CHAPTER IX.

Derivates of functions of one or two real variables.

§ 1. Some elementary theorems. The first part of this chapter (§§ 1—10) is devoted to studying the various relations between the derivates of a function of a real variable. With the help of the notion of extreme differentials introduced by Haslam-Jones, certain of these relations will subsequently be extended, in the second part of the chapter (§§ 11—14), to functions of two variables.

Accordingly, the term "function" will be restricted in the first part of this chapter to mean function of one real variable.

Before proceeding to the theorems directly connected with the Lebesgue theory, we shall establish in this § some elementary results.

We first observe that *a linear set E contains at most a finite number, or an enumerable infinity, of points which are isolated on one side at least.* To fix the ideas, let A be the set of the points of E which are isolated points of E on the right. For each integer n, let A_n denote the set of the points x of A such that the interval $[x, x+1/n]$ contains no point of E other than x. Then it is plain that, for each integer k, the interval $[k/n, (k+1)/n]$ can have at most one point in common with A_n. Hence each set A_n is at most enumerable, and the same is true of the set $A=\sum_n A_n$.

We say that a finite function F assumes at a point x_0 a *strict maximum* if there exists an open interval I containing x_0 such that $F(x)<F(x_0)$ for every point $x \epsilon I$ other than x_0. By symmetry we define a *strict minimum.*

(1.1) ***Theorem.*** *Given a finite function of a real variable F, each of the following sets is at most enumerable*:

(i) *the set of the points at which the function F assumes a strict maximum or minimum*;

(ii) *the set of the points x at which*

$$\limsup_{t\to x-} F(t) > \limsup_{t\to x+} F(t) \qquad \textit{or} \qquad \liminf_{t\to x-} F(t) < \liminf_{t\to x+} F(t);$$

(iii) *the set of the points x at which*

$$\overline{F}^+(x) < \underline{F}^-(x) \qquad \textit{or} \qquad \overline{F}^-(x) < \underline{F}^+(x).$$

Proof. *re* (i). Consider the set A of the points at which, for instance, the function F assumes a strict maximum, and let A_n denote, for each positive integer n, the set of the points x such that $F(t)<F(x)$ holds for each point $t\neq x$ of the interval $(x-1/n, x+1/n)$. We see at once that each set A_n is isolated, and therefore at most enumerable. Since $A=\sum_n A_n$, it follows that the set A is at most enumerable.

re (ii). Let us consider, for definiteness, the set B of the points x at which $\limsup_{t\to x-} F(t) > \limsup_{t\to x+} F(t)$. We denote, for each pair of integers p and q, by $B_{p,q}$ the set of the points x such that

$$\limsup_{t\to x-} F(t) > p/q > \limsup_{t\to x+} F(t).$$

Clearly each point of a set $B_{p,q}$ is, for that set, an isolated point on the right. Each of the sets $B_{p,q}$ is thus at most enumerable, and, since $B=\sum_{p,q} B_{p,q}$, the same is true of the whole set B.

re (iii). Consider the set C of the points x at which $\overline{F}^+(x)<\underline{F}^-(x)$, and denote, for each pair of integers $q>0$ and p, by $C_{p,q}$ the set of the points x at which $\overline{F}^+(x)<p/q<\underline{F}^-(x)$. Write $F_{p,q}(x)=F(x)-px/q$. We find $\overline{F}^+_{p,q}(x)<0<\underline{F}^-_{p,q}(x)$ at each point $x \epsilon C_{p,q}$, and this shows that the function $F_{p,q}$ assumes a strict maximum at each point of $C_{p,q}$. By the result just established, each set $C_{p,q}$ is at most enumerable, and consequently, the same is true of the whole set C.

It is sometimes convenient (*vide*, below, § 5) to appeal to a slightly more general form of the last part of Theorem 1.1, which concerns relative derivates (cf. Chap. IV, p. 108) and which reads thus:

(1.2) ***Theorem.*** *If U and F are two finite functions of a real variable, the set of the points t at which the derivative $U'(t)>0$ (finite or infinite) exists and at which $\overline{F}^+_U(t)<\underline{F}^-_U(t)$, is at most enumerable.*

This is proved in the same way as the corresponding part of Theorem 1.1. In fact, if we denote, for every pair of integers $q>0$ and p, by $C_{p,q}$ the set of the points x at which $\overline{F}_U^+(x)<p/q<\underline{F}_U^-(x)$, we see at once that the function $F(x)-(p/q)\cdot U(x)$ assumes at each point of $C_{p,q}$ a strict maximum. Therefore each set $C_{p,q}$ is at most enumerable.

For Theorem 1.1 and its various generalizations, *vide*: A. Denjoy [1, p. 147], B. Levi [1], A. Rosenthal [1], A. Schönflies [I, p. 158], W. Sierpiński [1; 2] and G. C. Young [1]. As regards the enumerability of the set of the points at which the function assumes a strict maximum or minimum, it is easily seen that this result remains valid for functions in any separable metrical space (cf. F. Hausdorff [I, p. 363]). Mention should be made also of the elegant generalizations of Theorem 1.1, obtained successively by H. Blumberg [1], M. Schmeiser [1] and V. Jarník [3].

§ 2. Contingent of a set. We have mentioned earlier (in Chapter IV, p. 133), that certain theorems on derivates of functions may be stated as propositions concerning metrical properties of sets in Euclidean spaces. In connection with these results, we shall state in this § some definitions which begin with some well-known notions of Analytical Geometry.

By the *direction* of a half-line l in a space $\boldsymbol{R}_m$ (where $m\geqslant 2$) we shall mean the system of the m direction cosines of l. The half-line issuing from a point a and having the direction θ will be denoted by $a\theta$. The half-line issuing from a point a and containing a point $b\neq a$ will be denoted by $\overrightarrow{ab}$.

If we interpret the system of the m direction cosines of a half-line as a point in $\boldsymbol{R}_m$ (situated on the surface of a unit sphere), we may regard the set of all directions in a Euclidean space as a complete, separable, metrical space (cf. Chap. II, § 2). It is then clear what is to be understood by the terms: convergence and limit of a sequence of directions, everywhere dense set of directions, etc. We shall say further that a sequence of half-lines $\{l_n\}$ issuing from the same point a *converges* to a half-line l issuing from a, if the sequence of the directions of the half-lines l_n converges to the direction of l.

Given a set E in a space $\boldsymbol{R}_m$, a half-line l issuing from a point $a\epsilon E$ will be called an *intermediate half-tangent* of E at a, if there exists a sequence $\{a_n\}$ of points of E distinct from a, converging to a and such that the sequence of half-lines $\{\overrightarrow{aa_n}\}$ converges to l. The set of all intermediate half-tangents of a set E at a point a is termed, following

G. Bouligand [I], the *contingent* of E at a and denoted by $\mathrm{contg}_E a$ (by the contingent of E at an isolated point of E, we shall understand the empty set). A straight line passing through a which is formed of two intermediate half-tangents of E at a is called *intermediate tangent* of E at a. Similarly a hyperplane h passing through the point a is called *intermediate tangent hyperplane* of E at a, if each half-line issuing from a and situated in h is an intermediate half-tangent of E at a. In $\boldsymbol{R}_2$ the notions of intermediate tangent hyperplane and intermediate tangent are plainly equivalent.

Given in the space $\boldsymbol{R}_m$ a hyperplane h, $a_1x_1+a_2x_2+\ldots+a_mx_m=b$, (cf. Chapter III, § 2) the two *half-spaces* (*half-planes* if $m=2$) $a_1x_1+a_2x_2+\ldots+a_mx_m\geqslant b$ and $a_1x_1+a_2x_2+\ldots+a_mx_m\leqslant b$, into which h divides $\boldsymbol{R}_m$, will be termed *sides* of the hyperplane h. In the case in which h is an intermediate tangent hyperplane of a set E at a point a and in which, further, the contingent $\mathrm{contg}_E a$ is wholly situated on one side of h, the side opposite to the latter is called *empty side* of h and the hyperplane h is termed *extreme tangent hyperplane* of E at a. The two sides of h may, of course, both be empty at the same time, and this occurs if the contingent of E at a coincides with the set of all half-lines issuing from a which lie in the hyperplane h itself. The hyperplane h is then termed *unique tangent hyperplane*, or simply, *tangent hyperplane*, of E at a.

For simplicity of wording, we shall restrict ourselves in the sequel to the case of sets situated either in the plane $\boldsymbol{R}_2$ or in the space $\boldsymbol{R}_3$. Needless to say, the extension to any space $\boldsymbol{R}_m$ presents no essential difficulty (an elegant statement, which sums up the results of §§ 3 and 13 of this chapter and which is valid for an arbitrary space $\boldsymbol{R}_m$, will be found in the note of F. Roger [2]).

As usual, the hyperplanes in $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ are termed *straight lines* and *planes* respectively. Moreover, in the case of plane sets we shall speak of *tangent* (intermediate, extreme, unique) in place of tangent straight line (intermediate, extreme, unique).

We shall discuss the case of the plane (§ 3) and that of the space (§ 13) separately, although the proofs of the fundamental theorems 3.6 and 13.7 which correspond to these two cases, are wholly analogous. The proof of the former is, however, more elementary, whereas the latter requires some subsidiary considerations connected with the notion of total differential (cf. below § 12).

§ 3. Fundamental theorems on the contingents of plane sets. For brevity, we shall say that the contingent of a plane set E at a point a is the *whole plane*, if it includes all half-lines issuing from this point. Similarly the contingent of E at a point a will be said to be a *half-plane*, if E has at this point an extreme tangent l and if $\mathrm{contg}_E a$ consists of all the half-lines issuing from a and situated on one side of l.

We shall see in this § that, given any plane set E, at each point a of E except at most in a subset of zero length, either 1° the contingent of E is the whole plane, or 2° it is a half-plane, or finally 3° the set E has a unique tangent. This result (together with the more precise result contained in Theorem 3.6) was first stated by A. Kolmogoroff and J. Verčenko [1;2]. It was rediscovered independently, and generalized to sets situated in any space $\boldsymbol{R}_m$, by F. Roger [2]. The proofs, together with some interesting applications of the theorem of Kolmogoroff and Verčenko, will be found in the notes of U. S. Haslam-Jones [2;3]. (For the first part of Theorem 3.6 cf. also A. S. Besicovitch [4].)

A finite function of a real variable F, defined on a linear set E, is said to fulfil the *Lipschitz condition* on E, if there exists a finite number N such that $|F(x_2)-F(x_1)|\leqslant N\cdot|x_2-x_1|$ whenever x_1 and x_2 are points of E. As we verify at once, we then have $\Lambda\{\mathrm{B}(F;E)\}\leqslant(N+1)\cdot|E|$ (for the notation, cf. Chap. II, § 8, and Chap. III, § 10). Thus, *if a function F fulfils the Lipschitz condition on a set E of finite [zero] outer measure, its graph* $\mathrm{B}(F;E)$ *on E is of finite [zero] length.*

It is also easy to see that any function which fulfils the Lipschitz condition on a linear set E, can be continued outside E so as to fulfil the Lipschitz condition on the whole straight line $\boldsymbol{R}_1$ and so as to be linear on each interval contiguous to $\bar{E}$.

(3.1) **Lemma.** *Let R be a plane set, θ a fixed direction and P the set of the points a of R at which* $\mathrm{contg}_R\, a$ *contains no half-line of direction θ. Then* (i) *the set P is the sum of a sequence of sets of finite length, and* (ii) *at each point a of P, except at most at those of a subset of length zero, the set R has an extreme tangent such that the side of the tangent containing the half-line $a\theta$ is its empty side.*

In the particular case in which θ is the direction of the positive semi-axis of y, the set P is expressible as the sum of an enumerable infinity of sets each of which is the graph of a function on a set on which the function fulfils the Lipschitz condition.

Proof. By changing, if necessary, the coordinate system, we may suppose in both parts of the theorem that θ is the direction of the positive semi-axis of y. Let us denote, for every positive integer n, by P_n the set of the points (x,y) of P such that the inequalities $|x'-x|\leqslant 1/n$ and $|y'-y|\leqslant 1/n$ imply $y'-y\leqslant n\cdot|x'-x|$ for every point (x',y') of R. Since there is no point a of P at which the contingent of R contains the half-line with the direction of the positive semi-axis of y, it is clear that $P=\sum_n P_n$. Let us now express each P_n as the sum of a sequence $\{P_{n,k}\}_{k=1,2,\ldots}$ of sets with diameters less than $1/n$. We shall then have $|y_2-y_1|\leqslant n\cdot|x_2-x_1|$ for every pair of points, (x_1,y_1) and (x_2,y_2), belonging to the same set $P_{n,k}$. Let $Q_{n,k}$ be the orthogonal projection of $P_{n,k}$ on the axis of x. We easily see that each point of $Q_{n,k}$ is the projection of a single point of $P_{n,k}$. Consequently, the set $P_{n,k}$ may be considered as the graph of a function $F_{n,k}$ on $Q_{n,k}$. Moreover we have $|F_{n,k}(x_2)-F_{n,k}(x_1)|\leqslant n\cdot|x_2-x_1|$ for each pair of points x_1 and x_2 of $Q_{n,k}$, i. e. the function $F_{n,k}$ fulfils the Lipschitz condition on $Q_{n,k}$ and therefore (cf. above p. 264) each set $P_{n,k}=\mathrm{B}(F_{n,k};Q_{n,k})$ is of finite length. Thus, since $P=\sum_{n,k}P_{n,k}$, we obtain the required expression of the set P as the sum of an at most enumerable infinity of sets of finite length, which are at the same time graphs of functions fulfilling the Lipschitz condition on sets situated on the x-axis.

It remains to examine the existence of an extreme tangent to the set R at the points of P. For this purpose, let us keep fixed for the moment a pair of positive integers n and k, and let $\widetilde{Q}_{n,k}$ be the set of the points of $Q_{n,k}$ which are points of outer density for $Q_{n,k}$ and at which the function $F_{n,k}$ is derivable with respect to the set $Q_{n,k}$. Since the set $Q_{n,k}-\widetilde{Q}_{n,k}$ is of measure zero (cf. Theorem 4.4, Chap. VII) and since the function $F_{n,k}$ fulfils the Lipschitz condition on $Q_{n,k}$, it follows that $\Lambda[\mathrm{B}(F_{n,k};Q_{n,k}-\widetilde{Q}_{n,k})]=0$.

We need, therefore, only prove that R has an extreme tangent at each point of the set $\mathrm{B}(F_{n,k};\widetilde{Q}_{n,k})$ and that, further, the side of this tangent which contains a half-line in the direction of the positive semi-axis of y is its empty side.

Let (ξ_0,η_0) be any point of $\mathrm{B}(F_{n,k};\widetilde{Q}_{n,k})$, and A_0 the derivative of $F_{n,k}$ at ξ_0 with respect to the set $Q_{n,k}$. Let ε be a positive number less than 1. Since ξ_0 is a point of outer density for the set $Q_{n,k}$, we can associate with each point (ξ,η), sufficiently close to (ξ_0,η_0), a point $\xi'\epsilon Q_{n,k}$ such that

(3.2) $|\xi'-\xi_0|\leqslant|\xi-\xi_0|$ and (3.3) $|\xi'-\xi|\leqslant\varepsilon\cdot|\xi-\xi_0|$

(for otherwise, the outer lower density of $Q_{n,k}$ at ξ_0 would not exceed $1-\varepsilon$).

Remembering now that $\eta_0=F_{n,k}(\xi_0)$, let us write for brevity

$$D_{n,k}(\xi')=F_{n,k}(\xi')-\eta_0-A_0\cdot(\xi'-\xi_0).$$

We shall have

$$\eta-\eta_0-A_0\cdot(\xi-\xi_0)=D_{n,k}(\xi')+[\eta-F_{n,k}(\xi')]+A_0\cdot(\xi'-\xi). \tag{3.4}$$

Now suppose that the point (ξ,η) belongs to R and that $|\xi-\xi_0|\leqslant 1/2n^2$ and $|\eta-\eta_0|\leqslant 1/2n^2$. By (3.3), we have $|\xi'-\xi|\leqslant 1/n$, and, by (3.2), $|F_{n,k}(\xi')-\eta_0|\leqslant n\cdot|\xi'-\xi_0|\leqslant n\cdot|\xi-\xi_0|\leqslant 1/2n$, so that $|F_{n,k}(\xi')-\eta|\leqslant 1/n$. Since the point $(\xi',F_{n,k}(\xi'))$ belongs to $P_{n,k}\subset P_n$, it follows from the definition of the set P_n that $\eta-F_{n,k}(\xi')\leqslant n\cdot|\xi-\xi'|$, and using (3.3) again, we derive from (3.4) that

$$\begin{aligned}\eta-\eta_0-A_0\cdot(\xi-\xi_0)&\leqslant|D_{n,k}(\xi')|+(n+|A_0|)\cdot|\xi'-\xi|\leqslant\\ &\leqslant|D_{n,k}(\xi')|+\varepsilon\cdot(n+|A_0|)\cdot|\xi-\xi_0|.\end{aligned} \tag{3.5}$$

Now as ξ, and therefore ξ', tends to ξ_0, the ratio $D_{n,k}(\xi')/(\xi'-\xi_0)$ tends to zero; the same is therefore true, on account of (3.3), of the ratio $D_{n,k}(\xi')/(\xi-\xi_0)$. Consequently, since ε is an arbitrary positive number less than 1, it follows from (3.5) that the upper limit of the ratio $[\eta-\eta_0-A_0\cdot(\xi-\xi_0)]/|\xi-\xi_0|$, as the point $(\xi,\eta)\,\epsilon\, R$ tends to (ξ_0,η_0), is non-positive. Further, since the line $y-\eta_0=A_0\cdot(x-\xi_0)$ is plainly an intermediate tangent of the set $\mathrm{B}(F_{n,k};Q_{n,k})\subset R$ at the point (ξ_0,η_0), we see that this line is an extreme tangent of the set R at this point and that the half-plane $y-\eta_0\geqslant A_0\cdot(x-\xi_0)$, which contains the half-line issuing from (ξ_0,η_0) in the direction of the positive semi-axis of y, is an empty side of this tangent.

This completes the proof.

(3.6) **Theorem.** *Given a plane set R, let P be a subset of R at no point of which the contingent of R is the whole plane. Then* (i) *the set P is the sum of an enumerable infinity of sets of finite length and* (ii) *at every point of P, except at those of a set of length zero, either the set R has a unique tangent or else the contingent of R is a half-plane.*

Proof. Let $\{\theta_n\}$ be an everywhere dense sequence of directions in the plane and, for each positive integer n, let P_n denote the set of the points of P at which the contingent of R does not contain the half-line of direction θ_n. We clearly have $P=\sum_n P_n$, and by the preceding lemma each set P_n, and therefore the whole set P, is the sum of a sequence of sets of finite length. Further, the same lemma shows that the set R has an extreme tangent at every point of P, except at most in a set of length zero.

Now let Q be the set of the points of P at which 1° there exists an extreme tangent which is not a unique tangent and 2° the contingent of R is not a half-plane. For each positive integer n, let Q_n denote the set of the points b of Q such that the half-line $b\theta_n$ is situated on the non-empty side of the extreme tangent of R at b, but does not belong to $\mathrm{contg}_R\, b$. Plainly $Q=\sum_n Q_n$. Now, by the preceding lemma, for every point $b\,\epsilon\, Q_n$, except at most those of a set of length zero, the half-line $b\theta_n$ is situated on the empty side of the extreme tangent at b. It follows that all the sets Q_n, and therefore also the whole set Q, are of length zero. Hence, at every point of P, except perhaps those of a subset of length zero, either there is a unique tangent or the contingent at this point is a half-plane.

(3.7) **Theorem.** *Given a plane set R, let P be a subset of R at every point of which the set R has an extreme tangent parallel to a fixed straight line D. Then the orthogonal projection of P on the line at right angles to D is of linear measure zero.*

Proof. We may clearly assume that the line D coincides with the axis of x. Let S and T denote, respectively, the sets of the points (ξ,η) of P for which the half-planes $y\geqslant\eta$ and $y\leqslant\eta$ are respectively the empty sides of the extreme tangents. Consider the former of these sets. By Lemma 3.1, the set S is the sum of a sequence of the sets $\mathrm{B}(F_n;Q_n)$, where the Q_n are sets on the x-axis and the F_n functions fulfilling the Lipschitz condition on these sets, respectively. We may suppose (cf. p. 264) that each function F_n is defined, and fulfils the Lipschitz condition, on the whole x-axis and is linear on the intervals contiguous to the set $\bar{Q}_n$.

This being so, we easily see that, for every n, the relation $\underline{F}_n^-(x)\geqslant 0\geqslant\overline{F}_n^+(x)$ holds at each point x of Q_n which is not an isolated point on any side for Q_n, i.e. (cf. §1, p. 260) at all the points of Q_n, except at most those of an enumerable set. Thus by Lemma 6.3, Chap. VII, we have $|F_n[Q_n]|=0$ for every positive integer n, and since the projection of S on the y-axis coincides with the sum of the sets $F_n[Q_n]$, this projection is itself of measure zero. By symmetry, the same is true of the projection of the set T on the y-axis, and this completes the proof.

As an immediate corollary, we derive from Theorem 3.6 the following proposition:

(3.8) *Given a plane set R, let P be a subset of R at each point a of which there exists a straight line through a which contains no half-line of* $\mathrm{contg}_R\, a$. *Then the set R has a unique tangent at all the points of P except at most those of a subset of length zero.*

This result can be easily extended to the space (cf. F. Roger [2]) as follows:

(3.9) *Given a set R in the space $\boldsymbol{R}_3$, let P be a subset of R at each point a of which there exists a plane through a which contains no half-line of* $\mathrm{contg}_R\, a$. *Then* (i) *the set P is the sum of an enumerable infinity of sets of finite length and* (ii) *the set R has a unique tangent at all the points of P except at most those of a subset of length zero.*

Proof. Let $\{\theta_n\}$ be an everywhere dense sequence of directions in the space $\boldsymbol{R}_3$. For each positive integer h, let $P_{n,h}$ denote the set of the points a of P such that $|\cos(\overrightarrow{ab}, \theta_n)|>1/h$ for every point b of R distant less than $1/h$ from a. We express each set $P_{n,h}$ as the sum of a sequence $\{P_{n,h,k}\}_{k=1,2,\ldots}$ of sets of diameter less than $1/h$. We then have

$$P=\sum_{n,h} P_{n,h}=\sum_{n,h,k} P_{n,h,k}. \tag{3.10}$$

Keeping, for the moment, the indices n, h, k fixed, we choose a new system of rectangular coordinates, taking for the positive semi-axis of z the half-line of direction θ_n. Let α, β and γ be, respectively, the three positive semi-axes of the new coordinate-system. For any set, or any point, Q, we denote by $Q^{(\alpha)}$, $Q^{(\beta)}$ and $Q^{(\gamma)}$ the orthogonal projections of Q on the planes $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$, normal to the axes α, β and γ respectively.

We have $|\cos(\overrightarrow{ab}, \gamma)|>1/h$ whenever $a\in P_{n,h,k}$, $b\in R$ and $0<\varrho(a,b)<1/h$. It follows at once that there is no point $P^{(\alpha)}_{n,h,k}$ at which the contingent of the plane set $R^{(\alpha)}$ contains a half-line at right-angles to the semi-axis γ. Hence, by (3.8), the set $P^{(\alpha)}_{n,h,k}$ is the sum of an at most enumerable infinity of sets of finite length, and the set $R^{(\alpha)}$ has a unique tangent at all the points of $P^{(\alpha)}_{n,h,k}$ except at most those of a set $M_{n,h,k}$ of length zero. Similarly, the set $R^{(\beta)}$ has a unique tangent at all the points of $P^{(\beta)}_{n,h,k}$ except at most those of a set $N_{n,h,k}$ of length zero. It follows that the set R has a unique tangent at each point a of $P_{n,h,k}$, except perhaps when $a^{(\alpha)}\in M_{n,h,k}$ or when $a^{(\beta)}\in N_{n,h,k}$. Now we easily see that the two ratios $\varrho(a,b)/\varrho(a^{(\alpha)},b^{(\alpha)})$ and $\varrho(a,b)/\varrho(a^{(\beta)},b^{(\beta)})$ remain bounded (by h) when a and b belong to the set $P_{n,h,k}$. It follows that the set of the exceptional points of $P_{n,h,k}$ at which the set R has no unique tangent is, with the sets $M_{n,h,k}$ and $N_{n,h,k}$, of length zero. For the same reason, since the set $P^{(\alpha)}_{n,h,k}$ is the sum of an at most enumerable infinity of sets of finite length, so is also the set $P_{n,h,k}$. This completes the proof, on account of the relation (3.10).

§ 4. Denjoy's theorems. We shall apply the results of the preceding § to establish certain important relations, valid almost everywhere, which connect the Dini derivates of any function whatsoever, and which are known by the name of *the Denjoy relations*. For simplicity of wording, we agree to call *opposite derivates* of a function F at a point x_0 the Dini derivates $\overline{F}^+(x_0)$ and $\underline{F}^-(x_0)$, or else $\underline{F}^+(x_0)$ and $\overline{F}^-(x_0)$.

We shall begin with some preliminary remarks.

Let F be a finite function defined in a neighbourhood J of a point x_0 and let B denote the graph of F on J. It is clear that if the function F is derivable at the point x_0, the set B has at $(x_0,F(x_0))$ a unique tangent not parallel to the axis of y. Similarly, if two opposite derivates of F are finite and equal at x_0, the set B has at $(x_0,F(x_0))$ an extreme tangent $y-F(x_0)=k\cdot(x-x_0)$, whose angular coefficient k is equal to the common value of these derivates. Conversely, if at the point $(x_0,F(x_0))$ the set B has the extreme tangent $y-F(x_0)=k\cdot(x-x_0)$ where $k\neq\infty$, then 1° $\overline{F}^+(x_0)=\underline{F}^-(x_0)=k$ in the case in which the half-plane $y-y_0\geqslant k\cdot(x-x_0)$ is an empty side of this tangent and $\limsup\limits_{x\to x_0} F(x)\leqslant F(x_0)$, and 2° $\underline{F}^+(x_0)=\overline{F}^-(x_0)=k$ in the case in which the half-plane $y-y_0\leqslant k\cdot(x-x_0)$ is an empty side and $\liminf\limits_{x\to x_0} F(x)\geqslant F(x_0)$.

In the enunciations of the theorems which follow, we shall frequently be concerned with exceptional sets E, connected with a function F and subject to the condition $\Lambda\{\mathrm{B}(F;E)\}=0$. This condition evidently implies both $|E|=0$ and $|F[E]|=0$, since the sets E and $F[E]$ are merely the orthogonal projections of the set $\mathrm{B}(F;E)$ on the x- and y-axes, respectively.

(4.1) ***Theorem.*** *If at each point of a set E one of the extreme unilateral derivates of a function F is finite, this derivate is equal to its opposite derivate at every point of E except perhaps at the points of a set E_1 of measure zero such that* $\Lambda\{\mathrm{B}(F;E_1)\}=0$.

Proof. We may clearly suppose that the same derivate, $\overline{F}^+(x)$ say, is the one which is finite throughout E. We thus have $\limsup\limits_{x\to x_0+} F(x)\leqslant F(x_0)$ at every point $x_0\in E$ and, on account of Theorem 1.1 (ii), we may even suppose that $\limsup\limits_{x\to x_0} F(x)\leqslant F(x_0)$ at every point x_0 of E. Now, when $x_0\in E$, the contingent of $\mathrm{B}(F;E)$ at the point $(x_0,F(x_0))$ contains no half-line situated in the half-plane $x\geqslant x_0$ and having angular coefficient exceeding $\overline{F}^+(x_0)$. Therefore, by Theorem 3.6, the set $\mathrm{B}(F;E)$ has an extreme tangent at each of its points $(x_0, F(x_0))$, except for those of a subset B_1 of length zero, and this tangent has the half-plane $y-y_0 \geqslant \overline{F}^+(x_0)\cdot(x-x_0)$ for its empty side. Hence, denoting by E_1 the orthogonal projection of B_1 on the x-axis, we see, from the remarks made at the beginning of this §, that at every point x of the set $E-E_1$ the derivates $\overline{F}^+(x)$ and $\underline{F}^-(x)$ are equal. This completes the proof since $\Lambda\{\mathrm{B}(F;E_1)\}=\Lambda(B_1)=0$.

(4.2) ***Theorem.*** *If at each point of a set E a finite function F has either two finite Dini derivates on the same side, or else a finite extreme bilateral derivate ($\overline{F}(x)$ or $\underline{F}(x)$), then the function F is almost everywhere derivable in E; moreover, denoting by E_0 the set of the points x of E at which the function F is not derivable, we have $\Lambda\{\mathrm{B}(F;E_0)\}=0$.*

Proof. It will suffice to consider separately the following two cases:

1º The function F has two Dini derivates on the same side finite at each point of E. We then have, by Theorem 4.1,

$$\overline{F}^+(x)=\underline{F}^-(x) \quad \textit{and} \quad \underline{F}^+(x)=\overline{F}^-(x) \tag{4.3}$$

at each point x of E, except perhaps those of a set E_0 such that $\Lambda\{\mathrm{B}(F;E_0)\}=0$. But the relations (4.3) imply the equality of all four Dini derivates at the point x, and since two of them are finite, by hypothesis, at each point x of E, the function F is derivable throughout $E-E_0$.

2º The function F has an extreme bilateral derivate finite at each point of E. By applying twice over Theorem 4.1, and making use of the obvious relations $\overline{F}^+(x) \geqslant \underline{F}^+(x)$ and $\overline{F}^-(x) \geqslant \underline{F}^-(x)$, we see that the four Dini derivates are finite and equal at each point of E, except perhaps at those of a set on which the graph of F is of zero length. This completes the proof.

Theorem 4.2 (in a slightly less complete form, it is true) has already been mentioned in Chap. VII, p. 236, as a corollary of Theorems 10.1 and 10.5, Chap. VII. We have also stated that (as a consequence of these same theorems) the set of the points at which a function has a unique derivative (even a unilateral derivative) infinite, is necessarily of measure zero. We can now extend this result by taking the modulus, as follows:

(4.4) ***Theorem.*** *For any finite function F, the set of the points x at which $\lim\limits_{h\to 0+}|F(x+h)-F(x)|/h=+\infty$, is of measure zero.*

Proof. Denoting the set of the points in question by A, we see at once that the graph of the function F has at every point of the set $\mathrm{B}(F;A)$, except perhaps at those of a set of length zero, an extreme tangent parallel to the y-axis. Thus, by Theorem 3.7, the set A, which is the projection of the set $\mathrm{B}(F;A)$ on the x-axis, is of measure zero and this completes the proof.

It results, in particular, from Theorems 4.1, 4.2 and 4.4 that *the Dini derivates of any finite function F satisfy one of the following four relations at almost every point x*: 1º $\overline{F}^+(x)=\overline{F}^-(x)=+\infty$, $\underline{F}^+(x)=\underline{F}^-(x)=-\infty$; 2º $\overline{F}^+(x)=\underline{F}^-(x)\neq\infty$, $\underline{F}^+(x)=-\infty$, $\overline{F}^-(x)=+\infty$, 3º $\underline{F}^+(x)=\overline{F}^-(x)\neq\infty$; $\overline{F}^+(x)=+\infty$, $\underline{F}^-(x)=-\infty$; 4º $\overline{F}^+(x)=\underline{F}^+(x)=\overline{F}^-(x)=\underline{F}^-(x)\neq\infty$. For direct proofs of this theorem, which was established first by Denjoy for continuous functions and then generalized to arbitrary functions, *vide*: A. Denjoy [1], G. C. Young [2], F. Riesz [7], J. Ridder [4], J. C. Burkill and U. S. Haslam-Jones [1], and H. Blumberg [2] (cf. also A. N. Singh [1]). A further discussion of the Denjoy relations will be found in the notes of V. Jarnìk [1] (for functions of one variable) and of A. S. Besicovitch [6] and A. J. Ward [4] (for functions of two variables). For Theorem 4.4 see S. Saks and A. Zygmund [1] (cf. also S. Banach [1]).

A part of the Denjoy relations has recently been generalized to differential coefficients of higher orders; see the important memoirs of A. Denjoy [9], J. Marcinkiewicz and A. Zygmund [1], and J. Marcinkiewicz [2].

We may now supplement Lemma 6.3, Chap. VII, by the following result:

(4.5) ***Theorem.*** *Let M be a finite number and F a finite function such that $|\overline{F}^+(x)| \leqslant M$ at every point x of a set E. Then $|F[E]| \leqslant M\cdot|E|$.*

Proof. Let E_1 denote the set of the points x of E at which $\underline{F}^-(x)\neq\overline{F}^+(x)$. By Theorem 4.1, we have $\Lambda\{\mathrm{B}(F;E_1)\}=0$ and therefore, $|F[E_1]|=0$. On the other hand, since $|\underline{F}^-(x)|=|\overline{F}^+(x)|\leqslant M$ at each point $x \in E-E_1$, it follows from Lemma 6.3, Chap. VII, that $|F[E-E_1]| \leqslant M\cdot|E|$, and this completes the proof.

An immediate consequence is the following criterion for a function to fulfil Lusin's condition (N) (Chap. VII, § 6):

(4.6) ***Theorem.*** *If a finite function F has at each point x of a set E a finite Dini derivate, the function necessarily fulfils the condition* (N) *on E.*

Proof. It is enough to show that if at each point x of a set H of measure zero the function F has one of its Dini derivates, $\overline{F}^+$ say, finite, then $|F[H]|=0$. For this purpose, let H_n be the set of the points $x \in H$ at which $|\overline{F}^+(x)| \leqslant n$. We have, by Theorem 4.5, $|F[H_n]| \leqslant n\cdot|H_n|=0$ for each positive integer n, and hence $|F[H]|=0$.

It is easy to see that the hypotheses of Theorem 4.5 imply that $\Lambda\{B(F;E)\}\leqslant(M+1)\cdot|E|$. This remark enables us to complete Theorem 6.5 of Chap. VII, as follows: *If the derivate $\overline{F}^+$ of a finite function F is finite at every point of a measurable set E, except at those of an enumerable subset, then the function F, together with its derivate $\overline{F}^+$, is measurable on E and we have*

$$|F[E]|\leqslant\int_E|\overline{F}^+(x)|\,dx \quad \text{and} \quad \Lambda\{B(F;E)\}=\int_E\{1+[\overline{F}^+(x)]^2\}^{1/2}\,dx.$$

We may note also the following consequence of Theorem 4.5: *If one of the four Dini derivates of a function F vanishes at every point of a set E, then $|F[E]|=0$.*

For functions $F(x)$ which are continuous, or more generally *continuous in the Darboux sense* (i.e. assume in each interval $[a,b]$ all the values between $F(a)$ and $F(b)$), we deduce at once the following result:

(4.7) ***Theorem.*** *If F is a finite function, continuous in the Darboux sense on an interval I, and if at each point of this interval, except those of an enumerable set, one at least of the four Dini derivates is equal to zero, then the function F is constant on I.*

*** § 5. Relative derivates.** The Denjoy relations can be extended in various ways to relative derivates of a function with respect to another function. Let us remark that, in accordance with the definition given in Chap. IV, p. 108, the extreme derivates of any function with respect to a finite function U are determined at each point which belongs to no interval of constancy of the function U; consequently, the set of values taken by the function U at the points at which the extreme derivates with respect to U remain indeterminate is at most enumerable.

In the sequel it will be useful to employ the notation adopted in Chap. IV, § 8. Let us recall in particular, that if C is a curve given by the equations $x=X(t)$, $y=Y(t)$, its graph on a linear set E (i.e. the set of the points $(X(t),Y(t))$ for $t\in E$) is denoted by $B(C;E)$.

(5.1) ***Lemma.*** *If C is a curve given by the equations $x=U(t)$, $y=F(t)$, the set E of the points t at which $\overline{F}_U(t)<+\infty$, may be expressed as the sum of a sequence of sets $\{E_n\}$ such that*

(∇) *for every n and for every open interval I of length less than $1/n$, the set $B(C;I)$ has a unique tangent at every point of $B(C;I\cdot E_n)$ except those of a set of length zero.*

Proof. Let us denote, for each positive integer n, by E_n the set of the points t such that, provided that the differences $F(t')-F(t)$ and $U(t')-U(t)$ do not vanish simultaneously, the inequality $|t'-t|\leqslant 1/n$ implies $[F(t')-F(t)]/[U(t')-U(t)]\leqslant n$. We see at once that, for any open interval I of length less than $1/n$, the contingent of $B(C;I)$ at a point of $B(C;I\cdot E_n)$ cannot contain half-lines of angular coefficient greater than n, and can be, therefore, neither a whole plane, nor a half-plane (cf. § 3, p. 264). The property (∇) of the sequence $\{E_n\}$ thus appears as a direct consequence of Theorem 3.6.

(5.2) ***Theorem.*** *If U and F are continuous functions and we have $\overline{F}_U(t)<+\infty$ at every point t of a set E, then there is a finite derivative $F'_U(t)$ at each point t of E, except at the points of a set H such that $|U[H]|=0$.*

Proof. Let C denote the curve given by the equations $x=U(t)$, $y=F(t)$. On account of Lemma 5.1, the set E is expressible as the sum of a sequence of sets $\{E_n\}$ which fulfil the condition (∇) of this lemma. Keeping fixed, for the moment, a positive integer n, let us consider an open interval I of length less than $1/n$. Let $B_n(I)$ denote the set of the points of $B(C;I\cdot E_n)$ at which the graph of the curve C on I either has no unique tangent, or else has a unique tangent parallel to the y-axis. Further, let $\widetilde{B}_n(I)$ be the projection of the set $B_n(I)$ on the x-axis. On account of the condition (∇) and Theorem 3.7, we have $|\widetilde{B}_n(I)|=0$. Now since U and F are continuous, it is clear that the derivative $F'_U(t)$ exists and is finite at each point $t\in I\cdot E_n$, provided that $U(t)$ does not belong to the set $\widetilde{B}_n(I)$. Hence, I being any open interval of length less than $1/n$, this derivative exists and is finite at each point $t\in E_n$, except at most at the points of a set H_n such that $|U[H_n]|=0$. This completes the proof, since $E=\sum_n E_n$.

(5.3) ***Theorem.*** *If U is a continuous function and F any finite function for which $\overline{F}_U(t)=0$ at each point t of a set E, then $|F[E]|=0$.*

Proof. Let C denote, as in the proof of the preceding theorem, the curve $x=U(t)$, $y=F(t)$, and let E be expressed as the sum of a sequence of sets $\{E_n\}$ subject to the condition (∇) of Lemma 5.1. Keeping fixed, for a moment, a positive integer n, let us consider any open interval I of length less than $1/n$. At each point of $B(C;I\cdot E_n)$, except those of a set of length zero, the set $B(C;I)$ then has a unique tangent, and since the function U is continuous and $\overline{F}_U(t)=0$ at each point $t\in E$, this tangent is parallel to the x-axis. It follows, by Theorem 3.7, that the set $F[I\cdot E_n]$, which coincides with the projection of the set $B(C;I\cdot E_n)$ on the y-axis, is of measure zero. Since I is any interval of length less than $1/n$, it follows that $|F[E_n]|=0$ for each positive integer n, and finally that $|F[E]|=0$.

The hypothesis of continuity of the function U is essential for the validity of Theorems 5.2 and 5.3 (the hypothesis of continuity of the function F, which is not required in Theorem 5.3, may, however, be removed also from Theorem 5.2). Let $F(t)=-t$ identically, and let $U(t)=t$ for irrational values and $U(t)=t+1$ for rational values of t. Denoting by E the set of irrational points of the interval $(0,1)$, we shall have at each point t of this set

$$\overline{F}_U(t)=\overline{F}_U^+(t)=\overline{F}_U^-(t)=0 \quad \text{and} \quad \underline{F}_U(t)=\underline{F}_U^+(t)=\underline{F}_U^-(t)=-1.$$

Nevertheless $|U[E]|=|F[E]|=1$. On the other hand, the hypothesis of continuity of the function U may be removed from Theorem 5.3, if we replace the condition $\overline{F}_U(t)=0$ by the more restrictive condition $F'_U(t)=0$. To see this, we shall first establish an elementary lemma.

(5.4) ***Lemma.*** *If U is a finite function on a set E, there exists a set $T\subset E$ such that the set $U[T]$ is at most enumerable and such that each point $\tau\in E-T$ is the limit of a sequence of points $\{t_i\}$ of E which fulfils the conditions* (i) $t_i>\tau$ *and* $U(t_i)\neq U(\tau)$ *for each* $i=1,2,\ldots$ *and* (ii) $\lim\limits_i U(t_i)=U(\tau)$.

Proof. Let T be the set of the points $\tau\in E$ none of which is the limit of a sequence $\{t_i\}$ of points of E subject to the conditions (i) and (ii) of the lemma. Let us denote, for each positive integer k, by T_k the set of the points τ of T for which there is no point $t\in E$ such that both $0<t-\tau<1/k$ and $0<|U(t)-U(\tau)|<1/k$. We have $T=\sum\limits_k T_k$. Plainly the function U cannot, on any portion of T_k of diameter less than $1/k$, assume two distinct values differing by less than $1/k$. It follows that each set $U[T_k]$ is at most enumerable, and the same is therefore true of the whole set $U[T]$.

(5.5) ***Theorem.*** *If U and F are any finite functions and $F'_U(t)=0$ or, more generally $\overline{F}_U^+(t)=\underline{F}_U^+(t)=0$, at each point t of a set E, then $|F[E]|=0$.*

Proof. Let C be the curve $x=U(t)$, $y=F(t)$, and let E_n denote, for each positive integer n, the set of the points t of E such that the inequality $0\leqslant t'-t\leqslant 1/n$ implies $|F(t')-F(t)|\leqslant|U(t')-U(t)|$ whatever be the point t'. We can express each set E_n as the sum of a sequence $\{E_{n,k}\}_{k=1,2,\ldots}$ of sets of diameter less than $1/n$.

Let us keep n and k fixed for the moment. It is clear that the contingent of the set $\mathrm{B}(C;E_{n,k})$ cannot, at any point of this set, contain a half-line whose angular coefficient exceeds the number 1. Consequently, denoting by $B_{n,k}$ the set of the points of $\mathrm{B}(C;E_{n,k})$ at which the set $\mathrm{B}(C;E_{n,k})$ has no unique tangent, we see from Theorem 3.6 that $\Lambda(B_{n,k})=0$.

Now the set $E_{n,k}$ contains, by Lemma 5.4, a subset $T_{n,k}$ such that $U[T_{n,k}]$ is at most enumerable and that each point $\tau\in E_{n,k}-T_{n,k}$ is the limit of a sequence $\{t_i\}$ of points of $E_{n,k}$ which fulfils the conditions (i) and (ii) of this lemma. Hence, the relations $\overline{F}_U^+(t)=\underline{F}_U^+(t)=0$ being satisfied, by hypothesis, at each point $t\in E_{n,k}$, the set $\mathrm{B}(C;E_{n,k})$ has a unique tangent parallel to the x-axis at each point of the set $\mathrm{B}(C;E_{n,k}-T_{n,k})-B_{n,k}$. Since $\Lambda(B_{n,k})=0$, it thus follows from Theorem 3.7 that the set $F[E_{n,k}-T_{n,k}]$, which coincides with the projection of the set $\mathrm{B}(C;E_{n,k}-T_{n,k})$ on the y-axis, is of measure zero. The same is therefore true of the set $F[E_{n,k}]$, for the set $F[T_{n,k}]$ is, with $U[T_{n,k}]$, at most enumerable. It follows at once that $|F[E]|=0$, since $E=\sum\limits_{n,k}E_{n,k}$.

We may mention an application of Theorems 5.3 and 5.5, which is connected with the following theorem of H. Lebesgue [II, p. 299]: *If the derivative of a continuous function F, with respect to a function U of bounded variation, is identically zero, then the function F is a constant.* J. Petrovski [1] and R. Caccioppoli [1] extended this theorem, in the case when the function U is continuous, by removing the hypothesis of bounded variation for U. At the same time, Petrovski remarked that it was sufficient for the validity of the theorem to suppose that the relation $F'_U(t)=0$ holds everywhere except in an enumerable set.

It is easy to see that this result is contained in each of the separate theorems 5.3 and 5.5. These theorems actually enable us to state the result of Petrovski and Caccioppoli in two slightly more general forms. Thus:

1° *Suppose that U and F are continuous functions and that at each point t, except at most those of an enumerable set, one at least of the four relations $\overline{F}_U(t)=0$, $\underline{F}_U(t)=0$, $\overline{F}_U^+(t)=\underline{F}_U^+(t)=0$ or $\overline{F}_U^-(t)=\underline{F}_U^-(t)=0$ is fulfilled. Then the function F is a constant.*

2° *Suppose that U is any finite function and F a continuous function, and let one of the relations $\overline{F}_U^+(t)=\underline{F}_U^+(t)=0$ or $\overline{F}_U^-(t)=\underline{F}_U^-(t)=0$ hold at each point t except at most those of an enumerable set. Then the function F is a constant.*

We observe further that, in both the statements 1° and 2°, we may replace the hypothesis of continuity of F by the hypothesis that F is continuous in the Darboux sense (cf. § 4, p. 272); moreover the condition that the exceptional set be at most enumerable may be replaced by the condition that the set of values assumed by the function F at the points of this set be of measure zero.

The Denjoy relations have a more complete extension to relative derivates when the function U of Theorem 5.2 is subjected to certain restrictions. Thus:

(5.6) ***Theorem.*** *Let U and F be finite functions, and suppose that, at each point t of a set E, the derivative $U'(t)$ (finite or infinite) exists and that $\overline{F}_U^+(t)<+\infty$. Then $\underline{F}_U^-(t)=\overline{F}_U^+(t)\neq\infty$ at each point t of E except at most the points of a set H such that $|U[H]|=0$.*

Proof. We may clearly restrict ourselves to the case in which the derivative $U'(t)$ is non-negative throughout E, and even, by Theorem 4.5, to the case in which (1^0) $U'(\tau)>0$ at each point τ of E. We then have $\limsup_{t\to\tau-} U(t)\leqslant U(\tau)\leqslant \liminf_{t\to\tau+} U(t)$ at each point $\tau\in E$, and consequently, on account of Theorem 1.1 (ii), we may suppose that (2^0) the function U is continuous at each point of E. This implies that we then have also $\limsup_{t\to\tau+} F(t)\leqslant F(\tau)$ at each point τ of E, and hence, appealing again to Theorem 1.1 (ii), we may suppose further that (3^0) $\limsup_{t\to\tau} F(t)\leqslant F(\tau)$ at each point $\tau\in E$. Finally by Theorem 1.2, we may suppose (4^0) $\underline{F}_U^-(\tau)\leqslant \overline{F}_U^+(\tau)$ at each point $\tau\in E$.

Let now C be the curve defined by the equations $x=U(t)$, $y=F(t)$. We denote, for each positive integer n, by E_n the set of the points $t\in E$ such that, for every point t', (i) the inequality $0<t'-t<1/n$ implies the two inequalities $U(t')>U(t)$ and $F(t')-F(t)<n\cdot[U(t')-U(t)]$, and (ii) the inequality $0<t-t'<1/n$ implies $U(t)>U(t')$.

Since, by hypothesis, $\overline{F}_U^+(t)<+\infty$ and since, by (1^0), $U'(t)>0$ at each point t of E, we see that $E=\sum_n E_n$.

Keeping a positive integer n fixed for the moment, let I be any open interval of length less than $1/n$. Whenever (ξ,η) is a point of $\mathrm{B}(C;I\cdot E_n)$, the contingent of the set $\mathrm{B}(C;I)$ at (ξ,η) clearly contains no half-line which is situated in the half-plane $x\geqslant\xi$ and which has an angular coefficient exceeding n. Let $D(I)$ denote the set of the points of the set $\mathrm{B}(C;I)$ at which this set has an extreme tangent, non-parallel to the y-axis, with an empty side containing the half-line in the direction of the positive semi-axis of y. Further, let $B_n(I)$ be the set of the points of $\mathrm{B}(C;I\cdot E_n)$ which do not belong to $D(I)$, and let $\widetilde{B}_n(I)$ be the projection of $B_n(I)$ on the x-axis. By Theorems 3.6 and 3.7, the set $\widetilde{B}_n(I)$ is of measure zero.

This being so, let t_0 be any point of the set $I\cdot E_n$ such that $U(t_0)$ does not belong to the set $\widetilde{B}_n(I)$. Let us denote by k_0 the angular coefficient of the extreme tangent to the set $\mathrm{B}(C;I)$ at the point $(U(t_0),F(t_0))$. It follows easily from the hypotheses (1^0), (2^0) and (3^0) that $\underline{F}_U^-(t_0)\geqslant k_0\geqslant \overline{F}_U^+(t_0)$, and this, in view of ($4^0$), leads to the relation $\underline{F}_U^-(t_0)=\overline{F}_U^+(t_0)\neq\infty$.

Thus, since I is any open interval of length less than $1/n$, we find that the last relation holds at each point t_0 of E_n other than those belonging to a set H_n such that $|U[H_n]|=0$. This completes the proof, since we have seen that $E=\sum_n E_n$.

In view of Theorem 7.2, Chap. VII, we derive from Theorem 5.6 the following theorem which has been established in a different way by A. J. Ward [3]:

(5.7) *Suppose that the function U is* VBG$_*$ *and let F be any finite function. Let E be a set at each point t of which we have either $\overline{F}_U^+(t)<+\infty$ or $\underline{F}_U^-(t)>-\infty$. Then the derivates $\underline{F}_U^-$ and $\overline{F}_U^+$ are finite and equal at all points of E except at most those of a set H such that $|U[H]|=0$.*

It will result from the considerations of § 6 (see, in particular, Theorem 6.2) that Theorem 5.7 remains valid for all continuous functions U which fulfil the condition (T_1). Nevertheless, its conclusion ceases to hold if we allow U to be any function which is VBG or even ACG. To see this, let G be a non-negative continuous function which is ACG on the interval $[0,1]$ and for which $G(t)=0$ and $\underline{G}^-(t)<-1$ at any point t of a perfect set E of positive measure (for the construction of such a function cf. Chap. VII, § 5, p. 224). Let us choose $U(t)=t+G(t)$ and $F(t)=t$. We shall then have at every point t of E, $U(t)=t$, $\underline{U}^+(t)\geqslant 1$ and $\underline{U}^-(t)<0<\overline{U}^-(t)$, so that $0\leqslant \underline{F}_U^+(t)\leqslant \overline{F}_U^+(t)\leqslant 1$, while $\underline{F}_U^-(t)=-\infty$ and $\overline{F}_U^-(t)=+\infty$. Nevertheless $|U[E]|=|E|>0$. (This example is due to Ward.)

***§ 6. The Banach conditions (T_1) and (T_2).** A finite function of a real variable F is said to fulfil the *condition* (T_1) on an interval I if almost every one of its values is assumed at most a finite number of times on I. A finite function F is said to fulfil the *condition* (T_2) on an interval I if almost every one of its values is assumed at most an enumerable infinity of times on I.

These two conditions were formulated by S. Banach [6]. We shall begin by studying the condition (T_1) and we shall establish a differential property which is equivalent to this condition in the case when F is continuous (*vide* below Theorem 6.2). Another equivalent condition, due to Nina Bary, will be established in § 8 (Theorem 8.3).

(6.1) ***Lemma.*** *Suppose that F is a continuous function and that E is a set at no point of which the function F has a derivative (finite or infinite). Suppose further that each point x of E is an isolated point of the set $\mathop{\mathrm{E}}_t[F(t)=F(x)]$. Then $\Lambda\{\mathrm{B}(F;E)\}=0$, and consequently $|E|=|F[E]|=0$.*

Proof. For each $x \epsilon E$ there exists a neighbourhood I such that, when $t \epsilon I$, the difference $F(t)-F(x)$ remains of constant sign as long as t remains on the same side of x; this difference then changes sign as t passes from one side of x to the other, except in the case in which the function F assumes a strict maximum or minimum at x. Therefore, if we denote by E_0 the set of the points at which the function assumes a strict maximum or minimum, we see at once that the four Dini derivates of F have the same sign at any point x of $E-E_0$. In other words, since, by hypothesis, the function F has no finite or infinite derivative at any point of E, we shall have at each point x of $E-E_0$ either $+\infty > \underline{F}(x) \geqslant 0$ or else $-\infty < \overline{F}(x) \leqslant 0$. Hence, by Theorem 4.2, $\Lambda\{B(F; E-E_0)\}=0$, and, since the set E_0 is at most enumerable (cf. Theorem 1.1), it follows that $\Lambda\{B(F;E)\}=0$.

(6.2) ***Theorem.*** *In order that a function F which is continuous on an interval I, fulfil the condition* (T_1) *on this interval, it is necessary and sufficient that the set of the values assumed by F at the points at which the function has no derivative (finite or infinite) be of measure zero.*

Proof. Denoting by Y the set of the values assumed an infinity of times by the function F on I, and denoting by E the set of the points of I at which F has no derivative, we have to prove that the relations $|Y|=0$ and $|F[E]|=0$ are equivalent.

1^0 $|Y|=0$ implies $|F[E]|=0$. Let X be the set of the points $x \epsilon I$ such that $F(x) \epsilon Y$. Then $F[X]=Y$, whence $|F[X]|=0$.

On the other hand, for each $x_0 \epsilon E-X$, the set of the points x such that $F(x)=F(x_0)$, is finite, and consequently an isolated set. It follows from Lemma 6.1 that $|F[E-X]|=0$, and hence finally that $|F[E]| \leqslant |F[X]|+|F[E-X]|=0$.

2^0 $|F[E]|=0$ implies $|Y|=0$. Let H denote the set of the points x at which $F'(x)=0$. By Theorem 4.5, we have $|F[H]|=0$.

Now let y_0 be any point of $Y-F[E]$, and let E_0 denote the set of the points x at which $F(x)=y_0$. The set E_0 being infinite and closed, let x_0 be a point of accumulation of E_0. Since the function F has a derivative at each point of E_0, we find that $F'(x_0)=0$; thus $x_0 \epsilon H$ and therefore $y_0 \epsilon F[H]$. It follows that $Y-F[E] \subset F[H]$, and hence that $|Y-F[E]|=0$. Thus $|F[E]|=0$ implies $|Y|=0$ and the proof is complete.

(6.3) ***Theorem.*** 1^0 *A continuous function which is* VBG_* *(in particular, one of bounded variation) on an interval I, necessarily fulfils the condition* (T_1) *on I.*

2^0 *A continuous function which is* VBG *on an interval I necessarily fulfils the condition* (T_2) *on I.*

Proof. On account of Theorem 7.2, Chap. VII, the first part of the theorem is an immediate corollary of Theorem 6.2. To establish 2^0, let us suppose that F is continuous and VBG on an interval I. The interval I is then expressible as the sum of a sequence $\{E_n\}$ of closed sets on each of which the function F is VB. We may clearly suppose that each E_n contains the end-points of the interval I. Let us denote, for each positive integer n, by F_n, the function which is equal to F on E_n and which is linear on the intervals contiguous to E_n. The functions F_n are plainly of bounded variation on I, and therefore, by 1^0, they fulfil the condition (T_1). It follows at once that the function F fulfils the condition (T_2) on I.

In the part of Theorem 6.3 (1^0) that applies to functions which are VBG_*, the continuity hypothesis for the function F is not a superfluous one (thus, the function $F(x)=\sin(1/x)$ for $x \neq 0$ and $F(0)=0$ is VBG_* and does not fulfil the condition (T_1) on $[0, 1]$). This hypothesis may however be replaced by a weaker one, which consists in supposing that the function F has no points of discontinuity other than of the first kind (i. e. that, at each point x, both the unilateral limits $F(x+)$ and $F(x-)$ exist). In particular, functions of bounded variation, whether continuous or not, all fulfil the condition (T_1) (and from this it follows easily that the continuity hypothesis may be removed altogether from the second part (2^0) of the theorem).

For functions of bounded variation, the condition (T_1) may also be deduced from the following general property of plane sets, established by W. Gross [1] (cf. J. Gillis [1]): *If E is a plane set and E_n denotes the set of the values of η such that the line $y=\eta$ contains at least n distinct points of the set E, then $\Lambda(E) \geqslant n \cdot |E_n|$.*

In connection with part 2^0 of Theorem 6.3, it may be noted further that functions which are VBG, or even ACG, need not fulfil the condition (T_1). An example is furnished by the function U considered in § 5, p. 277. The latter is also, as will follow from results to be established in § 7 (cf. in particular, Theorem 7.4), an example of a continuous function which is ACG, and consequently fulfils the condition (N), without fulfilling the condition (S) of Banach.

For continuous functions of bounded variation, the condition (T_1) is also a consequence of the following theorem of S. Banach [5] (cf. also N. Bary [3, p. 631]), which contains at the same time an important criterion for a continuous function to be of bounded variation:

(6.4) ***Theorem.*** *Let F be a continuous function on an interval $I_0=[a,b]$ and let $s(y)$ denote for each y the number (finite or infinite) of the points of I_0 at which F assumes the value y. Then the function $s(y)$ is measurable ($\mathfrak{B}$) and we have*

$$\int_{-\infty}^{+\infty} s(y)\,dy = \mathrm{W}(F;I_0). \tag{6.5}$$

Proof. For each positive integer n, let us put $I_1^{(n)}=[a,a+(b-a)/2^n]$ and $I_k^{(n)}=(a+(k-1)(b-a)/2^n, a+k(b-a)/2^n]$, when $k=2,3,...,2^n$. This defines a subdivision $\mathfrak{J}^{(n)}$ of the interval I_0 into 2^n subintervals, of which the first is closed and the others are half-open on the left. For $k=1,2,...,2^n$, let $s_k^{(n)}$ denote the characteristic function of the set $F[I_k^{(n)}]$, and let $s^{(n)}(y)=\sum_{k=1}^{2^n} s_k^{(n)}(y)$.

We see at once that the functions $s^{(n)}(y)$ constitute a non-decreasing sequence which converges at each point y to $s(y)$. Hence, the functions $s^{(n)}(y)$ being clearly measurable ($\mathfrak{B}$), so is also the function $s(y)$.

On the other hand, $\int_{-\infty}^{+\infty} s_k^{(n)}(y)dy=|F[I_k^{(n)}]|=\mathrm{O}(F;I_k^{(n)})$. Therefore, denoting by $W^{(n)}$ the sum of the oscillations of the function F on the intervals of the subdivision $\mathfrak{J}^{(n)}$, we obtain $\int_{-\infty}^{+\infty} s^{(n)}(y)=W^{(n)}$, and the relation (6.5) follows by making $n\to\infty$.

(6.6) ***Theorem.*** *If $F(x)$ is a continuous function which fulfils the condition (T_2) on an interval I_0, the set D of the points at which the derivative $F'(x)$ (finite or infinite) exists, is non-enumerably infinite.*

Moreover, if we write

$$P=\mathop{\mathrm{E}}_{x}[x\,\epsilon\,D;F'(x)\geqslant 0] \quad \textit{and} \quad N=\mathop{\mathrm{E}}_{x}[x\,\epsilon\,D;F'(x)\leqslant 0],$$

then, for each interval $I=[a,b]\subset I_0$, we have

$$-|F[N]|\leqslant F(b)-F(a)\leqslant |F[P]|. \tag{6.7}$$

Proof. We may, plainly, suppose that

$$F(a)\leqslant F(b), \tag{6.8}$$

since the other case may be discussed by changing the sign of the function F.

Let Y be the set of those values of F on I which are assumed by the function F at most an enumerable number of times on I. Denoting, for each y, by E_y the set of the points $x\,\epsilon\,I$ such that

$F(x)=y$, we shall show that with each point $y\,\epsilon\,Y$ we can associate a point $x_y\,\epsilon\,E_y$, in such a manner that (i) $\overline{F}(x_y)\geqslant 0$ and (ii) x_y is an isolated point of the set E_y.

For this purpose, we remark first that if the set E_y reduces to a single point, the latter may be chosen for our x_y. For, in that case, the condition (ii) is clearly fulfilled, while the condition (i) holds on account of the hypothesis (6.8).

Let us therefore consider the other case, in which the set E_y contains more than one point. Then, since the function F is continuous by hypothesis and $y\,\epsilon\,Y$, the set E_y is closed, and at most enumerable. This set, therefore, contains a pair of isolated points α,β between which it has no further points. (This is obvious, if the set E_y is finite. If E_y is infinite, its derived set (cf. Chap. II, p. 40) is itself closed, non-empty, and at most enumerable; the latter, therefore, contains an isolated point x_0. Thus near x_0 there are only isolated points of E_y. It will, therefore, suffice to choose, among the latter, any two consecutive points as our points α and β.) Consequently, at one at least of the points α and β, the upper derivate of $F(x)$ is non-negative. We choose this point as our x_y. We then see at once that the conditions (i) and (ii) are fulfilled.

This being established, let X denote the set of all the points x_y which are thus associated with the points $y\,\epsilon\,Y$. It follows from the conditions (i) and (ii) and from Lemma 6.1, that $|F[X-P]|=|F[X-D]|=0$, and so, by definition of the set X, that $|Y|=|F[X]|=|F[X\cdot P]|\leqslant|F[P]|$. Since the condition ($\mathrm{T}_2$) implies that $|F[I]|=|Y|$, we obtain, in view of (6.8), the inequality $-|F[N]|\leqslant 0\leqslant F(b)-F(a)\leqslant|F[I]|\leqslant|F[P]|$, i.e. the inequality (6.7).

Finally, since this relation holds for every subinterval $[a,b]$ of I_0, we see that, unless the function F is a constant, one at least of the sets $F[N]$ and $F[P]$ is of positive measure. The set $D=N+P$ is thus non-enumerably infinite, and this completes the proof.

(6.9) ***Theorem.*** *Let F be a continuous function which fulfils the condition (T_2) and let g be a finite summable function. Suppose further that $F'(x)\leqslant g(x)$ at each point x at which the derivative $F'(x)$ exists, except perhaps those of an enumerable set or, more generally, those of a set E such that $|F[E]|=0$. Then the function F is of bounded variation and, for each interval $[a,b]$, we have*

$$F(b)-F(a)\leqslant\int_a^b F'(x)\,dx. \tag{6.10}$$

Proof. Let P be the set of the points x of $[a,b]$ at which the derivative $F'(x)$ exists and is non-negative. Then, since at each point $x \in P-E$ we have $0 \leqslant F'(x) \leqslant g(x) < +\infty$, it follows from Theorem 6.5, Chap. VII, that $|F[P-E]| \leqslant \int_P F'(x)\,dx \leqslant \int_a^b |g(x)|\,dx$. On the other hand, by hypothesis, $|F[E]|=0$. Hence, on account of Theorem 6.6, $F(b)-F(a) \leqslant \int_a^b |g(x)|\,dx$ for each interval $[a,b]$, and, in consequence, F is a function of bounded variation whose function of singularities is monotone non-increasing. The inequality (6.10) follows at once.

In view of Theorem 6.3 (2^0), we may apply Theorem 6.9, in particular, to continuous functions F which are VBG. We also observe that Theorem 6.9, when F is of bounded variation, may be deduced from de la Vallée Poussin's Decomposition Theorem (Chap. IV, § 9).

Theorem 6.9 may be generalized further, by replacing the condition that the function g is summable, by the condition that the latter is $\mathfrak{D}_*$-integrable (the function F then shows itself to be VBG$_*$). We thus obtain a proposition similar to Theorem 7.3, Chap. VI. The proof of Theorem 6.9 in this generalized form is, however, more complicated.

***§ 7. Three theorems of Banach.** We have repeatedly emphasized the importance of Lusin's condition (N) in the theory of the Denjoy integrals. We shall show in this §, that every continuous function which fulfils the condition (N), also fulfils the condition (T_2). This result due to S. Banach [6] (cf. also N. Bary [3, p. 195]) renders Theorems 6.6 and 6.9 applicable to functions which fulfil the condition (N).

We shall also study another condition, introduced by S. Banach [6] and termed condition (S). We say that a finite function F fulfils the *condition* (S) on an interval I_0, if to each number $\varepsilon>0$ there corresponds an $\eta>0$ such that, for each measurable set $E \subset I_0$, the inequality $|E|<\eta$ implies $|F[E]|<\varepsilon$. (This condition is essentially more restrictive than the condition (N); cf. the remarks, p. 279, also G. Fichtenholz [4].)

(7.1) ***Lemma.*** *Given a function F which is continuous on an interval I, every closed set $E \subset I$ contains a measurable set A on which the function F assumes each value $y \in F[E]$ exactly once.*

Proof. With each $y \in F[E]$ we associate the lower bound x_y of the set of the points x of E at which $F(x)=y$, and we denote by A the set of all the points x_y which correspond in this way to the values $y \in F[E]$. Since the set E is closed, we plainly have $A \subset E$ and F assumes on A each of the values $y \in F[E]$ exactly once.

In order to establish the measurability of A, let us denote, for each positive integer n, by E_n, the set of the points $x \in E$ such that E contains at least one point t which is subject to the conditions $F(t)=F(x)$ and $x-t \geqslant 1/n$. We have $A=E-\sum_n E_n$, where E is closed by hypothesis, and where each E_n is closed by continuity of F. The set A is thus measurable and this completes the proof.

(7.2) ***Lemma.*** *Let F be a continuous function which fulfils the condition* (N) *on an interval I. Then*

(i) *every measurable set $E \subset I$ contains, for each $\varepsilon>0$, a measurable subset Q, such that $|F[E]-F[Q]|<\varepsilon$, and on which the function F assumes each of its values at most once;*

(ii) *every measurable set $E \subset I$ contains a measurable subset R, such that $|F[E]-F[R]|=0$, and on which the function F assumes each of its values at most an enumerable infinity of times.*

Proof. *re* (i). As a measurable set, E is the sum of a set H of measure zero and an ascending sequence of closed sets $\{E_n\}$. Since the function F fulfils the condition (N), we have $|F[H]|=0$, and hence, the sets $F[E_n]$ being measurable, there exists a positive integer n_0 such that $|F[E]-F[E_{n_0}]|<\varepsilon$. Now, by Lemma 7.1, there exists a closed set $Q \subset E_{n_0}$ such that each value $y \in F[E_{n_0}]$ is assumed exactly once by F on Q. This set Q plainly fulfils the conditions stated.

re (ii). In view of (i), there exists for each positive integer n a measurable set $Q_n \subset E$, such that $|F[E]-F[Q_n]|<1/n$, and on which the function F assumes each of its values at most once. Therefore, writing $R=\sum_n Q_n$, we see immediately that $|F[E]-F[R]|=0$ and that on R the function F assumes each of its values at most an enumerable infinity of times. This completes the proof.

We shall establish in this § three theorems due to Banach on functions which fulfil the conditions (N) or (S). The first of these theorems, which concerns functions fulfilling the condition (N), is as follows:

(7.3) *Theorem.* *Any continuous function F which fulfils the condition* (N) *on an interval I, necessarily fulfils also the condition* (T_2) *on I.*

Proof. Let us denote, for each measurable set $E \subset I$, by $\mathfrak{R}_E$ the class of all measurable sets $R \subset E$ which are subject to the following two conditions: (i) $|F[E]-F[R]|=0$, and (ii) each value $y \in F[E]$ is assumed by the function F at most enumerably often on R. By Lemma 7.2, the class $\mathfrak{R}_E$ is non-empty, however we choose the measurable set $E \subset I$. We shall denote, for any such set E, by μ_E the upper bound of the measures of the sets ($\mathfrak{R}_E$).

Consider, in particular, a sequence $\{H_n\}$ of sets ($\mathfrak{R}_I$) such that $\lim_n |H_n| = \mu_I$. Let $H = \sum_n H_n$ and let U be a set ($\mathfrak{R}_{I-H}$). We verify at once that $|U|=0$, whence on account of the condition (N), $|F[U]|=0$. Therefore $|F[I-H]|=|F[U]|=0$, so that almost every value $y \in F[I]$ is assumed by F only on the set H, and therefore at most enumerably often.

The second of the theorems of Banach concerns functions which fulfil the condition (S).

(7.4) *Theorem.* *In order that a continuous function F be subject to the condition* (S) *on an interval I, it is necessary and sufficient that F be subject on I to both the conditions* (N) *and* (T_1).

Proof. 1° Suppose that the function F fulfils the condition (S) on I. Since this condition clearly implies the condition (N), we need only prove that F fulfils the condition (T_1).

Suppose then, if possible, that the set of the values assumed infinitely often on I by the function F, is of positive outer measure. Since this set is measurable by Theorem 6.4, it contains a closed subset Y of positive measure. Let X denote the set of all the points $x \in I$ such that $F(x) \in Y$. The set X, plainly, is also closed.

We shall now define by induction a sequence of measurable sets $\{X_i\}$ subject to the following conditions: (i) $X_i \cdot X_j = 0$ whenever $i \neq j$, (ii) $|F[X_i]| \geqslant |Y|/2$ for $i=1,2,\ldots$, and (iii) the function F assumes each of its values at most once on each set X_i.

For this purpose, suppose defined the first k sets X_i for which the conditions (i), (ii) and (iii) are satisfied. Let $E_k = I - \sum_{i=1}^{k} X_i$. Since, on $\sum_{i=1}^{k} X_i$, the function F assumes each of its values at most a finite number of times, it follows that each value $y \in Y$ is necessarily assumed on the set E_k. By Lemma 7.2, this set therefore contains measurable subset X_{k+1} such that $|F[X_{k+1}]| \geqslant |F[E_k]|/2 \geqslant |Y|/2$ and that each value is assumed by F at most once on X_{k+1}. The sets $X_1, X_2, \ldots, X_{k+1}$ clearly fulfil the conditions (i), (ii) and (iii).

The sequence $\{X_i\}$ being thus defined, it follows from (i) that $\lim_i |X_i| = 0$, and hence, remembering that the function F fulfils the condition (S), we have also $\lim_i |F[X_i]| = 0$. But this clearly contradicts (ii), since $|Y|>0$.

2° Suppose now that the function F fulfils the conditions (N) and (T_1), but not the condition (S). We could then determine a positive number σ and a sequence of sets $\{E_k\}$ in I so that for $k=1,2,\ldots$,

$$(7.5) \qquad |E_k| < 1/2^k, \qquad \text{and} \qquad (7.6) \qquad |F[E_k]| > \sigma.$$

Let us write $E = \limsup_k E_k$ and $A = \limsup_k F[E_k]$. We easily see that, if $y \in A$, then either $y \in F[E]$, or else the value y is assumed by F on I infinitely often (in fact there exists an increasing sequence of positive integers $\{k_i\}$ and a sequence of points $\{x_i\}$, every two of which are distinct, such that $x_i \in E_{k_i}$ and $F(x_i) = y$ for $i=1,2,\ldots$).

Now, on account of (7.5), we have $|E|=0$, and therefore also $|F[E]|=0$. On the other hand, by (7.6), $|A| \geqslant \sigma > 0$. Thus $|A - F[E]| \geqslant \sigma$, and since, as we have just seen, each value $y \in A - F[E]$ is assumed by F infinitely often on I, this contradicts the hypothesis that the function F fulfils the condition (T_1).

We shall establish next a "differentiability theorem" for the functions which fulfil the condition (N):

(7.7) *Theorem.* *In order that a continuous function F be absolutely continuous on an interval I_0, it is necessary and sufficient that the function F fulfil simultaneously the condition* (N) *and the condition*

$$(7.8) \qquad \int_P F'(x)\,dx < +\infty,$$

where P denotes the set of the points at which the function F has a finite non-negative derivative.

Proof. Since the conditions of the theorem are obviously necessary (cf. Theorem 6.7, Chap. VII), let us suppose that the function F fulfils the condition (N) and the inequality (7.8). Let g be the function equal to $F'(x)$ for $x \in P$ and to 0 elsewhere. Then, if E denotes the set of the points x at which $F'(x) = +\infty$, we shall have $F'(x) \leqslant g(x)$ at every point x of $I_0 - E$ at which the derivative $F'(x)$ exists.

On the other hand, since $|E|=0$ (cf. Theorem 4.4, or Chap. VII, §10, p. 236), we have $|F[E]|=0$, and, since the function F fulfils, by Theorem 7.3, the condition (T_2), it follows from Theorem 6.9 that F is of bounded variation on I_0. This completes the proof, since, by Theorem 6.7, Chap. VII, every continuous function of bounded variation, which fulfils the condition (N) is absolutely continuous.

Theorem 7.7 (in a slightly less general form) was first proved by N. Bary [2; 3, p. 199]. It shows in particular that *every continuous function $F(x)$, which is subject to the condition* (N) *and whose derivative is non-negative at almost every point where $F(x)$ is derivable, is monotone non-decreasing*. This proposition contains an essential generalization of Theorem 6.2, Chap. VII.

Theorem 7.7 may, moreover, be generalized still further. *If a continuous function $F(x)$ fulfils the condition* (N) *and if the function $g(x)$, equal to $F'(x)$ wherever $F(x)$ is derivable and to 0 elsewhere, has a major function* (in the Perron sense), *then the function $F(x)$ is ACG_* i. e. an indefinite $\mathcal{P}$-integral.*

For the part played by the conditions (N), (T_1) and (T_2) in the theory of Denjoy integrals, cf. also J. Ridder [8].

From Theorem 7.7 we obtain the third theorem of Banach:

(7.9) ***Theorem.*** *Any function which is continuous and subject to the condition* (N) *on an interval, is derivable at every point of a set of positive measure.*

*§ 8. Superpositions of absolutely continuous functions.

Suppose given a bounded function G on an interval $[a,b]$, and a function H defined on the interval $[\alpha,\beta]$ where α and β denote respectively the lower and the upper bound of G on $[a,b]$. We call *superposition* of the functions G and H on $[a,b]$, the function $H(G(x))$. The function G is termed *inner function* and the function H *outer function* of this superposition.

If a function F is continuous and increasing on an interval $[a,b]$, the continuous increasing function G defined on the interval $[F(a),F(b)]$ so as to satisfy the identity $G(F(x))=x$ on $[a,b]$, will, as usual, be termed *inverse function* of F and denoted by F^{-1}.

It has long been known that the superposition of two absolutely continuous functions is not, in general, an absolutely continuous function. By means of the conditions discussed in the preceding §§, particularly the condition (S), Nina Bary and D. Menchoff succeeded in characterizing completely the class of functions expressible as superpositions of absolutely continuous functions. (Cf. also G. Fichtenholz [3].)

(8.1) ***Theorem.*** *Any function F which is continuous and subject to the condition* (T_1) *on an interval $[a,b]$ is expressible on this interval as a superposition of two continuous functions, of which the inner function is of bounded variation and the outer function is increasing and absolutely continuous.*

If, further, the function F fulfils the condition (N), *the inner function of this superposition is necessarily absolutely continuous also.*

Proof. Let α and β denote respectively the lower and upper bounds of F on $[a,b]$, and let $s_F(y)$ denote, for each y, the number (finite or infinite) of the points of the interval $[a,b]$ at which F assumes the value y. Since, by hypothesis, the function F is continuous and subject to the condition (T_1), we shall have $0<1/s_F(y)\leqslant 1$ for almost all the values y of the interval $[\alpha,\beta]$. Let us denote by U an indefinite integral of the function which is equal to $1/s_F(y)$ on $[\alpha,\beta]$ and to 1 elsewhere. We now write $G(x)=U[F(x)]$ for $x\in[a,b]$. We thus have $F(x)=U^{-1}[G(x)]$, and in order to establish the first part of the theorem, it is enough to show that (i) the function U^{-1} is absolutely continuous and (ii) the function G is of bounded variation.

Suppose, if possible, that the function U^{-1} (which is continuous and increasing together with U) is not absolutely continuous. Then (cf. Theorem 6.7, Chap. VII), there exists a set E of measure zero such that $|U^{-1}[E]|>0$. Writing $Q=U^{-1}[E]$, we thus have

$$(8.2) \qquad |Q|>0 \qquad \textit{and} \qquad |U[Q]|=0.$$

We may, plainly, suppose that the set E, and therefore the set Q, are sets $(\mathfrak{G}_\delta)$. Thus (cf. Theorem 13.3, Chap. III)

$$|U[Q]|=\int_Q U'(y)\,dy,$$

which renders the relations (8.2) contradictory, since almost everywhere $U'(y)=1/s_F(y)>0$ for $y\in[\alpha,\beta]$ and $U'(y)=1$ outside the interval $[\alpha,\beta]$.

In order to establish (ii), we shall make use of the criterion of Theorem 6.4. Denote for each z by $s_G(z)$ the number of the points of the interval $[a,b]$ at which the function G assumes the value z. Since the function U is increasing, we clearly have $s_G(U(y))=s_F(y)$ for each y, and $s_G(z)=0$ for each z outside the interval $[U(\alpha),U(\beta)]$. Hence, remembering that the function U is absolutely continuous, we obtain (cf. Theorem 15.1, Chap. I)

$$\int_{-\infty}^{+\infty} s_G(z)\,dz = \int_{U(\alpha)}^{U(\beta)} s_G(z)\,dz = \int_{\alpha}^{\beta} s_G(U(y))\,dU(y) =$$
$$= \int_{\alpha}^{\beta} s_F(y)\,U'(y)\,dy = \int_{\alpha}^{\beta} dy = \beta - \alpha$$

which shows, by Theorem 6.4, that the function G is of bounded variation.

Finally, if the given function F fulfils the condition (N), so does the function $G(x) = U(F(x))$, and the latter, since it is of bounded variation, is absolutely continuous by Theorem 6.7, Chap. VII. This completes the proof.

(8.3) ***Theorem.*** 1^0 *In order that a continuous function F be expressible as a superposition of two continuous functions of which the inner function is of bounded variation and the outer function is absolutely continuous, it is necessary and sufficient that F fulfil the condition* (T_1).

2^0 *In order that a continuous function be representable as a superposition of two absolutely continuous functions, it is necessary and sufficient that the function fulfil both the conditions* (T_1) *and* (N), *or what amounts to the same, the condition* (S).

Proof. Since it follows at once from Theorem 8.1 that these conditions are sufficient, we need only prove them necessary.

Let therefore $F(x) = H(G(x))$ on an interval $[a,b]$, where G is a function of bounded variation and H an absolutely continuous function. Let α and β be respectively the lower and the upper bound of G on $[a,b]$. Let E_G and E_H denote the sets of the values which the functions G and H assume infinitely often on the intervals $[a,b]$ and $[\alpha,\beta]$, respectively. Since the functions G and H fulfil the condition (T_1), we have $|E_G| = |E_H| = 0$, and since the function H is, moreover, absolutely continuous, we have also $|H[E_G]| = 0$. Now we see at once that each value which is assumed infinitely often on $[a,b]$ by the function F, belongs either to E_H, or to $H[E_G]$. The set of these values is thus of measure zero, and the function F fulfils the condition (T_1).

If, further, the function G is absolutely continuous (as well as H), then the function F is a superposition of two functions which fulfil the condition (N), and, therefore, itself fulfils this condition. This completes the proof.

Theorems 8.1 and 8.3 are due to Nina Bary [1; 3, pp. 208, 633] (cf. also S. Banach and S. Saks [1]). Part 2^0 of Theorem 8.3 was established a little earlier in a note of N. Bary and D. Menchoff [1] (cf. also N. Bary [3, p. 203]) in a form analogous to Theorem 6.2. Thus:

(8.4) ***Theorem.*** *In order that a function F which is continuous on an interval $[a,b]$ be on this interval a superposition of two absolutely continuous functions, it is necessary and sufficient that the set of the values assumed by the function F at the points of $[a,b]$ at which F is not derivable, be of measure zero.*

Proof. Let Q_F be the set of the points of $[a,b]$ at which F is not derivable. Suppose first that

1^0 $F(x) = H(G(x))$ on $[a,b]$, where H and G are absolutely continuous functions. Let Q_G and Q_H be respectively the sets of the points at which the functions G and H are not derivable. We have $|Q_G| = |Q_H| = 0$ and, consequently, $|F[Q_G]| = |H[Q_H]| = 0$. Now, we see at once that if the function F is not derivable at a point x, then either $x \in Q_G$, or $G(x) \in Q_H$. Therefore $F[Q_F] \subset F[Q_G] + H[Q_H]$, and hence, $|F[Q_F]| = 0$.

Conversely, suppose that

2^0 $|F[Q_F]| = 0$. By Theorem 6.2, the function then fulfils the condition (T_1). To show that F also fulfils the condition (N), consider any set of measure zero, E say, in $[a,b]$. Since the function F is derivable at each point of $E - Q_F$, we have, by Theorem 6.5, Chap. VII, $|F[E - Q_F]| = 0$, and since, by hypothesis, $|F[Q_F]| = 0$, we obtain $|F[E]| = 0$. The function F thus fulfils both the conditions (T_1) and (N), and is, therefore, by Theorem 8.3 (2^0), a superposition of two absolutely continuous functions.

It follows from Theorem 8.3 (2^0) that *a superposition of any finite number of absolutely continuous functions is expressible as a superposition of two absolutely continuous functions.* For the superposition of any finite number of functions which fulfil the condition (S), itself fulfils this condition.

The results exposed in this § have been the starting point of the important researches of Nina Bary [3] on the representation of continuous functions by means of superpositions of absolutely continuous functions. Let us cite two of her fundamental theorems: 1^0 *Every continuous function is the sum of three superpositions of absolutely continuous functions, and there exist continuous functions which cannot be expressed as the sum of two such superpositions.* 2^0 *Every continuous function which fulfils the condition* (N)*—or, more generally, every continuous function which is derivable at every point of a set which has positive measure in each interval—is the sum of two superpositions of absolutely continuous functions, and there exist continuous functions which fulfil the condition* (N), *but are not expressible as one superposition of absolutely continuous functions* (the function $U(x)$ discussed above in § 6, p. 279, is an example of such a function).

For further researches, *vide* N. Bary [4] and J. Todd [1; 2].

§ 9. The condition (D). We shall now establish, for the extreme approximate derivates, a theorem, analogous to Theorem 4.6, but whose proof depends on a different idea. It is convenient to formulate it, from the beginning, in a slightly more general manner.

Given two positive numbers N and ε, we shall say that a function F fulfils at a point x_0 *the condition* $(\mathrm{D}^+_{N,\varepsilon})$, if there exist positive numbers h, as small as we please, such that the difference between the outer measure of the set $\mathop{\mathrm{E}}_x[F(x)-F(x_0)\geqslant N\cdot(x-x_0);\ 0\leqslant x-x_0\leqslant h]$ and that of the set $\mathop{\mathrm{E}}_x[F(x)-F(x_0)\geqslant -N\cdot(x-x_0);\ 0\leqslant x-x_0\leqslant h]$ exceeds the number $h\varepsilon$ in absolute value. By symmetry, merely replacing $F(x)-F(x_0)$ and $x-x_0$ by $F(x_0)-F(x)$ and x_0-x respectively, we define the condition $(\mathrm{D}^-_{N,\varepsilon})$.

If, for a point x_0, there exists a pair of finite positive numbers N and ε such that the function F fulfils at this point the condition $(\mathrm{D}^+_{N,\varepsilon})$, or the condition $(\mathrm{D}^-_{N,\varepsilon})$, we say that F fulfils at x_0 the *condition* (D).

For measurable functions the condition (D) may be formulated more simply: *a measurable function F fulfils at a point x_0, the condition* (D), *if there exists a finite positive number N such that x_0 is not a point of dispersion for the set of the points x at which* $|F(x)-F(x_0)|\leqslant N\cdot|x-x_0|$.

(9.1) ***Theorem.*** *If any one of the four approximate extreme derivates of a function F is finite at a point x_0, then the function fulfils the condition* (D) *at this point.*

Proof. Suppose, to fix the ideas, that $|\overline{F}^+_{\mathrm{ap}}(x_0)|<+\infty$ and write $N=|\overline{F}^+_{\mathrm{ap}}(x_0)|+1$. Let E_1 and E_2 be the sets of the points x which are situated on the right of the point x_0 and which fulfil respectively the inequalities $F(x)-F(x_0)\geqslant N\cdot(x-x_0)$ and $F(x)-F(x_0)\geqslant -N\cdot(x-x_0)$. It follows at once from the definitions of approximate derivates (Chap. VII, § 3) that x_0 is a point of dispersion for the set E_1, while E_2 has at x_0 a positive upper outer density. Denoting the latter by δ, we see at once that the function F fulfils at x_0 the condition $(\mathrm{D}^+_{N,\varepsilon})$ whatever be the positive number $\varepsilon<\delta$.

(9.2) ***Lemma.*** *Let N and ε be finite positive numbers and suppose that a finite function F fulfils the condition $(\mathrm{D}^+_{N,\varepsilon})$ at each point of a set E. Then* $|F[E]|\leqslant(2N/\varepsilon)\cdot|E|$.

Proof. We shall first show that for every interval $I=[a,b]$,

$$(9.3)\qquad |I|\geqslant(\varepsilon/2N)\cdot|F[E\cdot I]|.$$

For this purpose, we write, for every y,

$$(9.4)\qquad H(y)=|\mathop{\mathrm{E}}_x[F(x)\geqslant y;\ x\in I]|.$$

The function H thus defined is non-increasing and bounded on the whole straight line $(-\infty,+\infty)$; we have, in fact, for every y,

$$(9.5)\qquad 0\leqslant H(y)\leqslant|I|.$$

Given an arbitrary point y_0 of $F[E\cdot I]$, which is distinct from $F(b)$ and at which the function H is derivable, let us consider a point $x_0\in E\cdot I$ such that $F(x_0)=y_0$. Plainly $x_0\neq b$. Let us write, for brevity,

$$A(h,k)=\mathop{\mathrm{E}}_x[F(x)\geqslant y_0+k\cdot(x-x_0);\ 0\leqslant x-x_0\leqslant h]$$

and

$$B(h,k)=\mathop{\mathrm{E}}_x[F(x)\geqslant y_0+kh;\ 0\leqslant x-x_0\leqslant h].$$

For every subinterval $[x_0,x_0+h]$ of I, we then have the relation $B(h,N)\subset A(h,N)\subset A(h,-N)\subset B(h,-N)$, whence it follows easily, on account of (9.4), that

$$H(y_0-Nh)-H(y_0+Nh)\geqslant|B(h,-N)|-|B(h,N)|\geqslant|A(h,-N)|-|A(h,N)|.$$

Now, since F fulfils, by hypothesis, the condition $(\mathrm{D}^+_{N,\varepsilon})$ at x_0, there exist positive values h, as small as we please, such that

$$|A(h,-N)|-|A(h,N)|\geqslant h\varepsilon,$$

and therefore $H(y_0-Nh)-H(y_0+Nh)\geqslant h\varepsilon$. Hence, $H'(y_0)\leqslant-\varepsilon/2N$ for every point $y_0\neq F(b)$ of $F[E\cdot I]$ at which the function H is derivable. Therefore, denoting, for each positive integer n, by Q_n the part of the set $F[E\cdot I]$ contained in the interval $[-n,n]$, we find, on account of (9.5), $|I|\geqslant|H(n)-H(-n)|\geqslant\varepsilon\cdot|Q_n|/2N$, from which the inequality (9.3) follows by making $n\to\infty$.

This being established, let η be a positive number and $\{I_k\}$ a sequence of intervals such that

$$(9.6)\qquad E\subset\sum_k I_k \quad\textit{and}\quad |E|+\eta\geqslant\sum_k|I_k|.$$

Since (9.3) holds for each interval I, it follows from (9.6) that $|E|+\eta\geqslant(\varepsilon/2N)\cdot\sum_k|F[E\cdot I_k]|\geqslant(\varepsilon/2N)\cdot|F[E]|$, whence, remembering that η is an arbitrary positive number, we see that $|F[E]|\leqslant(2N/\varepsilon)\cdot|E|$.

(9.7) ***Theorem.*** *If at each point of a set E, a finite function F fulfils the condition* (D) *(and so, in particular, if at each point of E the function F has one of its extreme approximate derivates finite), then the function F fulfils the condition* (N) *on E.*

Proof. Let H be any subset of E of measure zero, and let H_n denote, for each positive integer n, the set of the points x of H at which the function F fulfils the condition $(D^{+}_{n,1/n})$ or $(D^{-}_{n,1/n})$. We clearly have $H=\sum_n H_n$, and since, by Lemma 9.2, $|F[H_n]|\leqslant 4n^2\cdot|H_n|=0$, we obtain $|F[H]|=0$.

Theorem 9.7 enables us to complete Theorems 10.5 and 10.14 of Chap. VII, as follows:

(9.8) ***Theorem.*** 1^0 *Every finite function F which is continuous on a closed set E and which has at each point of E, except perhaps those of an enumerable subset, either two finite Dini derivates on the same side, or one finite extreme bilateral derivate, is* ACG_* *on E.*

2^0 *Every finite function F which is continuous on a closed set E and which has at each point of E, except perhaps those of an enumerable subset, either one finite Dini derivate, or one finite extreme approximate bilateral derivate, or finally two finite extreme approximate unilateral derivates on the same side, is* ACG *on E.*

Proof. By Theorems 10.1, 10.5, 10.8 and 10.14 of Chap. VII, the function F is VBG_* on E in case 1^0 and VBG on E in case 2^0. On the other hand, by Theorems 4.6 and 9.7, this function fulfils, in both cases, the condition (N) on E. Hence, by Theorems 6.8 and 8.8 of Chap. VII, the function is ACG_* on E in case 1^0, and ACG on E in case 2^0.

In the most important case in which the closed set E is an interval, Theorem 9.8 may further be stated in terms of Denjoy integrals. For this purpose, let us begin by noting the following proposition (cf. A. S. Besicovitch [2], and J. C. Burkill and U. S. Haslam-Jones [1]):

(9.9) ***Theorem.*** *If a finite function F is measurable on a set E and has at each point of this set one of its Dini derivates finite, then this derivate is, at almost all points of E, an approximate derivative of F.*

Proof. It follows from Theorem 10.8, Chap. VII, that the function F is VBG on E, and so, approximately derivable at almost all the points of E. Let us denote by E_1 the set of the points of E at which one at least of the opposite Dini derivates $\overline{F}^+$ and $\underline{F}^-$ is finite. Plainly, $\underline{F}^-(x)\leqslant F'_{ap}(x)\leqslant\overline{F}^+(x)$ at each point x at which the approximate derivative $F'_{ap}(x)$ exists, and therefore by Theorem 4.1, $\underline{F}^-(x)=\overline{F}^+(x)=F'_{ap}(x)$ at almost all points x of E_1. Similarly, we show that $\underline{F}^+(x)=\overline{F}^-(x)=F'_{ap}(x)$ at almost all the points x of E at which one of the derivates $\underline{F}^+$ and F^- is finite. This completes the proof.

(9.10) ***Theorem.*** 1^0 *If f is a finite function which, at each point of an interval I_0, except those of an enumerable set, is equal to an extreme bilateral derivate of a continuous function F, then the function f is $\mathfrak{D}_*$-integrable on I_0 and the function F is an indefinite $\mathfrak{D}_*$-integral of f.*

2^0 *If f is a finite function which, at each point of an interval I_0, except those of an enumerable set, is equal either to a Dini derivate, or to an extreme approximate bilateral derivate of a continuous function F, then the function f is $\mathfrak{D}$-integrable on I_0 and the function F is an indefinite $\mathfrak{D}$-integral of f.*

Proof. In view of Theorem 9.8, the function F is ACG_* in case 1^0, and ACG in case 2^0. Moreover, at almost all the points x of E, we have $F'(x)=f(x)$ in case 1^0, and by Theorem 9.9, $F'_{ap}(x)=f(x)$ in case 2^0. This proves the theorem.

Although Theorem 9.8 presents a formal analogy with Theorems 10.5 and 10.14 of Chap. VII, there is an essential difference between the result of this § and those of § 10, Chap. VII. We see, in the first place, that the criteria of Theorems 10.5 and 10.14 of Chap. VII concern functions which are given on quite arbitrary sets, whereas those of Theorem 9.8 are established only for closed sets. In the second place, if the derivates of a quite arbitrary function satisfy on a set E the conditions of Theorem 10.5, or of Theorem 10.14, of Chap. VII, then the set E can, by these theorems, be decomposed into a sequence of sets on which the function is absolutely continuous. On the contrary, Theorem 9.8 of this § does not enable us to draw any conclusion as to a similar decomposition of the set E (even when this set is an interval), unless the function considered is continuous.

Two examples will now be given to show that this feature of Theorem 9.8, which represents a restriction as compared with the results of § 10, Chap. VII, is essential for the validity of the theorem.

(i) Consider the function $F(x)=\sum_n[2^n x]/5^n$, where $[2^n x]$ denotes, as usual, the largest integer not exceeding $2^n x$. This function is increasing. Its lower right-hand derivate is finite everywhere, and even, as we easily see, vanishes identically. Nevertheless, there is no decomposition of the interval $J_0=[0,1]$ into a sequence of sets on which F is absolutely continuous, or even only uniformly continuous. In fact, no such decomposition can exist for a monotone function F whose points of discontinuity form a set everywhere dense in J_0.

For, if such a decomposition $\{E_1, E_2, ..., E_n, ...\}$ existed, one at least of the sets E_n would, by Baire's Theorem (Chap. II, § 9), be everywhere dense in an interval $I\subset J_0$. This is plainly impossible since the function F, monotone by hypothesis, is uniformly continuous on each set E_n and has points of discontinuity in the interior of I.

(ii) Let us now consider an example of a continuous function $F(x)$, increasing on the interval $J_0=[0,1]$, and which has its lower right-hand derivate zero at every point of a set E, without being ACG on E.

For this purpose, let us agree to call, for brevity, function attached to an interval $I=[a, b]$, any function $H(x)$, which is continuous and non-decreasing on I, and which fulfils the conditions:

(a) $H(x)$ is constant on each of the intervals I_k of a sequence $\{I_k\}$ of non-overlapping sub-intervals of I such that $|I|=\sum_k |I_k|$; the length of any sub-interval of I on which $H(x)$ is constant does not exceed $|I|/2$;

(b) $H(x)-H(a)\leqslant x-a$ and $H(b)-H(x)\leqslant b-x$ for every $x \epsilon I$.

Such a function is easily obtained, by slightly modifying the construction of the function $f(x)$, considered in Chap. III, § 13, p. 101.

This being so, we shall define by induction a sequence $\{F_n(x)\}$ of functions attached to the interval J_0, beginning with an arbitrary function $F_1(x)$ attached to this interval. Given the function F_n attached to J_0, let $\{I_k^{(n)}=[a_k^{(n)}, b_k^{(n)}]\}$ be a sequence of the intervals of constancy of F_n in the interval J_0. (By an interval of constancy of a function in J_0 we mean here any interval $I\subset J_0$ such that the function is constant on I without being constant on any sub-interval of J_0 which contains I and is distinct from I.) For each $k=1,2,...$, we determine a function $H_k^{(n)}(x)$ attached to the interval $I_k^{(n)}$, and we write

$$F_{n+1}(x)=\begin{cases} \sum_i^{(x)}[H_i^{(n)}(b_i^{(n)})-H_i^{(n)}(a_i^{(n)})] & \textit{for}\quad x\,\epsilon\, J_0-\sum_k I_k^{(n)} \\ H_k^{(n)}(x)-H_k^{(n)}(a_k^{(n)})+\sum_i^{(x)}[H_i^{(n)}(b_i^{(n)})-H_i^{(n)}(a_i^{(n)})] & \textit{for}\quad x\,\epsilon\, I_k^{(n)},\ k=1,2,..., \end{cases}$$

the sum $\sum_i^{(x)}$ being extended over all the values i such that $b_i^{(n)}\leqslant x$.

The sequence $\{F_n(x)\}$ being thus defined, let

$$F(x)=\sum_n F_n(x)/2^n. \tag{9.11}$$

The function $F(x)$ is clearly continuous, increasing, and singular on J_0. Consider the set $E=\prod_n\sum_k (I_k^{(n)})^\circ$, and let x_0 be any point of E. Then there exists a sequence $\{I_{k_n}^{(n)}\}_{n=1,2,...}$ of intervals each of which contains x_0 in its interior. Plainly, for each positive integer n, $F_j(b_{k_n}^{(n)})-F_j(x_0)=0$ if $j\leqslant n$, and $F_j(b_{k_n}^{(n)})-F_j(x_0)\leqslant b_{k_n}^{(n)}-x_0$ if $j>n$. Hence, by (9.11), $F(b_{k_n}^{(n)})-F(x_0)\leqslant(b_{k_n}^{(n)}-x_0)/2^n$ for each n, and therefore $\underline{F}^+(x_0)=0$.

Nevertheless, the function F is not ACG on E. To see this, suppose, if possible, that E is the sum of a sequence of sets E_n on each of which the function F is AC. Since the set J_0-E is the sum of a sequence of non-dense closed sets, one at least of the sets E_n is everywhere dense in a sub-interval I of J_0, and, since the function F is absolutely continuous on each E_n, this function would be so also on the whole interval I. This is clearly impossible, for the function F is singular and increasing.

§ 10. A theorem of Denjoy-Khintchine on approximate derivates. The considerations of the preceding § will now be completed by a theorem which establishes, for the extreme approximate derivates, relations similar to those which hold for Dini derivates (cf. § 4). This theorem was proved independently by A. Denjoy [6, p. 209] and by A. Khintchine [4; 5, p. 212] (cf. also J. C. Burkill and U. S. Haslam-Jones [1; 3]).

(10.1) ***Theorem.*** *If a finite function F is measurable on a set E and if, to each point x of E, there corresponds a measurable set $Q(x)$ such that* (i) *the lower unilateral density of $Q(x)$ at x is positive on at least one side of the point x and* (ii) $\overline{F}_{Q(x)}(x)<+\infty$ *or* $\underline{F}_{Q(x)}(x)>-\infty$, *then the function F is approximately derivable at almost all the points of E.*

Consequently, if a finite function F is measurable on a set E, then at almost every point of E either the function F is approximately derivable, or else $\overline{F}^+_{\mathrm{ap}}(x)=\overline{F}^-_{\mathrm{ap}}(x)=+\infty$ *and* $\underline{F}^+_{\mathrm{ap}}(x)=\underline{F}^-_{\mathrm{ap}}(x)=-\infty$.

Proof. In view of Lusin's Theorem (Chap. III, § 7), we may suppose that the set E is closed and that the function F is continuous on E. To fix the ideas, consider the set A of the points $x\,\epsilon\, E$ such that (i_1) the lower right-hand density of $Q(x)$ at x is positive and (ii_1) $\overline{F}_{Q(x)}(x)<+\infty$. We shall show that the function F is approximately derivable at almost all the points of A. By symmetry, this assertion will remain valid for each of the other three subsets of E, defined by a similar specification of the conditions (i) and (ii) of the theorem.

Let us denote by P the set of the points of A at which the function F is not approximately derivable, and suppose, if possible, that $|P|>0$. For each positive integer n, let A_n be the set of the points x of E such that the inequality $0\leqslant h\leqslant 1/n$ implies

$$|\mathop{\mathrm{E}}_t[F(t)-F(x)\leqslant n\cdot(t-x);\ t\,\epsilon\, E;\ x\leqslant t\leqslant x+h]|\geqslant h/n. \tag{10.2}$$

The sets A_n are closed. To see this, let us keep an index n fixed for the moment, and let $\{x_i\}_{i=1,2,...}$ be a sequence of points of the set A_n converging to a point x_0. Let $h\leqslant 1/n$ be a non-negative number, and, for brevity, let $E_i=\mathop{\mathrm{E}}_t[F(t)-F(x_i)\leqslant n\cdot(t-x_i);\ t\,\epsilon\, E;\ x_i\leqslant t\leqslant x_i+h]$ where $i=0,1,2,...$. We obtain $|E_i|\geqslant h/n$ for $i=1,2,...$, and since, by continuity of F on E, we have $E_0\supset\limsup_i E_i$, it follows (cf. Chap. I, Theorem 9.1) that $|E_0|\geqslant h/n$, which shows that $x_0\,\epsilon\, A_n$, i. e. that A_n is a closed set.

Let us now denote, for every pair of positive integers n and k, by $A_{n,k}$ the set of the points x of A_n such that the inequality $0\leqslant h\leqslant 1/k$ implies

$$(10.3)\qquad |\underset{t}{\mathrm{E}}[t\,\epsilon\,A_n;\ x-h\leqslant t\leqslant x]|\geqslant(1-\tfrac{1}{2}n^{-1})\cdot h.$$

We observe easily that the sets A_n, and therefore the sets $A_{n,k}$ also, cover the set A almost entirely. Hence, there exists a pair of positive integers n_0 and k_0 such that $|A_{n_0,k_0}\cdot P|>0$. Let R denote a portion of the set $A_{n_0,k_0}\cdot P$ such that

$$(10.4)\quad |R|>0,\qquad (10.5)\quad \delta(R)<1/n_0\qquad\text{and}\qquad (10.6)\quad \delta(R)<1/k_0.$$

Writing $G(x)=F(x)-(n_0+1)\cdot x$, we shall show that the function G is monotone non-increasing on R. Suppose therefore, if possible, that there exist two points a and b in R, where $a<b$, such that

$$(10.7)\qquad G(a)<G(b).$$

Let $J=[a,b]$. Since the set A_{n_0} is closed and the function G continuous on A_{n_0}, the function G attains, at a point c of the set $A_{n_0}\cdot J$, the lower bound of its values on this set. In virtue of (10.7) we have $c<b$. Since $c\,\epsilon\,A_{n_0}$, and since, by (10.5), $0\leqslant b-c\leqslant 1/n_0$, we may put $n=n_0$, $x=c$ and $h=b-c$ in the relation (10.2). We thus obtain

$$(10.8)\qquad |\underset{t}{\mathrm{E}}[G(t)-G(c)\leqslant-(t-c);\ t\,\epsilon\,E;\ c\leqslant t\leqslant b]|\geqslant(b-c)/n_0.$$

Again, since $b\,\epsilon\,R\subset A_{n_0,k_0}$ and since, by (10.6), $0\leqslant b-c\leqslant 1/k_0$, we may put $n=n_0$, $x=b$ and $h=b-c$ in (10.3). This gives

$$(10.9)\qquad |\underset{t}{\mathrm{E}}[t\,\epsilon\,A_{n_0};\ c\leqslant t\leqslant b]|\geqslant(1-\tfrac{1}{2}n_0^{-1})\cdot(b-c).$$

Now the sets which occur in the relations (10.8) and (10.9) are both measurable; it therefore follows from these relations that there exist, in the open interval (c,b), points $t\,\epsilon\,A_{n_0}$ for which $G(t)-G(c)\leqslant-(t-c)<0$. This is plainly impossible, since the function G attains its minimum on the set $A_{n_0}\cdot J$ at the point c.

The function G is thus monotone on the set R, and since it is, moreover, measurable (indeed continuous) on the closed set $E\supset R$, it follows that G is approximately derivable at almost all the points of R. On the other hand, however, since $R\subset P$ the function F is approximately derivable at no point of R, and, in view of (10.4), we arrive at a contradiction. This completes the proof.

By a slight modification of the proof, we may extend Theorem 10.1, in a certain way, to functions which need not be measurable. Let us agree to understand by *approximate derivability* of a finite function F at a point x_0, the existence of a set for which x_0 is a point of outer density and with respect to which the function F is derivable (if the function F is measurable, this notion of approximate derivability clearly agrees with the definition of Chap. VII, § 3). When approximate derivability is interpreted thus in the statement of Theorem 10.1, this theorem remains valid without the hypothesis that the function F be measurable on the set E (although the hypothesis concerning the measurability of the sets $Q(x)$ remains essential).

From Theorem 10.1, we may deduce the following proposition: *If, for a finite function F, we can make correspond to each point x of a set E, a measurable set $Q(x)$ whose lower right-hand density at x is positive, and with respect to which the function has an infinite derivative at x, then the set E is of measure zero.* This theorem is similar to Theorem 4.4, but only partially generalizes the latter. It is not actually possible to replace, in Theorem 4.4, the ordinary, by the approximate, limit, without also removing the modulus sign in the expression $|F(x+h)-F(x)|$. This rather unexpected fact was brought to light by V. Jarník [2], who showed that there exist continuous functions F for which the relation $\lim\limits_{h\to 0+}\mathrm{ap}\,|F(x+h)-F(x)|/h=+\infty$ holds at almost all points x.

Finally, let us note that Theorem 10.1 is frequently stated in the following form:

If a finite function F is measurable on a set E, then at almost every point x of E either (i) *the function F is approximately derivable, or else* (ii) *there exists a measurable set $R(x)$ whose right-hand and left-hand upper densities are both equal to 1 at x, and with respect to which the two upper unilateral derivates of F at x are $+\infty$ and the two lower derivates $-\infty$.*

It has been shown by A. Khintchine [4] (cf. also V. Jarník [2]) that *there exist continuous functions for which the case* (ii) *holds at almost every point x.*

§ 11. Approximate partial derivates of functions of two variables. The §§ which follow will be devoted to generalizations of the results of § 4 for functions of two real variables (their extension to any number of variables presents, as already said, no fresh difficulty). In this § we shall establish some subsidiary results.

Given a plane set Q and a number η, we shall understand by the *outer linear measure* of Q *on the line* $y=\eta$, the measure of the linear set $\underset{t}{\mathrm{E}}[(t,\eta)\,\epsilon\,Q]$. Similarly, we define the *outer linear measure* of Q *on a line* $x=\xi$, where ξ is any number. It follows from Fubini's Theorem in the form (8.6), Chap. III, that if Q is a measurable set whose linear measure on almost all the lines $y=\eta$ (i. e. on the lines $y=\eta$ for almost all values of η) is zero, then the set Q is of plane measure zero.

A point (x_0, y_0) will be termed *point of linear density* of a plane set Q *in the direction of the x-axis*, if x_0 is a point of density of the linear set $\underset{t}{\mathrm{E}}[(t, y_0)\,\epsilon\, Q]$. We define similarly the *points of linear density of Q in the direction of the y-axis.*

(11.1) ***Theorem.*** *Almost all points of any measurable plane set Q are points of linear density for it both in the direction of the x-axis and in that of the y-axis.*

Proof. We may clearly assume that the set Q is closed. Consider, to fix the ideas, the set D of the points of Q which are points of density of Q in the direction of the x-axis. Since the set $Q-D$ is of linear measure zero on each line $y=\eta$, the proof of the relation $|Q-D|=0$ reduces to showing that the set D is measurable.

In order to do this, we write, for each point (x,y) and each pair of numbers a and b,

$$E(x,y;\,a,b)=\underset{t}{\mathrm{E}}[(t,y)\,\epsilon\, Q;\; a\leqslant t\leqslant b],$$

and we denote, for each pair of positive integers n and k, by $Q_{n,k}$ the set of the points (x,y) of Q such that the inequalities $a<x<b$ and $b-a\leqslant 1/k$ imply $|E(x,y;\,a,b)|\geqslant(1-n^{-1})\cdot(b-a)$. Plainly $D=\prod\limits_n\sum\limits_k Q_{n,k}$.

We now remark that all the sets $Q_{n,k}$ are closed. To see this, we keep the indices n and k fixed for the moment, and consider an arbitrary sequence $\{(x_i,y_i)\}_{i=1,2,\dots}$ of points of $Q_{n,k}$ which converges to a point (x_0,y_0). Let a and b denote real numbers such that $a<x_0<b$ and $b-a\leqslant 1/k$. For every sufficiently large index i, we then have $a<x_i<b$, and so $|E(x_i,y_i;\,a,b)|\geqslant(1-n^{-1})\cdot(b-a)$. Now it is easy to see that $\limsup\limits_i E(x_i,y_i;\,a,b)\subset E(x_0,y_0;\,a,b)$; it therefore follows from Theorem 9.1, Chap. I, that $|E(x_0,y_0;\,a,b)|\geqslant(1-n^{-1})\cdot(b-a)$, and so, that $(x_0,y_0)\,\epsilon\, Q_{n,k}$.

Since the sets $Q_{n,k}$ are closed, D is a set $(\mathfrak{F}_{\sigma\delta})$ and this completes the proof.

If F is a finite function of two variables, the extreme approximate partial derivates of $F(x,y)$ with respect to x will be denoted by $\overline{F}^{+}_{\mathrm{ap}_x}$, $\underline{F}^{+}_{\mathrm{ap}_x}$, $\overline{F}^{-}_{\mathrm{ap}_x}$ and $\underline{F}^{-}_{\mathrm{ap}_x}$. If these derivates are equal at a point (x,y), their common value, i.e. the approximate partial derivative of F with respect to x, will be denoted by $F'_{\mathrm{ap}_x}(x,y)$. Analogous symbols will be used with respect to y. For the partial Dini derivates, we shall retain the notation of Chap. V, namely $\overline{F}^{+}_{x}$, $\underline{F}^{+}_{x}$, etc.

(11.2) ***Theorem.*** *If a finite function of two variables F is measurable on a set Q, its extreme approximate partial derivates are themselves measurable on Q.*

Proof. In view of Lusin's Theorem (Chap. III, § 7), we may suppose that the set Q is closed and that the function F is continuous on Q. Consider, to fix the ideas, the derivate $\overline{F}^{+}_{\mathrm{ap}_x}$. Let a be any finite number and let P be the set of the points (x,y) of Q at which $\overline{F}^{+}_{\mathrm{ap}_x}(x,y)\leqslant a$. We have to prove that the set P is measurable.

For this purpose, let D denote the set of the points of the set Q which are its points of linear density in the direction of the x-axis. Further, for every point (x,y) and every positive integer n, let $E_n(x,y)$ denote the set of the points t such that

$$t\geqslant x,\quad (t,y)\,\epsilon\, Q\quad \textit{and}\quad F(t,y)-F(x,y)\leqslant(a+n^{-1})\cdot(t-x).$$

We easily observe (cf. Chap. VII, § 3) that, in order that $\overline{F}^{+}_{\mathrm{ap}_x}(x_0,y_0)\leqslant a$ at a point $(x_0,y_0)\,\epsilon\, D$, it is necessary and sufficient that the point (x_0,y_0) be a point of right-hand density for every set $E_n(x_0,y_0)$, where $n=1,2,\dots$ Hence, denoting for every system of three positive integers n,k and p, by $Q_{n,k,p}$ the set of the points (x,y) of Q such that the inequality $0\leqslant h\leqslant 1/p$ implies $|E_n(x,y)\cdot[x,x+h]|\geqslant(1-k^{-1})\cdot h$, we have

$$P\cdot D=\prod_n\prod_k\sum_p Q_{n,k,p}. \tag{11.3}$$

Now the set Q is closed and the function F is continuous on Q, and by means of Theorem 9.1, Chap. I (cf. the proofs of Theorems 10.1 and 11.1) we easily prove that all the sets $Q_{n,k,p}$ are closed. Hence, by (11.3), the set $P\cdot D$ is measurable, and since, by Theorem 11.1, $|Q-D|=0$, we see that the set P is measurable also. This completes the proof.

It follows, in particular, from Theorem 11.2 that *the extreme approximate derivates of any finite measurable function of one real variable are themselves measurable functions.* We thus obtain a result analogous to Theorem 4.3, Chap. IV, which concerned the measurability of Dini derivates (cf. also Theorem 4.1, Chap. V, and the remark p. 171).

§ 12. Total and approximate differentials. A finite function of two real variables F is termed *totally differentiable*, or simply *differentiable*, at a point (x_0, y_0) if there exist two finite numbers A and B such that the ratio

$$(12.1)\quad [F(x,y)-F(x_0,y_0)-A\cdot(x-x_0)-B\cdot(y-y_0)]/[|x-x_0|+|y-y_0|]$$

tends to zero as $(x,y)\to(x_0,y_0)$. The pair of numbers $\{A,B\}$ is then termed *total differential* of the function F at the point (x_0,y_0) and we see at once that A and B are the partial derivatives of F at (x_0,y_0) with respect to x and to y respectively.

If, for a finite function of two variables F and for a point (x_0,y_0), there exist two finite numbers A and B such that the ratio (12.1) tends approximately to 0 as $(x,y)\to(x_0,y_0)$, the function F is termed *approximately differentiable* at (x_0,y_0) and the pair of numbers $\{A,B\}$ is called *approximate differential* of F at (x_0,y_0). The numbers A and B will be called *coefficients* of this differential.

We see at once that no function can have at a given point more than one differential, whether total or approximate.

The existence of a total differential of a function $F(x,y)$ at a point may be interpreted as the existence of a plane, tangent at this point to the surface $z=F(x,y)$ and non-perpendicular to the xy-plane. In this way the notion of total differentiability of functions of two variables corresponds exactly to the similar notion of derivability of functions of one variable. Nevertheless, whereas every function of bounded variation of one variable is almost everywhere derivable, a function of bounded variation (in the Tonelli sense), and even an absolutely continuous function, of two variables may be nowhere totally differentiable (cf. W. Stepanoff [3, p. 515]).

The coefficients of an approximate differential of a function at a point are not, in general, approximate partial derivatives of this function. Nevertheless they coincide with the latter almost everywhere, as results from the following theorem:

(12.2) ***Theorem.*** *In order that a finite function of two variables F, which is measurable on a set Q, be approximately differentiable at almost all the points of this set, it is necessary and sufficient that F be, almost everywhere in Q, approximately derivable with respect to each variable.*

When this is the case, the approximate partial derivates $F'_{\mathrm{ap}_x}(x,y)$ and $F'_{\mathrm{ap}_y}(x,y)$ are, at almost all the points (x,y) of Q, the coefficients of the approximate differential of F.

Proof. 1° Suppose that the function F is approximately differentiable at almost all the points of Q. We denote, for each positive integer n, by R_n the set of the points (ξ,η) of Q such that, for every square J containing (ξ,η), we have

$$(12.3)\quad \Big|\underset{(x,y)}{\mathrm{E}}[|F(x,y)-F(\xi,\eta)|\leqslant n\cdot\delta(J);\ (x,y)\,\epsilon\, J]\Big|\geqslant 3\cdot|J|/4$$

whenever $\delta(J)\leqslant 2/n$. Writing $R=\sum_n R_n$, we clearly have $|Q-R|=0$.

Let us now denote, for a general plane set E and any number η, by $E^{[\eta]}$ the linear set of the points ξ such that $(\xi,\eta)\,\epsilon\, E$. Keeping fixed, for the moment, a positive integer n_0 and a real number η_0, we consider any two points ξ_1 and ξ_2 of $R_{n_0}^{[\eta_0]}$ for which $0\leqslant\xi_2-\xi_1\leqslant 1/n_0$, and we denote by J_0 the square $[\xi_1,\xi_2;\ \eta_0,\eta_0+\xi_2-\xi_1]$. We then have $\delta(J_0)\leqslant 2/n_0$, and so, putting $n=n_0$, $J=J_0$, $\eta=\eta_0$ in (12.3), and choosing $\xi=\xi_1$ and $\xi=\xi_2$ successively, we see at once that the square J_0 contains points (x,y) for which we have at the same time

$$|F(\xi_1,\eta_0)-F(x,y)|\leqslant n_0\cdot\delta(J_0)\leqslant 2n_0\cdot(\xi_2-\xi_1)$$

and

$$|F(\xi_2,\eta_0)-F(x,y)|\leqslant n_0\cdot\delta(J_0)\leqslant 2n_0\cdot(\xi_2-\xi_1).$$

Hence $|F(\xi_2,\eta_0)-F(\xi_1,\eta_0)|\leqslant 4n_0\cdot|\xi_2-\xi_1|$, which shows that, for any fixed η, $F(x,\eta)$, as a function of x, is AC on each set $R_n^{[\eta]}$, and so VBG on the whole set $R^{[\eta]}$ (cf. Chap. VII, § 5). Now R is (with Q) a plane measurable set, so that the linear set $R^{[\eta]}$ is measurable for almost every η. Hence (cf. Theorem 4.3, Chap. VII), for almost all η, the function $F(x,\eta)$ is approximately derivable with respect to x at almost all the points of $R^{[\eta]}$. Since further, by Theorem 11.2, the set of the points of R at which the function F is approximately derivable with respect to one variable, is measurable, it follows at once that the function F is approximately derivable with respect to x at almost all the points of R, and so, at the same time, at almost all the points of Q. Similarly, we establish the corresponding result concerning approximate derivability of F with respect to y.

2° Suppose that the function F is approximately derivable, at almost all the points of Q, with respect to x and with respect to y. We shall show that the function F then has, at almost all the points of Q, an approximate differential with coefficients $F'_{\mathrm{ap}_x}(x,y)$ and $F'_{\mathrm{ap}_y}(x,y)$. On account of Theorem 11.2 and of Lusin's Theorem (Chap. III, § 7), we may suppose that

(a) the set Q is bounded and closed, (b) the function F is approximately derivable with respect to each variable at all the points of Q, and (c) the function F, and both its approximate partial derivatives, are continuous on Q.

This being so, we write, for each point (ξ,η) of Q and each point (x,y) of the plane,

$$(12.4)\quad \begin{aligned} D(\xi,\eta;x,y)&=|F(x,y)-F(\xi,\eta)-(x-\xi)\cdot F'_{\mathrm{ap}_x}(\xi,\eta)-(y-\eta)\cdot F'_{\mathrm{ap}_y}(\xi,\eta)|,\\ D_1(\xi,\eta;x)&=|F(x,\eta)-F(\xi,\eta)-(x-\xi)\cdot F'_{\mathrm{ap}_x}(\xi,\eta)|,\\ D_2(\xi,\eta;y)&=|F(\xi,y)-F(\xi,\eta)-(y-\eta)\cdot F'_{\mathrm{ap}_y}(\xi,\eta)|.\end{aligned}$$

Let ε and τ be any positive numbers. We shall begin by defining a positive number σ and a closed subset A of Q such that $|Q-A|<\varepsilon$ and such that, for any point (ξ,η),

$$\text{(i)}\quad \begin{gathered}|\mathop{\mathrm{E}}_x[D_1(\xi,\eta;x)\leqslant\tau\cdot|x-\xi|;\,(x,\eta)\,\epsilon\, Q;\,a\leqslant x\leqslant b]|\geqslant(1-\varepsilon)\cdot(b-a)\\ \textit{whenever}\quad (\xi,\eta)\,\epsilon\, A,\quad a\leqslant\xi\leqslant b\quad \textit{and}\quad b-a<\sigma.\end{gathered}$$

For this purpose, let us denote, for each positive integer n, by A_n the set of the points (ξ,η) of Q such that the inequality in the first line of (i) is fulfilled whenever $a<\xi<b$ and $b-a<1/n$. Since the set Q is closed and since the function F and its derivatives F'_{ap_x} and F'_{ap_y} are continuous on Q, it is easily seen that all the sets A_n are closed. On the other hand, the sets A_n form an ascending sequence and we immediately see that the set $Q-\lim_n A_n$ is of measure zero on each line $y=\eta$. Hence, this set being measurable, we have $|Q-\lim_n A_n|=0$. Consequently $|Q-A_{n_0}|<\varepsilon$ for a sufficiently large index n_0, and writing $\sigma=1/n_0$ and $A=A_{n_0}$ we find that the inequality $|Q-A|<\varepsilon$ and the condition (i) are both satisfied.

In exactly the same way, but replacing the set Q by A and interchanging the rôle of the coordinates x and y, we determine now a positive number $\sigma_1<\sigma$ and a closed subset B of A such that $|A-B|<\varepsilon$ and that for any point (ξ,η)

$$\text{(ii)}\quad \begin{gathered}|\mathop{\mathrm{E}}_y[D_2(\xi,\eta;y)\leqslant\tau\cdot|y-\eta|;\,(\xi,y)\,\epsilon\, A;\,a\leqslant y\leqslant b]|\geqslant(1-\varepsilon)\cdot(b-a)\\ \textit{whenever}\quad (\xi,\eta)\,\epsilon\, B,\quad a\leqslant\eta\leqslant b\quad \textit{and}\quad b-a<\sigma_1.\end{gathered}$$

Finally, let $\sigma_2<\sigma_1$ be a positive number such that

$$|F'_{\mathrm{ap}_x}(x_2,y_2)-F'_{\mathrm{ap}_x}(x_1,y_1)|<\tau$$

for any pair of points (x_1,y_1) and (x_2,y_2) of Q subject to the conditions $|x_2-x_1|<\sigma_2$ and $|y_2-y_1|<\sigma_2$.

This being so, let (ξ_0,η_0) be any point of B. Let $J=[a_1,\beta_1;\,a_2,\beta_2]$ denote any interval such that $(\xi_0,\eta_0)\,\epsilon\, J$ and $\delta(J)<\sigma_2<\sigma_1<\sigma$. We write:

$$E_2=\mathop{\mathrm{E}}_y[D_2(\xi_0,\eta_0;y)\leqslant\tau\cdot|y-\eta_0|;\,(\xi_0,y)\,\epsilon\, A;\,a_2\leqslant y\leqslant\beta_2],$$

and, for each y,

$$E_1(y)=\mathop{\mathrm{E}}_x[D_1(\xi_0,y;x)\leqslant\tau\cdot|x-\xi_0|;\,(x,y)\,\epsilon\, Q;\,a_1\leqslant x\leqslant\beta_1].$$

Then any point (x,y) such that $y\,\epsilon\, E_2$ and $x\,\epsilon\, E_1(y)$ belongs to the set $Q\cdot J$ and, for such a point, we have

$$\begin{aligned}D(\xi_0,\eta_0;x,y)&\leqslant D_1(\xi_0,y;x)+D_2(\xi_0,\eta_0;y)+|x-\xi_0|\cdot|F'_{\mathrm{ap}_x}(\xi_0,y)-F'_{\mathrm{ap}_x}(\xi_0,y_0)|\leqslant\\ &\leqslant 2\tau\cdot[|x-\xi_0|+|y-\eta_0|].\end{aligned}$$

On the other hand, it follows at once from (ii) and (i) respectively, that $|E_2|\geqslant(1-\varepsilon)\cdot(\beta_2-a_2)$, and $|E_1(y)|\geqslant(1-\varepsilon)\cdot(\beta_1-a_1)$ whenever $y\,\epsilon\, E_2$. Hence, $D(\xi_0,\eta_0;x,y)$ being a measurable (indeed continuous) function of the point (x,y) on $Q\cdot J$, it follows that the set of the points $(x,y)\,\epsilon\, Q\cdot J$ such that $D(\xi_0,\eta_0;x,y)\leqslant 2\tau\cdot[|x-\xi_0|+|y-\eta_0|]$ is of measure at least equal to $(1-\varepsilon)^2\,(\beta_1-a_1)\,(\beta_2-a_2)=(1-\varepsilon)^2\cdot|J|$. The point (ξ_0,η_0) here denotes any point of the set B, and J any interval, containing (ξ_0,η_0), whose diameter is sufficiently small. Therefore, since $|Q-B|\leqslant|Q-A|+|A-B|\leqslant 2\varepsilon$, where ε is at our disposal, we see that, for every positive number τ, almost every point (ξ,η) of Q is a point of density for the set of the points (x,y) of Q which fulfil the inequality $D(\xi,\eta;x,y)/[|x-\xi|+|y-\eta|]\leqslant 2\tau$; and in view of (12.4), this completes the proof.

We notice a similarity between the preceding proof and that of the "Density Theorem" (Chap. IV, § 10). Actually the result just established constitutes a direct generalization of the Density Theorem. To see this, we need only interpret, in the statement of Theorem 12.2, the function F as the characteristic function of the set Q (cf. the first edition of this book, p. 231).

The notion of approximate differential, together with Theorem 12.2, are due to W. Stepanoff [3]. There is, however, a slight difference between the definition adopted here and that of Stepanoff, so that, in its original form, as proved by Stepanoff, Theorem 12.2 generalizes Theorem 6.1, Chap. IV, rather than the Density Theorem of § 10, Chap. IV.

We conclude this § by mentioning the following theorem, which, in view of Theorems 9.9 and 11.2, is an immediate consequence of Theorem 12.2:

(12.5) ***Theorem.*** *Suppose that a finite function of two variables F which is measurable on a set Q, has at each point of Q at least one finite Dini derivate with respect to x and at least one finite Dini derivate with respect to y.*

Then the function F is approximately differentiable at almost every point of Q.

§ 13. Fundamental theorems on the contingent of a set in space. Following F. Roger [2], we shall now extend to sets in the space $\boldsymbol{R}_3$, the results obtained in § 3. The proofs will be largely a repetition of those of § 3 with the obvious verbal changes. We shall therefore present them in a slightly more condensed form.

Generalizing the definitions of § 3, p. 264, to functions of two variables, we shall say that a function $F(x,y)$ fulfils the *Lipschitz condition* on a plane set E, if there exists a finite constant N such that $|F(x_2, y_2)-F(x_1, y_1)| \leqslant N\cdot[|x_2-x_1|+|y_2-y_1|]$ for every two points (x_1, y_1) and (x_2, y_2) of E. We verify at once that the graph of the function F on E is then of finite area whenever $|E|<+\infty$, and of area zero whenever, in particular, $|E|=0$ (cf. Chap. II, § 8; more precisely, we have, for every set E, $\Lambda_2\{\mathrm{B}(F;E)\}\leqslant 4\cdot(1+N^2)\cdot|E|$).

In the sequel we shall make use of the following notation for limits relative to a set. If E is a set (in any space) and t_0 is a point of accumulation for E, the lower and upper limits of a function $F(t)$ as t tends to t_0 on E will be written $\liminf\limits_{t\to t_0}{}_E F(t)$ and $\limsup\limits_{t\to t_0}{}_E F(t)$ respectively. Their common value, when they are equal, will be written $\lim\limits_{t\to t_0}{}_E F(t)$.

(13.1) ***Lemma.*** *Let R be a set in the space $\boldsymbol{R}_3$, θ a fixed direction in this space and P the set of the points a of R at which* $\mathrm{contg}_R a$ *contains no half-line of direction θ. Then* (i) *the set P is the sum of a sequence of sets of finite area and* (ii) *at each point a of P, except at most at those of a subset of area zero, the set R has an extreme tangent plane, for which the side containing the half-line $a\theta$ is its empty side.*

In the particular case in which θ is the direction of the positive semi-axis of z, the set P is expressible as the sum of an enumerable infinity of sets each of which is the graph of a function on a plane set on which the function fulfils the Lipschitz condition.

Proof. We may clearly suppose (in the first part of the theorem also) that θ is the direction of the positive semi-axis of z. We denote, for every positive integer n, by P_n the set of the points (x,y,z) of P such that the inequalities $|x'-x|\leqslant 1/n$, $|y'-y|\leqslant 1/n$ and $|z'-z|\leqslant 1/n$ imply $z'-z\leqslant n\cdot[|x'-x|+|y'-y|]$ for every point (x',y',z') of R. We express, further, each P_n as the sum of a sequence $\{P_{n,k}\}_{k=1,2,\ldots}$ of sets with diameters less than $1/n$. For every pair of points (x_1,y_1,z_1) and (x_2,y_2,z_2) of the same set $P_{n,k}$, we thus have $|z_2-z_1|\leqslant n\cdot[|x_2-x_1|+|y_2-y_1|]$, and if we denote by $Q_{n,k}$ the orthogonal projection of $P_{n,k}$ on the xy-plane, we easily see that the set $P_{n,k}$ may be regarded as the graph of a function $F_{n,k}$ on $Q_{n,k}$. Plainly $|F_{n,k}(x_2,y_2)-F_{n,k}(x_1,y_1)|\leqslant n\cdot[|x_2-x_1|+|y_2-y_1|]$ for every two points (x_1,y_1) and (x_2,y_2) of $Q_{n,k}$. Thus $F_{n,k}$ fulfils the Lipschitz condition on $Q_{n,k}$ and hence (cf. p. 304) $\Lambda_2(P_{n,k})=\Lambda_2\{\mathrm{B}(F_{n,k};Q_{n,k})\}<+\infty$. Thus $P=\sum\limits_{n,k}P_{n,k}$ is the required expression of the set P.

It remains to discuss the existence of an extreme tangent plane to R at the points of P. For a fixed pair of positive integers n and k, the function $F_{n,k}$, which fulfils the Lipschitz condition on the set $Q_{n,k}$, can be continued at once, by continuity, on to the closure $\bar{Q}_{n,k}$ of this set, and then on to the whole plane by writing $F_{n,k}(x,y)=0$ outside $\bar{Q}_{n,k}$. On account of Theorem 12.2, the function $F_{n,k}$ is approximately differentiable at almost all the points of $\bar{Q}_{n,k}$. Hence, denoting by $\widetilde{Q}_{n,k}$ the subset of $Q_{n,k}$ consisting of the points of density of $Q_{n,k}$ at which $F_{n,k}$ is approximately differentiable, we see that $|Q_{n,k}-\widetilde{Q}_{n,k}|=0$ and hence, that $\Lambda_2\{\mathrm{B}(F_{n,k};Q_{n,k}-\widetilde{Q}_{n,k})\}=0$. We need, therefore, only show that R has an extreme tangent plane at each point of $\mathrm{B}(F_{n,k};\widetilde{Q}_{n,k})$, and that, further, the half-line with the direction of the positive semi-axis of z is contained in the empty side of this plane.

Let (ξ_0,η_0,ζ_0) be any point of $\mathrm{B}(F_{n,k};\widetilde{Q}_{n,k})$ and let $\{A_0,B_0\}$ be the approximate differential of $F_{n,k}$ at the point (ξ_0,η_0). Let $\varepsilon<1$ be any positive number, and let E_ε be the set of the points $(x,y)\in Q_{n,k}$ such that

$$|F_{n,k}(x,y)-F_{n,k}(\xi_0,\eta_0)-A_0\cdot(x-\xi_0)-B_0\cdot(y-\eta_0)|\leqslant\varepsilon\cdot[|x-\xi_0|+|y-\eta_0|].$$

Since the function $F_{n,k}$ is measurable, (ξ_0,η_0) is (cf. Chap. VII, § 3) a point of outer density for the set E_ε. Hence we can make correspond to each point (ξ,η,ζ), sufficiently near to (ξ_0,η_0,ζ_0), a point $(\xi',\eta')\in E_\varepsilon$ such that:

$$(13.2)\qquad |\xi'-\xi_0|\leqslant|\xi-\xi_0| \quad \textit{and} \quad |\eta'-\eta_0|\leqslant|\eta-\eta_0|,$$

$$(13.3)\qquad |\xi'-\xi|\leqslant\varepsilon\cdot|\xi-\xi_0| \quad \textit{and} \quad |\eta'-\eta|\leqslant\varepsilon\cdot|\eta-\eta_0|.$$

Remembering that $\zeta_0=F_{n,k}(\xi_0,\eta_0)$, we now write for brevity $D_{n,k}(\xi',\eta')=F_{n,k}(\xi',\eta')-\zeta_0-A_0\cdot(\xi'-\xi_0)-B_0\cdot(\eta'-\eta_0)$. We thus have

$$(13.4)\qquad \begin{aligned}&\zeta-\zeta_0-A_0\cdot(\xi-\xi_0)-B_0\cdot(\eta-\eta_0)=\\ &=D_{n,k}(\xi',\eta')+[\zeta-F_{n,k}(\xi',\eta')]+[A_0\cdot(\xi'-\xi)+B_0\cdot(\eta'-\eta)].\end{aligned}$$

This being so, let (ξ,η,ζ) be a point of R such that each of the differences $|\xi-\xi_0|$, $|\eta-\eta_0|$ and $|\zeta-\zeta_0|$ is less than, or equal to, $1/4n^2$. Then by (13.3), we have $|\xi'-\xi|\leqslant 1/n$ and $|\eta'-\eta|\leqslant 1/n$, while, by (13.2), $|F_{n,k}(\xi',\eta')-\zeta_0|\leqslant n\cdot[|\xi'-\xi_0|+|\eta'-\eta_0|]\leqslant 1/2n$, and so $|F_{n,k}(\xi',\eta')-\zeta|\leqslant 1/n$. Since the point $(\xi',\eta',F_{n,k}(\xi',\eta'))$ belongs to $\mathrm{B}(F;E_\varepsilon)\subset P_n$, it follows that $\zeta-F_{n,k}(\xi',\eta')\leqslant n\cdot[|\xi-\xi'|+|\eta-\eta'|]$, and, again making use of (13.3), we deduce from (13.4) that

$$(13.5)\qquad \begin{aligned}&\zeta-\zeta_0-A_0\cdot(\xi-\xi_0)-B_0\cdot(\eta-\eta_0)\leqslant\\ &\leqslant|D_{n,k}(\xi',\eta')|+(n+|A_0|)\cdot|\xi'-\xi|+(n+|B_0|)\cdot|\eta'-\eta|\leqslant\\ &\leqslant|D_{n,k}(\xi',\eta')|+\varepsilon\cdot(|A_0|+|B_0|+n)\cdot[|\xi-\xi_0|+|\eta-\eta_0|].\end{aligned}$$

We now observe that, since $(\xi',\eta')\in E_\varepsilon$, (13.2) implies

$$|D_{n,k}(\xi',\eta')|/[|\xi-\xi_0|+|\eta-\eta_0|]\leqslant|D_{n,k}(\xi',\eta')|/[|\xi'-\xi_0|+|\eta'-\eta_0|]\leqslant\varepsilon.$$

Hence, ε being an arbitrary positive number, we derive from (13.5)

$$(13.6)\qquad \limsup_{(\xi,\eta,\zeta)\to(\xi_0,\eta_0,\zeta_0)}{}_R\ [\zeta-\zeta_0-A_0\cdot(\xi-\xi_0)-B_0\cdot(\eta-\eta_0)]/[|\xi-\xi_0|+|\eta-\eta_0|]\leqslant 0.$$

Moreover, since $\{A_0,B_0\}$ is the approximate differential of the function $F_{n,k}$ at (ξ_0,η_0) and since the point (ξ_0,η_0) is a point of outer density for the set $Q_{n,k}$, the plane $z-\zeta_0-A_0\cdot(x-\xi_0)-B_0\cdot(y-\eta_0)=0$ is certainly an intermediate tangent plane (cf. § 2, p. 263) of R at the point (ξ_0,η_0,ζ_0). It is therefore, by (13.6), an extreme tangent plane at this point, with an empty side consisting of the half-space $z-\zeta_0\geqslant A_0\cdot(x-\xi_0)+B_0\cdot(y-\eta_0)$. This completes the proof.

We shall employ in space a terminology similar to that of the plane (cf. § 3, p. 264) and agree to say that the contingent of a set $E\subset\boldsymbol{R}_3$ at a point a of E is the *whole space* if it includes all the half-lines issuing from the point a; and again, that the contingent of E at a point a of E is a *half-space*, if E has an extreme tangent plane at a and if $\mathrm{contg}_E a$ consists of all the half-lines issuing from a which are situated on one side of this plane. We make use of these terms to state the analogue of Theorem 3.6:

(13.7) ***Theorem.*** *Given a set R in $\boldsymbol{R}_3$, let P be a subset of R at no point of which the contingent of R is the whole space. Then* (i) *the set P is the sum of an enumerable infinity of sets of finite area and* (ii) *at every point of P, except at those of a set of area zero, either the set R has a unique tangent plane, or else the contingent of R is a half-space.*

The proof of this statement, which follows directly from Lemma 13.1, is quite similar to that of Theorem 3.6. We need only replace, in the proof of the latter, the terms length, tangent and half-plane by area, tangent plane and half-space, respectively.

It only remains to extend to space, Theorem 3.7. This extension, in the form (13.11) in which we shall establish it, is essentially little more than an immediate, and almost trivial, consequence of Theorem 3.7. Its proof requires however some subsidiary considerations of the measurability of certain sets.

(13.8) ***Lemma.*** *If Q is a set $(\mathfrak{F}_{\sigma\delta})$ in $\boldsymbol{R}_3$, its orthogonal projection on the xy-plane is a measurable set.*

Proof. Let us denote generally, for every set E situated in $\boldsymbol{R}_3$, by $\Gamma(E)$ its projection on the xy-plane. In order to establish the measurability of the set $\Gamma(Q)$, it will suffice to show that for each $\varepsilon>0$ there exists a closed set $P\subset\Gamma(Q)$ such that $|P|\geqslant|\Gamma(Q)|-\varepsilon$.

We express Q as the product of a sequence $\{Q_n\}_{n=1,2,\ldots}$ of sets $(\mathfrak{F}_\sigma)$. It may clearly be assumed that the set Q is bounded and that, moreover, all the sets Q_n are situated in a fixed closed sphere S_0.

We shall define in $\boldsymbol{R}_3$, by induction, a sequence $\{F_n\}_{n=0,1,\ldots}$, of closed sets subject to the following conditions for $n=1,2,\ldots$: (i) $F_n\subset F_{n-1}$, (ii) $F_n\subset Q_n$ and (iii) $|\Gamma(F_n\cdot Q)|\geqslant|\Gamma(F_{n-1}\cdot Q)|-\varepsilon/2^n$.

For this purpose, we choose $F_0=S_0$ and we suppose that the next $r-1$ sets F_n have been defined. We have $Q\subset Q_r$, and so $F_{r-1}\cdot Q_r\cdot Q=F_{r-1}\cdot Q$, and since $F_{r-1}\cdot Q_r$ is, with Q_r, a set $(\mathfrak{F}_\sigma)$, there exists a closed set $F_r\subset F_{r-1}\cdot Q_r$ such that $|\Gamma(F_r\cdot Q)|\geqslant|\Gamma(F_{r-1}\cdot Q)|-\varepsilon/2^r$. This closed set F_r clearly fulfils (i), (ii) and (iii) for $n=r$.

Now let $F=\prod\limits_n F_n=\lim\limits_n F_n$. It follows from (ii) that $F\subset Q$, and therefore that $\Gamma(F)\subset\Gamma(Q)$. Further, $\Gamma(F)$ is a closed set, for, since $\{F_n\}$ is a descending sequence of closed and bounded sets, we easily see that $\Gamma(F)=\lim\limits_n\Gamma(F_n)$. Finally this last relation coupled with (iii) shows that $|\Gamma(F)|\geqslant\lim\limits_n|\Gamma(F_n\cdot Q)|\geqslant|\Gamma(F_0\cdot Q)|-\varepsilon=|\Gamma(Q)|-\varepsilon$, which completes the proof.

It would be easy to prove that the projection of a set $(\mathfrak{F}_{\sigma\delta})$ is the nucleus of a determining system formed of closed sets and thus to deduce Lemma 13.8 from Theorem 5.5, Chap. II. We have preferred, however, to give a direct elementary proof, based on a method due to N. Lusin [3]. The same argument shows that any continuous image of a set $(\mathfrak{F}_{\sigma\delta})$ is measurable.

It has been proved more generally (*vide*, for instance, W. Sierpiński [II, p. 149], or F. Hausdorff [II, p. 212]) that any continuous image of an analytic set (in particular, of a set measurable $(\mathfrak{B})$) situated in $\boldsymbol{R}_3$ is an analytic and, therefore, measurable set.

(13.9) ***Lemma.*** *Given a set R in $\boldsymbol{R}_3$, let Q be the set of the points (ξ,η,ζ) of R which fulfil the condition:*

(Δ) *the part of the contingent of R at the point (ξ,η,ζ), which is situated in the plane $x=\xi$, is wholly contained in one or other of the two half-spaces $y\geqslant\eta$ and $y\leqslant\eta$.*

Then the orthogonal projection of the set Q on the xy-plane is of plane measure zero.

Proof. We may clearly suppose that the set R is closed (for the contingent of any set R coincides, at all points of R, with that of the closure of R).

Let us denote generally, for any set E in $\boldsymbol{R}_3$ and any number ξ, by $E^{[\xi]}$ the set $\underset{(y,z)}{\mathrm{E}}[(\xi,y,z)\,\epsilon\, E]$. It follows from Theorem 3.6 that, for every ξ, the plane set $R^{[\xi]}$ has an extreme tangent, parallel to the z-axis at every point of $Q^{[\xi]}$ except those of a set of length zero. Hence, by Theorem 3.7, the projection of Q on the xy-plane is of linear measure zero on each line $x=\xi$ of this plane, and, in order to prove that this projection is of plane measure zero, we need only show that the latter is measurable.

Let us denote, for each pair of positive integers k and n, by $A_{k,n}$ the set of the points (ξ,η,ζ) of R such that the inequalities

(13.10) $\quad |x-\xi|+|y-\eta|+|z-\zeta|<1/n \quad$ *and* $\quad |x-\xi|<[|y-\eta|+|z-\zeta|]/n$

imply, for any point $(x,y,z)\,\epsilon\, R$, the inequality $y-\eta\leqslant[|x-\xi|+|z-\zeta|]/k$. Similarly, we shall denote by $B_{k,n}$ the set of the points (ξ,η,ζ) of R for which the inequalities (13.10) imply, for every point (x,y,z) of R, the inequality $y-\eta\geqslant-[|x-\xi|+|z-\zeta|]/k$. Writing

$$A=\prod_k\sum_n A_{k,n} \qquad \text{and} \qquad B=\prod_k\sum_n B_{k,n},$$

we find that $Q=A+B$. On the other hand, since the set R is, by hypothesis, closed, we observe at once that each set $A_{k,n}$, and likewise each set $B_{k,n}$, is closed. The sets A and B, and so the set Q also, are thus sets $(\mathfrak{F}_{\sigma\delta})$, and in view of Lemma 13.8, the projection of Q on the xy-plane is a measurable set.

(13.11) ***Theorem.*** *Given a set R in $\boldsymbol{R}_3$, let P be a subset of R at every point of which the set R has an extreme tangent plane parallel to a fixed straight line D. Then the orthogonal projection of P on the plane perpendicular to D is of plane measure zero.*

Proof. We may clearly suppose that the straight line D is the z-axis. Let us denote by P_1 the set of the points of P at which the extreme tangent plane, parallel, by hypothesis, to the z-axis is not, however, parallel to the yz-plane. Similarly, P_2 will denote the set of the points of P at which the extreme tangent plane is not parallel to the xz-plane. We then have $P=P_1+P_2$.

Now we observe at once that each point (ξ,η,ζ) of P_1 fulfils the condition (Δ) of Lemma 13.9. It therefore follows from this lemma, that the projection of P_1 on the xy-plane is of plane measure zero.

By symmetry, the same is true of the projection of the set P_2. The proof is thus complete.

§ 14. Extreme differentials. Let F be a finite function of two real variables. A pair of finite numbers $\{A,B\}$ will be called *upper differential* of F at a point (x_0,y_0) if, when we write $z_0=F(x_0,y_0)$, (i) the plane $z-z_0=A\cdot(x-x_0)+B\cdot(y-y_0)$ is an intermediate tangent plane of the graph of the function F at the point (x_0,y_0,z_0) and

$$\text{(ii)} \qquad \limsup_{(x,y)\to(x_0,y_0)}\frac{F(x,y)-F(x_0,y_0)-A\cdot(x-x_0)-B\cdot(y-y_0)}{|x-x_0|+|y-y_0|}=0.$$

These conditions may clearly be replaced by the following: (i_1) the plane $z-z_0=A\cdot(x-x_0)+B\cdot(y-y_0)$ is an extreme tangent plane of the graph of F at (x_0,y_0,z_0) with the empty side $z-z_0\geqslant A\cdot(x-x_0)+B\cdot(y-y_0)$, and ($\mathrm{ii}_1$) $\limsup\limits_{(x,y)\to(x_0,y_0)} F(x,y)\leqslant F(x_0,y_0)$.

The definition of *lower differential* is similar, and the two differentials, upper and lower, will be called *extreme differentials*.

If a function F has a total differential (cf. § 12, p. 300) at a point, this differential is both an upper and a lower differential of F at the point considered. Conversely, if a function F has at a point (x_0,y_0) both an upper and a lower differential, these are identical and then reduce to a total differential of F at (x_0,y_0).

For a finite function of one real variable F, the existence of an upper differential at a point x_0 is to be interpreted to mean that $\overline{F}^+(x_0)=\underline{F}^-(x_0)\neq\infty$ (in which case the number $\overline{F}^+(x_0)=\underline{F}^-(x_0)$ may be regarded as the upper differential of F at x_0). There is a similar interpretation for the lower differential of functions of one variable. This interpretation brings to light the relationship between the theorems of this § and those of § 4.

We propose to give an account of researches concerning the existence almost everywhere of total, approximate, or extreme differentials. These researches were begun by H. Rademacher [3], who established the first general sufficient condition in order that a continuous function be almost everywhere differentiable. W. Stepanoff [1; 3] later removed from Rademacher's reasoning certain superfluous hypotheses, and obtained a more complete result, valid for any measurable function: *In order that a function F which is measurable on a set E, should be differentiable almost everywhere in E, it is necessary and sufficient that the relation* $\limsup_{(x,y)\to(\xi,\eta)} |F(x,y)-F(\xi,\eta)|/[|x-\xi|+|y-\eta|]<+\infty$ *should hold at almost all the points (ξ,η) of E.* (Certain details of Stepanoff's proof, particularly those concerning measurability of the Dini partial derivates, have been subjected to criticism (cf. J. C. Burkill and U. S. Haslam-Jones [1].) U. S. Haslam-Jones [1] extended further the result of Stepanoff, and by introducing the notion of extreme differentials (which he called upper and lower derivate planes), obtained theorems analogous to those of Denjoy for functions of one variable. The researches of Haslam-Jones have been continued and completed by A. J. Ward [1; 4] who, in particular, removed the hypothesis of measurability in certain of Haslam-Jones's theorems.

We shall derive the results of Haslam-Jones from the theorems of the preceding § (cf. F. Roger [3]; direct proofs will be found in the memoirs of Haslam-Jones and Ward referred to, and in the first edition of this book).

In what follows, we shall make use of some subsidiary conventions of notation. If F is a function of two real variables and t denotes a point (x,y) of the plane $\boldsymbol{R}_2$, we shall frequently write $F(t)$ for $F(x,y)$. If $t_1=(x_1,y_1)$ and $t_2=(x_2,y_2)$ are two points of the plane, $|t_2-t_1|$ will denote the number $|x_2-x_1|+|y_2-y_1|$.

Given in the plane two distinct half-lines issuing from a point t_0, each of the two closed regions into which these half-lines divide the plane will be called *angle*. The point t_0 will be termed *vertex* of each of these angles.

We shall begin by proving a theorem somewhat analogous to Theorem 1.1 (ii).

(14.1) ***Theorem.*** *Let F be a finite function in the plane $\boldsymbol{R}_2$ and let E be a plane set, each point τ of which is the vertex of an angle $A(\tau)$ such that* $\limsup_{t\to\tau} {}_{A(\tau)} F(t) < \limsup_{t\to\tau} F(t)$. *Then the set E is of plane measure zero.*

Proof. Let us denote, for each pair of integers p and q, by $E_{p,q}$ the set of the points τ of E at which $\limsup_{t\to\tau} {}_{A(\tau)} F(t)<p/q<\limsup_{t\to\tau} F(t)$. For fixed p and q, we observe that no point $\tau\in E_{p,q}$ is a point of accumulation for the part of the set $E_{p,q}$ contained in the interior of the corresponding angle $A(\tau)$. Hence, no point of the set $E_{p,q}$ can be a point of outer density for this set. Each of the sets $E_{p,q}$ is thus of plane measure zero, and the same is therefore true of the whole set E.

As we easily see, in virtue of Theorem 3.6, each set $E_{p,q}$, and consequently the whole set E, is the sum of a sequence of sets of finite length (this of course, implies that E is of plane measure zero). Cf. A. Kolmogoroff and J. Verčenko [1].

(14.2) ***Theorem.*** *Let F be a finite function in the plane. Then*

(i) *if P is a plane set each point τ of which is the vertex of an angle $A(\tau)$ such that*

$$\lim_{t\to\tau} {}_{A(\tau)} |F(t)-F(\tau)|/|t-\tau|=+\infty, \tag{14.3}$$

the set P is necessarily of plane measure zero;

(ii) *if Q is a plane set each point τ of which is the vertex of an angle $A_0(\tau)$ such that*

$$\limsup_{t\to\tau} {}_{A_0(\tau)} [F(t)-F(\tau)]/|t-\tau|<+\infty, \tag{14.4}$$

the function F necessarily has an upper differential at almost all the points of Q;

(iii) *if R is a plane set each point τ of which is the vertex of two angles $A_1(\tau)$ and $A_2(\tau)$ such that*

$$\limsup_{t\to\tau} {}_{A_1(\tau)} [F(t)-F(\tau)]/|t-\tau|<+\infty$$

and

$$\liminf_{t\to\tau} {}_{A_2(\tau)} [F(t)-F(\tau)]/|t-\tau|>-\infty,$$

the function F is totally differentiable at almost all the points of R.

Proof. *re* (i). By Theorem 13.7 the set $\mathrm{B}(F;P)$ has, at each of its points except those of a subset of area zero, an extreme tangent plane. The latter is seen to be necessarily parallel to the z-axis. Hence, by Theorem 13.11, the set P, as the projection of $\mathrm{B}(F;P)$ on the xy-plane, is of plane measure zero.

re (ii). It clearly follows from (14.4) that, at each point τ of Q, we have $\limsup_{t\to\tau} {}_{A_0(\tau)} F(t)\leqslant F(\tau)$. Hence, by Theorem 14.1, we have $\limsup_{t\to\tau} F(t)\leqslant F(\tau)$ at all the points τ of Q, except at most those of a set Q_0 of measure zero.

Let us now denote by B the graph of the function F (on the whole plane). Let B_1 be the set of the points of $\mathrm{B}(F;Q)$ at which the set B has no extreme tangent plane, and B_2 the set of the points of $\mathrm{B}(F;Q)$ at which such a tangent plane exists, but is parallel to the z-axis. Finally, let Q_1 and Q_2 be the projections of the sets B_1 and B_2 respectively, on the xy-plane. On account of Theorem 13.7, we easily verify that $\Lambda_2(B_1)=0$, and so, that $|Q_1|=0$. Similarly. it follows at once from Theorem 13.11 that $|Q_2|=0$. Now, if (ξ,η) is any point of $Q-(Q_1+Q_2)$, the set B has at $(\xi,\eta,F(\xi,\eta))$ an extreme tangent plane of the form $z-\zeta=M(\xi,\eta)\cdot(x-\xi)+N(\xi,\eta)\cdot(y-\eta)$, where $M(\xi,\eta)$ and $N(\xi,\eta)$ are finite numbers. We observe further without difficulty that the half-space

$$z-\zeta\geqslant M(\xi,\eta)\cdot(x-\xi)+N(\xi,\eta)\cdot(y-\eta)$$

is an empty side of this plane. Hence (cf. p. 309), at each point (ξ,η) of the set $Q-(Q_0+Q_1+Q_2)$, the pair of numbers $\{M(\xi,\eta),N(\xi,\eta)\}$ is an upper differential of the function F. This completes the proof, since $|Q_0+Q_1+Q_2|=0$.

Finally, (iii) is an immediate consequence of (ii).

In the case in which the function F is measurable, we can complete part (i) of Theorem 14.2 (which itself generalizes Theorem 4.4). Thus, *if F is any measurable function of two variables, the set of the points (x,y) at which $\lim\limits_{h\to 0+}|F(x+h,y)-F(x,y)|/h=+\infty$, is of plane measure zero.*

This proposition plainly follows from Theorem 4.4, except for measurability considerations, essential to the proof, which seem to require general theorems on the measurability of the projections of sets $(\mathfrak{B})$ (cf. p. 308).

We conclude with the following theorem (cf. A. J. Ward [1] and the first edition of this book, p. 234) which, in view of Theorem 14.2 (i), (ii), may be regarded as an extension of Theorem 9.9 to the functions of two variables:

(14.5) ***Theorem.*** *If F is a finite function of two variables, which is measurable on a set E and which has an extreme differential at each point of a set $Q\subset E$, then this differential is, at the same time, an approximate differential of F at almost all the points of Q.*

Proof. On account of Lusin's theorem (Chap. III, § 7) we may clearly suppose that the set E is closed and that the function F is continuous on E. Let us suppose further, for definiteness, that the function F has an upper differential at each point of Q, and let us denote, for each positive integer n, by Q_n the set of the points t of Q such that, for every point t', $|t'-t|<1/n$ implies the inequality $F(t')-F(t)<n\cdot|t'-t|$. Finally, let each set Q_n be expressed as the sum of a sequence $\{Q_{n,k}\}_{k=1,2,...}$ of sets with diameters less than $1/n$. We shall have $Q=\sum\limits_{n,k}Q_{n,k}$.

We see at once that the function F fulfils the Lipschitz condition on each set $Q_{n,k}$, and therefore also on each set $\bar{Q}_{n,k}$. Hence, by Theorem 12.2, the function F has the approximate differential $\{F'_{\mathrm{ap}_x}(x,y),\ F'_{\mathrm{ap}_y}(x,y)\}$ at almost every point (x,y) of each set $\bar{Q}_{n,k}$, and therefore at almost every point (x,y) of the set Q.

Let us, on the other hand, denote, for each point (x,y) of Q, by $\{A(x,y),\ B(x,y)\}$ the upper differential of F at this point. It follows at once from the definition of upper differential, p. 309, that $\underline{F}_x^-(x,y)\geqslant A(x,y)\geqslant\overline{F}_x^+(x,y)$, and similarly $\underline{F}_y^-(x,y)\geqslant B(x,y)\geqslant\overline{F}_y^+(x,y)$, at each point (x,y) of Q. Hence, at each point (x,y) of Q at which the approximate partial derivates $F'_{\mathrm{ap}_x}(x,y)$ and $F'_{\mathrm{ap}_y}(x,y)$ exist, we have $A(x,y)=F'_{\mathrm{ap}_x}(x,y)$ and $B(x,y)=F'_{\mathrm{ap}_y}(x,y)$. The upper differential $\{A(x,y),\ B(x,y)\}$ of the function F thus coincides at almost all points (x,y) of Q with the approximate differential of F.