CHAPTER VIII.

Denjoy integrals.

§ 1. Descriptive definition of the Denjoy integrals. We shall base the study of the Denjoy integrals on their descriptive definition. The essential ideas have already been sketched in Chap. VII, § 1. We now complete them further as follows.

A function of a real variable f will be termed *$\mathfrak{D}$-integrable* on an interval $I=[a,b]$ if there exists a function F which is ACG on I and which has f for its approximate derivative almost everywhere. The function F is then called *indefinite $\mathfrak{D}$-integral* of f on I. Its increment $F(I)=F(b)-F(a)$ over the interval I is termed *definite $\mathfrak{D}$-integral of f* over I and is denoted by

$$(\mathfrak{D})\int_I f(x)\,dx \qquad \text{or} \qquad (\mathfrak{D})\int_a^b f(x)\,dx.$$

Similarly, a function f will be termed *$\mathfrak{D}_*$-integrable* on an interval $I=[a,b]$, if there exists a function F which is ACG$_*$ on I and which has f for its ordinary derivative almost everywhere. The function F is then called *indefinite $\mathfrak{D}_*$-integral* of f on I; the difference $F(I)=F(b)-F(a)$ is termed *definite $\mathfrak{D}_*$-integral* of f over I and denoted by $(\mathfrak{D}_*)\int_I f(x)\,dx$ or by $(\mathfrak{D}_*)\int_a^b f(x)\,dx$.

For uniformity of notation, the Lebesgue integral will frequently be called *$\mathfrak{L}$-integral.*

The integrals $\mathfrak{D}$ and $\mathfrak{D}_*$ are often given the names of *Denjoy integrals in the wide sense*, and *in the restricted sense*, respectively. The first of these is also termed *Denjoy-Khintchine integral* (cf. Chap. VII, § 1), and the second, *Denjoy-Perron integral* (for the latter, as we shall see below in § 3, is equivalent to the Perron integral considered in Chap. VI).

It is immediate, by Theorem 6.2, Chap. VII, that when a function is $\mathfrak{D}$- or $\mathfrak{D}_*$-integrable on an interval, its definite Denjoy integrals are uniquely determined on this interval (its indefinite integrals being determined except for an additive constant). More generally, *if two functions are equal almost everywhere and the one is integrable in the Denjoy sense (wide or restricted) on an interval* I_0, *then so is the other and the two functions have the same definite integral over* I_0. Another immediate consequence of the preceding definitions is the distributive property for Denjoy integrals. Thus, *if two functions* g *and* h *are* $\mathfrak{D}$- *or* $\mathfrak{D}_*$-*integrable on an interval* I, *the same is true of any linear combination* $ag+bh$ *of these functions, and we have*

$$(\mathfrak{D})\int_I [a\cdot g(x)+b\cdot h(x)]\,dx = a\cdot(\mathfrak{D})\int_I g(x)\,dx + b\cdot(\mathfrak{D})\int_I h(x)\,dx.$$

It follows from Theorem 10.14, Chap. VII, that *a continuous function which is approximately derivable at all points except, perhaps, at those of an enumerable set, is necessarily an indefinite* $\mathfrak{D}$-*integral of its approximate derivative.* Similarly, by Theorem 10.5. Chap. VII, *a continuous function which is derivable (in the ordinary sense) at all but an enumerable set of points, is an indefinite* $\mathfrak{D}_*$-*integral of its derivative.* The process of integration $\mathfrak{D}_*$ therefore includes that of Newton (cf. Chap. VI, § 1). The fundamental relations between the Denjoy and Lebesgue processes are given in the following

(1.1) ***Theorem.*** 1^0 *A function* f *which is* $\mathfrak{D}_*$-*integrable on an interval* I *is necessarily also* $\mathfrak{D}$-*integrable on* I *and we have* $(\mathfrak{D})\int_I f\,dx = (\mathfrak{D}_*)\int_I f\,dx$.

2^0 *A function* f *which is* $\mathfrak{L}$-*integrable on an interval* I *is necessarily* $\mathfrak{D}_*$-*integrable on* I *and we have* $(\mathfrak{D}_*)\int_I f\,dx = \int_I f\,dx$.

3^0 *A function which is* $\mathfrak{D}$-*integrable and almost everywhere non-negative on an interval* I *is necessarily* $\mathfrak{L}$-*integrable on* I.

Proof. 1^0 and 2^0 follow at once from the definitions of the Denjoy integrals and from the descriptive definition of the Lebesgue integral (Chap. VII, § 1). As regards 3^0, it is sufficient to recall the fact that, in view of Theorem 6.2, Chap. VII, a function which is ACG and whose approximate derivative is almost everywhere non-negative, is necessarily monotone non-decreasing, and therefore its derivative is summable.

Part 3^0 of Theorem 1.1 shows that for functions of constant sign the Denjoy processes are equivalent to that of Lebesgue (cf. Theorem 6.5, Chap. VI, for the corresponding result concerning the Perron integral). Hence, we derive the following further extension of Lebesgue's theorem on term by term integration of monotone sequences of functions (Chap. I, Theorem 12.6).

(1.2) ***Theorem.*** *Given a non-decreasing sequence* $\{f_n\}$ *of functions which are* $\mathfrak{D}$-*integrable on an interval* I *and whose* $\mathfrak{D}$-*integrals over* I *constitute a sequence bounded above, the function* $f(x)=\lim_n f_n(x)$ *is itself, necessarily,* $\mathfrak{D}$-*integrable on* I *and we have*

$$(\mathfrak{D})\int_I f(x)\,dx = \lim_n\,(\mathfrak{D})\int_I f_n(x)\,dx.$$

Exactly the same is true with $\mathfrak{D}_*$ *in place of* $\mathfrak{D}$ *in the hypothesis and conclusion.*

Proof. This theorem reduces at once to the theorem of Lebesgue just referred to, for we need only consider in place of the functions f_n, the functions $f_n - f_1$, which are integrable in the Denjoy sense and non-negative, and which are therefore integrable in the Lebesgue sense on account of Theorem 1.1 (3^0).

We shall show later on (Chap. IX, § 11) that the extreme approximate derivates of any measurable function are themselves measurable functions. This includes the result that any function which is $\mathfrak{D}$-integrable is measurable. In the meantime we give an independent proof of this last assertion.

(1.3) ***Theorem.*** *A function which is* $\mathfrak{D}$-*integrable is necessarily measurable and almost everywhere finite.*

Proof. Let f be $\mathfrak{D}$-integrable on an interval I and let F be its indefinite integral. The function F is therefore ACG on I, so that I is the sum of a sequence $\{E_n\}$ of closed sets on each of which F is AC. By Lemma 4.1, Chap. VII, there exists for each n a function F_n of bounded variation on I, which coincides with F on E_n. We therefore have almost everywhere on E_n the relation $f(x)=F'_{\mathrm{ap}}(x)=F'_n(x)$; and since the derivative of a function of bounded variation is measurable and almost everywhere finite, it follows that f is measurable and almost everywhere finite on each E_n and consequently on the whole interval I.

Finally, let us mention as an immediate consequence of Theorem 9.1, Chap. VII,

(1.4) ***Theorem.*** *If a function f is $\mathfrak{D}$-integrable on an interval I_0, then every closed subset of I_0 contains a portion Q such that the function f is summable on $\bar{Q}$ and such that the series of the definite $\mathfrak{D}$-integrals of f over the intervals contiguous to $\bar{Q}$ is absolutely convergent.*

Similarly, if the function f is $\mathfrak{D}_*$-integrable on I_0, then every closed subset of I_0 contains a portion Q such that the function f is summable on $\bar{Q}$ and such that the series of the oscillations of the indefinite $\mathfrak{D}_*$-integrals of f over the intervals contiguous to $\bar{Q}$ is convergent.

§ 2. Integration by parts.

We have already observed (Chap. VI, p. 210) that a slight modification of the definition of Lebesgue-Stieltjes integral leads to an indefinite integral which is an additive function of an interval. As this modification will be useful to us in the present §, we now formulate it explicitly.

Given a finite function g integrable in the Lebesgue-Stieltjes sense with respect to a function of bounded variation F on an interval $I=[a,b]$, we shall write

$$(\mathcal{S})\int_a^b g\,dF=\int_a^b g\,dF-\{g(a)\cdot[F(a)-F(a-)]+g(b)\cdot[F(b+)-F(b)]\}.$$

The number $(\mathcal{S})\int_a^b g\,dF$ will be called *definite $\mathcal{S}$-integral of g with respect to F* over I. As we see at once, this number (unlike the Lebesgue-Stieltjes integral) does not depend on the values taken by the function F outside the interval I, and for each point c of $[a,b]$ we have

$$(\mathcal{S})\int_a^c g\,dF+(\mathcal{S})\int_c^b g\,dF=(\mathcal{S})\int_a^b g\,dF.$$

(2.1) ***Theorem.*** *Let g be a bounded function integrable with respect to a monotone non-decreasing function F on an interval $[a,b]$. Then:*

(i) $(\mathcal{S})\int_a^b g\,dF=\mu\cdot[F(b)-F(a)]$, *where μ is a number between the bounds of the function g on $[a,b]$;*

(ii) *writing* $S(x)=(\mathcal{S})\int_a^x g\,dF$ *for* $a\leqslant x\leqslant b$, *we have* $S'(x)=g(x)\cdot F'(x)$ *at almost all points of continuity of the function g, and in fact at every point x where g is continuous and F derivable.*

Proof. Clearly (i) follows at once from the obvious inequality $m\cdot[F(b)-F(a)]\leqslant(\mathcal{S})\int_a^b g\,dF\leqslant M\cdot[F(b)-F(a)]$, where m and M are the lower and upper bounds of g on $[a,b]$. In order to establish (ii), consider a point x_0 at which g is continuous and F is derivable. We may suppose, by subtracting a constant from g if necessary, that $g(x_0)=0$. Denoting, for each interval J, by $\varepsilon(J)$ the upper bound of $|g(x)|$ on J, we have $|S(J)|/|J|\leqslant\varepsilon(J)\cdot F(J)/|J|$ and taking J to be an interval containing x_0 and of length tending to zero, we find $S'(x_0)=0=g(x_0)$. This completes the proof.

(2.2) ***Lemma.*** *Let F be a function of bounded variation on an interval $I_0=[a,b]$, G a continuous function on I_0, and H the function defined on I_0 by the formula*

$$(2.3)\qquad H(x)=F(x)\,G(x)-(\mathcal{S})\int_a^x G(t)\,dF(t)\qquad for\quad a\leqslant x\leqslant b.$$

Then, if the function G is ACG [ACG$_*$] on I_0, so is the function H.

Proof. We may clearly assume F to be monotone non-decreasing. Denoting by M_0 the upper bound of $|F(x)|$ on I_0, we shall begin by proving that for every interval $I\subset I_0$ we must have

$$(2.4)\qquad |H(I)|\leqslant M_0\cdot|G(I)|+\mathrm{O}(G;I)\cdot F(I)\quad and\quad \mathrm{O}(H;I)\leqslant 3M_0\cdot\mathrm{O}(G;I).$$

In fact, by Theorem 2.1 (i), we have, for every subinterval $J=[\alpha,\beta]$ of I,

$$\begin{aligned}H(\beta)-H(\alpha)&=[G(\beta)-G(\alpha)]\cdot F(\beta)+[F(\beta)-F(\alpha)]\cdot G(\alpha)-(\mathcal{S})\int_\alpha^\beta G(t)\,dF(t)\\&=[G(\beta)-G(\alpha)]\cdot F(\beta)+[F(\beta)-F(\alpha)]\cdot[G(\alpha)-\mu],\end{aligned}$$

where μ is a number between the bounds of G on J. Consequently, $|H(J)|\leqslant M_0\cdot|G(J)|+\mathrm{O}(G;J)\cdot F(J)$, and the first of the relations (2.4) follows by choosing $J=I$. On the other hand, we derive $|H(J)|\leqslant 3M_0\cdot\mathrm{O}(G;I)$ for every interval $J\subset I$, and hence the second relation (2.4).

Hence, since the function G is continuous, the function H is continuous also. Further, if $\{I_k\}$ is any finite sequence of non-overlapping intervals and if ω denotes the largest of the numbers $\mathrm{O}(G;I_k)$, we obtain from the relations (2.4)

$$\sum_k|H(I_k)|\leqslant M_0\cdot\sum_k|G(I_k)|+\omega\cdot F(I_0)\quad and\quad \sum_k\mathrm{O}(H;I_k)\leqslant 3M_0\cdot\sum_k\mathrm{O}(G;I_k).$$

The first of these inequalities implies that if the function G is AC on a set E, so is the function H, and consequently, that if the function G is ACG on the whole interval I_0, then the function H is also ACG on I_0. Similarly, the second of the above inequalities shows that if G is ACG$_*$ on I_0 then so is F, and this completes the proof.

We can now complete Theorem 14.8, Chap. III, which concerned integration by parts for the Lebesgue integral, by establishing a similar theorem for the Denjoy integrals:

(2.5) ***Theorem.*** *If $F(x)$ is a function of bounded variation and $g(x)$ a function $\mathfrak{D}$- or $\mathfrak{D}_*$-integrable on an interval $I_0=[a,b]$, then the function $F(x)\,g(x)$ is integrable on I_0 in the same sense, and moreover denoting by G the indefinite integral of g, we have*

$$(\mathfrak{D})\int_a^b F(x)g(x)\,dx = G(b)F(b)-G(a)F(a)-(\mathfrak{S})\int_a^b G(x)\,dF(x).$$

Proof. We shall establish the theorem for the $\mathfrak{D}$-integral. The proof for the $\mathfrak{D}_*$-integral is quite similar.

By Lemma 2.2, the function H defined by the formula (2.3) is ACG on I_0. Moreover, by Theorem 2.1 (ii), if we form the approximate derivative of both sides of (2.3), we obtain almost everywhere the relation $H'_{\mathrm{ap}}(x)=F(x)\,G'_{\mathrm{ap}}(x)=F(x)\,g(x)$. It follows that the function $F(x)g(x)$ is $\mathfrak{D}$-integrable on the interval I_0 and that $(\mathfrak{D})\int_a^b F(x)\,g(x)\,dx = H(b)-H(a)$. This last relation is equivalent to the one to be proved.

The idea of the above proof, which is directly based on the *descriptive* definition of the Denjoy integrals, is due to Zygmund. For another proof, depending on the constructive definition of these integrals, cf. for instance E. W. Hobson [I, p. 711]. For an interesting generalization of the theorem on integration by parts to the $\mathcal{PS}$-integral (cf. Chap. VI, § 8) *vide* A. J. Ward [3].

From Theorem 2.5, there follows easily the second mean value theorem for the Denjoy integral, which may be regarded as a generalization of Theorem 14.10, Chap. III.

(2.6) ***Theorem.*** *Given a non-decreasing function F on an interval $I_0=[a,b]$ and a function g which is $\mathfrak{D}$-integrable on I_0, there must exist a point ξ in I_0 such that*

$$(\mathfrak{D})\int_a^b g(x)F(x)\,dx = F(a)\cdot(\mathfrak{D})\int_a^\xi g(x)\,dx + F(b)\cdot(\mathfrak{D})\int_\xi^b g(x)\,dx.$$

Proof. Writing $G(x)=(\mathfrak{D})\int_a^x g(t)\,dt$, we have by Theorems 2.5 and 2.1 (i) the relation

$$(\mathfrak{D})\int_a^b g(x)\,F(x)\,dx = G(b)\,F(b)-(\mathfrak{S})\int_a^b G(x)\,dF(x)=$$
$$=G(b)F(b)-\mu\cdot[F(b)-F(a)]=\mu\cdot F(a)+[G(b)-\mu]\cdot F(b),$$

where μ is a number between the bounds of $G(x)$ on I. It follows that there exists a point ξ in I_0 such that $\mu=G(\xi)$, and the relation just obtained becomes

$$(\mathfrak{D})\int_a^b g(x)\,F(x)\,dx = F(a)\cdot G(\xi)+F(b)\cdot[G(b)-G(\xi)],$$

which, by definition of $G(x)$, reduces to the required formula.

§ 3. Theorem of Hake-Alexandroff-Looman.

The relations between the Denjoy integrals and those of Lebesgue and of Newton having already been obtained in § 1, we now proceed to establish an important result of Hake, Alexandroff and Looman, which asserts the equivalence of integration in the restricted Denjoy sense with Perron integration.

At the same time we shall show that in the definition of Perron integral (Chap. VI, § 6) we need only take account of the continuous major and minor functions. In order to make this assertion quite precise, let us agree to say that a function f is *$\mathcal{P}_0$-integrable* on an interval I_0 if 1° the function has continuous major and minor functions on I_0 and 2° denoting by U any continuous major function and by V any continuous minor function of f on I_0, the lower bound of the numbers $U(I_0)$ is equal to the upper bound of the numbers $V(I_0)$. The function f is then plainly $\mathcal{P}$-integrable on I_0, the definite $\mathcal{P}$-integral of f on I_0 being equal to this common bound. We have to prove the converse, i. e. that every function which is $\mathcal{P}$-integrable is also $\mathcal{P}_0$-integrable.

(3.1) ***Lemma.*** *If a function f is $\mathcal{P}_0$-integrable on each interval interior to an interval $[a,b]$ and if the definite $\mathcal{P}$-integral over the interval $[a+\varepsilon, b-\eta]$ tends to a finite limit as $\varepsilon\to 0+$ and $\eta\to 0+$, then the function f is $\mathcal{P}_0$-integrable on the whole interval $[a,b]$.*

Proof. It is clearly sufficient (by halving the given interval) to consider the case of a function f which is $\mathscr{P}_0$-integrable on each interval of the form $[a, b-\varepsilon]$ where $0<\varepsilon<b-a$. Let $P(x)=(\mathscr{P})\int_a^x f\,dx$ for $a\leqslant x<b$ and $p=P(b-)$.

We choose any positive number σ. Writing for symmetry $a_0=a$, we consider any increasing sequence of points $\{a_k\}_{k=0,1,\ldots}$ which converges to b. The function f being $\mathscr{P}_0$-integrable on each interval $[a_k, a_{k+1}]$, we easily define, on the half open interval $[a, b)$, a continuous function F such that F is a major function of f on each of the intervals $[a_k, a_{k+1}]$ and that $[F(x)-F(a_k)]-[P(x)-P(a_k)]<\sigma/2^k$ for $a_k\leqslant x\leqslant a_{k+1}$ and $k=0,1,\ldots$. By the second of these conditions the oscillation of the function F on the interval $[a_k, b)$ tends to 0 as $k\to\infty$, and therefore F has a finite limit $F(b-)$ at the point b. Writing $F(x)=F(a)+(x-a)^{1/3}$ for $x<a$, and $F(x)=F(b-)$ for $x\geqslant b$, we extend the definition of F to make this function continuous on the whole straight line $\boldsymbol{R}_1$, and the following conditions are then satisfied:

(3.2) $-\infty\neq \underline{F}(x)\geqslant f(x)$ *for* $a\leqslant x<b$ and (3.3) $F(b)-F(a)\leqslant p+2\sigma$.

Now let c be an interior point of $[a, b]$ such that the oscillation of F on $[c, b]$ is less than σ. For each point x of $[c, b]$, let $O(x)$ denote the oscillation of F on $[x, b]$. The function $O(x)$ is continuous and non-increasing on the interval $[c, b]$, and we extend its definition on to the whole straight line $\boldsymbol{R}_1$ by making $O(x)=O(c)$ for $x<c$ and $O(x)=O(b)=0$ for $x>b$. We now write $G(x)=F(x)-O(x)$ and $U(x)=G(x)+\sigma\cdot(x-b)^{1/3}/(b-a)^{1/3}$. Since the function $\sigma\cdot(x-b)^{1/3}/(b-a)^{1/3}-O(x)$ is non-decreasing, it follows at once from (3.2) that $-\infty\neq\underline{U}(x)\geqslant f(x)$ for $a\leqslant x<b$. Moreover, since $G(b)-G(x)$ is non-negative for each point x of the interval $[c, b]$, and 0 for $x\geqslant b$, we find $\underline{G}(b)\geqslant 0$, and therefore $\underline{U}(b)=+\infty$. Hence U is a continuous major function of f on the interval $[a, b]$ and fulfils, by (3.3), the inequality $U(b)-U(a)\leqslant F(b)-F(a)+2\sigma\leqslant p+4\sigma$. Similarly we define a function V which is a continuous minor function of f on $[a, b]$ and fulfils the condition $V(b)-V(a)\geqslant p-4\sigma$. It follows that the function f is $\mathscr{P}_0$-integrable on $[a, b]$. This completes the proof.

(3.4) ***Lemma.*** *Let Q be a closed and bounded set, a, b its bounds, $\{I_k=[a_k, b_k]\}$ the sequence of intervals contiguous to Q, and f a function which is summable on Q and $\mathscr{P}_0$-integrable on each interval contiguous to Q.*

Then, if the series of the oscillations of the indefinite $\mathscr{P}$-integrals of the function f on the intervals I_k converges, the function f is $\mathscr{P}_0$-integrable on the whole interval $[a, b]$ and we have

(3.5) $$(\mathscr{P})\int_a^b f\,dx=\int_Q f\,dx+\sum_k(\mathscr{P})\int_{a_k}^{b_k} f\,dx.$$

Proof. Let ε be a positive number and let K be a positive integer such that

(3.6) $$\sum_{k=K+1}^{\infty} O_k<\varepsilon,$$

where O_k denotes the oscillation of the indefinite $\mathscr{P}$-integral of f on the interval I_k. Denote by f_1 the function which agrees with f on the set Q and on the intervals I_k for $k\leqslant K$, and which is 0 elsewhere. By Theorem 3.2, Chap. VI, and by the hypotheses of the lemma, the function f_1 has a continuous major function U_1 and a continuous minor function V_1 such that

(3.7) $$U_1(b)-U_1(a)-\varepsilon\leqslant\int_Q f\,dx+\sum_{k=1}^{K}(\mathscr{P})\int_{I_k} f\,dx\leqslant V_1(b)-V_1(a)+\varepsilon.$$

We shall now define a continuous major function for $f-f_1$.

Let F_k be, for each k, a continuous major function of f on the interval I_k such that $F_k(a_k)=0$ and $\mathrm{O}(F_k; I_k)\leqslant 2O_k$, and let $A_k(x)$ and $B_k(x)$ denote, for any point $x\in I_k$, the oscillations of the function F_k on the intervals $[a_k, x]$ and $[x, b_k]$ respectively. We write $G(x)=F_k(x)+A_k(x)-[B_k(x)-B_k(a_k)]$ when $x\in I_k^\circ$ and $k>K$, and $G(x)=0$ elsewhere. Finally, for each x, we write

$$U_2(x)=G(x)+\sum_k^{(x)}G(b_k-),$$

where the summation $\sum_k^{(x)}$ is extended over the indices k for which $b_k\leqslant x$. Since, for every k, we have $G(a_k+)=G(a_k)=0$ and $\mathrm{O}(G; I_k)\leqslant 3\cdot\mathrm{O}(F_k; I_k)\leqslant 6\cdot O_k$, the function U_2 is continuous on the straight line $\boldsymbol{R}_1$, and since the function G vanishes identically on each interval I_k for $k\leqslant K$, we have by (3.6),

(3.8) $$U_2(b)-U_2(a)\leqslant 6\cdot\sum_{k=K+1}^{\infty}O_k\leqslant 6\varepsilon.$$

Now, for each k, we have $G(x)\geqslant 0$ and $G(b_k-)-G(x)\geqslant 0$ for every point $x\in I_k$. Therefore the increment of the function U_2 is non-negative on each interval containing points of the set Q, and consequently $\underline{U}_2(x)\geqslant 0=f(x)-f_1(x)$ at each point x of this set. Again, since the function G vanishes on each interval I_k for $k\leqslant K$, we have $\underline{U}_2(x)=0=f(x)-f_1(x)$, whenever $x\in I_k^\circ$ for $k\leqslant K$. Finally, since the function $A_k(x)-B_k(x)$ is non-decreasing on each I_k, we see that $-\infty\neq\underline{U}_2(x)\geqslant\underline{F}_k(x)\geqslant f(x)=f(x)-f_1(x)$ at each point $x\in I_k^\circ$ for $k>K$. Thus U_2 is a continuous major function of $f-f_1$ on $[a,b]$. Similarly, we determine a continuous minor function V_2 of $f-f_1$, subject to the condition $V_2(b)-V_2(a)\geqslant -6\varepsilon$ which corresponds to (3.8). Therefore, writing $U=U_1+U_2$ and $V=V_1+V_2$, we obtain a continuous major function U and a continuous minor function V for f on $[a,b]$, and if we denote by p the right-hand side of (3.5), we obtain from (3.6) and (3.7), $U(b)-U(a)-8\varepsilon\leqslant p\leqslant V(b)-V(a)+8\varepsilon$. The function f is thus $\mathscr{P}_0$-integrable on the interval $[a,b]$ and its definite $\mathscr{P}$-integral over this interval is given by the formula (3.5).

(3.9) ***Theorem.*** *A function f which is $\mathfrak{D}_*$-integrable on an interval I_0 is necessarily $\mathscr{P}_0$-integrable on I_0, and we have*

$$(\mathfrak{D}_*)\int_{I_0} f\,dx=(\mathscr{P})\int_{I_0} f\,dx.$$

Proof. Let F be an indefinite $\mathfrak{D}_*$-integral of f on I_0. We call an interval $I\subset I_0$ regular, if the function f is $\mathscr{P}_0$-integrable on I and if the function F is on I an indefinite $\mathscr{P}$-integral of f. Further, we call a point $x\in I_0$ regular, if each sufficiently small interval $I\subset I_0$ containing x is regular. Let P be the set of the non-regular points of I_0. We see at once that the set P is closed and that every subinterval of I_0 which contains no points of this set is regular. We have to prove that the set P is empty.

Suppose, if possible, that $P\neq 0$. By Lemma 3.1 we see easily that every interval contiguous to P is regular and that the set P therefore has no isolated points. On the other hand, by Theorem 9.1, Chap. VII, the set P contains a portion P_0 on which the function F is AC_*. Let J_0 be the smallest interval containing P_0. Since the set P has no isolated points, the same is true of any portion of P, and therefore $P\cdot J_0^\circ\neq 0$. It follows that in order to obtain a contradiction, which will justify our assertion, we need only prove that the interval J_0 is regular.

To show this, let J be any subinterval of J_0 and let Q be the set consisting of the points of the set $P\cdot J$ and of the end-points of J. We denote by $\{I_n\}$ the sequence of the intervals contiguous to Q and by G the function which coincides with F on Q and is linear on the intervals I_n. Plainly the function G is absolutely continuous on J. Therefore, since $G'(x)=F'(x)=f(x)$ at almost all points x of Q, and since $G(I_n)=F(I_n)$ for each n, we obtain

$$F(J)=G(J)=\int_J G'(x)\,dx=\sum_n F(I_n)+\int_Q f(x)\,dx. \tag{3.10}$$

Now the function f is summable on Q and $\mathscr{P}_0$-integrable on each interval I_n and moreover, F is an indefinite $\mathscr{P}$-integral of f on each of these intervals. The series of the oscillations of F on the intervals I_n being convergent, it follows, by Lemma 3.4, that the function f is $\mathscr{P}_0$-integrable on J and that, on account of (3.10), $F(J)=(\mathscr{P})\int_J f\,dx$. Therefore, since J is any subinterval of J_0, the interval J_0 is regular and this completes the proof.

(3.11) ***Theorem.*** *A function which is $\mathscr{P}$-integrable on an interval I_0 is necessarily $\mathfrak{D}_*$-integrable on I_0.*

Proof. Let f be a function $\mathscr{P}$-integrable on an interval I_0 and let P be its indefinite $\mathscr{P}$-integral. We shall show that the function P is an indefinite $\mathfrak{D}_*$-integral of f. Since $P'(x)=f(x)$ almost everywhere (cf. Theorem 6.1, Chap. VI), it is enough to show that the function P is ACG_* on I_0, i. e. that any closed set $Q\subset I_0$ contains a portion on which the function P is AC_*.

Let H be any major function of f. Since $\underline{H}(x)>-\infty$ at each point x of I_0, the function H is by Theorem 10.1, Chap. VII, VBG_* on I_0, and hence I_0 is expressible as the sum of a sequence of closed sets (cf. Theorem 7.1, Chap. VII) on each of which the function H is VB_*. It follows, by Baire's Theorem (Theorem 9.2, Chap. II) that the set Q contains a portion Q_0 on which the function H is VB_*. Since the difference $P-H$ is a monotone function, the function P is actually VB_* on Q_0. We shall show that P is further AC_* on Q_0.

For this purpose, we denote by $J_0=[a,b]$ the smallest interval containing Q_0. Let ε be any positive number and U a major function of f on I_0 such that

$$U(I_0)-P(I_0)<\varepsilon. \tag{3.12}$$

Let P_1 and U_1 denote the functions which coincide on $\bar{Q}_0$ with the functions P and U respectively, and which are linear on the intervals contiguous to $\bar{Q}_0$ and constant on the half-lines $(-\infty, a]$ and $[b, +\infty)$. The function P_1 is clearly of bounded variation. On the other hand, we see easily that $\underline{U}_1(x) > -\infty$ at every point, and that $\underline{U}_1(x) \geqslant P_1'(x)$ at almost all points, of the interval J_0. Therefore, writing $f_1(x) = P_1'(x)$ wherever the second of the above inequalities holds, and $f_1(x) = -\infty$ elsewhere, we see at once that the function $f_1(x)$ is summable on J_0 and has U_1 for a major function. It follows that $U_1(I) \geqslant \int_I f_1(x)\,dx$ for each interval $I \subset J_0$, and therefore that the function of singularities (cf. Chap. IV, p. 120) of U_1 is monotone non-decreasing on J_0. Let T_1 be the function of singularities of P_1. Since the function $P_1 - U_1$ is monotone non-increasing on J_0 and since, by (3.12), we have $0 \geqslant P_1(J_0) - U_1(J_0) = P(J_0) - U(J_0) \geqslant -\varepsilon$, it follows that $T_1(I) \geqslant -\varepsilon$ for each interval $I \subset J_0$; and ε being any positive number, this requires $T_1(I) \geqslant 0$ for every interval $I \subset J_0$. Similarly, by considering minor functions of f in place of major functions, we find $T_1(I) \leqslant 0$, and therefore, finally, $T_1(I) = 0$, for each interval $I \subset J_0$. The function P_1 is thus absolutely continuous on J_0. This requires the function P to be AC on the set Q_0 as well as VB_*, and therefore AC_* on this set on account of Theorem 8.8, Chap. VII. Thus every closed set $Q \subset I_0$ contains a portion Q_0 on which the function P is AC_*, and this completes the proof.

The first of the theorems proved in this §, which together establish the equivalence of the processes of $\mathcal{D}_*$-, $\mathcal{P}_0$- and $\mathcal{P}$-integration, was derived in 1921 by H. Hake [1] from the constructive definition of the integral $\mathcal{D}_*$ (*vide* below, § 5). The second theorem was obtained some years later by P. Alexandroff [1; 2] and H. Looman [4] independently. For an interesting extension of these results to Perron-Stieltjes integral, *vide* A. J. Ward [3].

It should, perhaps, be added that in their original definitions O. Perron [1] and O. Bauer [1] employed only continuous major and minor functions. The equivalence of the original Perron-Bauer definition with that of Chap. VI, § 6, has therefore been established here as a consequence of Theorems 3.9 et 3.11.

Let us remark further that in the definition of Perron integral, ordinary major and minor functions may be replaced by generalized continuous major and minor functions defined as follows. A function U is a *generalized continuous major function* of a function f on an interval I if 1° U is continuous and VBG_* on I, 2° the set of the values assumed by U at the points at which $U'(x) = -\infty$, is of measure zero, and 3° $\underline{U}(x) \geqslant f(x)$ at almost all points x. The definition of *generalized continuous minor functions* is obtained by symmetry.

We shall conclude this § with the following result, due to Marcinkiewicz:

(3.13) ***Theorem.*** *A measurable function f which has on I_0 at least one continuous major function and at least one continuous minor function, is necessarily $\mathcal{P}$-integrable on I_0.*

Proof. Let U and V be respectively a continuous major function and a continuous minor function of f on I_0. We shall call a point $x \in I_0$ regular if the function f is $\mathcal{P}$-integrable on each sufficiently small interval $I \subset I_0$ which contains x. Let Q be the set of the points x of I_0 which are not regular. The set Q is plainly closed and we see at once that the function f is $\mathcal{P}$-integrable on each subinterval of I_0 which contains no points of Q. Thus it has to be proved that $Q = 0$.

Suppose, if possible, that $Q \neq 0$. For every interval I on which the function f is $\mathcal{P}$-integrable, we have $U(I) \geqslant (\mathcal{P})\int_I f(x)\,dx \geqslant V(I)$. Therefore, if $[a, b]$ is an interval contiguous to Q, the definite $\mathcal{P}$-integral of f on the interval $[a+\varepsilon, b-\eta]$ interior to $[a, b]$ tends to a finite limit as $\varepsilon \to 0$ and $\eta \to 0$. By Lemma 3.1, the function f is thus $\mathcal{P}$-integrable on each interval contiguous to Q. It follows, in particular, that Q can have no isolated points.

Now let Q_0 be a portion of Q on which the functions U and V are both VB_*. Such a portion exists by Theorem 9.1, Chap. VII, since the functions U and V are VBG_* on I_0 on account of Theorem 10.1, Chap. VII. Let J_0 be the smallest interval containing Q_0. Since $\underline{U}(x) \geqslant f(x) \geqslant \overline{V}(x)$ everywhere on I_0, the function f is summable on $\bar{Q}_0$ together with the two derivatives $\underline{U}(x)$ and $\overline{V}(x)$. On the other hand, denoting by $\{I_n\}$ the sequence of the intervals contiguous to $\bar{Q}_0$ and by O_n the oscillation on I_n of the indefinite $\mathcal{P}$-integral of f, we shall have $O_n \leqslant \mathrm{O}(U; I_n) + \mathrm{O}(V; I_n)$ for every n, and so $\sum_n O_n < +\infty$. It follows by Lemma 3.4, that the function f is $\mathcal{P}$-integrable on the whole interval J_0. But this is clearly impossible, for since the set Q has no isolated points, the interval J_0 contains in its interior some points of Q. We thus arrive at a contradiction which completes the proof.

Just as in the definition of the Perron integral, we may replace, in Theorem 3.13, ordinary major and minor functions by generalized continuous ones (cf. above, p. 252). Nevertheless, the conditions of Theorem 3.13 differ from those of the definition of Chap. VI, § 6, in that continuity is essential. In fact, if we write $f(x) = 0$ for $x \leqslant 0$ and $f(x) = -1/x^2$ for $x > 0$, $U(x) = 0$ identically in $\boldsymbol{R}_1$ and $V(x) = 0$ for $x \leqslant 0$ and $V(x) = 1/x$ for $x > 0$, we see at once that U and V are respectively a major and a minor function of f; and yet f is evidently not $\mathcal{P}_0$-integrable on $[0, 1]$.

***§ 4. General notion of integral.** We shall deal in this § with some notions of a more abstract kind which we shall employ, in the next §, as a basis for the constructive definition of the Denjoy integrals.

Let $\mathfrak{C}$ be a functional operation by which there corresponds to each interval $I=[a,b]$ a class of functions defined on I, and to each function f of this class a finite real number. This class of functions will be called *domain of the operation $\mathfrak{C}$ on the interval I*, and the number associated with f will be denoted by $\mathfrak{C}(f;I)$.

An operation $\mathfrak{C}$ will be termed an *integral*, if the following three conditions are fulfilled:

(i) If a function f belongs to the domain of the operation $\mathfrak{C}$ on an interval I_0, the function belongs also to the domain of $\mathfrak{C}$ on any interval $I\subset I_0$, and $\mathfrak{C}(f;I)$ is a continuous additive function of the interval $I\subset I_0$.

(ii) If a function f belongs to the domain of the operation $\mathfrak{C}$ on two abutting intervals I_1 and I_2, the function belongs also to the domain of $\mathfrak{C}$ on the interval I_1+I_2.

(iii) A function f which vanishes identically on an interval I belongs to the domain of $\mathfrak{C}$ on I, and we have $\mathfrak{C}(f;I)=0$.

If $\mathfrak{C}$ is an integral, any function f which belongs to the domain of $\mathfrak{C}$ on an interval I_0 will be termed *$\mathfrak{C}$-integrable* on I_0 and the number $\mathfrak{C}(f;I_0)$ will be called *definite $\mathfrak{C}$-integral* of f on I_0. The function of an interval $I\subset I_0$, $\mathfrak{C}(f;I)$, which is additive and continuous on account of (i), will then be called *indefinite $\mathfrak{C}$-integral* of f on I_0 and its oscillation on I_0 (i. e. the upper bound of the numbers $|\mathfrak{C}(f;I)|$, where I denotes any subinterval of I_0) will be denoted by $O(\mathfrak{C};f;I_0)$.

Two integrals $\mathfrak{C}_1$ and $\mathfrak{C}_2$ will be termed *compatible*, if $\mathfrak{C}_1(f;I)=\mathfrak{C}_2(f;I)$ for every interval I and for every function f which is both $\mathfrak{C}_1$- and $\mathfrak{C}_2$-integrable on I.

We shall say that the integral $\mathfrak{C}_2$ *includes* the integral $\mathfrak{C}_1$, if the two integrals are compatible and if every function which is $\mathfrak{C}_1$-integrable is also $\mathfrak{C}_2$-integrable. When this is so we shall write $\mathfrak{C}_1\subset\mathfrak{C}_2$.

Given an integral $\mathfrak{C}$ and a function g which vanishes outside a bounded set E, it is evident that if g is $\mathfrak{C}$-integrable on an interval I_0 which contains E in its interior, then g is so also on any interval I which contains E, and we have $\mathfrak{C}(g;I)=\mathfrak{C}(g;I_0)$.

This fact justifies the following definition: we shall say that a function f is *$\mathfrak{C}$-integrable on a bounded set E*, if the function g which coincides with f on E and is 0 elsewhere, is $\mathfrak{C}$-integrable on each interval $I\supset E$. The number $\mathfrak{C}(g;I)$ is then independent of the choice of the interval $I\supset E$; we shall call this number *definite $\mathfrak{C}$-integral* of the function f *on the set E* and we shall denote it by $\mathfrak{C}(f;E)$.

Of the known processes of integration, all those which give rise to a continuous indefinite integral (for instance those of Lebesgue, Newton, Denjoy, etc.) are easily seen to be integrals according to the above definition. If, however, we wished to include also discontinuous integrals (e. g. that of W. H. Young cf. Chap. VII, p. 215) we should have to modify some details of the definition.

Given a function f on an interval I_0 and given an integral $\mathfrak{C}$, we shall say that a point $a\in I_0$ is a *$\mathfrak{C}$-singular point* of f in I_0 if there exist arbitrarily small intervals $I\subset I_0$ containing a on each of which the function f is not $\mathfrak{C}$-integrable. Denoting by S the set of these points, we see at once that the set S is closed and that the function f is $\mathfrak{C}$-integrable on every subinterval of I_0 which contains no points of S.

With each integral $\mathfrak{C}$ we now associate three "generalized" integrals $\mathfrak{C}^{\mathrm{C}}$, $\mathfrak{C}^{\mathrm{H}}$ and $\mathfrak{C}^{\mathrm{H}*}$, defined as follows.

Given any interval I_0, the domain of the operation $\mathfrak{C}^{\mathrm{C}}$ on I_0 is the class of all the functions f which fulfil the following two conditions:

(c^1) the set of the $\mathfrak{C}$-singular points of f in I_0 is finite (or empty);

(c^2) there exists a continuous additive function of an interval F on I_0 such that $F(I)=\mathfrak{C}(f;I)$ whenever I is a subinterval of I_0 which contains no $\mathfrak{C}$-singular point of f.

Since such a function F (if existent) is uniquely determined by the conditions (c^1) and (c^2), we can write $\mathfrak{C}^{\mathrm{C}}(f;I_0)=F(I_0)$.

The domain of the operation $\mathfrak{C}^{\mathrm{H}}$ on I_0 is defined as the class of the functions f which fulfil the following conditions:

(h^1) if S denotes the set of all $\mathfrak{C}$-singular points of f in I_0, the function f is $\mathfrak{C}$-integrable on the set S and on each of the intervals I_k contiguous to the set consisting of the points of S and of the end-points of I_0;

(h^2) $\sum_k|\mathfrak{C}(f;I_k)|<+\infty$ and, in the case in which the sequence $\{I_k\}$ is infinite, $\lim_k O(\mathfrak{C};f;I_k)=0$.

For any such function f, we write by definition:

$$\mathfrak{T}^{H}(f; I_0) = \sum_k \mathfrak{T}(f; I_k) + \mathfrak{T}(f; S).$$

Finally, we obtain the definition of the operation $\mathfrak{T}^{H_*}$ by replacing in the definition of the operation $\mathfrak{T}^{H}$ the condition (h^2) by the more restrictive condition:

$$(h^2_*) \qquad \sum_k O(\mathfrak{T}; f; I_k) < +\infty.$$

We verify at once that the operations $\mathfrak{T}^{C}$, $\mathfrak{T}^{H}$ and $\mathfrak{T}^{H_*}$ all fulfil the conditions (i), (ii) and (iii), p. 254. These operations are therefore integrals according to the definition, p. 254, and we evidently have $\mathfrak{T} \subset \mathfrak{T}^{C}$ and $\mathfrak{T} \subset \mathfrak{T}^{H_*} \subset \mathfrak{T}^{H}$. For brevity, we shall write $\mathfrak{T}^{CH}$ and $\mathfrak{T}^{CH_*}$ in place of $(\mathfrak{T}^{C})^{H}$ and $(\mathfrak{T}^{C})^{H_*}$ respectively.

The integral $\mathfrak{T}^{C}$ and the integrals $\mathfrak{T}^{H}$ and $\mathfrak{T}^{H_*}$ may be regarded respectively as the Cauchy and the Harnack generalizations of the integral $\mathfrak{T}$. They correspond, in fact, to the classical processes employed by Cauchy and Harnack to extend integration from bounded to unbounded functions of certain classes. The original process of Harnack actually corresponds to the operation $\mathfrak{T}^{H_*}$ rather than to the operation $\mathfrak{T}^{H}$. Cf. A. Harnack [1], E. W. Hobson [I, Chap. VIII] and A. Rosenthal [I, p. 1053].

If we were to add to the conditions (h^1) and (h^2) which characterize the generalized integral $\mathfrak{T}^{H}$, the condition that $\lim_k O(\mathfrak{T}; f; I_k)/\varrho(x, I_k) = 0$ for almost all $x \in S$, we should arrive at a generalized integral $\mathfrak{T}^{H'}$ intermediate between $\mathfrak{T}^{H}$ and $\mathfrak{T}^{H_*}$. By applying the process $\mathfrak{T}^{H'}$ in the constructive definitions of Denjoy integrals of the next §, we should then obtain an integral $\mathfrak{D}'$, intermediate between $\mathfrak{D}$ and $\mathfrak{D}_*$. Its descriptive definition is very simple: *a function f is $\mathfrak{D}'$-integrable if it is $\mathfrak{D}$-integrable and if its indefinite $\mathfrak{D}$-integral is almost everywhere derivable (in the ordinary sense)*. This integral has been discussed by A. Khintchine [1]; cf. also J. C. Burkill [1].

*§ 5. Constructive definition of the Denjoy integrals.

With the notation of the preceding §, we see at once that for each integral $\mathfrak{T} \subset \mathfrak{D}$, we have also $\mathfrak{T}^{C} \subset \mathfrak{D}$; similarly the relation $\mathfrak{T} \subset \mathfrak{D}_*$ implies $\mathfrak{T}^{C} \subset \mathfrak{D}_*$. It is not quite so obvious that the relations $\mathfrak{T} \subset \mathfrak{D}$ and $\mathfrak{T} \subset \mathfrak{D}_*$ imply respectively $\mathfrak{T}^{H} \subset \mathfrak{D}$ and $\mathfrak{T}^{H_*} \subset \mathfrak{D}_*$. This last assertion is a consequence of the following theorem which is analogous to Lemma 3.4.

(5.1) ***Theorem.*** *Let Q be a bounded closed set with the bounds a and b, and let $\{I_k\}$ be the sequence of intervals contiguous to Q; and suppose that f is a function $\mathfrak{D}$-integrable on the set Q as well as on each of the intervals I_k, and that (in the case in which the sequence $\{I_k\}$ is infinite)*

$$\sum_k \left|(\mathfrak{D})\int_{I_k} f\,dx\right| < +\infty \qquad \text{and} \qquad \lim_k O(\mathfrak{D}; f; I_k) = 0.$$

Then the function f is $\mathfrak{D}$-integrable on the whole interval $I = [a, b]$ and we have

$$(5.2) \qquad (\mathfrak{D})\int_I f\,dx = (\mathfrak{D})\int_Q f\,dx + \sum_k (\mathfrak{D})\int_{I_k} f\,dx.$$

If we suppose, further, that the function f is $\mathfrak{D}_*$-integrable on Q as well as on each of the intervals I_k and that $\sum_k O(\mathfrak{D}_*; f; I_k) < +\infty$, then the function f is $\mathfrak{D}_*$-integrable on I.

Proof. We shall prove the theorem for the $\mathfrak{D}$-integral. The case of the $\mathfrak{D}_*$-integral is similar.

Let $I(x)$ denote the interval $[a, x]$ where we suppose $x \in [a, b]$, and let

$$(5.3) \qquad F(x) = \sum_k (\mathfrak{D}) \int_{I_k \cdot I(x)} f(t)\,dt.$$

We shall show that the function F, thus defined, is ACG on the interval I. For this purpose, it will suffice to show that F is AC on the set Q, the function being evidently continuous on I and ACG on each of the intervals I_k.

Let $g(x)$ be the function equal to 0 for $x \in Q$ and to $\frac{1}{|I_k|} \cdot (\mathfrak{D})\int_{I_k} f(t)\,dt$ for $x \in I_k^{\circ}$ where $k = 1, 2, \ldots$. The function g is summable on I and if $G(x) = \int_a^x g(t)\,dt$, the function F clearly coincides with G on Q; F is thus AC on Q and therefore ACG on I.

This being so, we have $F'_{ap}(x) = G'(x) = g(x) = 0$ at almost all points x of Q, while it follows at once from (5.3) that $F'_{ap}(x) = f(x)$ at almost all points x of $I - Q$. Hence, F being ACG on I, it follows that the function equal to f on $I - Q$ and to 0 on Q has F for an indefinite $\mathfrak{D}$-integral. On the other hand, the function equal to f on Q and to 0 elsewhere is, by hypothesis, $\mathfrak{D}$-integrable on I. It follows that the function f itself is $\mathfrak{D}$-integrable on I, and that $(\mathfrak{D})\int_I f(x)\,dx = F(b) - F(a) + (\mathfrak{D})\int_Q f\,dx$, which, on account of (5.3), is equivalent to (5.2). This completes the proof.

We now pass on to the **constructive** definition of the Denjoy integrals. We begin by introducing the following notation.

Let $\{\mathfrak{T}^{\xi}\}$ be a sequence of integrals, in general transfinite, such that $\mathfrak{T}^{\xi}\subset\mathfrak{T}^{\eta}$ whenever $\xi<\eta$. We then denote by $\sum\limits_{\xi<\alpha}\mathfrak{T}^{\xi}$ the operation $\mathfrak{T}$ whose domain on each interval I is the sum of the domains of the operations $\mathfrak{T}^{\xi}$ for $\xi<\alpha$, and which is defined for every function f of its domain by the relation $\mathfrak{T}(f;I)=\mathfrak{T}^{\xi_0}(f;I)$, where ξ_0 is the least of the indices $\xi<\alpha$ such that f is $\mathfrak{T}^{\xi}$-integrable on I. It then follows, of course, that $\mathfrak{T}(f;I)=\mathfrak{T}^{\xi}(f;I)$ for every $\xi\geqslant\xi_0$, since by hypothesis $\mathfrak{T}^{\xi}$ then includes $\mathfrak{T}^{\xi_0}$.

This being so, let $\{\mathfrak{L}^{\xi}\}$ and $\{\mathfrak{L}^{\xi}_*\}$ be two transfinite sequences defined, by an induction starting with the Lebesgue integral $\mathfrak{L}$, as follows:

$$\mathfrak{L}^0=\mathfrak{L}^0_*=\mathfrak{L},$$

$$\mathfrak{L}^{\alpha}=\Big(\sum_{\xi<\alpha}\mathfrak{L}^{\xi}\Big)^{\mathrm{CH}}\qquad\textit{and}\qquad\mathfrak{L}^{\alpha}_*=\Big(\sum_{\xi<\alpha}\mathfrak{L}^{\xi}_*\Big)^{\mathrm{CH}_*}\qquad\textit{for}\quad\alpha>0.$$

Denoting by Ω the smallest ordinal number of the third class (cf. for instance, W. Sierpiński [I, p. 235]) we shall show that

$$\mathfrak{D}=\sum_{\xi<\Omega}\mathfrak{L}^{\xi}=\mathfrak{L}^{\Omega}\qquad\textit{and}\qquad\mathfrak{D}_*=\sum_{\xi<\Omega}\mathfrak{L}^{\xi}_*=\mathfrak{L}^{\Omega}_*.$$

We shall restrict ourselves to the case of the $\mathfrak{D}$-integral (that of the $\mathfrak{D}_*$-integral being quite similar).

Since $\mathfrak{L}\subset\mathfrak{D}$, we find at once by induction (cf. above, p. 256) that for every ξ, $\mathfrak{L}^{\xi}\subset\mathfrak{D}$, and so, obviously, $\sum\limits_{\xi<\Omega}\mathfrak{L}^{\xi}\subset\mathfrak{D}$. In order to change this last relation into one of identity, it is enough to show that every function f which is $\mathfrak{D}$-integrable on an interval $I_0=[a,b]$, is $\mathfrak{L}^{\xi}$-integrable on I_0 for some index $\xi<\Omega$.

Let S^{ξ} denote the set of the $\mathfrak{L}^{\xi}$-singular points of f in I_0. The sequence $\{S^{\xi}\}$, as a descending sequence of closed sets, is **stationary**, i. e. there exists an index $\nu<\Omega$ such that $S^{\nu}=S^{\nu+1}$. (For if not, there would exist for every $\xi<\Omega$ a point $x_{\xi}\in S^{\xi}-S^{\xi+1}$, and therefore also an interval I_{ξ} with rational end-points, containing the point x_{ξ} of S^{ξ} but without points in common with the closed set $S^{\xi+1}$, nor therefore, with any of the sets $S^{\xi+2}, S^{\xi+3},\dots$. We should thus obtain a transfinite sequence of type Ω of distinct intervals with rational end-points, and this is impossible.) We shall prove that $S^{\nu}=0$.

Suppose, if possible, that $S^{\nu}\neq0$. We see at once that the function f is $\mathfrak{L}^{\nu}$-integrable on each interval $I\subset I_0$ which contains no points of S^{ν}. It follows that the function f is $(\mathfrak{L}^{\nu})^{\mathrm{C}}$-integrable, and *a fortiori* $\mathfrak{L}^{\nu+1}$-integrable, on each interval contiguous to S^{ν}. Since $S^{\nu}=S^{\nu+1}$, it follows, in particular, that the set S^{ν} contains no isolated points.

The function f being, by hypothesis, $\mathfrak{D}$-integrable on I_0, the set S^{ν} (cf. Theorem 1.4) must contain a portion Q such that the function f is summable on $\bar{Q}$ and such that the series of the definite $\mathfrak{D}$-integrals of f over the intervals contiguous to $\bar{Q}$ converges absolutely. Since $\mathfrak{L}\subset(\mathfrak{L}^{\nu})^{\mathrm{C}}\subset\mathfrak{D}$, it follows at once that the function f is $(\mathfrak{L}^{\nu})^{\mathrm{CH}}$-integrable, i. e. $\mathfrak{L}^{\nu+1}$-integrable, on some interval J_0 containing Q. But this is clearly impossible, since, in view of the fact that the set S^{ν} has no isolated points, the interval J_0 certainly contains points of the set $S^{\nu}=S^{\nu+1}$ in its interior.

We thus have $S^{\nu}=0$, which establishes the $\mathfrak{L}^{\nu}$-integrability of f on I_0 and completes the proof.

Various definitions, constructive and descriptive, of Denjoy integrals will be found in the papers mentioned in Chap. VI, p. 207, and Chap. VII, pp. 214-215, as well as in the following treatises and memoirs: N. Lusin [I; 4], T. H. Hildebrandt [1], P. Nalli [I], E. Kamke [I], A. Kolmogoroff [2], H. Lebesgue [7; II, Chap. X], A. Rosenthal [1] and P. Romanowski [1].

For further extensions to functions of two or more variables, see also H. Looman [1] and M. Krzyżański [1].