CHAPTER VII.

Functions of generalized bounded variation.

§ 1. Introduction. The definition adopted in Chap. I (§ 10) as starting point of our exposition of the Lebesgue integral, connects the latter with the conception of definite integral due to Leibniz, Cauchy and Riemann (cf. Chap. I, § 1 and Chap. VI, § 1). On account of the results of § 7, Chap. IV, we may, however, also regard the Lebesgue integral as a special modification of that of Newton (cf. Chap. VI, § 1) and define it as follows:

(L) *A function of a real variable f is integrable if there exists a function F such that* (i) $F'(x)=f(x)$ *at almost all points x and* (ii) *F is absolutely continuous.*

The function F (then uniquely determined apart from an additive constant) is the indefinite integral of the function f.

A definition of integral is usually called descriptive when it is based on differential properties of the indefinite integral and therefore connected with the Newtonian notion of primitive; this is the case of the definition (L) of the Lebesgue integral. In the note of F. Riesz [9] the reader will find an elementary and elegant account of the fundamental properties of the Lebesgue integral based on a descriptive definition differing slightly from the one given above (an account based directly on the definition (L) is given in the first edition of this book).

By contrast to the descriptive definitions, we call constructive the definitions of integral which are based on the conception of definite integral of Leibniz-Cauchy, i. e. on approximation by the usual finite sums. Thus for instance, the classical definition given by H. Lebesgue [1] in his Thesis may be regarded as constructive (the reader will find a very suggestive explanation of this definition in the note by H. Lebesgue [8]); cf. also the definitions of Lebesgue integral given in the following memoirs: W. H. Young [3], T. H. Hildebrandt [1], F. Riesz [1] and A. Denjoy [7; 8].

As is readily seen, the definition (L) constitutes a modification of that of the integral of Newton, in two directions: firstly, a generalization which enables us to disregard sets of measure zero in the fundamental relation $F'(x)=f(x)$; and secondly, an essential restriction, which excludes all but the absolutely continuous functions from the domain of continuous primitive functions considered. Some such restriction is, in fact, indispensable, unless we give up the principle of unicity for the integral: to see this it is enough to consider, for instance, singular functions which are continuous and not constant, and whose derivatives vanish almost everywhere (cf. Chap. III, § 13, p. 101).

But although the condition (ii) cannot be wholly removed from the definition (L), it is possible to replace it by much weaker conditions, and the corresponding generalizations of the notion of absolute continuity give rise to extensions of the Lebesgue integral, known as the integrals $\mathfrak{D}_*$ and $\mathfrak{D}$ of Denjoy.

We shall treat in this Chapter two generalizations of absolutely continuous functions: the functions which are generalized absolutely continuous in the restricted sense or ACG_*, and those which are generalized absolutely continuous in the wide sense or ACG. If, in the definition (L), we replace the condition (ii) by the conditions that the function F is ACG_* or ACG respectively, we obtain the descriptive definitions of the integrals $\mathfrak{D}_*$ and $\mathfrak{D}$. It must be added however that the second of these definitions requires a simultaneous generalization of the notion of derivative, to which is assigned the name of approximate derivative (or asymptotic derivative) and which corresponds to approximate continuity (*vide* Chap. IV, § 10). A function which is ACG (unlike those which are absolutely continuous or which are ACG_*) may in fact fail, at each point of a set of positive measure, to be derivable in the ordinary sense, and yet be almost everywhere derivable in the approximate sense. Therefore, in order to obtain the definition of the integral $\mathfrak{D}$ from the definition (L), it is necessary not only to modify the condition (ii) as explained above, but also to replace in the condition (i) the ordinary by the approximate derivative.

The integrals $\mathfrak{D}_*$ and $\mathfrak{D}$ will be studied in the next chapter; the preliminary discussion of their definitions just given, is intended to emphasize the important part played by the generalizations of the notion of absolute continuity, which are treated in this chapter. The results of which an account is given in the following §§ are essentially due to Denjoy, Lusin and Khintchine. The first definition of the integral $\mathfrak{D}_*$ was given in notes dating from 1912 by A. Denjoy [2; 3] who employed the constructive method based on a transfinite process (*vide* Chap. VIII, § 5). These notes at once attracted the attention of N. Lusin [2] who originated the descriptive theory of this integral. Finally, A. Khintchine [1; 2] and A. Denjoy [4] defined, independently and almost at the same time, the process of integration $\mathfrak{D}$ as a generalization of the integral $\mathfrak{D}_*$. A systematic account of these researches may be found in the memoir of A. Denjoy [6].

As shown by W. H. Young [6] the generalization of the Denjoy integrals can be carried still further if we give up, partially at least, the continuity of the indefinite integral. For subsequent researches in this direction, *vide* J. C. Burkill [5; 6; 7], J. Ridder [6; 7], M. D. Kennedy and S. Pollard [1], S. Verblunsky [1], and J. Marcinkiewicz and A. Zygmund [1].

Except in a few general definitions in § 3, we shall consider in this chapter only functions of a real variable. As therefore we shall be employing in $\boldsymbol{R}_1$ notions established in the preceding chapters for arbitrary spaces $\boldsymbol{R}_m$, it will be convenient to add a few complementary definitions.

We shall say that a point a is a *right-hand point of accumulation* for a linear set E, if each interval $[a, a+h]$, where $h>0$, contains an infinity of points of E. A point of E which is not a right-hand point of accumulation for the set E is termed *isolated on the right* of this set. The definitions of *left-hand points of accumulation* and of points *isolated on the left* are obtained by symmetry.

Similarly, for each linear set E, in addition to the densities defined in § 10, Chap. IV, we define at each point x four unilateral densities: two *outer right-hand*, *upper* and *lower*, and two *outer left-hand*, *upper* and *lower*, *densities* of E. We shall understand by these four numbers the values of four corresponding Dini derivates of the measure-function (cf. Chap. IV, § 6) of E at the point x. If at a point x, two of these densities on the same side (right or left) are equal to 1, the point x is termed *unilateral (right- or left-hand) point of outer density* for the set E. The term "outer" is omitted from these expressions if the set E is measurable.

Finally, we shall extend the notation of linear interval and denote, for each point a of $\boldsymbol{R}_1$, by $(-\infty, a)$, $(-\infty, a]$, $(a, +\infty)$ and $[a, +\infty)$ the half-lines $x<a$, $x\leqslant a$, $x>a$ and $x\geqslant a$ respectively.

***§ 2. A theorem of Lusin.** While discussing the significance of the condition (ii) in the definition (L) of an integral, we remarked that a continuous function which is almost everywhere derivable is by no means determined (apart from the additive constant) when we are given its derivative almost everywhere. It is, however, of greater interest that, for a function f, the property of being almost everywhere the derivative of a continuous function, itself represents no restriction at all, except, of course, in so far as it implies that the function f is measurable and almost everywhere finite (this last assertion follows, for instance, from the corollaries to Theorem 10.1, p. 236). We shall prove this result, which is due to N. Lusin [I; 4] (cf. also E. W. Hobson [II, p. 284]), by means of two lemmas.

(2.1) ***Lemma.*** *If g is a function summable on an interval $[a,b]$, there exists, for each $\varepsilon>0$, a continuous function G such that* (i) $G'(x)=g(x)$ *almost everywhere on* $[a,b]$, (ii) $G(a)=G(b)=0$, *and* (iii) $|G(x)|\leqslant\varepsilon$ *at every point x of $[a,b]$.*

Proof. Let $H(x)$ be the indefinite integral of $g(x)$. We insert in $[a,b]$ a finite sequence of points $a=a_0<a_1<...<a_n=b$ such that the oscillation of H is less than ε on each of the intervals $[a_i, a_{i+1}]$ where $i=0, 1, ..., n-1$. Let F (cf. (13.4), Chap. III, p. 101) be a function which is continuous and singular on $[a,b]$, monotone on each interval $[a_i, a_{i+1}]$ and coincides with the function H at the end-points of these intervals. Writing $G=H-F$, we shall have (i) $G'(x)=H'(x)-F'(x)=H'(x)=g(x)$ at almost all the points x of $[a,b]$, (ii) $G(a)=G(b)=0$, and finally (iii) $|G(x)|=|H(x)-F(x)|\leqslant\varepsilon$ on each interval $[a_i, a_{i+1}]$, and therefore on the whole interval $[a,b]$.

(2.2) ***Lemma.*** *If g is a function which is summable on an interval $J=[a,b]$ and if P is a closed set in J, there exists for each $\varepsilon>0$ a continuous function G such that* (i) $G'(x)=g(x)$ *at almost all the points x of $J-P$,* (ii) $G(x)=0$ *and* $G'(x)=0$ *at all the points x of P and* (iii) $|G(x+h)|\leqslant\varepsilon\cdot|h|$ *for every x of P and every h.*

Proof. Let us represent the open set $J^\circ-P$ as the sum of a sequence $\{I_k=(a_k,b_k)\}_{k=1,2,...}$ of non-overlapping open intervals, and insert in each interval I_k an increasing sequence of points $\{a_k^{(i)}\}_{i=...,-1,0,1,...}$ infinite in both directions and tending to a_k or b_k according as $i\to-\infty$ or $i\to+\infty$. Let us further denote, for each $k=1,2,...,$ and $i=0,\pm1,\pm2,...,$ by $\varepsilon_k^{(i)}$ the smaller of the numbers $\varepsilon\cdot(a_k^{(i)}-a_k)/(k+|i|)$ and $\varepsilon\cdot(b_k-a_k^{(i+1)})/(k+|i|)$. Lemma 2.1 enables us to determine in each open interval I_k a continuous function G_k such that $G_k'(x)=g(x)$ almost everywhere on I_k, $G(a_k^{(i)})=0$ for $i=0,\pm1,\pm2,...,$ and $|G_k(x)|\leqslant\varepsilon_k^{(i)}$ when $a_k^{(i)}\leqslant x\leqslant a_k^{(i+1)}$. If we now write $G(x)=G_k(x)$ for $x\,\epsilon\, I_k$ and $k=1,2,...,$ and $G(x)=0$ elsewhere on $\boldsymbol{R}_1$, we see at once that the function G is continuous and fulfils the required conditions (i), (ii) and (iii).

(2.3) ***Lusin's Theorem.*** *If f is a function which is measurable and almost everywhere finite on an interval $J=[a,b]$, there always exists a continuous function F such that $F'(x)=f(x)$ almost everywhere on J.*

Proof. We shall define by induction a sequence of continuous functions $\{G_n\}_{n=0,1,...}$, each of these functions being almost everywhere derivable, and a sequence of closed sets $\{P_n\}_{n=0,1,...}$ in J, such that, writing $Q_n=\sum_{k=0}^{n}P_k$ and $F_n=\sum_{k=0}^{n}G_k$, the following conditions will be satisfied for $n=1,2,....$

(a) $F_n'(x)=f(x)$ *for* $x\,\epsilon\, Q_n$,

(b) $G_n(x)=0$ *for* $x\,\epsilon\, Q_{n-1}$,

(c) $|G_n(x+h)|\leqslant|h|/2^n$ *for every* $x\,\epsilon\, Q_{n-1}$ *and every h*,

(d) $|J-Q_n|\leqslant1/n$.

For this purpose, we choose $G_0(x)=0$ identically and $P_0=0$, and we suppose that for $n=0,1,...,r$ the closed sets P_n and the continuous functions G_n, almost everywhere derivable, have been defined so as to satisfy the conditions (a), (b), (c) and (d) for each $n\leqslant r$. Since the function f is measurable and almost everywhere finite, and since the function F_r is almost everywhere derivable, we can determine a measurable subset E_r of $J-Q_r$ such that

$$|J-Q_r-E_r|<1/(r+1), \tag{2.4}$$

and such that the derivative $F_r'(x)$ exists at each point x of the set E_r and is bounded, together with the function $f(x)$, on this set. Hence by Lemma 2.2, we can determine a continuous function G_{r+1}, almost everywhere derivable, in such a manner that (i) $G_{r+1}'(x)=f(x)-F_r'(x)$ at almost all points of $E_r\subset J-Q_r$, (ii) $G_{r+1}(x)=G_{r+1}'(x)=0$ at all points of Q_r, and (iii) $|G_{r+1}(x+h)|\leqslant|h|/2^{r+1}$ for every $x\,\epsilon\, Q_r$ and every h.

Now it follows from the first of these conditions and from (2.4), that there exists a closed set $P_{r+1}\subset E_r$ such that:

(2.5) $|J-Q_r-P_{r+1}|<1/(r+1)$, (2.6) $G_{r+1}'(x)=f(x)-F_r'(x)$ *for* $x\,\epsilon\, P_{r+1}$,

and we easily verify, on account of (2.6), (ii), (iii) and (2.5), that the conditions (a), (b), (c) and (d), still remain valid for $n=r+1$.

Let us now write:

(2.7) $F(x)=\lim\limits_k F_k(x)=\sum\limits_k G_k(x)$, (2.8) $Q=\lim\limits_k Q_k=\sum\limits_k P_k$.

In view of the condition (c), the series occurring in (2.7) converges uniformly, and the function F is therefore continuous. Let x_0 be any point of Q. Then for every sufficiently large integer n we have $x_0 \epsilon Q_n$, and since

$$\frac{F(x_0+h)-F(x_0)}{h}=\frac{F_n(x_0+h)-F_n(x_0)}{h}+\sum_{k=n+1}^{\infty}\frac{G_k(x_0+h)-G_k(x_0)}{h},$$

we find, on account of the conditions (a), (b) and (c), that

$$\limsup_{h\to 0}\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x_0)\right|\leqslant 1/2^n,$$

and so, that $F'(x_0)=f(x_0)$. Now it follows from the condition (d) that $|J-Q|=0$; we therefore have $F'(x)=f(x)$ at almost all the points x of J, and this completes the proof.

Theorem 2.3 remains valid for any space $\boldsymbol{R}_m$:

If f is a measurable function which is almost everywhere finite in a space $\boldsymbol{R}_m$, there exists an additive continuous function of an interval F such that $F'(x)=f(x)$ almost everywhere in $\boldsymbol{R}_m$.

The proof is almost the same as that of Theorem 2.3. We may also, in the foregoing statement, replace the ordinary derivative $F'(x)$ by the strong derivative (*vide* Chap. IV, § 2, p. 106), but the proof is then more elaborate.

It may be remarked further that Lusin's theorem in the form (2.3), is obvious if the function f is summable; for f is then almost everywhere the derivative of its indefinite integral. But this is no longer so when we wish to determine a function F with a strong derivative almost everywhere equal to f (cf. Chap. IV, p. 132). Nevertheless, it can be shown that *given in a space $\boldsymbol{R}_m$ any summable function of a point f, there always exists an additive continuous function of an interval, of bounded variation, F, such that $F'_s(x)=f(x)$ almost everywhere in $\boldsymbol{R}_m$.*

Lusin's method is applicable in several other arguments. It has been used, for instance, by J. Marcinkiewicz [1], to derive the theorem:

There exists a continuous function of a real variable F which has the following property: with each measurable function f, almost everywhere finite, there can be associated a sequence of positive numbers $\{h_n\}$ tending to 0 such that

$$\lim_n\,[F(x+h_n)-F(x)]/h_n=f(x)$$

at almost all the points x.

§ 3. Approximate limits and derivatives. Given any function F defined in the neighbourhood of a point x_0 of a space $\boldsymbol{R}_m$, we shall call *approximate upper limit* of F at x_0 the lower bound of all the numbers y ($+\infty$ included) for which the set $\underset{x}{\mathrm{E}}[F(x)>y]$ has x_0 as a point of dispersion (cf. Chap. IV, § 10). Similarly, the *approximate lower limit* of the function F at the point x_0 is the upper bound of the numbers y for which the set $\underset{x}{\mathrm{E}}[F(x)<y]$ has x_0 as a point of dispersion. These two approximate limits of F at x_0 are called also *extreme* approximate limits and denoted by $\limsup_{x\to x_0}\operatorname{ap} F(x)$ and $\liminf_{x\to x_0}\operatorname{ap} F(x)$ respectively. When they are equal, their common value is termed *approximate limit* of F at x_0 and denoted by $\lim_{x\to x_0}\operatorname{ap} F(x)$.

It is easily seen that if E is a measurable set for which x_0 is a point of density, then, in the preceding definitions of extreme approximate limits, the sets $\underset{x}{\mathrm{E}}[F(x)>y]$ and $\underset{x}{\mathrm{E}}[F(x)<y]$ may be replaced by the sets $\underset{x}{\mathrm{E}}[F(x)>y;\ x \epsilon E]$ and $\underset{x}{\mathrm{E}}[F(x)<y;\ x\epsilon E]$ respectively. Hence

(3.1) ***Theorem.*** *If two functions coincide on a measurable set E, their approximate extreme limits coincide at almost all points of E, and in fact at every point of density of E.*

We see further that *if x_0 is a point of density for a measurable set E and if the limit of $F(x)$ exists as x tends to x_0 on E, then this limit is at the same time the approximate limit of F at the point x_0.* Therefore, if a function F is approximately continuous (cf. Chap. IV, p. 131) at a point x_0, we must have $F(x_0)=\lim_{x\to x_0}\operatorname{ap} F(x)$.

If x_0 is a point of density for a measurable set E and if, further, the function F is measurable on E, it is easily seen that the approximate upper limit of F at x_0 is the lower bound of the numbers y for which the set $\underset{x}{\mathrm{E}}[F(x)\leqslant y;\ x\epsilon E]$ has x_0 as a point of density. It follows, by the definition of approximate lower limit, that *with the same hypotheses on the set E and on the function F, in order that $l=\lim_{x\to x_0}\operatorname{ap} F(x)$, it is necessary and sufficient that for each $\varepsilon>0$ the set $\underset{x}{\mathrm{E}}[l-\varepsilon\leqslant F(x)\leqslant l+\varepsilon;\ x\epsilon E]$ should have the point x_0 as a point of density.*

Let us remark finally that the following inequalities hold between approximate and ordinary extreme limits:

$$\liminf_{x\to x_0} F(x)\leqslant \liminf_{x\to x_0}\operatorname{ap} F(x)\leqslant \limsup_{x\to x_0}\operatorname{ap} F(x)\leqslant \limsup_{x\to x_0} F(x); \tag{3.2}$$

and hence the approximate limit exists and is equal to the ordinary limit, wherever the latter exists.

In order to understand better the meaning of the definitions of approximate limits, it may be remarked that the definitions of the ordinary limits are expressible in a very similar form. Thus the upper limit of $F(x)$ at x_0 may be defined as the lower bound of all the numbers y for which x_0 is not a point of accumulation for the set $\underset{x}{\mathrm{E}}[F(x)>y]$. The inequality (3.2) then becomes obvious.

For functions of a real variable, in addition to the approximate limits defined above, and which in this case we call *bilateral*, we introduce also four *unilateral* approximate limits. The *approximate upper right-hand limit* of a function F at a point x_0 is the lower bound of the numbers y for which the set $\underset{x}{\mathrm{E}}[F(x)>y;\ x>x_0]$ has x_0 as a point of dispersion. This limit is written $\limsup_{x\to x_0+} \mathrm{ap}\, F(x)$. The three other approximate extreme unilateral limits are defined and denoted similarly.

These generalizations of the notion of limit lead very naturally to parallel generalizations of derivates. Thus, given a finite function of a real variable F, we define at each point x_0 the *approximate right-hand upper derivate* $\overline{F}^{+}_{\mathrm{ap}}(x_0)$ and *lower derivate* $\underline{F}^{+}_{\mathrm{ap}}(x_0)$, the *approximate left-hand upper derivate* $\overline{F}^{-}_{\mathrm{ap}}(x_0)$ and *lower derivate* $\underline{F}^{-}_{\mathrm{ap}}(x_0)$, and the *approximate bilateral upper derivate* $\overline{F}_{\mathrm{ap}}(x_0)$ and *lower derivate* $\underline{F}_{\mathrm{ap}}(x_0)$, as the corresponding approximate extreme limits of the ratio $[F(x)-F(x_0)]/(x-x_0)$ as $x\to x_0$. When all these derivates are equal (or, what comes to the same, when $\overline{F}_{\mathrm{ap}}(x_0)=\underline{F}_{\mathrm{ap}}(x_0)$), their common value is called *approximate derivative* of F at x_0 and is denoted by $F'_{\mathrm{ap}}(x_0)$; if further, this derivative is finite, the function F is said to be *approximately derivable* at x_0.

For some further generalizations, such as "preponderant derivates" ("nombres dérivés prépondérants"), and for a deeper study of the properties of approximate derivates, the reader should consult A. Denjoy [6] and A. Khintchine [5].

The properties of bilateral approximate limits, discussed above, can be taken over, with the obvious formal modifications, so as to apply to unilateral approximate limits. In particular, Theorem 3.1 may be completed as follows:

(3.3) ***Theorem.*** *If two functions of a real variable coincide on a measurable set E, their approximate extreme limits and their approximate derivates coincide respectively at almost all points of E, and in fact at every point of density of E.*

Also, if a function F is measurable on a set E, we have $F'_{\mathrm{ap}}(x)=F'_E(x)$ at almost all the points x of E at which the function F has a derivative with respect to the set E.

§ 4. Functions VB and VBG. We shall denote by $\mathrm{V}(F;E)$, and call *weak variation* of a finite function $F(x)$ on a set E, the upper bound of the numbers $\sum_i |F(b_i)-F(a_i)|$ where $\{[a_i,b_i]\}$ is any sequence of non-overlapping intervals whose end-points belong to E. If $\mathrm{V}(F;E)<+\infty$, the function F is said to be *of bounded variation in the wide sense* on the set E, or, simply, *of bounded variation* on E, or VB on E.

In the special case in which the set E is a closed interval, we clearly have $\mathrm{V}(F;E)=\mathrm{W}(F;E)$, i. e. the weak variation of the function F on E then coincides with its absolute variation in the sense of Chap. III, § 13.

The definition of functions of bounded variation in the wide sense on a set thus constitutes a generalization (for functions of a real variable) of that of functions of bounded variation on an interval. If E is a linear figure formed of disconnected intervals we only get the inequality $\mathrm{V}(F;E)\geqslant\mathrm{W}(F;E)$, but it is easy to see that even then the relation $\mathrm{W}(F;E)<+\infty$ always implies $\mathrm{V}(F;E)<+\infty$.

Plainly, *every function which is* VB *on a set E is bounded on E and is* VB *on each subset of E*. Again, *any function F which is continuous on a set E and* VB *on a set $A\subset E$ everywhere dense in E* (cf. Chap. II, § 2) *is* VB *on the whole set E* (for then $\mathrm{V}(F;E)=\mathrm{V}(F;A)$). Finally, if F and G are two functions which are bounded on a set E and M denotes the upper bound of the absolute values of these functions on E, we have $\mathrm{V}(aF+bG;E)\leqslant|a|\cdot\mathrm{V}(F;E)+|b|\cdot\mathrm{V}(G;E)$ for each pair of constants a and b, and $\mathrm{V}(F\cdot G;E)\leqslant M\cdot[\mathrm{V}(F;E)+\mathrm{V}(G;E)]$ (cf. Chap. III, p. 97). Hence *every linear combination, with constant coefficients, of two functions which are* VB *on a set, and the product of the two functions, are themselves* VB *on this set.*

A function $F(x)$ is said to be *of generalized bounded variation in the wide sense* on a set E, or simply, *of generalized bounded variation* on E, or again, for short, VBG on E, if E is the sum of a finite or enumerable sequence of sets on each of which $F(x)$ is VB. From what has just been proved for functions which are VB we see at once that *every linear combination of two functions which are* VBG *on a set, and the product of the two functions, are themselves* VBG *on this set.*

(4.1) ***Lemma.*** *In order that a function F be bounded and non-decreasing [of bounded variation] on a set E, it is necessary and sufficient that F coincide on E with a function which is bounded and non-decreasing [of bounded variation] on the whole straight line $\boldsymbol{R}_1$.*

Proof. Let us denote for each x, by $E_{(x)}$ the set of the points of E which belong to the interval $(-\infty, x]$. We shall consider two cases separately.

1° The function F is bounded and non-decreasing on E. For each x, let $G(x)$ denote the upper bound of the function F on the set $E_{(x)}$, or else the lower bound of the function F on E, according as $E_{(x)} \neq 0$ or $E_{(x)} = 0$. The function G thus defined is evidently bounded and non-decreasing on the whole straight line $\boldsymbol{R}_1$ and coincides with the function F on E.

2° The function F is VB on E. For each point x, let $V(x) = \mathrm{V}(F; E_{(x)})$ if $E_{(x)} \neq 0$, and $V(x) = 0$ if $E_{(x)} = 0$. We see at once that the function $V(x)$ is monotone and bounded on the whole straight line $\boldsymbol{R}_1$ and that $V(x) - F(x)$ is non-decreasing and bounded on E. Hence, by what has just been proved in 1°, there exists a function $G(x)$ which is bounded and non-decreasing on $\boldsymbol{R}_1$ and which coincides on E with $V(x) - F(x)$. We have therefore $F(x) = V(x) - G(x)$ for every $x \in E$, and since the function $V(x) - G(x)$, as difference of two bounded monotone functions, is clearly of bounded variation on $\boldsymbol{R}_1$, this completes the proof.

(4.2) ***Theorem.*** *Let F be a function which is measurable on a set E and which is* VB *on a set $E_1 \subset E$. Then* (i) *F is approximately derivable at almost all points of E_1 and* (ii) *there exists a measurable set E_2 such that $E_1 \subset E_2 \subset E$ and that F is* VB *on E_2.*

Proof. By Lemma 4.1, there exists a function G which coincides with F on the set E_1 and which is of bounded variation on the whole straight line $\boldsymbol{R}_1$. Let E_2 be the set of the points x of E at which $F(x) = G(x)$. Then since F is, by hypothesis, measurable on E, the set E_2 must be measurable. Moreover, as $E_1 \subset E_2 \subset E$, the function F is, with G, of bounded variation on E_2, and by Lebesgue's Theorem 5.4, Chap. IV, and Theorem 3.3, the finite approximate derivative $F'_{\mathrm{ap}}(x) = G'(x)$ exists at almost all the points x of E_2.

Theorem 4.2 leads at once to the following theorem, which for the Denjoy integral takes the place of Lebesgue's Theorem on derivability of functions of bounded variation:

(4.3) ***Theorem of Denjoy-Khintchine.*** *A function which is measurable and* VBG *on a set is approximately derivable at almost all points of this set.*

Finally, if we make use of Theorem 9.1, Chap. IV, and Lemma 4.1, we may complete Theorem 4.2 as follows:

(4.4) ***Theorem.*** *A function F which is* VB *on a set E, is derivable with respect to the set E at almost all points of E. Moreover, if N denotes the set of the points at which the derivative $F'_E(x)$ (finite or infinite) does not exist, then the graph of the function F on N is of length zero and consequently the set of the values taken by F on N is of measure zero; in symbols $\Lambda\{\mathrm{B}(F; N)\} = |F[N]| = 0$.*

For an extension of Theorem 4.3 to functions of two variables, *vide* V. G. Čelidze [1].

§ 5. Functions AC and ACG.

A finite function F will be termed *absolutely continuous in the wide sense* on a set E, or *absolutely continuous* on E, or simply AC on E, if given any $\varepsilon > 0$ there exists an $\eta > 0$ such that for every sequence of non-overlapping intervals $\{[a_k, b_k]\}$ whose end-points belong to E, the inequality $\sum_k (b_k - a_k) < \eta$ implies $\sum_k |F(b_k) - F(a_k)| < \varepsilon$.

A function F will be termed *generalized absolutely continuous function in the wide sense* on a set E, or *generalized absolutely continuous function* on E, or finally ACG on E, if F is continuous on E and if E is the sum of a finite or enumerable sequence of sets E_n on each of which F is AC.

These definitions generalize that of functions absolutely continuous on a linear interval (cf. Chap. III, §§ 12, 13) and allow us to generalize certain fundamental properties of the latter. We see at once, by the arguments of the preceding §, that *every linear combination of two functions which are* AC [ACG] *on a bounded set, and the product of such functions, are themselves* AC [ACG] *on this set.* Further, *every function which is* AC *on a bounded set E is* VB *on E.* In fact, if F is such a function, there exists an $\eta_0 > 0$ such that $\mathrm{V}(F; E \cdot I) \leqslant 1$ for each interval I of length $< \eta_0$. It follows that F is bounded on E. Let M be the upper bound of the absolute values of F on E, and let J be an interval containing E; then, J is the sum of a finite number of non-overlapping intervals $J_1, J_2, \ldots, J_p$ each of which is of length $< \eta_0$, and we find $\mathrm{V}(F; E) \leqslant \sum_k \mathrm{V}(F; E \cdot J_k) + 2pM < +\infty$.

It follows at once that any function which is ACG on a set E (bounded or unbounded) is VBG on E, and therefore, by the theorem of Denjoy-Khintchine given in the preceding §, *every function which is* ACG *on a measurable set is approximately derivable at almost all points of this set.*

Nevertheless we can construct an example of a function which is ACG on an interval and which is not derivable in the ordinary sense at the points of a set of positive measure.

For this purpose, let H denote a bounded, perfect, non-dense set of positive measure, with the bounds a and b. Let $I=[a,b]$ and let $\{I_n=[a_n,b_n]\}$ be the sequence of the intervals contiguous to H. We denote further by ϱ_n the length of the largest subinterval of $[a,b]$ which does not overlap the first n intervals $I_1, I_2, \dots, I_n$ of this sequence. Plainly

$$\lim_n |I_n| = 0 \quad \textit{and} \quad \lim_n \varrho_n = 0. \tag{5.1}$$

Now let c_n denote for each $n=1,2,\dots$, the centre of the interval I_n, and let F be the function defined on the interval I by the following conditions: 1° $F(x)=0$ for $x \in H$; 2° $F(c_n)=|I_n|+\varrho_n$ for $n=1,2,\dots$; 3° the function F is linear in each of the intervals $[a_n, c_n]$ and $[c_n, b_n]$ where $n=1,2,\dots$. Thus defined, the function F is continuous on I by (5.1) and is AC on H and on each I_n; since $I=H+\sum_n I_n$, it follows that F is ACG on I.

We shall show that F is not derivable at any point $x \in H$. In fact, since F vanishes on H, we have

$$\underline{F}(x) \leqslant 0 \leqslant \overline{F}(x) \quad \textit{for every} \quad x \in H. \tag{5.2}$$

If therefore a point x_0 is a left-hand end-point of an I_n, there can be no derivative $F'(x_0)$ since it is clear that $\overline{F}(x_0)=\overline{F}^+(x_0)>0$ and therefore, by (5.2), that $\overline{F}(x_0) \neq \underline{F}(x_0)$. Similarly, $\underline{F}(x_0)<0\leqslant\overline{F}(x_0)$ if x_0 is a right-hand end-point of an interval I_0.

If, on the other hand, $x_0 \in H$, $x_0 \neq a_n$ and $x_0 \neq b_n$ for $n=1,2,\dots$, denote by i_n the suffix of that interval of the system $I_1, I_2, \dots, I_n$ which is nearest to x_0. Then $\lim_n i_n=+\infty$ and $0<|c_{i_n}-x_0|<|I_{i_n}|+\varrho_n$, and so, by the definition of $F(x)$, we have $F(c_{i_n})-F(x_0)=|I_{i_n}|+\varrho_{i_n}\geqslant|I_{i_n}|+\varrho_n>|c_{i_n}-x_0|$. Since $\lim_n c_{i_n}=x_0$, it follows that either $\overline{F}(x_0)\geqslant 1$ or $\underline{F}(x_0)\leqslant -1$, which by (5.2), proves that F is not derivable at x_0.

Let us remark, in conclusion, that *a function F which is continuous on a set E and which is* AC *on a subset of E everywhere dense in E, is* AC *on the whole set E.*

§ 6. Lusin's condition (N).

A finite function F is said to fulfil the *condition* (N) on a set E, if $|F[H]|=0$ for every set $H \subset E$ of measure zero (for the notation cf. Chap. III, p. 100). Clearly, *a function which fulfils the condition* (N) *on each of the sets of a finite or enumerable sequence, also fulfils this condition on the sum of these sets.*

The condition (N) was introduced by N. Lusin [I, p. 109], who was the first to recognize the importance of this condition in the theory of the integral. It is easy to see that in the domain of continuous functions the condition (N) is necessary and sufficient in order that the function should transform every measurable set into a measurable set (cf. H. Rademacher [1] and H. Hahn [I, p. 586]). Among the more recent researches devoted to the condition (N) and to other similar conditions (cf., below, Chap. IX) the reader should consult above all N. Bary [3].

(6.1) ***Theorem.*** *A function which is* ACG *on a set necessarily fulfils the condition* (N) *on this set.*

Proof. Since each set on which a function is ACG is the sum of a sequence of sets on which the function is AC, it will suffice to prove that $|F[H]|=0$ whenever H is a set of measure zero and F a function AC on H.

For this purpose, let ε be any positive number. We denote, for brevity, by $M(E)$ and $m(E)$ respectively the upper and lower bounds of F on E, when E is any subset of H, and we write $M(E)=m(E)=0$ in the case in which $E=0$. Since the function F is AC on H, there exists a number $\eta>0$ such that $\sum_k [M(H\cdot I_k)-m(H\cdot I_k)]<\varepsilon$ for every sequence of non-overlapping intervals $\{I_k\}$ which satisfies the condition $\sum_k |I_k|<\eta$. Now since the set H is of measure zero, we can determine a sequence of non-overlapping intervals $\{I_k\}$ which satisfies this last condition and which covers, at the same time, the whole set E. Therefore, since $|F[H\cdot I_k]|\leqslant M(H\cdot I_k)-m(H\cdot I_k)$ for each k, it follows that $|F[H]|\leqslant\varepsilon$. Hence, ε being arbitrary, $|F[H]|=0$.

It follows from Theorem 7.8 (1°), Chap. IV, that every function which is absolutely continuous on an interval and whose derivative is almost everywhere non-negative, is monotone non-decreasing. With the help of Theorem 6.1, this result can be extended to functions which are ACG and we have:

(6.2) ***Theorem.*** *Every function $F(x)$ which is* ACG *on an interval I and for which we have almost everywhere in this interval $F'_{\mathrm{ap}}(x)\geqslant 0$, or more generally, $\overline{F}^+(x)\geqslant 0$, is monotone non-decreasing.*

In particular therefore, if the approximate derivative of a function which is ACG *on an interval vanishes almost everywhere on this interval, then the function is a constant.*

Proof. Let ε be any positive number and let $G(x)=F(x)+\varepsilon x$. The function G is then ACG on the interval I (together with the function F), and moreover, we have $\overline{G}^+(x)=\overline{F}^+(x)+\varepsilon\geqslant\varepsilon>0$ at almost all the points x of I. Hence, denoting by H the set of the points x at which $\overline{G}^+(x)\leqslant 0$, we have $|H|=0$, and this implies, by Theorem 6.1, that $|G[H]|=0$. Thus the set $G[H]$ cannot contain any non-degenerate interval, and by Theorem 7.1, Chap. VI, the function $G(x)=F(x)+\varepsilon x$ is non-decreasing on I. It follows at once, by making $\varepsilon\to 0$, that the function F is itself non-decreasing.

If we analyze the preceding proof, we notice that the hypothesis of generalized absolute continuity of $F(x)$ has been used only to show that every function of the form $F(x)+\varepsilon x$, where $\varepsilon>0$, fulfils the condition (N). It is remarkable that the condition (N) need not remain satisfied when we add a linear function to a function fulfilling the condition, even when this last function is restricted to be continuous (*vide* S. Mazurkiewicz [1]). For this reason it is not enough to suppose in the preceding proof that the function $F(x)$ merely fulfils the condition (N).

Nevertheless, Theorem 6.2 itself does remain true for arbitrary functions which fulfil the condition (N). The theorems which will be proved in Chap. IX, § 7, include a more general result, namely that *every continuous function which fulfils the condition* (N) *and whose derivative is non-negative at almost all the points at which it exists, is monotone non-decreasing.*

We shall show (*vide*, below, Theorem 6.8) that for continuous functions of generalized bounded variation on closed sets, the converse of Theorem 6.1 is true, i. e. that in this case the condition (N) is equivalent to generalized absolute continuity. Similarly, for continuous functions of bounded variation the condition (N) is equivalent to absolute continuity in the ordinary sense.

We shall begin with a lemma which will also prove useful elsewhere.

(6.3) ***Lemma.*** *If, for a finite function F, the inequalities $\overline{F}^{+}(x)\leqslant M$ and $\underline{F}^{-}(x)\geqslant -M$, where M is a finite non-negative number, hold at each point x of a set D, then $|F[D]|\leqslant M\cdot|D|$.*

Proof. Let ε be any positive number. Let D_n denote for each positive integer n the set of the points x of D for which we have $F(t)-F(x)\leqslant(M+\varepsilon)\cdot|t-x|$ whenever $|t-x|\leqslant 1/n$. The sets D_n evidently constitute an ascending sequence and we see easily that $D=\lim\limits_n D_n$.

With each D_n we can associate a sequence of intervals $\{I_k^{(n)}\}_{k=1,2,\ldots}$ which covers D_n and fulfils the condition

$$\sum_k |I_k^{(n)}|\leqslant|D_n|+\varepsilon, \tag{6.4}$$

and in which, further, no $I_k^{(n)}$ has length greater than $1/n$. By definition of D_n, we therefore have, for every pair x_1, x_2 of points of $D_n\cdot I_k^{(n)}$, the inequality $|F(x_2)-F(x_1)|\leqslant(M+\varepsilon)\cdot|x_2-x_1|\leqslant(M+\varepsilon)\cdot|I_k^{(n)}|$, so that $|F[D_n\cdot I_k^{(n)}]|\leqslant(M+\varepsilon)\cdot|I_k^{(n)}|$. In view of the inequality (6.4) it therefore follows that, for every n,

$$|F[D_n]|\leqslant\sum_k|F[D_n\cdot I_k^{(n)}]|\leqslant(M+\varepsilon)\cdot\sum_k|I_k^{(n)}|\leqslant(M+\varepsilon)\cdot(|D_n|+\varepsilon);$$

and, by making first $n\to\infty$ and then $\varepsilon\to 0$, we derive $|F[D]|\leqslant M\cdot|D|$.

(6.5) ***Theorem.*** *If a function F is derivable at every point of a measurable set D, then*

$$|F[D]|\leqslant\int_D|F'(x)|\,dx. \tag{6.6}$$

Proof. We may clearly assume that the set D is bounded. Given any $\varepsilon>0$, let D_n denote, for each positive integer n, the set of the points $x\in D$, at which $(n-1)\varepsilon\leqslant|F'(x)|<n\cdot\varepsilon$. We then have, by the preceding lemma,

$$|F[D]|\leqslant\sum_{n=1}^{\infty}|F[D_n]|\leqslant\sum_{n=1}^{\infty}n\varepsilon\cdot|D_n|\leqslant\int_D|F'(x)|\,dx+\varepsilon\cdot|D|,$$

and hence, ε being arbitrary, the inequality (6.6).

The formula (6.6) remains true when we replace in it the derivative $F'(x)$ by any Dini derivate, provided however that we restrict the latter to be finite in D. The proof then becomes rather more elaborate and requires certain general theorems on derivates which will be established later (*vide* Chap. IX, § 4).

(6.7) ***Theorem.*** *In order that a function $F(x)$ which is continuous and* VB *on a bounded closed set E, be* AC *on E, it is necessary and sufficient that $F(x)$ fulfil the condition* (N) *on this set.*

Proof. In view of Theorem 6.1, it remains to be shown that the condition is sufficient.

Suppose then that F fulfils the condition (N) on E. Let a_0 and b_0 be the bounds of E, and let G denote the function which coincides with F at the points of E and is linear in the intervals contiguous to E. The function G is evidently continuous and of bounded variation, and fulfils the condition (N) on the whole interval $[a_0, b_0]$.

Given any subinterval $I=[a,b]$ of $[a_0,b_0]$, let us denote by D the set of the points of I, at which the function G is derivable, and write $H=I-D$. Plainly $|H|=0$, and therefore also $|G[H]|=0$.

On the other hand, since the interval with the end-points $G(a)$ and $G(b)$ is contained in $G[I]$, we have by Theorem 6.5

$$|G(b)-G(a)|\leqslant|G[D]|+|G[H]|=|G[D]|\leqslant\int_a^b|G'(x)|\,dx.$$

Since this inequality is valid for every subinterval $I=[a,b]$ of $[a_0,b_0]$ and since by Theorem 7.4, Chap. IV, the derivative $G'(x)$ is summable on $[a_0,b_0]$, it follows that the function G is AC on $[a_0,b_0]$, and therefore that F is AC on the set E, where F and G coincide.

It is easy to see that the same argument leads to a more general theorem: *in order that a continuous function F which is continuous on an interval I_0 be absolutely continuous on this interval, it is necessary and sufficient that F fulfil the condition* (N) *on I_0 and that its derivative exist almost everywhere on I_0 and be summable on I_0.* This theorem will again be generalized in Chap. IX, § 7.

(6.8) ***Theorem.*** *In order that a function F which is continuous and* VBG *on a closed set E be* ACG *on E, it is necessary and sufficient that F fulfil the condition* (N) *on this set.*

Proof. In view of Theorem 6.1, we need only prove the condition (N) sufficient. Now, since F is VBG on the set E, this set is expressible as the sum of a sequence of bounded sets $\{E_n\}$ such that the function F is VB on each E_n. By continuity of F on the closed set E, we may suppose (cf. § 4, p. 221) that each set E_n is closed. Since further F fulfils the condition (N) on E, it follows from Theorem 6.7 that the function F is AC on each E_n, and therefore ACG on E.

§ 7. Functions VB$_*$ and VBG$_*$. We shall denote by $V_*(F;E)$ and term *strong variation* of a finite function F on a set E, the upper bound of the sums $\sum_k O(F;I_k)$ where $\{I_k\}$ is any sequence of non-overlapping intervals whose end-points belong to E (in accordance with Chap. III, p. 60, $O(F;I_k)$ denotes the oscillation of F on the interval I_k). If $V_*(F,E)<+\infty$, the function F will be said to be *of bounded variation in the restricted sense* on the set E, or VB$_*$ on E.

Following the order of the definitions of § 4, we shall say further that a finite function is *of generalized bounded variation in the restricted sense*, or simply, is VBG$_*$ on a set E, if E is the sum of a finite or enumerable sequence of sets on each of which the function is VB$_*$.

In the special case in which the set E is a closed interval, we clearly have $V_*(F;E)=V(F;E)=W(F;E)$. It is easy to see that we always have $V(F;E)\leqslant V_*(F;E)$; so that *every function which is* VB$_*$ *on a set, is* VB *on this set*, and consequently, *every function which is* VBG$_*$ *on a set, is* VBG *on this set.* We next observe (by using trivial inequalities for the VB$_*$ case, and thence passing on to the VBG$_*$ case) that *every linear combination, with constant coefficients, of two functions which are* VB$_*$ [VBG$_*$], *and also the product of two such functions, are themselves* VB$_*$ [VBG$_*$].

Let us observe that, for a function, the property of being VB, VBG, AC, or ACG, on a set E depends solely on the behaviour of the function on E; whereas the property of being VB$_*$ or VBG$_*$ on E depends on the behaviour of the function on the whole of an interval containing the set E. In other words, of two functions which coincide on a set E, one may be VB$_*$ or VBG$_*$ on E and the other not. The same remark applies to the property of being AC$_*$ or ACG$_*$ with which we shall be concerned in the next §.

We have remarked in § 4, p. 221, that a function which is continuous on a set E and which is VB on an everywhere dense subset of E, is necessarily VB on E. A similar result is true for functions which are VB$_*$, the assumption of continuity of the given function being now superfluous. We have in fact:

(7.1) ***Theorem.*** *Every finite function F which is* VB$_*$ *on a bounded set E is equally so on the closure $\bar{E}$ of this set.*

Proof. Let a and b denote the bounds of E and therefore also of $\bar{E}$. Let $a=a_0<a_1<\ldots<a_n=b$ be any finite sequence of points of $\bar{E}$; we write $I=[a,b]$ and $I_k=[a_{k-1},a_k]$ for $k=1,2,\ldots,n$. We shall say that an interval I_k is of the first class if it contains points of E, and otherwise of the second class. The intervals I_1 and I_n are clearly of the first class, and we see easily that, if an interval I_k is of the second class, then both the adjacent intervals I_{k-1} and I_{k+1} are certainly of the first class.

Let us denote by $1=i_0<i_1<\ldots<i_r=n$ the suffixes of the intervals I_k of the first class and by $j_0<j_1<\ldots<j_s$ those of the second. With each interval I_{i_h} of the first class we associate a point $b_h\,\epsilon\, I_{i_h}\cdot E$ and we write $J_h=[b_{h-1},b_h]$ for $h=1,2,\ldots,r$. It is easy to see that

$$\sum_{h=0}^{r} O(F;I_{i_h})\leqslant O(F;I_1)+O(F;I_n)+2\cdot\sum_{h=1}^{r} O(F;J_h)$$

and

$$\sum_{h=0}^{s} O(F;I_{j_h})\leqslant\sum_{h=1}^{r} O(F;J_h).$$

Hence, $\sum_{k=1}^{n} O(F;I_k)\leqslant 3\cdot\sum_{h=1}^{r} O(F;J_h)+2\cdot O(F;I)\leqslant 3\cdot[V_*(F;E)+O(F;I)]$, and therefore $V_*(F;\bar{E})\leqslant 3\cdot[V_*(F;E)+O(F;I)]<+\infty$. This completes the proof.

(7.2) ***Theorem.*** *If a function F is* VBG_* *on a set E, then F is derivable at almost all points of this set; and further if N denotes the set of the points x of E at which the function has no derivative, finite or infinite, then* $|F[N]|=\Lambda\{\mathrm{B}(F;N)\}=0$.

Proof. We may clearly suppose that the set E is bounded and that the function F is VB_* on E. Moreover, by Theorem 7.1, we may suppose that the set E is closed.

Let therefore a and b denote the bounds of E on the left and on the right, and $\{I_n\}_{n=1,2,...}$ the sequence of the intervals contiguous to E. Writing m_n and M_n respectively for the lower and upper bounds of F on I_n, we define two functions $m(x)$ and $M(x)$ on $[a,b]$ making $m(x)=m_n$ and $M(x)=M_n$ for $x \epsilon I_n^\circ$ where $n=1,2,\ldots$, and $m(x)=M(x)=F(x)$ for $x \epsilon E$. The two functions $m(x)$ and $M(x)$ thus defined are plainly of bounded variation on the whole interval $[a,b]$ and coincide with $F(x)$ on the set E. Therefore, denoting by N_0 the set of the points $x \epsilon E$ at which either one at least of the (finite or infinite) derivatives $M'(x)$ and $m'(x)$ does not exist, or both exist without being equal, we find by Theorem 9.1, Chap. IV, that

$$|F[N_0]|=|\Lambda\{\mathrm{B}(F;N_0)\}|=0. \tag{7.3}$$

On the other hand, $m(x)=F(x)=M(x)$ at every point x of E, while $m(x)\leqslant F(x)\leqslant M(x)$ on the whole interval $[a,b]$. It follows that the derivative $F'(x)=m'(x)=M'(x)$ exists at each point x of E, except at most those of the set N_0 which is subject to the relation (7.3). Finally, since the functions $m(x)$ and $M(x)$ are derivable almost everywhere on the interval $[a,b]$, the function F must be derivable at almost all points of E, and this completes the proof.

Theorem 7.2 (for continuous functions and in a slightly less complete form) was first proved by Denjoy and by Lusin, independently. It plays in the theory of the Denjoy-Perron integral (*vide*, below, Chap. VIII) a part similar to that of Lebesgue's Theorem (Chap. IV, § 5) in the theory of the Lebesgue integral. A corresponding part is played in the theory of the Denjoy-Khintchine integral by Theorem 4.3. But the latter is stated in terms of approximate derivation (cf. the example of p. 224) whereas Theorem 7.2, which requires no modification of the notion of derivative, is, for functions of a real variable, a direct generalization of Lebesgue's Theorem.

§ 8. Functions AC_* and ACG_*. A finite function F is said to be *absolutely continuous in the restricted sense* on a bounded set E, or to be AC_* on E, if F is bounded on an interval containing E and if to each $\varepsilon>0$ there corresponds an $\eta>0$ such that, for every finite sequence of non-overlapping intervals $\{I_k\}$ whose end-points belong to E, the inequality $\sum_k|I_k|<\eta$ implies $\sum_k \mathrm{O}(F;I_k)<\varepsilon$.

A function will be termed *generalized absolutely continuous* on a set E, or ACG_* on E, if the function is continuous on E and if the set E is expressible as the sum of a sequence of bounded sets on each of which the function is AC_*.

In the case in which the set E is an interval, the class of functions AC_* on E coincides with that of the functions which are absolutely continuous on E in the ordinary sense. Every function which is AC_* on an arbitrary set E is AC on E, and every function which is ACG_* on E is ACG on E. On the other hand, *any function which is* AC_* *on a bounded set is* VB_* *on this set*, and therefore, *any function which is* ACG_* *on a set is* VBG_* *on this set.* To see this, let F be AC_* on a bounded set E. We can then determine a positive number η_0 such that $\mathrm{V}_*(F;E\cdot I)\leqslant 1$ for every interval I of length less than η_0. Let J be the smallest interval containing E, let M be the upper bound of $|F(x)|$ on J, and suppose J expressed as the sum of a finite number of non-overlapping intervals $J_1,J_2,\ldots,J_p$ each of length less than η_0. We shall then have

$$\mathrm{V}_*(F;E)\leqslant\sum_{k=1}^{p}\mathrm{V}_*(F;E\cdot J_k)+2Mp\leqslant(2M+1)\cdot p<+\infty,$$

and this shows that the function F is VB_* on E.

Thus a function which is AC_* on a bounded set E is both AC and VB_* on this set, and similarly a function which is ACG_* on E is both ACG and VBG_* on E. The converse also is true, provided that the set E is restricted to be closed (*vide*, below, Theorem 8.8). Instead of giving a special proof of this result, we shall establish some more general theorems about the relations between the notions

VB, AC, VB_*, AC_*, VBG, ACG, VBG_* and ACG_*.

(8.1) ***Lemma.*** *Let E denote a bounded closed set, $\{J_k\}$ the sequence of the intervals contiguous to E, and I_0 the smallest interval containing E. Then, for any function F which is finite on I_0, we have*

$$\mathrm{O}(F;I_0)\leqslant\mathrm{V}(F;E)+2\cdot\sum_k\mathrm{O}(F;J_k). \tag{8.2}$$

Proof. Let M, m and M_0, m_0 be the bounds (upper, lower) of F, on E and on I_0 respectively. Let M_0' be any finite number less than M_0, and x_0 a point of I_0 such that $M_0' \leqslant F(x_0)$. If we have $x_0 \epsilon E$, this inequality implies $M_0' \leqslant M$, while if x_0 belongs to an interval, J_{k_0} say, of the sequence $\{J_k\}$, $M_0' \leqslant M + \mathrm{O}(F; J_{k_0})$. Hence

$$M_0 \leqslant M + \sum_k \mathrm{O}(F; J_k), \tag{8.3}$$

and similarly

$$m_0 \geqslant m - \sum_k \mathrm{O}(F; J_k). \tag{8.4}$$

On subtracting (8.4) from (8.3), we obtain, since $M - m \leqslant \mathrm{V}(F; E)$, the relation (8.2).

(8.5) ***Theorem.*** *In order that a function F which is* VB [AC] *on a bounded closed set E, be* VB$_*$ [AC$_*$] *on E, it is necessary and sufficient that the series of its oscillations on the intervals contiguous to E be convergent.*

Proof. The necessity of these conditions is obvious (cf. above p. 231); we have therefore only to prove them sufficient.

Let then $\{J_k\}$ denote the sequence of the intervals contiguous to E, and suppose that

$$\sum_k \mathrm{O}(F; J_k) < +\infty. \tag{8.6}$$

We shall consider the two cases separately:

1° The function F is VB on E, i. e. $\mathrm{V}(F; E) < +\infty$. Then by Lemma 8.1, we have for every sequence $\{I_n\}$ of non-overlapping intervals whose end-points belong to E,

$$\sum_n \mathrm{O}(F; I_n) \leqslant \sum_n \mathrm{V}(F; E \cdot I_n) + 2 \cdot \sum_k \mathrm{O}(F; J_k) \leqslant \mathrm{V}(F; E) + 2 \cdot \sum_k \mathrm{O}(F; J_k).$$

It follows by (8.6) that $\mathrm{V}_*(F; E) < +\infty$, i. e. that the function F is VB$_*$ on E.

2° The function F is AC on E. Then, given any $\varepsilon > 0$, there exists a number $\eta > 0$ such that, for every sequence of non-overlapping intervals $\{I_n\}$ whose end-points belong to E, the inequality $\sum_n |I_n| < \eta$ implies $\sum_n \mathrm{V}(F; E \cdot I_n) < \varepsilon/2$. Now by (8.6), there exists a positive integer k_0 such that

$$\sum_{k=k_0+1}^{\infty} \mathrm{O}(F; J_k) < \varepsilon/4. \tag{8.7}$$

Denote by η_0 the smallest of the k_0+1 numbers $\eta, |J_1|, |J_2|, \ldots, |J_{k_0}|$, and let $\{\tilde{I}_n\}$ be any sequence of non-overlapping intervals with end-points in E, the sum of whose lengths is less than η_0. None of these intervals $\tilde{I}_n$ can contain one of the first k_0 intervals of the sequence $\{J_n\}$, and it follows from (8.7) and from Lemma 8.1, that $\sum_n \mathrm{O}(F; \tilde{I}_n) \leqslant \sum_n \mathrm{V}(F; E \cdot \tilde{I}_n) + \varepsilon/2 \leqslant \varepsilon$. Therefore the function F is AC$_*$ on E, and this completes the proof.

(8.8) ***Theorem.*** *In order that a function F be* AC$_*$ [ACG$_*$] *on a bounded closed set E, it is necessary and sufficient that F be both* VB$_*$ *and* AC [VBG$_*$ *and* ACG] *on E.*

Proof. The necessity of these conditions is obvious, so that we have only to prove them sufficient.

Now, if the function F is both VB$_*$ and AC on E, it follows at once from Theorem 8.5 that F is AC$_*$ on E. If on the other hand, F is VBG$_*$ and ACG on E, we can express the set E as the sum of a sequence of sets $\{E_n\}$ on each of which F is both VB$_*$ and AC. Since F is ACG, and so continuous, on the set E, which is by hypothesis closed, F is AC on the closure $\bar{E}_n$ of each E_n. Similarly, by Theorem 7.1, F is VB$_*$ on each $\bar{E}_n$. Therefore by what has just been proved, F is AC$_*$ on each of the sets $\bar{E}_n$ and so, ACG$_*$ on the set E.

Theorem 8.8 ceases to hold if we remove the restriction that the set E is closed. Let E be the set of irrational points, and $\{a_n\}_{n=1,2,\ldots}$ the sequence of rational points, of the interval $[0, 1]$; and let $F(x) = 0$ for $x \epsilon E$, and $F(a_n) = 1/2^n$ for $n = 1, 2, \ldots$. The function F thus defined is evidently VB$_*$ and AC on E. To show that F is not AC$_*$, nor even ACG$_*$, on E, suppose that the set E is the sum of a sequence of sets $\{E_n\}$ on each of which F is AC$_*$. By Baire's Theorem (Chap. II, Theorem 9.2), one at least of the sets E_n would be everywhere dense in a (non-degenerate) subinterval of $[0, 1]$. But this is plainly impossible, since every subinterval of $[0, 1]$ contains, in its interior, points of discontinuity of the function F.

§ 9. Definitions of Denjoy-Lusin.

The definitions which we have adopted in this chapter for the classes of functions VBG, ACG, VBG$_*$ and ACG$_*$ are based on the ideas of A. Khintchine [3]. Rather different definitions were given by N. Lusin [I] and A. Denjoy [6], which are equivalent to those of Khintchine when we restrict ourselves to continuous functions. We give them here, in the form of necessary and sufficient conditions, in the following theorem.

(9.1) ***Theorem.*** *In order that a function which is continuous on a closed set E, be* VBG [VBG$_*$, ACG, ACG$_*$] *on E, it is necessary and sufficient that every closed subset of E contain a portion on which the function is* VB [VB$_*$, AC, AC$_*$].

Proof. We shall deal only with the VBG case, the proof for the other three cases being quite similar.

1° The condition is necessary. Let F be a function which is continuous and VBG on E. We can then express the set E as the sum of a sequence of sets $\{E_n\}$ on each of which the function F is VB and, by continuity of F, the sets E_n may be supposed closed. Then by Baire's Theorem (Chap. II, § 9), every closed subset of E has a portion P contained wholly in one of the sets E_n. The function F, which is VB on each E_n, is thus certainly VB on P.

2° The condition is sufficient. Suppose that F is a continuous function on E and that every closed subset of E contains a portion on which F is VB. Let $\{I_n\}$ be the sequence of all the open intervals I with rational end-points such that F is VBG on $E\cdot I$. Let $Q=\sum_n E\cdot I_n$ and $H=E-Q$. Plainly F is VBG on Q and we need only prove that the set H is empty.

Suppose therefore that $H\neq 0$. Since H is clearly a closed set, there exists, by hypothesis, an open interval J such that $H\cdot J\neq 0$ and that the function F is VB on $H\cdot J$. We may evidently assume that the end-points of J are rational. Therefore, the function F, which is VBG on the set Q, is also VBG on the set $E\cdot J\subset H\cdot J+Q$. This requires J to belong to the sequence of intervals $\{I_n\}$ and we have a contradiction, since the set H, by definition, has no points in common with any of the intervals I_n.

Theorem 9.1 shows in particular that every continuous function which is VBG on an interval I is at the same time VB on some subinterval of I. It follows that for every continuous function which is VBG on an interval I, there exists an everywhere dense system of subintervals on each of which the function is almost everywhere derivable, although this function may, as shown in § 5, have no derivative at the points of a set of positive measure.

§ 10. Criteria for the classes of functions VBG$_*$, ACG$_*$, VBG and ACG. A series of theorems enabling us to distinguish certain types of functions of generalized bounded variation and certain types of generalized absolutely continuous functions, are due to A. Denjoy [6].

(10.1) ***Theorem.*** *If $F(x)$ is a function which fulfils at all points of a set, except at most those of an enumerable subset, one of the inequalities*

(10.2) $$\overline{F}(x)<+\infty \quad or \quad \underline{F}(x)>-\infty,$$

then the function $F(x)$ is VBG$_*$ on this set.

Proof. It is enough to show that the set E of the points at which we have, say, $\overline{F}(x)<+\infty$, is the sum of an enumerable infinity of sets on each of which F is VB$_*$.

For any positive integer n, let E_n denote the set of the points x of E such that for every t,

(10.3) $$0<|t-x|\leqslant 1/n \quad implies \quad [F(t)-F(x)]/(t-x)\leqslant n.$$

Further, for each integer i, let E_n^i denote the part of E_n situated in the interval $[i/n, (i+1)/n]$, and a_n^i, b_n^i the lower and upper bounds of those of the E_n^i which are not empty. We have clearly $E=\sum_{n=1}^{\infty}E_n=\sum_{n=1}^{\infty}\sum_{i=-\infty}^{+\infty}E_n^i$.

Let now $F_n(x)=F(x)-nx$. For every point $x\in E_n$ and for every point t which fulfils the first of the inequalities (10.3), we then have $[F_n(t)-F_n(x)]/(t-x)\leqslant 0$. In particular, given any pair of points x_1, x_2 (where $x_1\leqslant x_2$) of E_n^i, we obtain

(10.4) $$F_n(a_n^i)\geqslant F_n(x_1)\geqslant F_n(x_2)\geqslant F_n(b_n^i),$$

and for every t such that $x_1\leqslant t\leqslant x_2$ we find that $F_n(x_1)\geqslant F_n(t)\geqslant F_n(x_2)$. This last relation implies that, for every interval $I=[\alpha,\beta]$ whose end-points belong to the set E_n^i, we have $O(F_n;I)=F_n(\alpha)-F_n(\beta)$, and therefore by (10.4), for every sequence $\{I_j=[\alpha_j,\beta_j]\}$ of such intervals (which do not overlap),

$$\sum_j O(F_n;I_j)=\sum_j[F_n(\alpha_j)-F_n(\beta_j)]\leqslant F_n(a_n^i)-F_n(b_n^i).$$

The function $F_n(x)$, and therefore also the function $F(x)=F_n(x)+nx$, is thus VB$_*$ on every set E_n^i and this completes the proof.

(10.5) ***Theorem.*** *If $F(x)$ is a function which fulfils at all points of a set E, except, perhaps, at those of an enumerable set, one at least of the conditions*

(10.6) $$-\infty<\underline{F}^+(x)\leqslant\overline{F}^+(x)<+\infty \quad or \quad -\infty<\underline{F}^-(x)\leqslant\overline{F}^-(x)<+\infty,$$

then the set E is the sum of an at most enumerable infinity of sets on each of which the function F is AC$_*$.

If, therefore, we are given further that $F(x)$ is continuous on E, then $F(x)$ is ACG$_*$ on E.

Proof. It is enough to show that if at every point x of a set A the two extreme right-hand derivates $\bar{F}^+(x)$ and $\underline{F}^+(x)$ are finite, then A is expressible as the sum of an at most enumerable infinity of sets on each of which the function F is AC_*.

Let A_n denote, for each positive integer n, the set of the points $x \epsilon A$ such that, for every t,

$$(10.7) \qquad 0 \leqslant t-x \leqslant 1/n \quad \textit{implies} \quad |F(t)-F(x)| \leqslant n\cdot(t-x);$$

and, for each integer i, let A_n^i denote the common part of A_n and of the interval $[i/n, (i+1)/n]$. Plainly $A=\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty}\sum_{i=-\infty}^{+\infty} A_n^i$.

Now, if $I=[x_1, x_2]$ is any interval whose end-points belong to A_n^i, we have, for every $t \epsilon I$, the inequality $0\leqslant t-x_1 \leqslant 1/n$, and so, on account of (10.7), $|F(t)-F(x_1)| \leqslant n\cdot(t-x_1) \leqslant n\cdot|I|$. This gives us $O(F;I)\leqslant 2n\cdot|I|$; and therefore for any finite sequence $\{I_j\}$ of such intervals, $\sum_j O(F;I_j) \leqslant 2n\cdot\sum_j |I_j|$. It follows that the function F is AC_* on each of the sets A_n^i, and this completes the proof.

Theorem 10.5 shows, in particular, that every function which is continuous and everywhere derivable (even only unilaterally) is ACG_*. Nevertheless as we saw in Chap. VI, p. 187, such a function need not be absolutely continuous.

In view of Theorem 7.2, we may state also the following corollary of Theorems 10.1 and 10.5: *A function F which fulfils at each point of a set E one at least of the inequalities* (10.2) *or* (10.6), *is derivable at almost all points of E.* In particular therefore, *the set of the points at which a function has (on one side at least) its derivative infinite, is of measure zero.* These statements will be generalized in Chap. IX, §4.

Theorems 10.1 and 10.5 contain sufficient conditions in order that a function be VBG_* or ACG_*, but these conditions are clearly not necessary. Nevertheless, by employing the notion of derivates relative to a function (cf. Chap. IV, p. 108), it is easy to establish conditions similar to those of the preceding theorems, the conditions being this time both sufficient and necessary. Thus, as shown by A. J. Ward [3]:

In order that a finite function F be VBG_* on a set E, it is necessary and sufficient that there exist a bounded increasing function U such that the extreme derivates of F with respect to U are finite at each point of E except, perhaps, those of an enumerable set.

1° In order to establish the necessity of the condition, let us suppose first that the function F is VB_* on E. In view of Theorem 7.1 we may assume that the set E is bounded and closed. Let $[a, b]$ be the smallest interval containing E, and, for each point x of the interval $[a, b]$, let $V_1(x)$ and $V_2(x)$ denote the strong variations of F (cf. § 7) on the parts of E contained in the intervals $[a, x]$ and $[x, b]$ respectively. Finally for each x of $[a, b]$, let $V(x)=V_1(x)-V_2(x)+x$. The function V thus defined is increasing and finite on $[a, b]$, and can therefore be continued as a bounded increasing function on the whole straight line $\boldsymbol{R}_1$. We see at once that throughout the set E, except at most at the points a and b, the derivates of the function F with respect to V are finite and indeed cannot exceed in absolute value the number 1.

Suppose now given any function F which is VBG_* on E. The set E is then expressible as the sum of a sequence $\{E_n\}$ of sets on each of which the function F is VB_*. Consequently, by what has just been proved, there exists for each n a bounded increasing function V_n with respect to which the function F possesses finite derivates at each point of the set E_n except at most at the bounds of this set. Therefore, denoting by M_n the upper bound of $|V_n(x)|$ and writing $U(x)=\sum_n V_n(x)/2^n M_n$, we see at once that the function U thus defined is increasing and bounded and that at each point of E, except perhaps those of an enumerable set, the function F possesses finite derivates with respect to U.

2° The condition is sufficient. Let F be a finite function having at each point of E, except perhaps at those of an enumerable subset, finite derivates with respect to a bounded increasing function U. For each positive integer n, let E_n denote the set of the points x of E for which the inequality $|t-x|\leqslant 1/n$ implies $|F(t)-F(x)|\leqslant n\cdot|U(t)-U(x)|$; and let each E_n be expressed as the sum of a sequence $\{E_n^i\}_{i=1,2,\ldots}$ of sets of diameter less than $1/n$. We see easily (as in the proof of Theorem 10.1) that the function F is VB_* on each set E_n^i, and since the sets E_n^i plainly cover all but an enumerable subset of E, it follows at once that the function F is VBG_* on E. This completes the proof.

If we analyze the first part of the above argument, we see that if the function F is VBG_* on E and moreover bounded on an interval containing the set $\bar{E}$ in its interior, there exists an increasing bounded function U with respect to which the function F has its derivates finite at each point of E. Moreover, if the function F is continuous on an interval containing $\bar{E}$ in its interior, the function U may be defined in such a way as to be itself continuous (cf. the proof of Lemma 3.4, Chap. VIII). Finally, it can be shown that *in order that a function F be ACG_* on an open interval I, it is necessary and sufficient that there exist an increasing and absolutely continuous function with respect to which the function F has its derivates finite at every point of I.*

(10.8) ***Theorem.*** *If at every point x of a set E, except perhaps at the points of an enumerable subset, a function F fulfils any one of the inequalities*

$$(10.9) \qquad \bar{F}^+(x)<+\infty, \quad \underline{F}^+(x)>-\infty, \quad \bar{F}^-(x)<+\infty, \quad \underline{F}^-(x)>-\infty,$$

$$(10.10) \qquad \bar{F}_{ap}(x)<+\infty, \quad \underline{F}_{ap}(x)>-\infty,$$

then F is VBG on E.

Proof. We need only consider the case of the first of the inequalities (10.9) and that of the first of the inequalities (10.10). It is therefore sufficient to show that each of the sets $A=\underset{x}{\mathrm{E}}[\overline{F}^{+}(x)<+\infty]$ and $B=\underset{x}{\mathrm{E}}[\overline{F}_{\mathrm{ap}}(x)<+\infty]$ is expressible as the sum of an enumerable infinity of sets on each of which F is of bounded variation.

Consider first the set A. Given any positive integer n, let A_n denote the set of all the points $x \epsilon A$ such that, for every t,

(10.11) $\quad 0 \leqslant t-x \leqslant 1/n \quad$ *implies* $\quad F(t)-F(x) \leqslant n\cdot(t-x)$,

and by A_n^i, for each integer i, the part of A_n contained in the interval $[i/n, (i+1)/n]$. Let $F_n(x)=F(x)-nx$.

For every pair x_1, x_2 of points of A_n^i, where $x_1 \leqslant x_2$, we have $0 \leqslant x_2-x_1 \leqslant 1/n$, and so, by (10.11), $F(x_2)-F(x_1) \leqslant n\cdot(x_2-x_1)$, i. e. $F_n(x_2)-F_n(x_1) \leqslant 0$. The function $F_n(x)$ is thus monotone non-increasing on each set A_n^i, and it follows that A_n^i is expressible as the sum of a sequence of sets $\{A_n^{i,j}\}_{j=1,2,\ldots}$ on each which $F_n(x)$ is monotone and bounded. The function $F(x)=F_n(x)+nx$ is then plainly of bounded variation on each of the sets $A_n^{i,j}$, and moreover we have $A=\sum_{n=1}^{\infty} A_n=\sum_{n=1}^{\infty}\sum_{i=-\infty}^{+\infty}\sum_{j=1}^{\infty} A_n^{i,j}$.

Consider now the set B. From the definitions of approximate upper limit and approximate upper derivate (cf. § 3, p. 220), it follows at once that to each point $x \epsilon B$ we can make correspond a positive integer n such that the set $\underset{\xi}{E}\left[\dfrac{F(\xi)-F(x)}{\xi-x} \geqslant n\right]$ has the point x as a point of dispersion. Therefore, denoting by B_n the set of the points $x \epsilon B$ such that the inequality $0 \leqslant h \leqslant 1/n$ implies both the inequalities

(10.12) $\quad |\underset{\xi}{\mathrm{E}}[F(\xi)-F(x) \geqslant n\cdot(\xi-x);\ x \leqslant \xi \leqslant x+h]| \leqslant h/3$

and

(10.13) $\quad |\underset{\xi}{\mathrm{E}}[F(x)-F(\xi) \geqslant n\cdot(x-\xi);\ x-h \leqslant \xi \leqslant x]| \leqslant h/3$,

we have $B=\sum_{n=1}^{\infty} B_n$. We denote further, for every integer i, by B_n^i the part of B_n contained in the interval $[i/n, (i+1)/n]$ and we write, as before, $F_n(x)=F(x)-nx$.

The main part of the proof consists in showing that, for every i, the function $F_n(x)$ is monotone on B_n^i.

For this purpose, let x_1, x_2 be any pair of points of a B_n^i, and let $x_1<x_2$. We plainly have $0<x_2-x_1 \leqslant 1/n$, so that by writing $x=x_1$ and $h=x_2-x_1$ in (10.12), we obtain

$$|\underset{\xi}{\mathrm{E}}[F(\xi)-F(x_1) \geqslant n\cdot(\xi-x_1);\quad x_1 \leqslant \xi \leqslant x_2]| \leqslant (x_2-x_1)/3.$$

Similarly, from (10.13) with $x=x_2$ and $h=x_2-x_1$, we derive

$$|\underset{\xi}{\mathrm{E}}[F(x_2)-F(\xi) \geqslant n\cdot(x_2-\xi);\quad x_1 \leqslant \xi \leqslant x_2]| \leqslant (x_2-x_1)/3.$$

The two inequalities thus obtained show that the interval $[x_1, x_2]$ contains a point ξ_0 such that

$$F(\xi_0)-F(x_1)<n\cdot(\xi_0-x_1) \qquad \textit{and} \qquad F(x_2)-F(\xi_0)<n\cdot(x_2-\xi_0).$$

By adding these two inequalities term by term, we obtain $F(x_2)-F(x_1)<n\cdot(x_2-x_1)$, and so finally $F_n(x_2)-F_n(x_1)<0$.

We have thus shown that the function $F_n(x)$ is monotone decreasing on each B_n^i. It follows that B_n^i is expressible as the sum of a sequence of sets $\{B_n^{i,j}\}_{j=1,2,\ldots}$ on each of which $F_n(x)$ is monotone and bounded, and on which the function $F(x)=F_n(x)+nx$ is therefore of bounded variation. Moreover, we have $B=\sum_{n=1}^{\infty} B_n=\sum_{n=1}^{\infty}\sum_{i=-\infty}^{+\infty}\sum_{j=1}^{\infty} B_n^{i,j}$. This completes the proof.

On account of Theorem 4.2, it follows immediately from Theorem 10.8 that *any measurable function which satisfies one of the inequalities* (10.9) or (10.10) *at each point of a set E, is approximately derivable at almost all points of E*. This proposition will be generalized and completed in Chap. IX (§§ 9 and 10).

(10.14) ***Theorem.*** *If two extreme approximate derivates on the same side are finite for a function $F(x)$ at every point of a set E, except at most in an enumerable subset, then the set E is the sum of a sequence of sets on each of which $F(x)$ is absolutely continuous.*

Consequently, if the function $F(x)$ is further given to be continuous on E, then $F(x)$ is ACG *on E.*

Proof. It is clearly enough to show, for instance, that the set $A=\underset{x}{\mathrm{E}}[-\infty<\underline{F}_{\mathrm{ap}}^{+}(x) \leqslant \overline{F}_{\mathrm{ap}}^{+}(x)<+\infty]$ is the sum of an at most enumerable infinity of sets on each of which F is AC.

Now we can make correspond, to each point $x \in A$, a positive integer n such that x is a point of dispersion for the set $\underset{\xi}{E}[|F(\xi)-F(x)| \geqslant n \cdot (\xi - x)]$ (cf. § 3, p. 220). Hence, denoting by A_n the set of the points $x \in A$ such that, for every h, the inequality $0 \leqslant h \leqslant 2/n$ implies

$$(10.15) \qquad \left|\underset{\xi}{\mathrm{E}}[|F(\xi)-F(x)| \geqslant n \cdot (\xi - x);\ x \leqslant \xi \leqslant x+h]\right| \leqslant h/4,$$

we have $A = \sum_{n=1}^{\infty} A_n$; and, denoting as before by A_n^i (for each integer i) the part of A_n contained in the interval $[i/n, (i+1)/n]$, we obtain $A = \sum_{n=1}^{\infty} \sum_{i=-\infty}^{+\infty} A_n^i$.

Consider now any two points x_1 and x_2 of A_n^i, where $x_1 < x_2$, and let $x_3 = 2x_2 - x_1$.

We have, on the one hand, $0 < x_3 - x_1 = 2 \cdot (x_3 - x_2) \leqslant 2/n$, so that by writing $x = x_1$ and $h = x_3 - x_1$ in (10.15), we obtain the inequality

$$\left|\underset{\xi}{\mathrm{E}}[|F(\xi)-F(x_1)| \geqslant n \cdot (\xi - x_1);\ x_1 \leqslant \xi \leqslant x_3]\right| \leqslant (x_3 - x_1)/4 = (x_3 - x_2)/2,$$

and *a fortiori*

$$(10.16) \qquad \left|\underset{\xi}{\mathrm{E}}[|F(\xi)-F(x_1)| \geqslant n \cdot (\xi - x_1);\ x_2 \leqslant \xi \leqslant x_3]\right| \leqslant (x_3 - x_2)/2.$$

On the other hand, we have $0 \leqslant x_3 - x_2 = x_2 - x_1 \leqslant 1/n$, and so by (10.15) with $x = x_2$ and $h = x_3 - x_2$, we find

$$(10.17) \qquad \left|\underset{\xi}{\mathrm{E}}[|F(\xi)-F(x_2)| \geqslant n \cdot (\xi - x_2);\ x_2 \leqslant \xi \leqslant x_3]\right| \leqslant (x_3 - x_2)/4.$$

The inequalities (10.16) and (10.17) show that there exists a point ξ_0 in $[x_2, x_3]$ such that we have at the same time

$$|F(\xi_0)-F(x_1)| < n \cdot (\xi_0 - x_1) \leqslant n \cdot (x_3 - x_1) = 2n \cdot (x_2 - x_1),$$
$$|F(\xi_0)-F(x_2)| < n . (\xi_0 - x_2) \leqslant n \cdot (x_3 - x_1) = 2n \cdot (x_2 - x_1),$$

and this requires $|F(x_2)-F(x_1)| < 4n \cdot |x_2 - x_1|$. This last inequality is thus established for every pair of points x_1, x_2 of any one of the sets A_n^i, and it follows at once that F is AC on each of the sets A_n^i. This completes the proof.

Theorem 10.8 shows in particular that a continuous function which is everywhere approximately derivable, even unilaterally, is necessarily ACG.

In Chap. IX, § 9, we shall give two further criteria for a function to be ACG_* or ACG.