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## PREFACE.

This edition differs from the first<sup>1)</sup> by the new arrangement of the contents of several chapters, some of which have been completed by more recent results, and by the suppression of a number of errors, obligingly pointed out by Mr. V. Jarnik, which formed the object of the two pages of Errata in the first edition. It is probable that fresh errors have slipped in owing to modifications of the text, but the reader would certainly find many more, if the author had not received the valuable help of Messrs J. Todd, A. J. Ward and A. Zygmund in reading the proofs. Also, Mr. L. C. Young has greatly exceeded his rôle of translator in his collaboration with the author. To all these I express my warmest thanks.

This volume contains two Notes by S. Banach. The first of them, on Haar's measure, is the translation (with a few slight modifications) of the note already contained in the French edition of this book. The second, which concerns the integration in abstract spaces, is published here for the first time and completes the considerations of Chapter I.

The numbers given in the bibliographical references relate to the list of cited works which will be found at the end of the book. The asterisks preceding certain titles indicate the parts of the book which may be omitted on first reading.

S. Saks.

Warszawa-Żoliborz, July. 1937.

<sup>1)</sup> S. Saks, *Théorie de l'Intégrale*, Monografie Matematyczne, Volume II, Warszawa 1933.

## FROM THE PREFACE TO THE FIRST EDITION.

The modern theory of real functions became distinct from classical analysis in the second half of the 19-th century, as a result of researches, unsystematic at first, which dealt with the foundations of the Differential Calculus or which concerned the discovery of functions whose properties appeared to be very strange and unexpected.

The distrust with which this new field of investigation was regarded is typified by the attitude of H. Poincaré who wrote: “*Autrefois quand on inventait une fonction nouvelle, c’était en vue de quelque but pratique; aujourd’hui on les invente tout exprès pour mettre en défaut les raisonnements de nos pères et on n’en tirera jamais que éclat*”.

This view was by no means isolated. Ch. Hermite, in a letter to T. J. Stieltjes, expressed himself in even stronger terms: “*Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions qui n’ont pas de dérivées*”. Researches dealing with non-analytic functions and with functions violating laws which one hoped were universal, were regarded almost as the propagation of anarchy and chaos where past generations had sought order and harmony. Even the first attempts to establish a positive theory were rather sceptically received: it was feared that an excessively pedantic exactitude in formulating hypotheses would spoil the elegance of classical methods, and that discussions of details would end by obscuring the main ideas of analysis. It is true that the first researches hardly went beyond the traditional, formal apparatus, fixed by Cauchy and Riemann, which was difficult to adapt to the requirements of the new problems. Nevertheless, these researches succeeded in opening the way to applications of the Theory of Sets to Analysis, and — to quote H. Lebesgue’s inaugural lecture at the Collège de France — “*the great authority of Camille Jordan gave to the new school a valuable encouragement which amply compensated the few reproofs it had to suffer*”.

R. Baire, E. Borel, H. Lebesgue — these are the names which represent the Theory of Real Functions, not merely as an object of researches, but also as a method, names which at the same time recall the leading ideas of the theory. The names of Baire and Borel will be always associated with the method of classification of functions and sets in a transfinite hierarchy by means of certain simple operations to which they are subjected. Excellent accounts

of this subject are to be found in the treatises: Ch. J. de la Vallée Poussin, *Fonctions d’ensemble, Intégrale de Lebesgue, Classes de Baire*, 1916, F. Hausdorff, *Mengenlehre*, 1927, H. Hahn, *Theorie der reellen Funktionen*, 1933 (recent edition), C. Kuratowski, third volume of the present collection, and finally in the book of W. Sierpiński, *Topologja ogólna* (in Polish), and its English translation, *General Topology*, to be published in 1934 by the Toronto University Press.

The other line of researches, which arises directly from the study of the foundations of the Integral Calculus, is still more intimately connected with the great trains of thought of Analysis in the last century. On several occasions attempts were made to generalize the old process of integration of Cauchy-Riemann, but it was Lebesgue who first made real progress in this matter. At the same time, Lebesgue’s merit is not only to have created a new and more general notion of integral, nor even to have established its intimate connection with the theory of measure: the value of his work consists primarily in his theory of derivation which is parallel to that of integration. This enabled his discovery to find many applications in the most widely different branches of Analysis and, from the point of view of method, made it possible to reunite the two fundamental conceptions of integral, namely that of definite integral and that of primitive, which appeared to be forever separated as soon as integration went outside the domain of continuous functions.

The theory of Lebesgue constitutes the subject of the present volume. While distinguishing it from that of Baire, we have no wish to erect an artificial barrier between two streams of thought which naturally intermingle. On the contrary, we shall have frequent occasion, particularly in the last chapters of this book, to show explicitly how Lebesgue’s theory comes to be bound up not only with the results, but also with the very methods, of the theory of Baire. Is not the idea of Denjoy integration at bottom merely a striking adaptation of the idea which guided Baire? Where Baire, by repeated application of passage to the limit, widened the class of functions, Denjoy constructed a transfinite hierarchy of methods of integration starting with that of Lebesgue and whose successive stages are connected by two operations: one corresponding exactly to the generalized integral of Cauchy and the other to the generalized integral of Harnack-Jordan.

Now that the Theory of Real Functions, while losing perhaps a little of the charm of its first youth, has ceased to be a "new" science, it seems superfluous to discuss its importance. It is known that the theory has brought to light regularity and harmony, unheralded by the older methods, concerning, for instance, the existence of a limit, a derivative, or a tangent. It is enough to mention the theorems, now classical, on the behaviour of a power series on, or near, the boundary of its circle of convergence. Also, many branches of analysis, to cite only Harmonic Analysis, Integral Equations, Functional Operations, have lost none of their elegance where they have been inspired by methods of the Theory of Real Functions. On the contrary, we have learnt to admire in the arguments not only cleverness of calculation, but also the generality which, by an apparent abstraction, often enables us to grasp the real nature of the problem.

The object of the preceding remarks has been to indicate the place occupied by the subject of this volume in the Theory of Real Functions<sup>1)</sup>. Let us now say a few words about the structure of the book. It embodies the greater part of a course of lectures delivered by the author at the University of Warsaw (and published in Polish in a separate book<sup>2)</sup>), which has been modified and completed by several chapters. The reader need only be acquainted with a few elementary principles of the Theory of Sets, which are to be found in most courses of lectures on elementary analysis. Actually a summary of the elements of the theory of sets of points is given in one of the opening paragraphs.

Several pages of the book are inspired by suggestions and methods which I owe to the excellent university lectures of my teacher, W. Sierpiński, the influence of whose ideas has often guided my personal researches. Finally, I wish to express my warmest thanks to all those who have kindly assisted me in my task, particularly to my friend A. Zygmund, who undertook to read the manuscript. I thank also Messrs C. Kuratowski and H. Steinhaus for their kind remarks and bibliographical indications.

S. Saks.

Warszawa, May, 1933.

<sup>1)</sup> In this preface, I made no attempt to write a history of the early days of the theory, and still less to settle questions of priority of discovery. But, since an English Edition of this book is appearing now, I think I ought to mention the name of W. H. Young, whose work on the theory of integration started at the same period as that of Lebesgue.

<sup>2)</sup> *Zarys teorji całki*, Warszawa 1930, Wydawnictwo Kasy im. Mianowskiego, Instytutu Popierania Nauki.

## E R R A T A

The first numbers refer to pages, the second to lines; starred numbers denote lines counted from the foot.

for:

read:

4,14*	P. J. Daniell [2],	P. J. Daniell [1; 4; 5],
4,12*	S. Bochner [1],	G. Birkhoff [1], S. Bochner [1], N. Dunford [2; 3],
6,14*	$a/0 = +\infty$ .	$a/0 = \pm\infty$ according as $a \geq 0$ or $a < 0$ .
17,9*	W. Sierpiński [3],	W. Sierpiński [14],
30,13*	P. J. Daniell [2]	P. J. Daniell [4]
44,13*	spaces.	spaces. (For various examples of Carathéodory measures in metrical spaces see also A. Haar [1] and A. Appert [1].)
44,17—19	all the sets $X$ for which $I'(X)=0$ (in particular, it includes the empty set).	the empty set and all the sets $X$ for which $I'(X)=0$ .
93,9	authors.	authors. Cf. also N. Dunford [1].
110,3	$(4m+1)^m \cdot a^{-1} \sum_n  E_n  \leq (4m+1)^m \cdot a^{-1}$	$(4m+1)^m \cdot a^{-1} \sum_n  E_n  \leq (4m+1)^m \cdot a^{-1} \cdot  S $
163,3*	T. Radó [1; 1; 4]	T. Radó [1; 1; 4]; also E. J. McShane [1], C. B. Morrey [1] and T. Radó [5]

Now let  $F$  be an  $\mathfrak{L}$ -integral which for functions belonging to  $\mathfrak{C}$  coincides with the functional  $f$  subject to (2). We then have to  $F(x) = \lim_n F(x_n) = \lim_n f_n(x_n)$ . If further  $f_n$  is represented by the formula (4), then

$$F(x) = \lim_{\substack{n \\ \vartheta_1^2 + \dots + \vartheta_n^2 \leq 1}} \int \dots \int x(\vartheta_1, \dots, \vartheta_n, 0, 0, \dots) \varphi_n(\vartheta_1, \dots, \vartheta_n) d\vartheta_1 \dots d\vartheta_n$$

and, in particular, if  $\varphi_n$  is given by (5),

$$F(x) = \lim_{\substack{n \\ \vartheta_1^2 + \dots + \vartheta_n^2 \leq 1}} \int \dots \int x(\vartheta_1, \dots, \vartheta_n, 0, 0, \dots) \frac{d\vartheta_1 \dots d\vartheta_n}{2^n \sqrt{1-\vartheta_1^2} \dots \sqrt{1-\vartheta_1^2 - \dots - \vartheta_{n-1}^2}}.$$

This formula defines explicitly a certain  $\mathfrak{L}$ -integral for all functions bounded and continuous in  $H$ .

The above considerations may be extended to certain spaces of the type (B) (cf. S. Banach [I, Chap. V]), e. g. the spaces  $l^{(p)}$ ,  $L^{(p)}$  with  $p > 1$ .

## BIBLIOGRAPHY.

**Adams, C. R. and Clarkson, J. A.** [1] On definitions of bounded variation for functions of two variables, Trans. Amer. Math. Soc., 35, 824—854 (1933). — [2] Properties of functions  $f(x, y)$  of bounded variation, ibid., 36, 711—730 (1934).

**Alexandroff, P.** [1] Über die Äquivalenz des Perronschen und des Denjoy-schen Integralbegriffes, Math. Zeitschr., 20, 213—222 (1924). — [2] L'intégration au sens de M. Denjoy considérée comme recherche des fonctions primitives, Rec. Math. Soc. Math. Moscou, 31, 465—476 (1924).

**Appert, A.** [1] Mesures normales dans les espaces distanciés, Bull. Soc. Math. France, 60, 1—36 (1936).

**Auerbach, H.** [1] Démonstration nouvelle d'un théorème de M. Banach sur les fonctions dérivées des fonctions mesurables, Fundam. Math., 7, 263 (1925).

**Baire, R.** [1] Sur les fonctions de variables réelles, Ann. Mat. Pura e Appl. (3), 3, 1—122 (1899).

**Banach, S.** [1] Théorie des opérations linéaires, Monografje Matematyczne 1, Warszawa 1932. — [1] Sur les ensembles de points où la dérivée est infinie, C. R. Acad. Sci. Paris, 173, 457—459 (1921). — [2] Sur les fonctions dérivées des fonctions mesurables, Fundam. Math., 3, 128—132 (1922). — [3] Sur un théorème de M. Vitali, ibid., 5, 130—136 (1924). — [4] Sur une classe de fonctions d'ensemble, ibid., 6, 170—188 (1924). — [5] Sur les lignes rectifiables et les surfaces dont l'aire est finie, ibid., 7, 225—237 (1925). — [6] Sur une classe de fonctions continues, ibid., 8, 166—173 (1926). — [7] Ueber additive Massfunktionen in abstrakten Mengen, ibid., 15, 97—101 (1930).

**Banach, S. et Kuratowski, C.** [1] Sur une généralisation du problème de la mesure, Fundam. Math., 14, 127—131 (1929).

**Banach, S. et Saks, S.** [1] Sur les fonctions absolument continues des fonctions absolument continues, Fundam. Math., 11, 113—116 (1928).

**Bary, N.** [1] Sur la représentation analytique d'une classe de fonctions continues, C. R. Acad. Sci. Paris, 183, 469—471 (1926). — [2] Sur les fonctions jouissant de la propriété (N), ibid., 189, 441—443 (1929). — [3] Mémoire sur la représentation finie des fonctions continues, Math. Ann., 103, 185—248 and 598—653 (1930). — [4] Sur une classification des fonctions continues à partir des fonctions à variation bornée, Rec. Math. Soc. Math. Moscou, 40, 326—370 (1933).

**Bary, N. et Menchoff, D.** [1] Sur l'intégrale de Lebesgue-Stieltjes et les fonctions absolument continues des fonctions absolument continues, Ann. Mat. Pura e Appl. (4), 5, 19—54 (1928).

**Bauer, H.** [1] Der Perronsche Integralbegriff und seine Beziehung zum Lebesgueschen, Monatshefte Math. Phys., 26, 153—198 (1915).

**Besicovitch, A. S.** [1] On the fundamental geometrical properties of linearly measurable plane sets of points, *Math. Ann.*, **98**, 422—464 (1928). — [2] Discussion der stetigen Funktionen im Zusammenhang mit der Frage über ihre Differenzierbarkeit, *Bull. Acad. Sci. URSS*, **97**—**122**, 527—540 (1925). — [3] On sufficient conditions for a function to be analytic, and on behaviour of analytic functions in the neighborhood of non-isolated singular points, *Proc. London Math. Soc.* (2), **32**, 1—9 (1931). — [4] On tangents to general sets of points, *Fundam. Math.*, **22**, 49—53 (1934). — [5] On differentiation of Lebesgue double integrals, *ibid.*, **25**, 209—216 (1936). — [6] On differentiation of functions of two variables, *Math. Zeitschr.*, **41**, 402—404 (1936).

**Birkhoff, G.** [1] Integration of functions with values in a Banach space, *Trans. Amer. Math. Soc.*, **38**, 357—378 (1935).

**Blumberg, H.** [1] A theorem on arbitrary functions of two variables with applications, *Fundam. Math.*, **16**, 17—24 (1930). — [2] The measurable boundaries of an arbitrary function, *Acta Math.*, **65**, 263—282 (1935).

**Bochner, S.** [1] Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind, *Fundam. Math.*, **20**, 262—276 (1933). — [2] Eine Bemerkung zum Satz von Fubini, *ibid.*, **20**, 277—280 (1933). — [3] Absolut-additive abstrakte Mengenfunktionen, *ibid.*, **21**, 211—213 (1933).

**Borel, E.** [1] *Leçons sur la théorie des fonctions*, Paris 1898. — [1] Sur l'intégration des fonctions non bornées et sur les définitions constructives, *Ann. Ecole Norm.*, **36**, 71—91, (1919).

**Bourbaki, N.** [1] Sur un théorème de Carathéodory et la mesure dans les espaces topologiques, *C. R. Acad. Sci. Paris*, **201**, 1309—1311 (1935).

**Bouligand, G.** [1] *Introduction à la Géométrie infinitésimale directe*, Paris 1932.

**Burkill, J. C.** [1] The fundamental theorem of Denjoy integration, *Proc. Cambridge Philos. Soc.*, **21**, 659—663 (1923). — [2] Functions of intervals, *Proc. London Math. Soc.* (2), **22**, 275—310 (1924). — [3] The expressions of area as an integral, *ibid.* (2), **22**, 311—336 (1924). — [4] The derivates of functions of intervals, *Fundam. Math.*, **5**, 321—327 (1924). — [5] The approximately continuous Perron integral, *Math. Zeitschr.*, **34**, 270—278 (1931). — [6] The Cesàro-Perron integral, *Proc. London Math. Soc.* (2), **34**, 314—322 (1932). — [7] The Cesàro-Perron scale of integration, *ibid.*, **39**, 541—552 (1935).

**Burkill, J. C. and Haslam-Jones, U. S.** [1] The derivates and approximate derivates of measurable functions, *Proc. London Math. Soc.* (2), **32**, 346—355 (1931). — [2] Note on the differentiability of functions of two variables, *Journ. London Math. Soc.*, **7**, 297—305 (1932). — [3] Relative measurability and the derivates of non-measurable functions, *Quart. Journ. Math.*, Oxford Ser., **4**, 233—239 (1933).

**Busemann, H. und Feller, W.** [1] Zur Differentiation der Lebesgueschen Integrale, *Fundam. Math.*, **22**, 226—256 (1934).

**Caccioppoli, R.** [1] Sul lemma fondamentale del calcolo integrale, *Atti Mem. Accad. Sci. Padova*, **50**, 93—98 (1934).

**Carathéodory, C.** [I] Vorlesungen über reelle Funktionen, Leipzig-Berlin 1918. — (II) Vorlesungen über reelle Funktionen, Leipzig-Berlin, 2. Aufl. 1927. — [1] Ueber das lineare Mass von Punktmengen — eine Verallgemeinerung des Längenbegriffs, *Nachr. Ges. Wiss. Göttingen*, 404—426 (1914).

**Cauchy, A.** [I] Oeuvres complètes, Paris 1882—1899.

**Celidze, V. G.** [1] Ueber derivierte Zahlen einer Funktion zweier Variablen, *C. R. Acad. Sci. URSS*, **15**, 13—15 (1937).

**Clarkson, J. A.** [1] Uniformly convex spaces, *Trans. Amer. Math. Soc.*, **40**, 396—414 (1936).

**Cohen, L. W.** [1] A new proof of Lusin's theorem, *Fundam. Math.*, **9**, 122—123 (1927).

**Currier, A. E.** [1] Proof of the fundamental theorems on second-order cross partial derivatives, *Trans. Amer. Math. Soc.*, **35**, 245—253 (1933).

**Daniell, P. J.** [1] A general form of integral, *Ann. of Math.* (2), **19**, 279—294 (1917—18). — [2] Integrals in an infinite number of dimensions, *ibid.* (2), **20**, 281—288 (1918). — [3] Functions of limited variation in an infinite number of dimensions, *ibid.*, **21**, 30—38 (1919). — [4] Stieltjes derivatives, *Bull. Amer. Math. Soc.*, **26**, 444—448 (1919). — [5] Further properties of the general integral, *Ann. of Math.* (2), **21**, 203—220 (1920).

**Denjoy, A.** [1] Mémoire sur les nombres dérivés des fonctions continues, *Journ. Math. Pures et Appl.* (7), **1**, 105—240 (1915). — [2] Une extension de l'intégrale de M. Lebesgue, *C. R. Acad. Sci. Paris*, **154**, 859—862 (1912). — [3] Calcul de la primitive de la fonction dérivée la plus générale, *ibid.*, **154**, 1075—1078 (1912). — [4] Sur la dérivation et son calcul inverse, *ibid.*, **162**, 377—380 (1916). — [5] Sur les fonctions dérivées sommables, *Bull. Soc. Math. France*, **43**, 161—248 (1915). — [6] Mémoire sur la totalisation des nombres dérivés non-sommables, *Ann. Ecole Norm.*, **33**, 127—222 (1916); **34**, 181—238 (1917). — [7] Sur l'intégration riemannienne, *C. R. Acad. Sci. Paris*, **169**, 219—220 (1919). — [8] Sur la définition riemannienne de l'intégrale de Lebesgue, *ibid.*, **193**, 695—698 (1931). — [9] Sur l'intégration des coefficients différentiels d'ordre supérieur, *Fundam. Math.*, **25**, 273—326 (1935).

**Dini, U.** [1] Fondamenti per la teoria delle funzioni di variabili reali, Pisa 1878.

**Dunford, N.** [1] On a theorem of Plessner, *Bull. Amer. Math. Soc.*, **41**, 356—358 (1935). — [2] Integration in general analysis, *Trans. Amer. Math. Soc.*, **37**, 441—453 (1935) [Corrections, *ibid.*, **38**, 600—601 (1936)]. — [3] Integration and linear operations, *ibid.*, **40**, 474—484 (1936).

**Egoroff, D. Th.** [1] Sur les suites des fonctions mesurables, *C. R. Acad. Sci. Paris*, **152**, 244—246 (1911).

**Evans, G. C.** [1] On potentials of positive mass, I, *Trans. Amer. Math. Soc.*, **37**, 226—253 (1935).

**Faber, G.** [1] Über stetige Funktionen II., *Math. Ann.*, **69**, 372—433 (1910).

**Fatou, P.** [1] Séries trigonométriques et séries de Taylor, *Acta Math.*, **30**, 335—400 (1906).

**Feller, W.** [1] Bemerkungen zur Masstheorie in abstrakten Räumen, *Bull. Int. Acad. Yougosl.*, **28**, 30—45 (1934).

**Fichtenholz, G.** [1] Sur une fonction de deux variables sans intégrale double, *Fundam. Math.*, **6**, 30—36 (1924). — [2] Sur une généralisation de l'intégrale de Stieltjes, *C. R. Acad. Sci. URSS*, **3**, 95—100 (1936). — [3] Note sur les fonctions absolument continues, *Rec. Math. Soc. Math. Moscou*, **31**, 286—295 (1923). — [4] Sur un problème de M. Banach, *Fundam. Math.*, **10**, 302—304 (1927).

**Fichtenholz, G. et Kantorovitch, L.** [1] Sur les opérations dans l'espace des fonctions bornées, *Studia Math.*, **5**, 69—98 (1934).

**Fréchet, M.** [1] Sur l'intégrale d'une fonctionnelle étendue à un ensemble abstrait, *Bull. Soc. Math. France*, **43**, 249—267 (1915). — [2] Note on the area of a surface, *Proc. London Math. Soc.* (2), **24**, XLVIII (1926). — [3] Sur l'aire des surfaces polyédrales, *Ann. Soc. Polon. Math.*, **3**, 1—3 (1925). — [4] Sur le prolongement des fonctionnelles semi-continues et sur l'aire des surfaces courbes, *Fundam. Math.*, **7**, 210—224 (1925). — [5] Sur quelques définitions possibles de l'intégrale de Stieltjes, *Duke Math. Journ.*, **2**, 283—395 (1936).

- Fubini, G.** [1] Sugli integrali multipli, *Atti Accad. Naz. Lincei, Rend.*, **16**, 608—614 (1907). — [2] Sulla derivazione per serie, *ibid.*, **24**, 204—206 (1915).
- Geöcze, Z. de** [1] Quadrature des surfaces courbes, *Math. Naturwiss. Ber. Ungarn*, **26**, 1—88 (1910).
- Gillis, J.** [1] On linearly measurable plane sets of points, *C. R. Soc. Sci. Varsovie*, **27**, 49—70 (1936).
- Goldowsky, G.** [1] Note sur les dérivées exactes, *Rec. Math. Soc. Math. Moscou*, **35**, 35—36 (1928).
- Goursat, E.** [1] Sur la définition générale des fonctions analytiques d'après Cauchy, *Trans. Amer. Math. Soc.*, **1**, 14—16 (1900).
- Gowurin, M.** [1] Ueber die Stieltjessche Integration abstrakter Funktionen, *Fundam. Math.*, **27**, 254—268 (1936).
- Gross, W.** [1] Über das Flächenmass von Punktmenigen, *Monatshefte Math. Phys.*, **29**, 145—176 (1918).
- Haar, A.** [1] Der Massbegriff in der Theorie der kontinuierlichen Gruppen, *Ann. of Math.* (2), **34**, 147—169 (1933).
- Hahn, H.** [I] Theorie der reellen Funktionen, I. Band, Berlin 1921. — [II] Reelle Funktionen, I. Teil, Leipzig 1932. — [1] Ueber den Fundamentalsatz der Integralrechnung, *Monatshefte Math. Phys.*, **16**, 161—166 (1905). — [2] Ueber die Multiplikation total-additiver Mengenfunktionen, *Ann. Scuola Norm. Sup. Pisa* **3**, 429—452 (1933).
- Hake, H.** [1] Ueber de la Vallée Poussins Ober- und Unterfunktionen einfacher Integrale und die Integraldefinition von Perron, *Math. Ann.*, **83**, 119—142 (1921).
- Hardy, G. H. and Littlewood, J. E.** [1] Some properties of fractional integrals, *Math. Zeitschr.*, **27**, 565—606 (1928). — [2] A maximal theorem with function-theoretic applications, *Acta Math.*, **54**, 81—116 (1930).
- Harnack, A.** [1] Die allgemeinen Sätze über den Zusammenhang der Funktionen einer reellen Variablen mit ihren Ableitungen, II, *Math. Ann.*, **24**, 217—252 (1884).
- Haslam-Jones, U. S.** [1] Derivate planes and tangent planes of a measurable function, *Quart. Journ. Math., Oxford Ser.*, **3**, 120—132 (1932). — [2] Tangential properties of a plane set of points, *ibid.*, **7**, 116—123 (1936). — [3] The discontinuities of an arbitrary function of two variables, *ibid.*, **7**, 184—190 (1936).
- Hausdorff, F.** [I] Grundzüge der Mengenlehre, Leipzig 1914. — [II] Mengenlehre, 3. Aufl., Berlin 1935. — [1] Dimension und äusseres Mass, *Math. Ann.*, **79**, 157—179 (1919).
- Heffter, L.** [1] Zum Beweis des Cauchy-Goursatschen Integralsatzes, *Nachr. Ges. Wiss. Göttingen*, 312—316 (1903).
- Hildebrandt, T. H.** [1] On integrals related to and extensions of the Lebesgue integrals, *Bull. Amer. Math. Soc.* (2), **24**, 113—144 (1917), 177—202 (1918). — [2] On the interchange of limit and Lebesgue integral for a sequence of functions, *Trans. Amer. Math. Soc.*, **33**, 441—443 (1931).
- Hille, J. and Tamarkin, J. D.** [1] Remarks on a known example of a monotone continuous function, *Amer. Math. Monthly*, **36**, 255—264 (1929).
- Hobson, E. W.** [I] The theory of functions of a real variable and the theory of Fourier's series, Vol. I, 3d edition, Cambridge 1927. — [II] The theory of functions of a real variable and the theory of Fourier's series, Vol. II, 2d edition, Cambridge 1926.
- Izumi, S.** [1] On the F. Riesz' lemma, *Tôhoku Math. Journ.*, **42**, 65—66 (1936).
- Jarník, V.** [1] Über die Differenzierbarkeit stetiger Funktionen, *Fundam. Math.*, **21**, 48—58 (1933). — [2] Sur les nombres dérivés approximatifs, *ibid.*, **22**, 4—16 (1934). — [3] Sur les fonctions de deux variables réelles, *ibid.*, **27**, 147—150 (1936).

- Jeffery, R. L.** [1] The integrability of a sequence of functions, *Trans. Amer. Math. Soc.*, **33**, 433—440 (1931). — [2] Non-absolutely convergent integrals with respect to functions of bounded variation, *ibid.*, **34**, 645—675 (1932). — [3] Derived numbers with respect to functions of bounded variation, *ibid.*, **36**, 749—758 (1934).
- Jessen, B.** [1] Abstrakt Maal- og Integralteori, *Mat. Tidsskr. B*, **73**—84 (1934); **60**—74 (1935). — [2] The theory of integration in a space of an infinite number of dimensions, *Acta Math.*, **63**, 249—323 (1934).
- Jessen, B., Marcinkiewicz, J. and Zygmund, A.** [1] Note on the differentiability of multiple integrals, *Fundam. Math.*, **25**, 217—234 (1935).
- Kamke, E.** [I] Das Lebesgue'sche Integral, Leipzig 1925. — [I] Zur Definition der approximativ stetigen Funktionen, *Fundam. Math.*, **10**, 431—438 (1927).
- Kellogg, O. D.** [1] An example in potential theory, *Proc. Amer. Acad. Arts Sci.*, **58**, 527—533 (1923).
- Kempisty, S.** [1] Sur la méthode triangulaire du calcul de l'aire d'une surface courbe, *Bull. Soc. Math. France*, **64**, 119—132 (1936).
- Kennedy, M. D. and Pollard, S.** [1] Upper and lower integrals, *Math. Zeitschr.*, **39**, 432—454.
- Khintchine, A.** [1] Sur une extension de l'intégrale de M. Denjoy, *C. R. Acad. Sci. Paris*, **162**, 287—291 (1916). — [2] Sur le procédé d'intégration de M. Denjoy, *Rec. Math. Soc. Math. Moscou*, **30**, 543—557 (1918). — [3] Sur la dérivation asymptotique, *C. R. Acad. Sci. Paris*, **164**, 142—144 (1917). — [4] Recherches sur la structure des fonctions mesurables, *Rec. Math. Soc. Math. Moscou*, **31**, 265—285 and 377—433 (1924). — [5] Recherches sur la structure des fonctions mesurables, *Fundam. Math.*, **9**, 212—279 (1927).
- Kolmogoroff, A.** [1] Untersuchungen über den Integralbegriff, *Math. Ann.*, **103**, 654—696 (1930). — [2] La définition axiomatique de l'intégrale, *C. R. Acad. Sci. Paris*, **180**, 110—111 (1925). — [3] Beiträge zur Masstheorie, *Math. Ann.*, **107**, 351—366 (1932).
- Kolmogoroff, A. und Verčenko, J.** [1] Ueber Unstetigkeitspunkte von Funktionen zweier Veränderlichen, *C. R. Acad. Sci. URSS*, **1**, 1—3, 105—107 (1934). — [2] Weitere Untersuchungen über Unstetigkeitspunkte von Funktionen zweier Veränderlichen, *ibid.*, **4**, 361—364 (1934).
- Kondô, M.** [1] Sur les notions de catégorie et de mesure dans la théorie des ensembles de points, *Journ. Fac. Sci. Hokkaido Univ.*, **4**, 123—180 (1936).
- Krzyżanowski, M.** [1] Sur les fonctions absolument continues généralisées de deux variables, *C. R. Acad. Sci. Paris*, **198**, 2058—2060 (1934).
- Kuratowski, C.** [I] Topologie I (Espaces métrisables, espaces complets), *Monografie Matematyczne* **4**, Warszawa-Lwów 1933.
- Kuratowski, C. et Ulam, S.** [1] Quelques propriétés topologiques du produit combinatoire, *Fundam. Math.*, **19**, 247—251 (1932).
- Lampariello, G.** [1] Sulle superficie continue che ammettono area finita, *Atti Accad. Naz. Lincei, Rend.*, **3**, 294—298 (1926).
- Lebesgue, H.** [I] Leçons sur l'intégration et la recherche des fonctions primitives, Paris 1904. — [II] Leçons sur l'intégration et la recherche des fonctions primitives, 2-me éd., Paris 1928. — [1] Intégrale, Longueur, Aire, *Ann. Mat. Pura e Appl.* (3), **7**, 231—359 (1902). — [2] Sur les fonctions dérivées, *Atti Accad. Naz. Lincei, Rend.*, **15**, 3—8 (1906). — [3] Encore une observation sur les fonctions dérivées, *ibid.*, **16**, 92—100 (1907). — [4] Sur la recherche des fonctions primitives, *ibid.*, **16**, 283—290 (1907). — [5] Sur l'intégration des fonctions continues, *Ann. Ecole Norm. (3)*, **27**, 361—450 (1910). — [6] Remarques sur les théories de la mesure et de l'intégration, *ibid.*, **35**, 191—250 (1918). — [7] Sur la recherche des fonctions primitives, *Acta Math.*, **49**, 245—262 (1926). — [8] Sur le développement de la notion d'intégrale, *Mat. Tidsskr. B*, **54**—74 (1926).

**Levi, B.** [1] Ricerche sulle funzioni derivate, Atti Accad. Naz. Lineei, Rend., 15<sub>1</sub>, 433—438 (1906).

**Looman, H.** [1] Sur la totalisation des dérivées des fonctions continues de plusieurs variables indépendantes, Fundam. Math., 4, 246—285 (1923). — [2] Ueber die Cauchy-Riemannschen Differentialgleichungen, Nachr. Ges. Wiss. Göttingen, 97—108 (1923). — [3] Ueber eine Erweiterung des Cauchy-Goursat'schen Integralsatzes, Nieuw. Arch. Wiskde (2), 14, 234—239 (1925). — [4] Ueber die Perronsche Integraldefinition, Math. Ann., 93, 153—156 (1925).

**Lusin, N.** [1] Intégrale et série trigonométrique (in Russian), Moscow 1915. — [II] Leçons sur les ensembles analytiques, Paris 1930. — [1] Sur les propriétés des fonctions mesurables, C. R. Acad. Sci. Paris, 154, 1688—1690 (1912). — [2] Sur les propriétés de l'intégrale de M. Denjoy, ibid., 155, 1475—1478 (1912). — [3] Sur les ensembles analytiques, Fundam. Math., 10, 1—95 (1927). — [4] Sur la notion de l'intégrale, Annali Mat. Pura e Appl. (3), 26, 77—129 (1917).

**Lusin, N. et Sierpiński, W.** [1] Sur quelques propriétés des ensembles (A), Bull. Acad. Sci. Cracovie, 35—48 (1918).

**Lomnicki, Z. et Ulam, S.** [1] Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités, Fundam. Math., 23, 237—278 (1934).

**Marcinkiewicz, J.** [1] Sur les nombres dérivés, Fundam. Math., 24, 305—308 (1935). — [2] Sur les séries de Fourier, ibid., 27, 38—69 (1936).

**Marcinkiewicz, J. and Zygmund, A.** [1] On the differentiability of functions and summability of trigonometrical series, Fundam. Math., 26, 1—43 (1936).

**Mazur, S.** [1] O metodach sumowalności, Księga pamiątkowa I Polskiego Zjazdu Matematycznego, Lwów 1927, 102—107.

**Mazurkiewicz, S.** [1] Sur les fonctions qui satisfont à la condition (N), Fundam. Math., 16, 348—352 (1930).

**McShane, E. J.** [1] Integrals over surfaces in parametric form, Ann. of Math. (2), 34, 815—838 (1933).

**Menchoff, D.** [1] Les conditions de monogénéité, Paris 1936. — [1] Sur la généralisation des conditions de Cauchy-Riemann, Fundam. Math., 25, 59—97 (1935). — [2] Sur la monogénéité asymptotique, Rec. Math. Soc. Math. Moscow, 1, 189—210 (1936).

**Milicer-Grużewska, H.** [1] Sur la continuité de la variation, C. R. Soc. Sci. Varsovie, 21, 165—176 (1928).

**Montel, P.** [1] Sur les suites infinies de fonctions, Ann. Ecole Norm. (3), 24, 233—334 (1907). — [2] Sur les différentielles totales et les fonctions monogènes, C. R. Acad. Sci. Paris, 156, 1820—1822 (1913).

**Morera, G.** [1] Sulla definizione di funzione di una variabile complessa, Atti Accad. Sci. Torino, 37, 99—102 (1902).

**Morrey, C. B.** [1] A class of representations of manifolds I, Amer. J. Math., 55, 683—707 (1933).

**Nalli, P.** [1] Esposizione e confronto critico delle diverse definizioni proposte per l'integrale definita di una funzione limitata o no, Palermo 1914.

**Neubauer, M.** [1] Ueber die partiellen Derivierten unstetiger Funktionen, Monatshefte Math. Phys., 38, 139—146 (1931).

**Nikodym, O.** [1] Sur la mesure des ensembles plans dont tous les points sont rectilinéairement accessibles, Fundam. Math., 10, 116—168 (1927). — [2] Sur une généralisation des intégrales de M. Radon, ibid., 15, 131—179 (1930).

**Osgood, W. F.** [1] Zweite Note über analytische Funktionen mehrerer Veränderlichen, Math. Ann., 53, 461—464 (1900).

**Perron, O.** [1] Ueber den Integralbegriff, S.-B. Heidelberg. Akad. Wiss., 16 (1914).

**Petrovsky, J.** [1] Sur l'unicité de la fonction primitive par rapport à une fonction continue arbitraire, Rec. Math. Soc. Moscou, 41, 48—58 (1934).

**Plessner, A.** [1] Eine Kennzeichnung der total-stetigen Funktionen, Journ. Reine u. Angew. Math., 160, 26—32 (1929).

**Pollard, S.** [1] The Stieltjes integral and its generalisations, Quart. Journ. Math., Oxford Ser., 49, 87—94 (1920).

**Pompeiu, T.** [1] Sur la continuité des fonctions de variable complexe, Ann. Fac. Sci. Univ. Toulouse (2), 7, 264—315 (1905).

**Possel, R. de** [1] Sur la dérivation abstraite des fonctions d'ensembles, C. R. Acad. Sci. Paris, 201, 579—581 (1935).

**Rademacher, H.** [1] Eineindeutige Abbildungen und Messbarkeit, Monatshefte Math. Phys., 27, 183—291 (1916). — [2] Bemerkungen zu den Cauchy-Riemannschen Differentialgleichungen und zum Moreraschen Satz, Math. Zeitschr., 4, 177—185 (1919). — [3] Ueber partielle und totale Differenzierbarkeit I, Math. Ann., 79, 340—359 (1919). — [4] Ueber partielle und totale Differenzierbarkeit II, ibid., 81, 52—63 (1920).

**Radò, T.** [1] On the problem of Plateau, Ergebnisse der Mathematik, Berlin 1933. — [1] Sur l'aire des surfaces courbes, Acta Litt. Sci. Szeged, 3, 131—169 (1927). — [2] Sur le calcul des surfaces courbes, Fundam. Math., 10, 197—210 (1927). — [3] Sur un problème relatif à un théorème de Vitali, ibid., 11, 228—229 (1928). — [4] Ueber das Flächenmass rektifizierbarer Flächen, Math. Ann., 100, 445—479 (1928). — [5] O polu powierzchni krzywych, Mathesis Polonica, 7, 1—18 (1932). — [6] A remark on the area of surfaces, Amer. J. Math., 58, 598—606 (1936).

**Radon, J.** [1] Theorie und Anwendungen der absolut additiven Mengenfunktionen, S.-B. Akad. Wiss. Wien, 122, 1295—1438 (1913).

**Rajchman, A. et Saks, S.** [1] Sur la dérivalibilité des fonctions monotones, Fundam. Math. 4, 204—213 (1923).

**Ridder, J.** [1] Ueber den Cauchyschen Integralsatz für reelle und komplexe Funktionen, Math. Ann., 102, 132—156 (1929). — [2] Ueber stetige, additive Intervallfunktionen in der Ebene, Nieuw Arch. Wiskde (2), 16, 55—69 (1929). — [3] Ueber additive Intervallfunktionen, ibid., 16, 60—75 (1930). — [4] Ueber Derivierten und Ableitungen, C. R. Soc. Sci. Varsovie, 23, 1—11 (1930). — [5] Ueber den Perronschen Integralbegriff und seine Beziehung zu den  $R$ ,  $L$ - und  $D$ -Integralen, Math. Zeitschr., 34, 234—269 (1931). — [6] Ueber approximativ stetige Denjoy-Integrale, Fundam. Math., 21, 1—10 (1933). — [7] Ueber die gegenseitigen Beziehungen verschiedener approximativ stetiger Denjoy-Perron-Integrale, ibid., 22, 136—162 (1934). — [8] Ueber die  $T$ - und  $N$ -Bedingungen und die approximativ stetigen Denjoy-Perron-Integrale, ibid., 22, 163—179 (1934). — [9] Ueber Perron-Stieltjessche und Denjoy-Stieltjessche Integrationen, Math. Zeitschr., 40, 127—160 (1935). — [10] Ueber Denjoy-Perron Integration von Funktionen zweier Variablen, C. R. Soc. Sci. Varsovie, 28, 5—16 (1935). — [11] Ueber die gegenseitigen Beziehungen einiger "trigonometrischer" Integrationen, Math. Zeitschr., 42, 322—336 (1937).

**Riesz, F.** [1] Sur l'intégrale de Lebesgue, *Acta Math.*, **42**, 191—205 (1920). — [2] Sur le théorème de M. Egoroff et sur les opérations fonctionnelles linéaires, *Acta Litt. Sci. Szeged.*, **1**, 18—26 (1922). — [3] Elementarer Beweis des Egoroff schen Satzes, *Monatshette Math. Phys.*, **35**, 243—248 (1928). — [4] Sur les fonctions sousharmoniques et leur rapport à la théorie du potentiel, II, *Acta Math.*, **54**, 321—360 (1930). — [5] Sur un théorème de maximum de MM. Hardy et Littlewood, *Journ. London Math. Soc.*, **7**, 10—13 (1932). — [6] Sur l'existence de la dérivée des fonctions monotones et sur quelques problèmes qui s'y rattachent, *Acta Litt. Sci. Szeged.*, **5**, 208—221 (1932). — [7] Sur l'existence de la dérivée des fonctions d'une variable réelle et des fonctions d'intervalle, *Verhandl. Internat. Math. Kongress Zürich 1932*, I, 258—269. — [8] Sur les points de densité au sens fort, *Fundam. Math.*, **22**, 221—225 (1934). — [9] Sur l'intégrale de Lebesgue comme l'opération inverse de la dérivation, *Ann. Scuola Norm. Sup. Pisa (2)*, **5**, 191—212 (1936).

**Roger, F.** [1] Sur quelques applications métriques de la notion de contingent bilatéral, *C. R. Acad. Sci. Paris*, **201**, 28—30 (1935). — [2] Sur la relation entre les propriétés tangentes et métriques des ensembles cartésiens, *ibid.*, **201**, 871—873 (1935). — [3] Sur l'extension à la structure locale des ensembles cartésiens les plus généraux des théorèmes de M. Denjoy sur les nombres dérivés des fonctions continues, *ibid.*, **202**, 377—380 (1936).

**Romanowski, P.** [1] Essai d'une exposition de l'intégrale de Denjoy sans nombres transfinis, *Fundam. Math.*, **19**, 38—44 (1932).

**Rosenthal, A.** [1] Neuere Untersuchungen über Funktionen reeller Veränderlichen (Sonderabdruck aus der Encyclopädie der Mathematischen Wissenschaften), Leipzig-Berlin 1923—[1] Ueber die Singularitäten der reellen ebenen Kurven, *Math. Ann.*, **73**, 480—521 (1913).

**Roussel, A.** [1] Primitive de seconde espèce, *C. R. Acad. Sci. Paris*, **187**, 926—927 (1928).

**Ruziewicz, S.** [1] Sur les fonctions qui ont la même dérivée et dont la différence n'est pas constante, *Fundam. Math.*, **1**, 148—151 (1920); 2-me éd. 1937.

**Saks, S. et Zygmund, A.** [1] Sur les faisceaux des tangentes à une courbe, *Fundam. Math.*, **6**, 117—121 (1924). — [2] On functions of rectangles and their application to the analytic functions, *Ann. Scuola Norm. Sup. Pisa*, **3**, 1—6 (1934).

**Schauder, J.** [1] The theory of surface measure, *Fundam. Math.*, **8**, 1—48 (1926).

**Schmeiser, M.** [1] Some properties of arbitrary functions, *Fundam. Math.*, **22**, 70—76 (1934).

**Schoenflies, A.** [1] Die Entwicklung der Lehre von den Punktmannigfaltigkeiten, Bericht, erstattet der Deutschen Mathematiker-Vereinigung 1900.

**Sierpiński, W.** [1] Hypothèse du continu, *Monografie Matematyczne* **4**, Warszawa-Lwów 1934. — [II] Introduction to General Topology (translated by C. C. Krieger), Toronto 1934. — [1] Démonstration de la dénombrabilité des valeurs extrémiales d'une fonction, *C. R. Soc. Sci. Varsovie*, **5**, 232—237 (1912). — [2] Sur l'ensemble des points angulaires d'une courbe  $y=f(x)$ , *Bull. Acad. Sci. Cracovie*, 850—855 (1912). — [3] Un exemple élémentaire d'une fonction croissante qui a presque partout une dérivée nulle, *Giorn. Mat. Battaglini* (3), **7**, 314—334 (1916). — [4] Un lemme métrique, *Fundam. Math.*, **4**, 201—203 (1923). — [5] Sur un problème concernant les ensembles mesurables superficiellement, *ibid.*, **1**, 112—115 (1920). — [6] Démonstration de quelques théorèmes sur les fonctions mesurables, *ibid.*, **3**, 314—321 (1922). — [7] Sur la densité linéaire des ensembles plans, *ibid.*, **9**, 172—185 (1927). — [8] Sur les fonctions dérivées des fonctions discontinues, *ibid.*, **3**, 123—127 (1922). — [9] Sur une généralisation de la notion de continuité

approximative, *ibid.*, **4**, 124—127 (1923). — [10] Démonstration élémentaire du théorème sur la densité des ensembles, *ibid.*, **4**, 167—171 (1923). — [11] Démonstration d'un théorème sur les fonctions additives d'ensemble, *ibid.*, **5**, 262—264 (1924). — [12] Sur la mesurabilité des ensembles analytiques, *C. R. Soc. Sci. Varsovie*, **22**, 155—159 (1929). — [13] Sur une opération sur les familles d'ensembles, *ibid.*, **22**, 163—167 (1929). — [14] Remarque sur le théorème de M. Egoroff, *ibid.*, **20**, 84—87 (1928). — [15] Sur les constituantes des ensembles analytiques, *Fundam. Math.*, **21**, 29—34 (1933).

**Sierpiński, W. et Szpilrajn, E.** [1] Remarque sur le problème de la mesure, *Fundam. Math.*, **26**, 256—261 (1936).

**Singh, A. N.** [I] The theory and construction of non-differentiable functions, Lucknow 1935.

**Souslin, M.** [1] Sur une définition des ensembles mesurables  $B$  sans nombres transfinis, *C. R. Acad. Sci. Paris*, **164**, 88—91 (1917).

**Steinhaus, H.** [1] Sur l'existence de la dérivée, *Bull. Acad. Sci. Cracovie*, 62—65 (1919). — [2] Sur la probabilité de la convergence des séries, *Studia Math.*, **2**, 21—39 (1930).

**Stepanoff, W.** [1] Ueber totale Differenzierbarkeit, *Math. Ann.*, **90**, 318—320 (1923). — [2] Sur une propriété caractéristique des fonctions mesurables, *Rec. Math. Soc. Math. Moscou*, **30**, 487—489 (1924). — [3] Sur les conditions de l'existence de la différentielle totale, *ibid.*, **32**, 511—526 (1925).

**Szpilrajn, E.** [1] Remarques sur les fonctions complètement additives d'ensembles, *Fundam. Math.*, **22**, 303—311 (1934).

**Todd, J.** [1] Superpositions of functions (I), *Journ. London Math. Soc.*, **10**, 166—171 (1935). — [2] Superpositions of functions (II), *Proc. London Math. Soc.* (2), **41**, 433—439 (1936).

**Tonelli, L.** [1] Sulla rettificazione delle curve, *Atti Accad. Sci. Torino*, **43**, 399—416 (1908). — [2] Sull'integrazione per parti, *Atti Accad. Naz. Lincei* (5), **18**, 246—253 (1909). — [3] Successioni di curve e derivazione per serie, *ibid.*, **25**, 22—30, 85—91 (1916). — [4] Sul differenziale dell'arco di curva, *ibid.*, **25**, 207—213 (1916). — [5] Sur la quadrature des surfaces, *C. R. Acad. Sci. Paris*, **182**, 1198—1200 (1926). — [6] Sulla quadratura delle superficie, *Atti Accad. Naz. Lincei* (6), **3**, 357—363, 445—450 and 633—658 (1926). — [7] Su un polinomio d'approssimazione e l'area di una superficie, *ibid.*, **5**, 313—318 (1927). — [8] Sulle derivate esatte, *Mem. Istit. Bologna* (8), **8**, 13—15 (1930/31).

**Ulam, S.** [1] Zur Masstheorie in der allgemeinen Mengenlehre, *Fundam. Math.*, **16**, 140—150 (1930). — [2] Zum Massbegriff in Produkträumen, *Verhandl. Internat. Math. Kongress Zürich 1932*, II, 118—119.

**Vallée Poussin, Ch. J. de la** [I] Intégrales de Lebesgue. Fonctions d'ensemble. Classes de Baire, Paris 1916 (2-me éd., Paris 1936). — [1] Sur l'intégrale de Lebesgue, *Trans. Amer. Math. Soc.*, **16**, 435—501 (1915).

**Verblunsky, S.** [1] On the theory of trigonometric series, VII, *Fundam. Math.*, **23**, 193—236 (1934).

**Vitali, G.** [1] Sulle funzioni integrali, *Atti Accad. Sci. Torino*, **40**, 753—766 (1905). — [2] Una proprietà delle funzioni misurabili, *Istit. Lombardo Rend.* (2), **38**, 599—603 (1905). — [3] Sui gruppi di punti e sulle funzioni di variabili reali, *Atti Accad. Sci. Torino*, **43**, 75—92 (1908). — [4] Analisi delle funzioni a variazione limitata, *R. C. Circ. Mat. Palermo*, **46**, 388—408 (1922). — [5] Sulle funzioni continue, *Fundam. Math.*, **8**, 175—188 (1926).

**Volterra, V.** [1] Sui principii del calcolo integrale, *Giorn. Mat. Battaglini*, **19**, 333—372 (1881).

**Ward, A. J.** [1] On the differential structure of real functions, Proc. London Math. Soc. (2), 39, 339—362 (1935). — [2] On the differentiation of the additive functions of rectangles, Fundam. Math., 26, 167—182 (1936). — [3] The Perron-Stieltjes integral, Math. Zeitschr., 41, 578—604 (1936). — [4] The linear derivates and approximate linear derivates of a function of two variables, Proc. London Math. Soc. (2), 42, 266—273 (1936). — [5] On the derivation of additive functions of intervals in  $m$ -dimensional space, Fundam. Math., 28, 265—279 (1937). — [6] A sufficient condition for a function of intervals to be monotone, ibid., 29, 22—25 (1937). — [7] A certain function of rectangles, C. R. Soc. Sci. Varsovie (1937) (to appear).

**Wazewski, T.** [1] Kontinua prostowalne w związkach z funkcjami i odwzorowaniami absolutnie ciągłe, Ann. Soc. Polon. Math., 3, Suppl. 9—49 (1927).

**Wiener, N. and Young, R. C.** [1] The total variation of  $g(x+h)-g(x)$ , Trans. Amer. Math. Soc., 35, 327—340 (1933).

**Wolff, J.** [1] Ueber die Loomansche Erweiterung eines Satzes von Pompeiu, Nieuw Arch. Wiskde (2), 14, 337—339 (1925).

**Young, G. C.** [1] A note on derivates and differential coefficients, Acta Math., 37, 141—154 (1916). — [2] On the derivates of a function, Proc. London Math. Soc. (2), 15, 360—384 (1916).

**Young, G. C. and Young, W. H.** [1] On the existence of a differential coefficient, Proc. London Math. Soc. (2), 9, 325—335 (1911).

**Young, L. C.** [I] The theory of integration, Cambridge 1927. — [1] Note on the theory of measure, Proc. Cambridge Philos. Soc., 26, 88—93 (1930).

**Young R. C.** [1] On Riemann integration with respect to a continuous increment, Math. Ann., 29, 217—233 (1928). — [2] Functions of  $\mathcal{E}$  defined by addition or functions of intervals in  $n$ -dimensional formulation, ibid., 29, 171—216 (1928).

**Young, W. H.** [1] Zur Lehre der nicht abgeschlossenen Punktmengen, Ber. Verh. Sächs. Akad. Leipzig, 55, 287—293 (1903). — [2] Integration with respect to a function of bounded variation, Proc. London Math. Soc. (2), 13, 109—150 (1914). — [3] On the general theory of integration, Philos. Trans. Roy. Soc. London, 204, 221—252 (1905). — [4] On the area of surfaces, Proc. Roy. Soc. London (A), 96, 72—81 (1920). — [5] On the triangulation method of defining the area of a surface, Proc. London Math. Soc. (2), 19, 117—152 (1921). — [6] On non-absolutely convergent, not necessarily continuous integrals, ibid., 16, 175—218 (1918). — [7] The progress of mathematical analysis in the 20th century, ibid., 24, 421—434 (1926).

**Zygmund, A.** [I] Trigonometrical Series, Monografie Matematyczne, Warszawa—Lwów 1935. — [1] On the differentiability of multiple integrals, Fundam. Math., 23, 143—149 (1934) [Corrigenda, ibid., 25, 234 (1935)].

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