a} = |\boldsymbol{a}| \cos\varphi.$$

Hence by (I)

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|\,\mathrm{Proj}_{a}\boldsymbol{b} = |\boldsymbol{b}|\,\mathrm{Proj}_{b}\boldsymbol{a}. \tag{1}$$

Therefore: *the scalar product is equal to the product of the length of one vector by the projection of the other on the direction of the first.*

Distributive law. From the definition of a scalar product we have

$$(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c} = |\boldsymbol{c}|\,\mathrm{Proj}_{c}(\boldsymbol{a}+\boldsymbol{b}).$$

Since the $\mathrm{Proj}_{c}(\boldsymbol{a}+\boldsymbol{b}) = \mathrm{Proj}_{c}\boldsymbol{a} + \mathrm{Proj}_{c}\boldsymbol{b}$, it follows that

$$(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c} = |\boldsymbol{c}|\,\mathrm{Proj}_{c}\boldsymbol{a} + |\boldsymbol{c}|\,\mathrm{Proj}_{c}\boldsymbol{b}.$$

But $|\boldsymbol{c}|\,\mathrm{Proj}_{c}\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{c}$ and $|\boldsymbol{c}|\,\mathrm{Proj}_{c}\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{c}$, therefore

$$(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{c} + \boldsymbol{b}\cdot\boldsymbol{c}. \tag{II}$$

Proceeding similarly, we obtain

$$(\boldsymbol{a}-\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{c}. \tag{III}$$

Hence the distributive law with respect to multiplication holds for sums and differences. From these result the usual laws of multiplying sums by sums.

For example:

$$(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{c}+\boldsymbol{d}) = (\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c} + (\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{d} = \\ = \boldsymbol{a}\cdot\boldsymbol{c} + \boldsymbol{b}\cdot\boldsymbol{c} + \boldsymbol{a}\cdot\boldsymbol{d} + \boldsymbol{b}\cdot\boldsymbol{d}.$$

Associative law. Let m denote any number. Then $(m\boldsymbol{a})\cdot\boldsymbol{b} = |\boldsymbol{b}|\cdot\mathrm{Proj}_{b}(m\boldsymbol{a}) = m\boldsymbol{b}\,\mathrm{Proj}_{b}\boldsymbol{a}$, whence $(m\boldsymbol{a})\cdot\boldsymbol{b} = m(\boldsymbol{a}\cdot\boldsymbol{b})$.

Now let m and n denote numbers. By the preceding formula $(m\boldsymbol{a})\cdot(n\boldsymbol{b}) = m\boldsymbol{a}\cdot(n\boldsymbol{b}) = mn(\boldsymbol{a}\cdot\boldsymbol{b})$, hence

$$(m\boldsymbol{a})\cdot(n\boldsymbol{b}) = (mn)(\boldsymbol{a}\cdot\boldsymbol{b}). \tag{IV}$$

From these follow the usual laws of multiplying a polynomial by a polynomial.

For example:

$$(2\boldsymbol{a}-3\boldsymbol{b})\cdot 5\boldsymbol{c} = 10\boldsymbol{a}\cdot\boldsymbol{c} - 15\boldsymbol{b}\cdot\boldsymbol{c},$$
$$(4\boldsymbol{a}-2\boldsymbol{b})\cdot(3\boldsymbol{c}+\boldsymbol{d}) = 12\boldsymbol{a}\cdot\boldsymbol{c} - 6\boldsymbol{b}\cdot\boldsymbol{c} + 4\boldsymbol{a}\cdot\boldsymbol{d} - 2\boldsymbol{b}\cdot\boldsymbol{d}.$$

Square of a vector. The scalar product $\boldsymbol{a}\cdot\boldsymbol{a}$ is denoted by $\boldsymbol{a}^2$. Since $\boldsymbol{a}^2 = |\boldsymbol{a}|\cdot|\boldsymbol{a}|\cos 0$, then $\boldsymbol{a}^2 = |\boldsymbol{a}|^2$ and therefore $\boldsymbol{a} = \sqrt{\boldsymbol{a}^2}$.

Hence we obtain:

$$(\boldsymbol{a}+\boldsymbol{b})^2 = (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a}^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}^2,$$
$$(\boldsymbol{a}-\boldsymbol{b})^2 = (\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}^2, \tag{V}$$
$$(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}^2 - \boldsymbol{b}^2.$$

The first two formulae can be written in the following form:

$$|\boldsymbol{a}+\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}|\cos\varphi + |\boldsymbol{b}|^2,$$
$$|\boldsymbol{a}-\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\varphi + |\boldsymbol{b}|^2. \tag{VI}$$

These formulae express the so-called *theorem of Carnot* known from trigonometry.

Analytic representation of a scalar product. Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote unit vectors (p. 7). From the definition of a scalar product we obtain:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1, \quad \boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{i}\cdot\boldsymbol{k} = \boldsymbol{j}\cdot\boldsymbol{k} = 0. \tag{2}$$

Representing $\boldsymbol{a}$ and $\boldsymbol{b}$ in the form $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$, $\boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$ (p. 7), we can write $\boldsymbol{a}\cdot\boldsymbol{b}$ in the form

$$\boldsymbol{a}\cdot\boldsymbol{b} = (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k})\cdot(b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}).$$

Performing the indicated multiplication and using formulae (2) we get

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_xb_x + a_yb_y + a_zb_z. \tag{VII}$$

The above formula enables one to find the scalar product of two vectors when their components are known.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other, then $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ and therefore

$$a_xb_x + a_yb_y + a_zb_z = 0. \tag{VIII}$$

Conversely, if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other provided they are different from zero. Therefore formula (VIII) represents the *condition of perpendicularity* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (different from zero).

§ 8. Vector product. The *vector product* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the vector $\boldsymbol{c}$ which satisfies the following conditions:

(1) Length. If φ denotes the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\varphi. \tag{I}$$

(2) Direction. The vector $\boldsymbol{c}$ is perpendicular to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Hence if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ radiate from the same point, then the vector $\boldsymbol{c}$ is perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 23).

(3) Sense. The sense of the system of vectors $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ agrees with that of the chosen coordinate system, i. e. the system is left-handed.

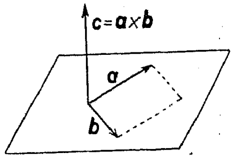

Fig. 23.

We denote the vector product by

$$\boldsymbol{a} \times \boldsymbol{b}.$$

From (I) it follows that $|\boldsymbol{c}|$ is zero, if and only if

$$\boldsymbol{a} = 0 \text{ or } \boldsymbol{b} = 0 \text{ or } \varphi = 0 \text{ or } \varphi = \pi.$$

Therefore: *the vector product is zero, if and only if one of its factors is zero, or if the factors are parallel to each other.*

Conditions (1) and (2) are obviously dropped if the vector product is zero. In particular we have

$$\boldsymbol{a} \times \boldsymbol{a} = 0. \tag{II}$$

Remark. The absolute value of the vector product is $|\boldsymbol{a}||\boldsymbol{b}| \sin \varphi$ (formula (I)). This expression represents the area of a parallelogram constructed on vectors correspondingly equal to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and radiating from one point (Fig. 23).

Change of order of factors. If the order of the factors is altered, then we get the product

$$\boldsymbol{b} \times \boldsymbol{a}.$$

The product $\boldsymbol{a} \times \boldsymbol{b}$ has (by the definition of a vector product) the same length and direction as $\boldsymbol{b} \times \boldsymbol{a}$ but an opposite sense. Hence

$$\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b}). \tag{III}$$

Therefore: *a change in the order of the factors changes the sign before the vector product.*

Associative law. On the basis of the definition of a vector product it is easy to demonstrate the following relations (where m and n denote numbers):

$$m(\boldsymbol{a} \times \boldsymbol{b}) = (m\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (m\boldsymbol{b}), \tag{IV}$$

$$(m\boldsymbol{a}) \times (n\boldsymbol{b}) = (mn)(\boldsymbol{a} \times \boldsymbol{b}). \tag{V}$$

For example:

$$3\boldsymbol{a} \times \boldsymbol{b} = 3(\boldsymbol{a} \times \boldsymbol{b}), \quad 2\boldsymbol{a} \times 3\boldsymbol{b} = 6(\boldsymbol{a} \times \boldsymbol{b}).$$

Distributive law with respect to a sum. The following formulae hold for vector products:

$$\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{c} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b}, \quad (\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}. \tag{VI}$$

We shall now derive the first formula. We can obviously suppose that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ have a common origin O.

For the time being let us assume that $|\boldsymbol{c}| = 1$. Pass a plane Π through O perpendicular to $\boldsymbol{c}$. Let

$$\boldsymbol{s} = \boldsymbol{a} + \boldsymbol{b} \tag{1}$$

and denote the projections of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{s}$ on the plane Π by $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{s}'$ (Fig. 24). We obviously have

$$\boldsymbol{s}' = \boldsymbol{a}' + \boldsymbol{b}'. \tag{2}$$

Let φ denote the angle between $\boldsymbol{c}$ and $\boldsymbol{a}$. Therefore $|\boldsymbol{a}'| = |\boldsymbol{a}| \sin \varphi = |\boldsymbol{a}||\boldsymbol{c}| \sin \varphi$, since we assumed that $|\boldsymbol{c}| = 1$. Hence

$$|\boldsymbol{a}'| = |\boldsymbol{c} \times \boldsymbol{a}| \text{ and similarly } |\boldsymbol{b}'| = |\boldsymbol{c} \times \boldsymbol{b}|, \quad |\boldsymbol{s}'| = |\boldsymbol{c} \times \boldsymbol{s}|. \tag{3}$$

Now rotate $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{s}'$ through 90° in the plane Π about O from left to right with respect to a person whose feet are at the origin and whose head is at the terminus of $\boldsymbol{c}$. We thus obtain $\boldsymbol{a}''$, $\boldsymbol{b}''$, $\boldsymbol{s}''$. By (2)

$$\boldsymbol{s}'' = \boldsymbol{a}'' + \boldsymbol{b}'', \tag{4}$$

$$|\boldsymbol{a}''| = |\boldsymbol{a}'|, \quad |\boldsymbol{b}''| = |\boldsymbol{b}'|, \quad |\boldsymbol{s}''| = |\boldsymbol{s}'|. \tag{5}$$

The vector $\boldsymbol{a}''$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{c}$; the sense of the system of vectors $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{a}'')$ is left-handed. Moreover, since $|\boldsymbol{a}''| = |\boldsymbol{c} \times \boldsymbol{a}|$ by (3) and (5), it follows $\boldsymbol{a}'' = \boldsymbol{c} \times \boldsymbol{a}$ and similarly $\boldsymbol{b}'' = \boldsymbol{c} \times \boldsymbol{b}$, $\boldsymbol{s}'' = \boldsymbol{c} \times \boldsymbol{s}$. Therefore in virtue of (4) and (1) we obtain

$$\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{c} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b}.$$

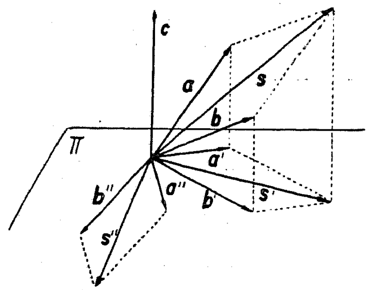

Fig. 24.

We obtained the above relation by assuming that $|\boldsymbol{c}| = 1$. We shall now prove it for the general case. Let $\boldsymbol{h}$ be a unit vector agreeing in direction with $\boldsymbol{c}$. Then

$$|\boldsymbol{h}| = 1 \text{ and } \boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{h}, \tag{6}$$

whence according to the associative law

$$\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = |\boldsymbol{c}|\boldsymbol{h} \times (\boldsymbol{a} + \boldsymbol{b}) = |\boldsymbol{c}|\{\boldsymbol{h} \times (\boldsymbol{a} + \boldsymbol{b})\}. \tag{7}$$

But from the formula proved on the assumption that $|\boldsymbol{c}| = 1$, and from the associative law we have in succession:

$$|\boldsymbol{c}|\{\boldsymbol{h} \times \boldsymbol{a} + \boldsymbol{h} \times \boldsymbol{b}\} = |\boldsymbol{c}|(\boldsymbol{h} \times \boldsymbol{a}) + |\boldsymbol{c}|(\boldsymbol{h} \times \boldsymbol{b}) =$$
$$= (|\boldsymbol{c}|\boldsymbol{h}) \times \boldsymbol{a} + (|\boldsymbol{c}|\boldsymbol{h}) \times \boldsymbol{b},$$

whence by (6) and (7) we obtain in all generality:

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}.$$

The second of the relations (VI) we can obtain from the first by applying formula (III) as follows

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = -\{\mathbf{c} \times (\mathbf{a} + \mathbf{b})\} = -\{\mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}\} = \\ = -(\mathbf{c} \times \mathbf{a}) - (\mathbf{c} \times \mathbf{b}) = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}.$$

From (VI) follows easily

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d}. \quad \text{(VII)}$$

For example:

$$(2\mathbf{a} - 3\mathbf{b}) \times (5\mathbf{c} + 2\mathbf{d}) = 10\mathbf{a} \times \mathbf{c} + 4\mathbf{a} \times \mathbf{d} - 15\mathbf{b} \times \mathbf{c} - 6\mathbf{b} \times \mathbf{d},$$
$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = \mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} = -2\mathbf{a} \times \mathbf{b},$$
$$(3\mathbf{a} + 2\mathbf{b}) \times (5\mathbf{a} - 2\mathbf{b}) = -16\mathbf{a} \times \mathbf{b}.$$

Components of a vector product. Denoting unit vectors by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ we have:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0, \quad (8)$$

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i},$$
$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}. \quad (9)$$

Setting

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}, \quad \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k},$$

we obtain

$$\mathbf{a} \times \mathbf{b} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}).$$

Performing the multiplication and using (8) and (9) we get:

$$\mathbf{a} \times \mathbf{b} = (a_yb_z - a_zb_y)\,\mathbf{i} + (a_zb_x - a_xb_z)\,\mathbf{j} + (a_xb_y - a_yb_x)\,\mathbf{k}.$$

For $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ we therefore have

$$c_x = a_yb_z - a_zb_y, \quad c_y = a_zb_x - a_xb_z, \quad c_z = a_xb_y - a_yb_x. \quad \text{(VIII)}$$

§ 9. Product of several vectors. 1° Let us first consider the product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Setting $\mathbf{r} = \mathbf{b} \times \mathbf{c}$, we obtain

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{r} = a_xr_x + a_yr_y + a_zr_z.$$

Since $r_x = b_yc_z - b_zc_y$ etc.,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_x(b_yc_z - b_zc_y) + a_y(b_zc_x - b_xc_z) + a_z(b_xc_y - c_xb_y).$$

The above formula can be written in the form

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \quad \text{(I)}$$

From well-known properties of determinants it follows easily

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \quad \text{(II)}$$

Suppose that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ have their initial points at the origin of the coordinate system. From analytic geometry it is known that the volume V of a parallelepiped having edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$, is 1/6 of the determinant (I). Hence $V = \frac{1}{6}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Therefore: *the necessary and sufficient condition that the vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (*having a common origin*) *be in the same plane is that* $V = 0$, *or that* $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

However, if we do not assume that $\mathbf{a}, \mathbf{b}, \mathbf{c}$, have a common origin, then — as is quite evident — the condition $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ *is the necessary and sufficient condition that the vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be parallel to the same plane.*

2° Let us now consider the product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Let us denote this product by $\mathbf{u}$ and set $\mathbf{r} = \mathbf{b} \times \mathbf{c}$. Then

$$u_x = a_yr_z - a_zr_y = a_y(b_xc_y - b_yc_x) - a_z(b_zc_x - b_xc_z).$$

Adding and subtracting $a_xb_xc_x$ we obtain

$$u_x = b_x(a_xc_x + a_yc_y + a_zc_z) - c_x(a_xb_x + a_yb_y + a_zb_z);$$

hence $u_x = b_x(\mathbf{a} \cdot \mathbf{c}) - c_x(\mathbf{a} \cdot \mathbf{b})$ and similarly $u_y = b_y(\mathbf{a} \cdot \mathbf{c}) - c_y(\mathbf{a} \cdot \mathbf{b})$, $u_z = b_z(\mathbf{a} \cdot \mathbf{c}) - c_z(\mathbf{a} \cdot \mathbf{b})$. Therefore

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} \cdot (\mathbf{a} \cdot \mathbf{b}). \quad \text{(III)}$$

3° From (I), (II) and (III) follow the relations:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\ (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\ (\mathbf{b} \cdot \mathbf{c}), \quad \text{(IV)}$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{b}[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})] - \mathbf{a}[\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})]. \quad \text{(V)}$$

§ 10. Vector functions. If to each number t in the interval (t', t'') there corresponds a vector $\mathbf{w}$, then we say that a *vector function* is defined in the interval (t', t'') and we write

$$\mathbf{w} = \mathbf{F}(t). \quad (1)$$

The components w_x, w_y, w_z are also functions (i. e. scalar functions) of the variable t. Therefore:

$$w_x = f(t), \quad w_y = \varphi(t), \quad w_z = \psi(t). \quad (2)$$

The three preceding functions define the vector function (1) precisely.

Limit. The vector function (1) is said to have *the limit* $\mathbf{w}_0$ *as* t *tends to* t_0, and we write

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{w}_0,$$

when

$$\lim_{t\to t_0} f(t) = w_{0,x}, \quad \lim_{t\to t_0} \varphi(t) = w_{0,y}, \quad \text{and} \quad \lim_{t\to t_0} \psi(t) = w_{0,z}.$$

Continuity. The vector function (1) is *continuous at* t_0, if $\lim\limits_{t\to t_0} \mathbf{F}(t) = \mathbf{w}_0$, where $\mathbf{w}_0 = \mathbf{F}(t_0)$.

The following relations obviously hold:

$$\lim_{t\to t_0} f(t) = f(t_0), \quad \lim_{t\to t_0} \varphi(t) = \varphi(t_0), \quad \lim_{t\to t_0} \psi(t) = \psi(t_0).$$

The functions f, φ, ψ are therefore continuous at $t = t_0$. Conversely, if f, φ, ψ are continuous at t_0, then $\mathbf{w} = \mathbf{F}(t)$ is also continuous at t_0.

Derivative. Let Δt denote the increment of the variable t, and $\Delta\mathbf{w}$ the corresponding increment of the vector $\mathbf{w}$. Then $\mathbf{w} + \Delta\mathbf{w} = \mathbf{F}(t + \Delta t)$, and $\Delta\mathbf{w} = \mathbf{F}(t + \Delta t) - \mathbf{F}(t)$, whence

$$\frac{\Delta\mathbf{w}}{\Delta t} = \frac{\mathbf{F}(t+\Delta t) - \mathbf{F}(t)}{\Delta t}.$$

The limit $\lim\limits_{\Delta t\to 0} \dfrac{\Delta\mathbf{w}}{\Delta t}$ is called the *derivative* of the function $\mathbf{F}(t)$ at the point t.

We denote the derivative by $\dfrac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}$, $\mathbf{w}'$ or $\mathbf{F}'(t)$.

Since $\Delta\mathbf{w}$ has components

$$\Delta w_x = f(t+\Delta t) - f(t), \quad \Delta w_y = \varphi(t+\Delta t) - \varphi(t),$$
$$\Delta w_z = \psi(t+\Delta t) - \psi(t),$$

it follows that

$$w'_x = f'(t), \quad w'_y = \varphi'(t), \quad w'_z = \psi'(t).$$

Higher ordered derivatives are defined in the usual manner: the second derivative as the derivative of the first derivative, the third derivative as the derivative of the second derivative etc. We denote higher ordered derivatives by

$$\frac{\mathrm{d}^2\mathbf{w}}{\mathrm{d}t^2}, \ \frac{\mathrm{d}^3\mathbf{w}}{\mathrm{d}t^3}, \ \ldots \text{ or } \mathbf{w}'', \mathbf{w}''', \ldots \text{ etc.}$$

It is easy to show that

$$w''_x = f''(t), \quad w''_y = \varphi''(t), \quad w''_z = \psi''(t) \text{ etc.}$$

If the functions $\mathbf{w} = \mathbf{F}(t)$ and $\mathbf{v} = \mathbf{\Phi}(t)$ possess derivatives, then the following relations obtain:

$$\frac{\mathrm{d}(\mathbf{w}\pm\mathbf{v})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} \pm \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{I}$$

$$\frac{\mathrm{d}(m\mathbf{w})}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} \quad \text{(where } m \text{ is a number)}, \tag{II}$$

$$\frac{\mathrm{d}(\mathbf{w}\cdot\mathbf{v})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}\cdot\mathbf{v} + \mathbf{w}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{III}$$

$$\frac{\mathrm{d}(\mathbf{w}\times\mathbf{v})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}\times\mathbf{v} + \mathbf{w}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{IV}$$

We shall demonstrate for instance relation (III). We have $\Delta(\mathbf{w}\cdot\mathbf{v}) = (\mathbf{w}+\Delta\mathbf{w})\cdot(\mathbf{v}+\Delta\mathbf{v}) - \mathbf{w}\cdot\mathbf{v}$; hence

$$\frac{\Delta(\mathbf{w}\cdot\mathbf{v})}{\Delta t} = \frac{\Delta\mathbf{w}}{\Delta t}\cdot\mathbf{v} + \mathbf{w}\cdot\frac{\Delta\mathbf{v}}{\Delta t} + \Delta\mathbf{w}\cdot\frac{\Delta\mathbf{v}}{\Delta t},$$

whence, upon passing to the limit we obtain (III).

Vector functions of many variables. We can also consider vector functions of many variables. For instance, the vector function

$$\mathbf{w} = \mathbf{F}(\xi, \eta, \zeta)$$

is a function of the three variables ξ, η, ζ. The projections of $\mathbf{w}$ are then defined by certain functions

$$w_x = f(\xi,\eta,\zeta), \quad w_y = \varphi(\xi,\eta,\zeta), \quad w_z = \psi(\xi,\eta,\zeta).$$

The limit, continuity and partial derivatives of vector functions of several variables can easily be given by analogy with the case for one variable.

§ 11. Moment of a vector. Moment of a vector with respect to a point. Let us suppose that a vector $\overline{AB}$ and a point O are given. *The moment of the vector* $\overline{AB}$ *with respect to the point* O is defined as the vector $\mathbf{M}$ satisfying the following conditions:

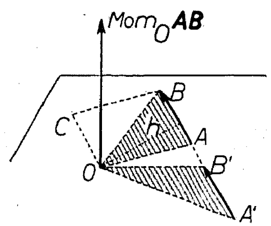

Fig. 25.

(1) $|\mathbf{M}|$ is equal to twice the area of the triangle OAB or

$$|\mathbf{M}| = |\overline{AB}|\cdot h,$$

where h denotes the distance of the point O from $\overline{AB}$.

(2) The direction of the vector $\boldsymbol{M}$ is perpendicular to the plane passing through O and $\overline{AB}$.

(3) The system of vectors $(\overline{AB}, \overline{OA}, \boldsymbol{M})$ has a sense agreeing with that of the coordinate system, i. e. a left sense.

We shall denote the moment of a vector $\overline{AB}$ with respect to the point O by the symbol

$$\mathrm{Mom}_o\overline{AB}.$$

The moment is zero only in the case when $\overline{AB} = 0$ or when the prolongation of the vector $\overline{AB}$ passes through O. If the moment is zero then conditions (2) and (3) are dropped.

For equipollent vectors we can establish the following

Theorem 1. *Equipollent vectors have equal moments with respect to the same point.*

Proof. By hypothesis $\overline{AB} \equiv \overline{A'B'}$. Therefore $\overline{AB}$ and $\overline{A'B'}$ are equal and lie on the same line. It is easy to verify that the moments of both vectors with respect to O have the same direction and sense. They also have the same length because triangles OAB and $OA'B'$ have equal areas (equal bases and a common altitude). Hence $\mathrm{Mom}_o\overline{AB} = \mathrm{Mom}_o\overline{A'B'}$, q. e. d.

Moment as a vector product. Let us consider the vector product $\overline{AB} \times \overline{OA}$. Let us note that the preceding product has the same direction and sense as the $\mathrm{Mom}_o\overline{AB}$. We also have $|\overline{AB} \times \overline{OA}| = |\mathrm{Mom}_o\overline{AB}|$ because the absolute value of the vector is equal to the area of the parallelogram $OABC$ (Fig. 25) and hence to twice the area of the triangle ABC. Therefore

$$\mathrm{Mom}_o\overline{AB} = \overline{AB} \times \overline{OA}.$$

Had we taken the equipollent vector $\overline{A'B'}$ instead of the vector $\overline{AB}$, then we would have

$$\mathrm{Mom}_o\overline{A'B'} = \overline{A'B'} \times \overline{OA'}.$$

By the preceding theorem

$$\mathrm{Mom}_o\overline{AB} = \overline{A'B'} \times \overline{OA'} = \overline{AB} \times \overline{OA}.$$

Therefore: *if A' is an arbitrary point of the line on which the vector $\overline{AB}$ lies, then*

$$\mathrm{Mom}_o\overline{AB} = \overline{AB} \times \overline{OA'}.$$

Theorem 2. *If two equal vectors have equal moments with respect to a point, then they are equipollent.*

Proof. By hypothesis $\overline{AB} = \overline{A'B'}$ and $\mathrm{Mom}_o\overline{AB} = \mathrm{Mom}_o\overline{A'B'}$. Therefore $\overline{AB} \times \overline{OA} = \overline{A'B'} \times \overline{OA'}$, hence $\overline{AB} \times \overline{OA} = \overline{AB} \times \overline{OA'}$ and therefore $\overline{AB} \times (\overline{OA} - \overline{OA'}) = 0$. Moreover, since $\overline{OA} - \overline{OA'} = \overline{A'A}$, it follows

$$\overline{AB} \times \overline{A'A} = 0.$$

But $\overline{AB} \times \overline{A'A} = \mathrm{Mom}_{A'}\overline{AB}$; hence

$$\mathrm{Mom}_{A'}\overline{AB} = 0.$$

It follows from this that the point A' lies on the prolongation of the vector $\overline{AB}$. Since, in addition, $\overline{AB}$ is parallel to $\overline{A'B'}$, then vectors $\overline{AB}$ and $\overline{A'B'}$ lie on the same line.

Moment of a sum of vectors having a common origin. Let us assume that $\overline{AB}$ and $\overline{AC}$ (both having initial points at A) are given. Let $\overline{AD}$ be their sum. We have

$$\mathrm{Mom}_o\overline{AD} = \overline{AD} \times \overline{OA} = (\overline{AB} + \overline{AC}) \times \overline{OA};$$

consequently

$$\mathrm{Mom}_o\overline{AD} = \overline{AB} \times \overline{OA} + \overline{AC} \times \overline{OA},$$

and therefore

$$\mathrm{Mom}_o\overline{AD} = \mathrm{Mom}_o\overline{AB} + \mathrm{Mom}_o\overline{AC}.$$

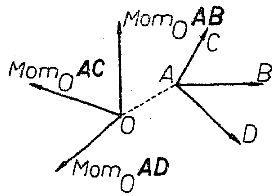

Fig. 26.

We obtain a similar formula for the sum of several vectors. Hence: *the sum of the moments of several vectors having a common origin is equal to the moment of their sum having the same origin.*

Components of a moment. The position of a vector $\boldsymbol{a}$ is defined if ist projections and the coordinates x, y, z of an arbitrary point A of the line l on which the vector $\boldsymbol{a}$ lies are given.

Let x_0, y_0, z_0 be the coordinates of the point O. We have

$$\mathrm{Mom}_o\boldsymbol{a} = \boldsymbol{a} \times \overline{OA}.$$

The projections of the vector $\overline{OA}$ are $x - x_0$, $y - y_0$, $z - z_0$. Hence, denoting the moment with respect to O by $\boldsymbol{M}$, we obtain:

$$M_x = a_y(z - z_0) - a_z(y - y_0), \quad M_y = a_z(x - x_0) - a_x(z - z_0), \quad M_z = a_x(y - y_0) - a_y(x - x_0). \tag{I}$$

If, in particular, the origin of the coordinate system is O, then $x_0 = 0$, $y_0 = 0$, $z_0 = 0$, and therefore

$$M_x = a_y z - a_z y, \quad M_y = a_z x - a_x z, \quad M_z = a_x y - a_y x. \qquad \text{(II)}$$

Suppose that $\boldsymbol{a} \equiv \boldsymbol{a}'$. Then, denoting the moments of the vectors with respect to an arbitrary point by $\boldsymbol{M}$ and $\boldsymbol{M}'$, we have $\boldsymbol{a} = \boldsymbol{a}'$, $\boldsymbol{M} = \boldsymbol{M}'$ or

$$a_x = a'_x, \quad a_y = a'_y, \quad a_z = a'_z, \quad M_x = M'_x, \quad M_y = M'_y, \quad M_z = M'_z.$$

Conversely, if the above relations hold, then $\boldsymbol{a} = \boldsymbol{a}'$, $\boldsymbol{M} = \boldsymbol{M}'$ and therefore by theorem 2, p. 16, the vectors $\boldsymbol{a}$ and $\boldsymbol{a}'$ are equipollent.

Therefore: *the projections of the vector* $\boldsymbol{a}$ *and the projections of the moment* $\boldsymbol{M}$ *with respect to an arbitrary point determine the length, direction, sense, and position of* $\boldsymbol{a}$.

Moment of a vector with respect to a line. Let the vector $\boldsymbol{a}$ and the line l be given. Through an arbitrary point O on the line l pass a plane Π perpendicular to l. Form the projection $\boldsymbol{a}'$ of the vector $\boldsymbol{a}$ on the plane Π.

The moment of the vector $\boldsymbol{a}'$ with respect to O is called as *the moment of the vector* $\boldsymbol{a}$ *with respect to* l and it is denoted by the symbol

$$\mathrm{Mom}_l \boldsymbol{a}.$$

Obviously $\mathrm{Mom}_l \boldsymbol{a}$ does not depend on the choice of point O.

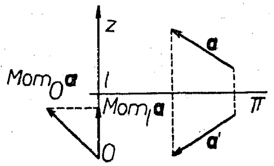

Fig. 27.

$\mathrm{Mom}_l \boldsymbol{a}$ is zero only in the following cases:

1° when $\boldsymbol{a} = 0$,

2° when $\boldsymbol{a} \parallel l$, because then $\boldsymbol{a}' = 0$,

3° when $\boldsymbol{a}$ produced cuts l, because then $\boldsymbol{a}'$ produced passes through O.

If d denotes the distance of $\boldsymbol{a}$ from l and φ the angle between $\boldsymbol{a}$ and l, then it is easy to show that

$$|\mathrm{Mom}_l \boldsymbol{a}| = d|\boldsymbol{a}| \sin\varphi. \qquad \text{(III)}$$

Let us choose the line l as the z-axis and the plane Π as the xy-plane. Let $\boldsymbol{M} = \mathrm{Mom}_O \boldsymbol{a}$ and $\boldsymbol{L} = \mathrm{Mom}_l \boldsymbol{a}$. Since $\boldsymbol{a}'$ has the projections $a'_x = a_x$, $a'_y = a_y$, $a'_z = 0$, then $L_x = 0$, $L_y = 0$ and $L_z = a_x y - a_y x$, where x, y, z are the coordinates of the initial point of $\boldsymbol{a}$. We then see that $M_z = L_z$.

Therefore: $\mathrm{Mom}_l \boldsymbol{a}$ *is the projection on the line* l *of the moment of the vector* $\boldsymbol{a}$ *with respect to an arbitrary point of this line.*

II. SYSTEMS OF VECTORS

§ 12. Total moment of a system of vectors. Let

$$\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$$

be a given system of vectors. Let us denote *the sum of the system* (i. e. the sum of the vectors of the system) by $\boldsymbol{s}$. Thus

$$\boldsymbol{s} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \ldots + \boldsymbol{a}_n.$$

Choose an arbitrary point O.

The *total moment* or briefly *the moment of the system with respect to* O is defined as the sum of the moments of the separate vectors with respect to O. We shall denote it by

$$\boldsymbol{M}_o.$$

We therefore have

$$\boldsymbol{M}_o = \mathrm{Mom}_o \boldsymbol{a}_1 + \mathrm{Mom}_o \boldsymbol{a}_2 + \ldots + \mathrm{Mom}_o \boldsymbol{a}_n.$$

The total moment we sometimes also denote by

$$\mathrm{Mom}_o(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n).$$

Let us select another point O'.

We have

$$\boldsymbol{M}_{o'} = \mathrm{Mom}_{o'} \boldsymbol{a}_1 + \mathrm{Mom}_{o'} \boldsymbol{a}_2 + \ldots + \mathrm{Mom}_{o'} \boldsymbol{a}_n.$$

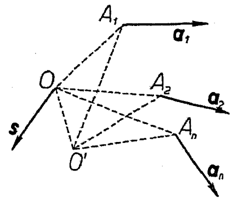

Fig. 28

Since $\mathrm{Mom}_0 \boldsymbol{a}_1 = \boldsymbol{a}_1 \times \overline{O'A_1}$, where A_1 is the initial point of $\boldsymbol{a}_1$ etc., it follows that

$$\boldsymbol{M}_{o'} = \boldsymbol{a}_1 \times \overline{O'A} + \boldsymbol{a}_2 \times \overline{O'A_2} + \ldots$$

But $\overline{O'A_1} = \overline{O'O} + \overline{OA_1}$ etc. hence

$$\boldsymbol{M}_{o'} = \boldsymbol{a}_1 \times (\overline{O'O} + \overline{OA_1}) + \boldsymbol{a}_2 \times (\overline{O'O} + \overline{OA_2}) + \ldots$$

After performing the multiplication we obtain:

$$\boldsymbol{M}_{o'} = (\boldsymbol{a}_1 \times \overline{O'O} + \boldsymbol{a}_2 \times \overline{O'O} + \ldots) + (\boldsymbol{a}_1 \times \overline{OA_1} + \boldsymbol{a}_2 \times \overline{OA_2} + \ldots). \qquad (1)$$

But $\boldsymbol{a}_1 \times \overline{O'O} + \boldsymbol{a}_2 \times \overline{O'O} + \ldots = (\boldsymbol{a}_1 + \boldsymbol{a}_2 + \ldots) \times \overline{O'O} = \boldsymbol{s} \times \overline{O'O}$.

The sum enclosed in the second parenthesis of (1) represents the moment of the system with respect to O. Therefore

$$\boldsymbol{M}_{o'} = \boldsymbol{s} \times \overline{O'O} + \boldsymbol{M}_o. \qquad \text{(I)}$$

The product $\boldsymbol{s} \times \overline{O'O}$ is the moment with respect to O' of the sum of the system of vectors with initial point at O.

Hence: *if we change the point with respect to which we find the total moment of the system, then this moment changes by the moment of the sum of the system whose initial point is at the old point taken with respect to the new point.*

The following corollaries are consequences of the preceding theorem:

1. *If the sum of the system is zero, then the total moment is constant* (i. e. it does not depend on the point with respect to which it is determined).

Because if $\mathbf{s} = 0$, then $\mathbf{s} \times \overline{O'O} = 0$, and hence $\mathbf{M}_{o'} = \mathbf{M}_o$.

2. *If the total moments with respect to three non-collinear points are equal, then the sum of the system of vectors is zero.*

For let us assume that the total moments with respect to the non-collinear points O, O', O'' are equal. Then $\mathbf{M}_o = \mathbf{M}_{o'} = \mathbf{M}_{o''}$, whence $\mathbf{s} \times \overline{O'O} = 0$ and $\mathbf{s} \times \overline{O''O} = 0$. Hence, if $\mathbf{s} \neq 0$, then $s \parallel OO'$ and $\mathbf{s} \parallel O''O$, which is impossible when O, O', O'', are non-collinear.

3. *If the point with respect to which the total moment is determined is moved along a line parallel to the sum of the system, then the moment does not undergo a change.*

For if $\mathbf{s} \parallel \overline{O'O}$, then $\mathbf{s} \times \overline{O'O} = 0$ and hence $\mathbf{M}_{o'} = \mathbf{M}_o$.

4. *The scalar product of the total moment by the sum of the system is constant* (i. e. it is independent of the point with respect to which it is determined).

For let us multiply both sides of (I) scalarly by $\mathbf{s}$. We obtain $\mathbf{s} \cdot \mathbf{M}_{o'} = \mathbf{s} \cdot (\mathbf{s} \times \overline{O'O}) + \mathbf{s} \cdot \mathbf{M}_o$, but $\mathbf{s} \times \overline{O'O} \perp \mathbf{s}$; therefore $\mathbf{s} \cdot (\mathbf{s} \times \overline{O'O}) = 0$, whence

$$\mathbf{s} \cdot \mathbf{M}_{o'} = \mathbf{s} \cdot \mathbf{M}_o.$$

The scalar product of the total moment by the sum is called the *parameter* of the system.

5. *The projection of the moment on the direction of the sum is constant in magnitude* (under the assumption that the sum is different from zero).

For by corollary 4 and the definition of a scalar product we have $|\mathbf{s}| \operatorname{Proj}_{\mathbf{s}} \mathbf{M}_{o'} = |\mathbf{s}| \operatorname{Proj}_{\mathbf{s}} \mathbf{M}_o$, whence

$$\operatorname{Proj}_{\mathbf{s}} \mathbf{M}_{o'} = \operatorname{Proj}_{\mathbf{s}} \mathbf{M}_o.$$

§ 13. Parameter. We shall presently determine the parameter (i. e. the scalar product of the total moment by the sum) for certain systems appearing frequently in mechanics.

A *central* system is one in which the prolongations of the separate vectors all pass through a fixed point O called the *centre* (Fig. 29).

The moment of the system with respect to the centre is zero, because the moment of each vector is zero. Therefore the parameter is zero.

Hence: *the parameter of a central system is zero.*

A *plane* system is one in which every vector lies in the same plane Π (Fig. 30).

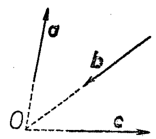

Fig. 29.

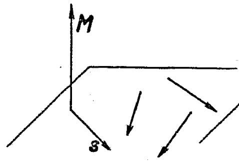

Fig. 30.

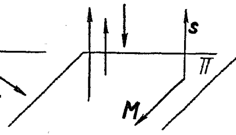

Fig. 31.

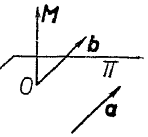

Fig. 32.

The total moment of the system with respect to an arbitrary point O in the plane Π is perpendicular to Π, because the moments of the individual vectors with respect to O are perpendicular to Π. Since the sum lies in the plane Π, then the sum is perpendicular to the total moment. It follows that the parameter is zero.

Therefore: *the parameter of a plane system is zero.*

A *parallel* system is one in which all the vectors are parallel (Fig. 31).

If the sum $\mathbf{s}$ is zero, then the parameter is obviously zero. Let us assume then, that $\mathbf{s} \neq 0$. Let O be an arbitrary point. The moments of the separate vectors with respect to O lie in the plane perpendicular to the vectors of the system and passing through O. Therefore the total moment also lies in the plane Π. Since $\mathbf{s} \perp \Pi$, then $\mathbf{s}$ is perpendicular to the total moment and consequently the parameter is zero.

Hence: *the parameter of a plane system is zero.*

Let us now assume that vectors $\mathbf{a}$ and $\mathbf{b}$ are *skew* (i. e. do not lie in the same plane). Let O be the initial point of the vector $\mathbf{b}$ (Fig. 32).

The moment $\mathbf{M}$ of the system $(\mathbf{a}, \mathbf{b})$ with respect to O is obviously equal to $\operatorname{Mom}_o \mathbf{a}$. The parameter $K = \mathbf{M} \cdot \mathbf{s} = \mathbf{M} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{M} \cdot \mathbf{a} + \mathbf{M} \cdot \mathbf{b}$. But $\mathbf{M} = \operatorname{Mom}_o \mathbf{a}$ is perpendicular to the plane Π which passes through O and the vector $\mathbf{a}$. Since $\mathbf{a}$ lies in Π and $\mathbf{b}$ does not, the moment $\mathbf{M}$ is perpendicular to $\mathbf{a}$, but not to $\mathbf{b}$, and consequently from the last equality $K = \mathbf{M} \cdot \mathbf{b} \neq 0$.

Therefore: *the parameter of a system consisting of two skew vectors is different from zero.*

§ 14. Equipollent systems. Two systems of vectors $(\mathbf{a}_1, \mathbf{a}_2, \ldots)$ and $(\mathbf{a}'_1, \mathbf{a}'_2, \ldots)$ are said to be *equipollent* if they have equal sums and equal total moments with respect to every point.

If we have a system $(\mathbf{a})$ consisting of only one vector $\mathbf{a}$ and a system $(\mathbf{a}')$ consisting of only one vector $\mathbf{a}'$, then — as follows from theorem 2, p. 16 — the necessary and sufficient condition that systems $(\mathbf{a})$ and $(\mathbf{a}')$ be equipollent is that $\mathbf{a} \equiv \mathbf{a}'$. Therefore, in this case, the notion of equipollence of systems coincides with the notion of equipollence of vectors.

In the general case we have the following theorems:

1. *If two systems have equal sums and equal total moments with respect to a certain point, then these systems are equipollent.*

This follows from formula (I), p. 19. For if the moments with respect to the point O are equal and the sums are equal, then the moments with respect to every point O' will be equal, since in replacing point O by O' they undergo equal changes in both systems.

2. *If two systems have equal moments with respect to three non-collinear points, then these systems are equipollent.*

Because if we denote the points with respect to which the total moments of both systems are equal by O_1, O_2, O_3 and the sums of these systems by $\mathbf{s}$ and $\mathbf{s}'$, then from fromula (I), p. 19, we shall obtain $\mathbf{s} \times \overline{O_1O_2} = \mathbf{s}' \times \overline{O_1O_2}$ and $\mathbf{s} \times \overline{O_1O_3} = \mathbf{s}' \times \overline{O_1O_3}$, whence

$$(\mathbf{s} - \mathbf{s}') \times \overline{O_1O_2} = 0 \quad \text{and} \quad (\mathbf{s} - \mathbf{s}') \times \overline{O_1O_3} = 0.$$

Were $\mathbf{s} - \mathbf{s}' \neq 0$, then we should have $\mathbf{s} - \mathbf{s}' \parallel \overline{O_1O_2}$ and $\mathbf{s} - \mathbf{s}' \parallel \overline{O_1O_3}$, which is impossible because O_1, O_2, O_3 are non-collinear. Hence $\mathbf{s} - \mathbf{s}' = 0$, or $\mathbf{s} = \mathbf{s}'$, whence, by the preceding theorem, the equipollence of the systems follows.

That *equipollent systems have equal parameters* is an immediate consequence of the definition of a parameter.

The converse of this statement is obviously false.

Systems equipollent to zero. If the sum of a system is zero, then — as we know — the total moment is constant. If the sum of the system is zero and the total moment is zero, then the system is said to be a *system equipollent to zero.*

A system equipollent to zero is equipollent to a zero vector.

In order to ascertain whether a system is equipollent to zero it is sufficient to see whether its sum and moment with respect to some arbitrary point are equal to zero.

It follows easily from theorem 2, p. 20, that *a system is equipollent to zero if the total moment with respect to three non-collinear points is zero.*

System of three vectors equipollent to zero. *If a system consisting of three vectors is equipollent to zero, then the prolongations of these vectors pass through one point (or the vectors are parallel).*

Let us suppose that the system of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is equipollent to zero. The total moment with respect to A (the initial point of $\mathbf{a}$) is therefore zero, whence $\mathrm{Mom}_A\mathbf{b} + \mathrm{Mom}_A\mathbf{c} = 0$, and hence $\mathrm{Mom}_A\mathbf{b} = -\mathrm{Mom}_A\mathbf{c}$. From this it follows that the vectors $\mathbf{b}$ and $\mathbf{c}$ lie in the plane Π passing through A. Since $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then $\mathbf{a} = -\mathbf{b} - \mathbf{c}$, and therefore $\mathbf{a}$ also lies in the plane Π. Let O denote the point of intersection of $\mathbf{a}$ and $\mathbf{b}$. Since the total moment of the system with respect to O is reduced to the moment of the vector $\mathbf{c}$ with respect to O, $\mathrm{Mom}_0\mathbf{c} = 0$ and hence $\mathbf{c}$ also passes through O. Finally, if $\mathbf{a} \parallel \mathbf{b}$, then $\mathbf{a} \parallel \mathbf{c}$ also, because $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ (Fig. 29).

§ 15. Vector couple. A *vector couple* is a system consisting of two parallel vectors $\mathbf{a}$ and $-\mathbf{a}$ oppositely directed and of equal length.

Since the sum of a vector couple is zero, the moment of the couple is constant. Computing it with respect to the initial point of $\mathbf{a}$, we see that the moment of the vector $\mathbf{a}$ is zero, but the moment of the vector $-\mathbf{a}$ is perpendicular to the plane of the couple and equal in magnitude to the area of the parallelogram constructed on the vectors forming the couple.

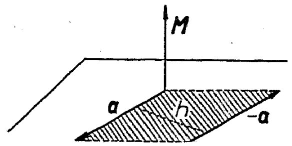

Fig. 33.

Therefore: *the moment of a couple is perpendicular to the plane of the couple and in magnitude equal to the area of the parallelogram constructed on the vectors of the couple.*

If the vectors of the couple lie on the same straight line, then obviously the moment is zero.

If h denotes the distance between vectors $\mathbf{a}$ and $-\mathbf{a}$ and $\mathbf{M}$ the moment of the couple, then

$$|\mathbf{M}| = |\mathbf{a}| \cdot h. \tag{1}$$

Corresponding to a given vector $\mathbf{M}$ there can always be found a couple whose moment is equal to $\mathbf{M}$. On the plane perpendicular to $\mathbf{M}$ it is sufficient to select a parallelogram whose area is equal to $|\mathbf{M}|$. The opposite sides, suitably directed, form the sought for couple. Clearly, the problem can be solved in an infinite number of ways.

Two couples whose moments are equal form an equipollent system. Hence, if a couple is arbitrarily translated or rotated in the plane of the couple, then an equipollent couple is obtained.

§ 16. Reduction of a system of vectors. Let a system S consisting of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be given. We shall consider the problem of determining the simplest system equipollent to S.

Let O be an arbitrary point. Denote the sum of the system S by $\boldsymbol{s}$ and the total moment with respect to O by $\boldsymbol{M}$. Let us consider the system R consisting of the couple $(\boldsymbol{a}, -\boldsymbol{a})$ whose moment equals $\boldsymbol{M}$ and the vector $\boldsymbol{s}$ with initial point at O. Systems R and S are obviously equipollent because they have equal sums $\boldsymbol{s}$ and equal moments $\boldsymbol{M}$ with respect to O.

Therefore: *every system of vectors is equipollent to a system consisting of a sum with initial point at an arbitrary point O and a couple whose moment is equal to the moment of the system with respect to O.*

The latter is the so-called *reduction theorem*. The point O is called the *centre of reduction*.

The couple $(\boldsymbol{a}, -\boldsymbol{a})$ can be chosen so that the point O is the initial point of $-\boldsymbol{a}$. Let us replace the vectors $\boldsymbol{s}$ and $-\boldsymbol{a}$ by their sum $\boldsymbol{b}$ whose initial point is at O (Fig. 34). The system consisting of $\boldsymbol{a}$ and $\boldsymbol{b}$ is obviously equipollent to system S.

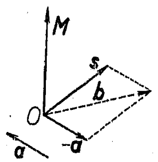

Fig. 34.

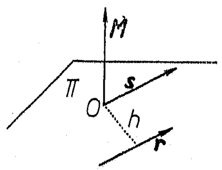

Fig. 35.

Hence: *every system of vectors is equipollent to a system of two vectors one of which has its origin at an arbitrary point.*

Every system of vectors is therefore equipollent to a certain system composed of a vector and a couple or two vectors. Let us now consider conditions under which a given system is equipollent to only one vector or one couple.

Let us examine in succession cases in which the parameter is different from zero and equal to zero.

1° Parameter different from zero. A system consisting of one vector or one couple is a plane system and hence its parameter $K = 0$. Therefore, if the parameter of the system S is different from zero, then the system S cannot be equipollent to one vector or one couple because equipollent systems have equal parameters.

Let us assume now that the system S whose parameter $K \neq 0$ is equipollent to the system R consisting of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The parameter R of the system is therefore also different from zero. It follows from this that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ cannot lie in one plane and are therefore skew (*vide* § 13, p. 21).

Hence: *if the parameter of a system is different from zero, then the system is equipollent to a system of two skew vectors.*

2° Parameter equal to zero, sum different from zero. Let us suppose that the parameter K of the system S is zero but the sum $\boldsymbol{s} \neq 0$. Select an arbitrary point O and denote the moment of the system S with respect to O by $\boldsymbol{M}$. Since $K = \boldsymbol{M} \cdot \boldsymbol{s} = 0$, then $\boldsymbol{M} \perp \boldsymbol{s}$. Pass a plane Π through O and perpendicular to $\boldsymbol{M}$ (Fig. 35). On Π we can choose a vector $\boldsymbol{r}$ equal to the vector $\boldsymbol{s}$ and such that $\mathrm{Mom}_o \boldsymbol{r} = \boldsymbol{M}$. The distance h from $\boldsymbol{r}$ to O is obtained from $|\boldsymbol{M}| = h|\boldsymbol{r}|$. It is easy to see that system S is equipollent to the vector $\boldsymbol{r}$.

Therefore: *if the parameter of a system is equal to zero but the sum is different from zero, then the system is equipollent to one vector.*

The vector $\boldsymbol{r}$, to which the entire system S is equipollent, is called the *resultant vector* or briefly the *resultant* of the system S.

The sum is not to be confused with the resultant. The sum has only a definite length, direction and sense; the resultant has in addition a definite position, i. e. the line on which it lies.

3° Parameter and sum equal to zero. Finally, let us assume that the parameter as well as the sum of the system are equal to zero. From the reduction theorem it follows that the system is equipollent to a couple. Since the sum is zero, the total moment $\boldsymbol{M}$ is constant.

If $\boldsymbol{M} \neq 0$, then the couple is the simplest system equipollent to the given one. If $\boldsymbol{M} = 0$, and as by the hypothesis the sum is equal to zero then the system is equipollent to zero, i. e. to a zero vector.

Therefore: *a system whose parameter and sum are equal to zero is equipollent to a couple of vectors or to a zero vector, depending on whether the total moment is different from zero or equal to zero.*

The above results are compiled in the following table:

Parameter	Sum	Moment	Simplest equipollent system
$K \neq 0$	—	—	vector and couple or two skew vectors
$K = 0$	$\boldsymbol{s} \neq 0$	—	resultant vector
	$\boldsymbol{s} = 0$	$\boldsymbol{M} \neq 0$	couple
	$\boldsymbol{s} = 0$	$\boldsymbol{M} = 0$	zero vector

The following theorems are easy consequences of the preceding:

1. *If the moment of a system with respect to a certain point O is zero, then the system has a resultant with its initial point at O.*

2. *A central system has a resultant whose initial point is at the centre.*

These theorems follow from the reduction theorem (p. 24) if we take point O (or the centre of the system) respectively, as the centre of reduction.

3. *A plane system either has a resultant or is equipollent to a couple.*

4. *A parallel system either has a resultant or is equipollent to a couple.*

Theorems 3 and 4 are obtained at once from the table because in both cases K is zero.

§ 17. Central axis. Wrench. Let S be a given system having a sum different from zero. Let us determine the geometric locus of points with respect to which the total moment is parallel to $\mathbf{s}$ (or $=0$).

For this purpose choose an arbitrary point O. Let $\mathbf{M}_o = \overline{OA}$ be the total moment of the system with respect to point O and $\overline{OB}$ the projection of $\mathbf{M}_o$ on $\mathbf{s}$.

Let us now determine the point O' with respect to which the moment of the sum $\mathbf{s}$ with initial point at O is equal to $\overline{AB}$. Such a point is found at a distance d from O on a line perpendicular at O to $\overline{AB}$ and $\mathbf{s}$, where d satisfies the condition:

$$d \cdot |\mathbf{s}| = |\overline{AB}|.$$

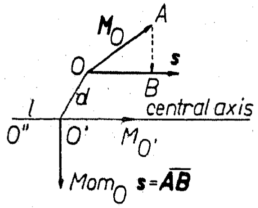

Fig. 36.

Therefore $\mathrm{Mom}_{o'}\mathbf{s} = \overline{AB}$ or

$$\mathbf{s} \times \overline{O'O} = \overline{AB},$$

and hence by (I), p. 19

$$\mathbf{M}_{o'} = \mathbf{s} \times \overline{O'O} + \mathbf{M}_o,$$

whence

$$\mathbf{M}_{o'} = \overline{AB} + \overline{OA} = \overline{OB}.$$

Therefore $\mathbf{M}_{o'}$ is parallel to $\mathbf{s}$ (or $=0$, when $\mathbf{M}_o \perp \mathbf{s}$).

Let us pass a line l through O' parallel to the sum $\mathbf{s}$. The relation $\mathbf{s} \parallel \overline{O'O''}$ holds for an arbitrary point O'' of line l; hence $\mathbf{s} \times \overline{O'O''} = 0$, whence $\mathbf{M}_{o'} = \mathbf{M}_o$ (p. 20, corollary 3).

Therefore: *the total moment with respect to an arbitrary point of l is parallel to $\mathbf{s}$ (or $=0$).*

Points not on the line l do not possess the above mentioned property, because if the moment $\mathbf{M}_{o_1}$ for some point O_1 is parallel to $\mathbf{s}$ or equal to zero, then by theorem 5, p. 20, the $\mathrm{Proj}_s\mathbf{M}_{o_1} = \mathrm{Proj}_s\mathbf{M}_{o'}$. Hence $\mathbf{M}_{o_1} = \mathbf{M}_{o'}$. By formula (I), p. 19, it follows that $\mathbf{s} \times \overline{O'O_1} = 0$, or that $\mathbf{s} \parallel \overline{O'O_1}$. Therefore point O^1 lies on l.

We have thus proved that the sought for geometric locus is a line parallel to $\mathbf{s}$. This line is called the *central axis* of the system.

Therefore the central axis of the szstem is a straight line with the property, that the total moment with respect to an arbitrary point of this line is parallel to the sum or equal to zero.

Hence: *a system whose sum is different from zero possesses one (and only one) central axis.*

A system consisting of a vector and a couple whose moment is parallel to the vector is called a *wrench.*

In particular, a vector or a couple is called a wrench.

Selecting a point on the central axis, we see by the reduction theorem, p. 24, that the system is reduced to a wrench. If the sum of the system is zero, then the system is reduced to a couple and hence also to a wrench.

Therefore: *every system is equipollent to a certain wrench.*

§ 18. Centre of parallel vectors. Let a system of parallel vectors $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$, whose sum is different from zero, be given. Denote a unit vector parallel to the vectors of the system by $\mathbf{w}$. The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ can be represented in the form

$$\mathbf{a}_1 = a_1\mathbf{w}, \quad \mathbf{a}_2 = a_2\mathbf{w}, \quad \ldots, \quad \mathbf{a}_n = a_n\mathbf{w},$$

where by $a_1, a_2, \ldots, a_n$ we denote the lengths of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Therefore $\mathbf{s} = (a_1 + a_2 + \ldots + a_n)\,\mathbf{w}$. Since $\mathbf{s} \neq 0$, then $a_1 + a_2 + \ldots + a_n \neq 0$.

Select an arbitrary point O' and denote the vectors $\overline{O'A_1}, \overline{O'A_2}, \ldots, \overline{O'A_n}$ by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, where $A_1, A_2, \ldots, A_n$ are the initial points of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Hence

$$\mathbf{M}_{o'} = a_1\mathbf{w} \times \mathbf{r}_1 + a_2\mathbf{w} \times \mathbf{r}_2 + \ldots + a_n\mathbf{w} \times \mathbf{r}_n,$$

or

$$\mathbf{M}_{o'} = \mathbf{w} \times \Sigma a_i\mathbf{r}_i. \qquad (1)$$

Choose a point O such that

$$\mathbf{r} = \overline{O'O} = \frac{\Sigma a_i\mathbf{r}_i}{\Sigma a_i}. \qquad (2)$$

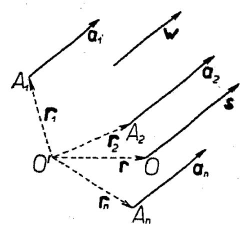

Fig. 37.

From (I), p. 19,

$$\mathbf{M}_o = \mathbf{s} \times \overline{OO'} + \mathbf{M}_{o'}. \qquad (3)$$

Since

$$s \times \overline{OO'} = (\Sigma a_i w) \times (-r) = -w \times r\, \Sigma a_i,$$

therefore according to (2), $s \times \overline{OO'} = -w \times \Sigma a_i r_i$. Hence by (1) and (3) it follows that $M_o = 0$.

The resultant of the system therefore passes through O (theorem 1, p. 26).

Let us note that according to (2) the position of the point O does not depend on the direction w of the vectors a_i. Therefore, if the vectors a_i are turned about their points of application the resultant will again pass through O.

The point O is called the *centre of the system* $a_1, a_2, \ldots, a_n$.

If the coordinates of the initial points A_i are denoted by x_i, y_i, z_i, those of the centre—by x_0, y_0, z_0, then selecting point O' as the origin of the system, we obtain by (2)

$$x_0 = \frac{\Sigma a_i x_i}{\Sigma a_i}, \quad y_0 = \frac{\Sigma a_i y_i}{\Sigma a_i}, \quad z_0 = \frac{\Sigma a_i z_i}{\Sigma a_i}. \tag{4}$$

§ 19. Elementary transformations of a system. The following transformations of a system of vectors are termed *elementary*:

(a) adding to the system (or removing from it) two vectors equal in magnitude, opposite in sense and lying on the same line;

(b) adding to the system (or removing from it) several vectors having a common origin and a sum equal to zero.

Elementary transformations obviously do not change the sum or the moment of the system. Therefore, if we apply elementary transformations to a system, we always obtain systems equipollent to it. Elementary transformations play an important role in the theory of rigid bodies.

It is easy to show that by means of elementary transformations we can:

1. *translate the point of application of a vector to an arbitrarily chosen point of the line on which the vector lies;*

2. *replace several vectors having a common origin by their sum having the same origin;*

3. *replace one vector by several vectors having the same origin as the given vector and having a sum equal to that of the given vector.*

Proof. 1. Suppose that among the vectors of the given system there is a vector a whose initial point is A.

Select an arbitrary point B of the line l on which a lies. Introduce to the system two vectors a and $-a$ whose initial points are at B. We have

thus carried out elementary transformation (a). Remove now from the system the vectors: a (whose origin is A) and $-a$. This will be elementary transformation (b). The operations which we have carried out on the system are equivalent to the translation of the point of application of vector a from A to B (Fig. 38).

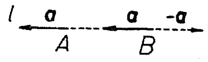

Fig. 38.

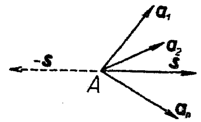

Fig. 39.

2. Suppose that the point A is the origin of the vectors $a_1, a_2, \ldots, a_n$. Add to the system two vectors whose common origin is A: $s = a_1 + a_2 + \ldots + a_n$, and $-s$ (elementary transformation (a)). Now remove the vectors $a_1, a_2, \ldots, a_n, -s$ (elementary transformation (b)). The operations which we have performed are equivalent to the replacement of the vectors $a_1, a_2, \ldots, a_n$ by their sum s (Fig. 39).

3. is proved similarly.

We shall now prove the following theorems:

Theorem I. *By means of elementary transformations every system of vectors can be reduced to a system of three vectors equipollent to the given system.*

Proof. Suppose that we have a system of vectors $(a_1, a_2, \ldots, a_n)$ whose points of application are $A_1, A_2, \ldots, A_n$, respectively. Select three non-collinear points L, M, N in such a way that none of the points $A_1, A_2, \ldots, A_n$ will lie on the plane passing through L, M, N.

Since lines A_1L, A_1M and A_1N do not lie in the same plane, therefore the vector a_1, can be replaced by three vectors u_1, v_1, w_1 with common origin A_1 lying on A_1L, A_1M, A_1N, while obviously $a_1 = u_1 + v_1 + w_1$ (Fig. 40). The vectors u_1, v_1, w_1 can be translated along the lines on which they lie to the points L, M, N, respectively. In this way we have replaced the vector a_1 by the vectors u_1, v_1, w_1 whose points of application are at L, M, N. Similarly, we replace each one of the vectors $a_2, \ldots, a_n$ by three vectors whose points of application are at L, M, N.

We now replace the vectors with origin at L by their sum u with origin also at L. Similarly, vectors with origins at M and N are replaced by sums v and w whose origins are M and N, respectively.

In this manner, by means of the elementary transformations, we have reduced the given system to a system consisting of three vectors, q. e. d.

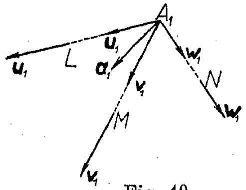

Fig. 40.

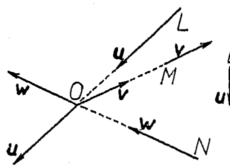

Fig 41.

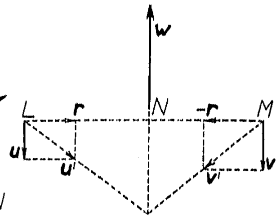

Fig. 42.

Theorem 2. *By means of elementary transformations a system equipollent to zero can be reduced to a zero vector.*

Proof. Assume that the system $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ is equipollent to zero. According to theorem 1 it can be replaced, by means of elementary transformations, by a system consisting of the three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ with points of application at L, M, N, respectively. The system $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is equipollent to zero because it is equipollent to the given system (for elementary transformations do not alter the sum or moment).

According to the theorem on p. 22, the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are either parallel or their prolongations are concurrent at O (Fig. 41). In the second case we can translate the points of application of the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ to O and then remove these vectors since their sum is zero.

Assume then that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are parallel (Fig. 42). Were $\boldsymbol{u}+\boldsymbol{v}=0$, then obviously $\boldsymbol{w}=0$. The system would then be reduced to the couple $\boldsymbol{u}, \boldsymbol{v}$. Since the moment is zero, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ would lie on the same line; since, besides $\boldsymbol{u}+\boldsymbol{v}=0$, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ could be removed. Therefore, let $\boldsymbol{u}+\boldsymbol{v} \neq 0$. Add two vectors $\boldsymbol{r}$ and $-\boldsymbol{r}$ lying on the line LM and having points of application at L and M, respectively. The vectors $\boldsymbol{u}$ and $\boldsymbol{r}$ with origin at L can be replaced by their sum $\boldsymbol{u}'$ with its point of application also at L. Similarly, the vectors $\boldsymbol{v}$ and $-\boldsymbol{r}$ can be replaced by their sum $\boldsymbol{v}'$ whose point of application is at M. The vectors $\boldsymbol{u}'$ and $\boldsymbol{v}'$ are not parallel; hence the vectors $\boldsymbol{u}', \boldsymbol{v}'$ and $\boldsymbol{w}$ can be removed as before. Thus the system equipollent to zero has been reduced to a zero vector by means of the elementary transformations, q. e. d.

Theorem 3. *If two systems of vectors are equipollent, then by means of elementary transformations one system can be transformed into the other.*

Proof. Suppose that the system of vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ with points of application at $A_1, A_2, \ldots, A_n$ is equipollent to the system of vectors $(\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r)$ with points of application at $B_1, B_2, \ldots, B_r$.

To the first system add the vectors $\boldsymbol{b}_1, -\boldsymbol{b}_1$ with origin at B_1, the vectors $\boldsymbol{b}_2, -\boldsymbol{b}_2$ with origin at B_2 etc. Since the vectors

$$\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, -\boldsymbol{b}_1, -\boldsymbol{b}_2, \ldots, -\boldsymbol{b}_r$$

form a system equipollent to zero, therefore by theorem 2 this system can be removed by means of elementary transformations, i. e. replaced by a zero vector. After the removal there remains the system $(\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r)$.