CHAPTER VI.

Major and minor functions.

§ 1. Introduction. Major and minor functions (defined in § 3 of this chapter) were first introduced by Ch. J. de la Vallée Poussin in his study of the properties of the Lebesgue integral and those of additive functions of a set. Entirely equivalent notions (of "Ober"- and "Unterfunktionen") were introduced independently by O. Perron [1], who based on them a new definition of integral, which does not require the theory of measure. Although, in its original form, this definition concerned only integration of bounded functions, its extension to unbounded functions was easy and led, as shown by O. Bauer [1], to a process of integration more general than that of Lebesgue. Moreover, as we shall see in § 6, the Perron integral may be regarded as a synthesis of two fundamental conceptions of integration: one corresponding to the idea of definite integral as limit of certain approximating sums, and the other to that of indefinite integral understood as a primitive function.

It is usual to associate these two conceptions of integration with the names of Leibniz and Newton. In accordance with this distinction (which is largely a matter of convention) we shall call a function of a real variable F *indefinite integral*, or *primitive*, *of Newton* for a function f, if F has everywhere its derivative finite and equal to f. The function f will then be termed *integrable in the sense of Newton*, and the increment of the function F on an interval I_0, will be called *definite integral of Newton* of f on I_0. As is seen immediately, this definition implies that any function which is integrable in the sense of Newton is everywhere finite. This restriction is essential (cf. the example of § 7, p. 206) for the unicity of integration in the sense of Newton, which then follows from classical theorems of Analysis, or, if we like, from Theorem 3.1, or from Theorem 7.1 of this chapter.

The theory of the integral was first developed on Newtonian lines. This is easily accounted for if we think how much simpler the inverse

of the operation of derivation must have seemed than the notion of definite integral as defined by Leibniz. It was A. Cauchy [I, t. 4, p. 122] who returned to the idea of Leibniz in order to apply it to integration of continuous functions, for which the methods of Cauchy and Newton are actually completely equivalent. This equivalence disappears, however, as soon as we pass on, with Riemann, to integration of discontinuous functions. In fact, even in the domain of bounded functions to which the Riemann process applies, there exist on the one hand (as we see at once) functions which are integrable in the sense of Riemann but have no primitive, and on the other hand (as shown by V. Volterra [1]; cf. also H. Lebesgue [II, p. 100]) functions which have a primitive but are not integrable in the Riemann sense. Also the Lebesgue process of integration does not include the integral of Newton, not even when the functions to be integrated are everywhere finite.

Thus, the function $F(x)=x^2\sin(\pi/x^2)$ for $x\neq 0$, completed by writing $F(0)=0$, has in the whole interval $[0, 1]$ a finite derivative which vanishes for $x=0$ and which is bounded on every interval $[\varepsilon, 1]$, where $0<\varepsilon<1$. On every interval $[\varepsilon, 1]$ the function $F(x)$ is therefore absolutely continuous. On the other hand, on the whole interval $[0,1]$ the function is not even of bounded variation. Hence $F'(x)$ is not summable on $[0,1]$, since its indefinite Lebesgue integral could then differ only by an additive constant from $F(x)$ on $[0,1]$, and this is impossible.

We have thus been led to the problem of determining a process of integration which includes both that of Lebesgue and that of Newton. As an application of the method of major and minor functions, we shall consider in this chapter (§§ 6 and 7) the solution of this problem constituted by the Perron integral. Another solution, the Denjoy integrals, will be treated in Chapter VIII.

The notions of major and minor functions, and their applications to Lebesgue integration, will be discussed here for arbitrary spaces $\boldsymbol{R}_m$. In defining the Perron integral, however, we shall limit ourselves to functions of one real variable. Although recently various authors have treated the extension of this integral to Euclidean spaces of any number of dimensions, the present state of the theory does not allow us to decide as to the importance of this generalization. On the contrary, in the domain of functions of a real variable, the method of major and minor functions as a means of generalizing the notion of integral has already repeatedly shown its fruitfulness. In the memoir of J. Marcinkiewicz and A. Zygmund [1], the reader will find new applications of this method in connection with certain fundamental problems of the theory of trigonometrical series (cf. also J. Ridder [11]).

§ 2. Derivation with respect to normal sequences of nets. Given a regular sequence $\mathfrak{N}=\{\mathfrak{N}_k\}$ of nets of intervals (*vide* Chap. III, § 2) in a space $\boldsymbol{R}_m$ and a function of an interval F in $\boldsymbol{R}_m$, we shall call *upper derivate* of F at a point x *with respect to the sequence of nets* $\mathfrak{N}$ the upper limit of the ratio $F(Q)/|Q|$ as $\delta(Q)\to 0$, where Q denotes any interval containing x and belonging to one of the nets of the sequence $\mathfrak{N}$. By symmetry we define similarly the *lower derivate* of F at x *with respect to the sequence of nets* $\mathfrak{N}$. We shall denote these two derivates by $(\mathfrak{N})\,\overline{F}(x)$ and $(\mathfrak{N})\,\underline{F}(x)$. When they are equal at a point x, their common value will be denoted by $(\mathfrak{N})\,F'(x)$ and called *derivative* of F at x *with respect to the sequence of nets* $\mathfrak{N}$.

These definitions are similar to those given in § 15, Chap. IV, in connection with derivation of additive functions of a set $(\mathfrak{B})$ in a metrical space. It should be observed, however, that additive functions of a set $(\mathfrak{B})$ correspond to additive functions of an interval of bounded variation, whereas in the present § we treat derivation of additive functions of an interval without supposing them *a priori* of bounded variation. For this reason it will be necessary to impose certain restrictions on the nets considered in this §, and to distinguish a class of nets which we shall call, for brevity, normal nets. The latter are, in point of fact, the nets occurring most frequently in applications (cf., for instance, Chap. III, p. 58).

A system of intervals will be called a *normal net* in the space $\boldsymbol{R}_m$, when it consists of the closed intervals $[a_k^{(1)}, a_{k+1}^{(1)}; a_k^{(2)}, a_{k+1}^{(2)}; \ldots; a_k^{(m)}, a_{k+1}^{(m)}]$ for $k=0, \pm 1, \pm 2, \ldots$, which are determined by systems of numbers $a_k^{(i)}$ subject to the conditions $a_k^{(i)} < a_{k+1}^{(i)}$ for $i=1, 2, \ldots, m$ and $k=\ldots, -1, 0, +1, \ldots$, and $\lim_{k\to\pm\infty} a_k^{(i)} = \pm\infty$. A regular sequence of normal nets will be termed *normal sequence*.

(2.1) ***Theorem.*** *Let $\mathfrak{N}=\{\mathfrak{N}_k\}$ be a normal sequence of nets, $g(x)$ a function which is summable in the space $\boldsymbol{R}_m$ and F a continuous additive function of an interval such that* (i) $(\mathfrak{N})\underline{F}(x) > -\infty$ *at every point x, except at most those of an enumerable set, and* (ii) $F'(x) \geqslant g(x)$ *at almost all the points x at which the function F is derivable in the ordinary sense.*

Then for every interval I, we have

$$F(I) \geqslant \int_I g(x)\,dx; \tag{2.2}$$

i. e. F is a function of bounded variation, whose function of singularities is monotone non-negative.

Proof. Consider the points in every neighbourhood of which there exist intervals I for which the inequality (2.2) is false, and let P denote the set of these points. The set P is evidently closed, and we see easily that the relation (2.2) must hold for every interval I such that $I^\circ \subset CP$. For if this were not the case, we could determine first an interval $I \subset CP$ such that $F(I) < \int_I g(x)dx$, and then, by the method of successive subdivisions, a descending sequence $\{I_n\}$ of subintervals of I such that $\delta(I_n)\to 0$ as $n\to\infty$ and that $F(I_n) < \int_{I_n} g(x)\,dx$ for $n=1, 2, \ldots$. Therefore, denoting by a the common point of the intervals I_n, we should have $a \in P$, which is clearly impossible.

It follows that in order to establish the validity of the inequality (2.2) for all intervals I, we need only prove that $P=0$. Suppose therefore, if possible, that $P \neq 0$. Let us denote, for any pair of positive integers k and h, by $N_{k,h}$ the sum of all the intervals I of the net $\mathfrak{N}_k$ for which $F(I) > -h\cdot|I|$. Therefore by writing $N_h = \prod_{k=h}^{\infty} N_{k,h}$, we obtain a sequence $\{N_h\}$ of closed sets whose sum, according to condition (i), covers the whole space except for an at most enumerable set. Consequently, on account of Baire's Theorem (Chap. II, Theorem 9.2), the set P contains a portion which either consists of a single point, or else is contained in a set N_h. The former case is excluded since it is evident from the continuity and additivity of the function F that the set P contains no isolated points. Therefore there exists a positive integer h_0 and an open sphere S such that $0 \neq P\cdot S \subset N_{h_0}$. Let us write $H(I) = F(I) + h_0\cdot|I| + \int_I |g(x)|\,dx$ where I is any interval. We shall have $H(I) \geqslant 0$ for any interval I such that $I^\circ \subset CP$, as well as for any interval I belonging to a net $\mathfrak{N}_k$ of index $k \geqslant h_0$ and having points of the set N_{h_0} in its interior. Therefore $H(I) \geqslant 0$ for any interval $I \subset S$ belonging to a net $\mathfrak{N}_k$ of index $k \geqslant h_0$, and consequently, by additivity and continuity of H, we have $H(I) \geqslant 0$, i. e. $F(I) \geqslant -h_0\cdot|I| - \int_I |g(x)|\,dx$, for any interval $I \subset S$ whatsoever. It follows at once that the function F is of bounded variation in S and that the function of singularities of F (cf. Chap. IV, p. 120) is monotone non-negative in S.

Hence, by condition (ii), $F(I)\geqslant\int_I F'(x)\,dx\geqslant\int_I g(x)\,dx$ for every interval $I\subset S$. But since $P\cdot S\neq 0$ we thus arrive at a contradiction and this completes the proof.

As an immediate corollary of Theorem 2.1, we have

(2.3) ***Theorem.*** *If $\mathfrak{N}$ is a normal sequence of nets in the space $\boldsymbol{R}_m$ and if F is a continuous additive function of an interval such that:* (i) $-\infty<(\mathfrak{N})\underline{F}(x)\leqslant(\mathfrak{N})\overline{F}(x)<+\infty$ *for each point x except at most the points of an enumerable set,* (ii) *the (ordinary) derivative $F'(x)$ is summable on each portion of the set of the points at which this derivative exists; then the function F is almost everywhere derivable and is the indefinite integral of its derivative.*

For Theorems 2.1 and 2.3 cf. J. Ridder [2]. Let us remark that in the case where the function F is of bounded variation, these theorems are included in Theorem 15.7, Chap. IV, which concerns derivation of additive functions of a set in an abstract metrical space.

It follows easily from Theorem 15.12, Chap. IV, that $F'(x)=(\mathfrak{N})F'(x)$ almost everywhere for any regular sequence of nets of intervals $\mathfrak{N}$ and for any additive function of an interval F which is continuous and of bounded variation. This remark enables us to replace condition (ii) of Theorem 2.1 by the following: (ii-*bis*) $F'(x)=(\mathfrak{N})F'(x)\geqslant g(x)$ *at almost all the points x at which the two derivatives $F'(x)$ and $(\mathfrak{N})F'(x)$ exist, are finite and equal.* Similarly we may modify condition (ii) of Theorem 2.3.

As it follows from an example due to A. J. Ward [7], the inequality $(\mathfrak{N})\underline{F}(x)>-\infty$ in condition (i) of Theorem 2.1 cannot be replaced by $(\mathfrak{N})\overline{F}(x)>-\infty$.

§ 3. Major and minor functions.

Before introducing the fundamental definitions of the theory of the Perron integral, we shall prove

(3.1) ***Theorem.*** *If an additive function of an interval F (not necessarily continuous) has a non-negative lower derivate at each point x of an interval I_0, then $F(I_0)\geqslant 0$.*

Proof. Let ε be any positive number and write $G(I)=F(I)+\varepsilon\cdot|I|$ for every interval I. Then $\underline{G}(x)\geqslant\varepsilon>0$ at each point $x\in I_0$. Suppose that $G(I_0)\leqslant 0$. By the method of successive subdivisions, we could then determine a descending sequence $\{I_n\}$ of intervals similar to I_0, such that $G(I_n)\leqslant 0$ for $n=0,1,2,\ldots$ and that $\delta(I_n)\to 0$ as $n\to\infty$. Therefore, denoting by x_0 the common point of the intervals I_n, we should have $\underline{G}(x_0)\leqslant 0$ which is impossible. Hence $G(I_0)>0$, and this gives $F(I_0)>-\varepsilon\cdot|I_0|$ for each $\varepsilon>0$, and finally $F(I_0)\geqslant 0$.

An additive function of an interval F is termed *major* [*minor*] *function* of a function of a point f on a figure R_0 if, at every point x of this figure, $-\infty\neq\underline{F}_s(x)\geqslant f(x)$ $[+\infty\neq\overline{F}_s(x)\leqslant f(x)]$. It follows at once from Theorem 3.1 that if the functions of an interval U and V are respectively a major and a minor function of a function f on a figure R_0, their difference $U-V$ is monotone non-negative on R_0.

(3.2) ***Theorem.*** *If f is a summable function, then, for each $\varepsilon>0$, the function f has an absolutely continuous major function U and an absolutely continuous minor function V such that, for each interval I,*

$$(3.3)\qquad 0\leqslant U(I)-\int_I f(x)\,dx\leqslant\varepsilon \quad\text{and}\quad 0\leqslant\int_I f(x)\,dx-V(I)\leqslant\varepsilon.$$

Proof. On account of the theorem of Vitali-Carathéodory (Chap. III, Theorem 7.6) we can associate with the function f two summable functions, one a lower semi-continuous function g and the other an upper semi-continuous function h, such that (i) $-\infty\neq g(x)\geqslant f(x)\geqslant h(x)\neq+\infty$ at every point x and that (ii) $\int_I[g(x)-f(x)]\,dx<\varepsilon$ and $\int_I[f(x)-h(x)]\,dx<\varepsilon$ for every interval I. Therefore, if we denote by U and V the indefinite integrals of the functions g and h respectively, we find by Theorem 2.2, Chap. IV, that $\underline{U}_s(x)\geqslant g(x)\geqslant h(x)\geqslant\overline{V}_s(x)$, and so, on account of (i), that $-\infty\neq\underline{U}_s(x)\geqslant f(x)$ and $+\infty\neq\overline{V}_s(x)\leqslant f(x)$ at each point x. Finally, (ii) then implies the relations (3.3) and this completes the proof.

Theorem 3.2 can easily be inverted. Thus: *in order that a function of a point f be summable, it is necessary and sufficient that for each $\varepsilon>0$ there exist two absolutely continuous functions of an interval U and V, the one a major and the other a minor function of f, which fulfil the condition $U(I)-V(I)<\varepsilon$ for every interval I.* (These absolutely continuous functions may clearly be replaced by functions of bounded variation, and if the function f is supposed measurable, then, of course, for its summability there suffices the existence of two functions of bounded variation, one of which is a major and the other a minor function of f.)

* § 4. Derivation with respect to binary sequences of nets.

The theorems of § 2 concerned derivation of additive functions with respect to any normal sequence of nets of intervals. For certain purposes however, more special sequences of nets are required. We shall say that a normal sequence $\{\mathfrak{N}_k\}_{k=1,2,\ldots}$ of nets in the space $\boldsymbol{R}_m$ is *binary*, if the net $\mathfrak{N}_{k+1}$ (where $k=1,2,\ldots$) is obtained by subdividing each interval N of the net $\mathfrak{N}_k$ into 2^m equal intervals similar to N.

An application of this notion may be found in the following theorem which is proved similarly to Lemma 11.8 of Chap. IV: *If $\mathfrak{N}$ is a binary sequence of nets, any additive function of an interval F is derivable with respect to $\mathfrak{N}$ at almost all the points at which either* $(\mathfrak{N})\underline{F}(x)>-\infty$ *or* $(\mathfrak{N})\overline{F}(x)<+\infty$.

Another application, of particular interest, is due to A. S. Besicovitch [3] who, by using derivation with respect to a binary sequence of nets, established a theorem on complex functions (*vide* below § 5). The substance of Besicovitch's result is contained in Theorem 4.4 below. We must first, however, give some subsidiary definitions.

For definiteness, just as in § 11, Chap. IV, we shall fix in the space $\boldsymbol{R}_m$ a binary sequence of nets $\mathfrak{Q}=\{\mathfrak{Q}_k\}$, where $\mathfrak{Q}_k$ denotes, for $k=1, 2, \ldots$, the net formed by the cubes

$$[p_1/2^k, (p_1+1)/2^k;\ p_2/2^k, (p_2+1)/2^k;\ \ldots;\ p_m/2^k, (p_m+1)/2^k]$$

where $p_1, p_2, \ldots, p_m$ are arbitrary integers; it goes without saying that in Theorem 4.4 this sequence may be replaced by any binary sequence whatsoever.

Given a non-negative number α, we shall say that a function of an interval F fulfils the *condition* (l_α^+) [*condition* (l_α^-)] at a point x, if $\liminf_{\delta(I)\to 0} F(I)/[\delta(I)]^\alpha \geqslant 0$ [$\limsup_{\delta(I)\to 0} F(I)/[\delta(I)]^\alpha \leqslant 0$], where I is any interval containing x. If a function f fulfils the condition (l_α^+) [(l_α^-)] at every point of a figure R, we shall say simply that f fulfils this condition on R. Finally, we shall say that a function fulfils the *condition* (l_α) at a point, or on a figure, if it fulfils simultaneously the conditions (l_α^+) and (l_α^-).

We recall further the notation $\Lambda_\alpha(E)$ for the α-dimensional measure of a set E (cf. Chap. II, p. 53).

(4.1) ***Lemma.*** *Given a set E in the space $\boldsymbol{R}_m$, together with a positive integer k_0 and a non-negative number $\alpha<m$, there exists for each $\varepsilon>0$ a sequence $\{Q_n\}$ of intervals belonging to the nets $\mathfrak{Q}_k$ for $k\geqslant k_0$, which fulfils the following conditions:*

(i) $$\sum_n [\delta(Q_n)]^\alpha \leqslant (4m)^m\cdot[\Lambda_\alpha(E)+\varepsilon];$$

(ii) *to each point x of E there corresponds a positive integer $k\geqslant k_0$ such that all the intervals of the net $\mathfrak{Q}_k$ which contain the point x belong to the sequence $\{Q_n\}$.*

Proof. Let us cover E by a sequence $\{E_i\}_{i=1,2,\ldots}$ of sets such that $0<\delta(E_i)<1/2^{k_0+1}$ for $i=1, 2, \ldots$ and such that

(4.2) $$\sum_i [\delta(E_i)]^\alpha \leqslant \Lambda_\alpha(E)+\varepsilon.$$

Let us denote by k_i, for each $i=1, 2, \ldots$, a positive integer such that

(4.3) $$1/2^{k_i} > \delta(E_i) \geqslant 1/2^{k_i+1}.$$

We easily see that $k_i>k_0$ for every i, and that each net $\mathfrak{Q}_{k_i}$, for $i=1,2,\ldots$, can contain at most 2^m intervals having points in common with E_i. Let $\{Q_n\}_{n=1,2,\ldots}$ be the sequence of all the intervals belonging to the nets $\mathfrak{Q}_{k_1}, \mathfrak{Q}_{k_2}, \ldots, \mathfrak{Q}_{k_i}, \ldots$ and having points in common with the sets $E_1, E_2, \ldots, E_i, \ldots$ respectively. The sequence $\{Q_n\}$ clearly fulfils the condition (ii). Moreover, we find on account of (4.3),

$$\sum_n [\delta(Q_n)]^\alpha \leqslant 2^m\cdot\sum_i m^\alpha\cdot 2^{-\alpha k_i} \leqslant 2^m m^\alpha 2^\alpha\cdot\sum_i 2^{-\alpha(k_i+1)} \leqslant (4m)^m\cdot\sum_i [\delta(E_i)]^\alpha,$$

and this by (4.2) gives at once the condition (i).

(4.4) ***Theorem.*** *Suppose that F is a continuous additive function of an interval in the space $\boldsymbol{R}_m$ and fulfils the condition (l_α) where $0\leqslant\alpha<m$, and that g is a summable function. Suppose further that* (i) $(\mathfrak{Q})\underline{F}(x)>-\infty$ *at every point x except at most those of a set E expressible as the sum of an enumerable infinity of sets of finite measure (Λ_α), and that* (ii) $(\mathfrak{Q})\underline{F}(x)\geqslant g(x)$ *at almost all points x; then*

(4.5) $$F(I_0)\geqslant\int_{I_0} g(x)\,dx$$

for every interval I_0.

Proof. Since the function F is continuous, it will suffice to prove (4.5) in the case in which the interval I_0 belongs to one of the nets $\mathfrak{Q}_k$, to the net $\mathfrak{Q}_{k_0}$, say. Further by changing, if necessary, the values of g on a set of measure zero, we can assume that the inequality $(\mathfrak{Q})\underline{F}(x)\geqslant g(x)$ holds at every point x.

Let ε be a positive number and let V be a minor function of g (cf. § 3, particularly Theorem 3.2) such that

(4.6) $$V(I_0)>\int_{I_0} g(x)\,dx-\varepsilon.$$

Let us write $G(I)=F(I)-V(I)+\varepsilon\cdot|I|$, where I denotes any interval. We shall have $(\mathfrak{Q})\underline{G}(x)\geqslant(\mathfrak{Q})\underline{F}(x)-\overline{V}(x)+\varepsilon\geqslant\varepsilon>0$ at every point x except at most at the points of E. Finally, since $\overline{V}_s(x)<+\infty$ at every point x, the function V fulfils the condition (l_α^-) and the function G therefore fulfils the condition (l_α^+).

Let us now represent the set E as the sum of a sequence $\{E_i\}_{i=1,2,...}$ of sets of finite measure (Λ_α), and denote, for each pair of positive integers i and n, by $R_{i,n}$ the set of the points x such that the inequality $G(I)>-\varepsilon\cdot[\delta(I)]^\alpha/2^i[\Lambda_\alpha(E_i)+1]$ holds whenever I is an interval containing x and belonging to one of the nets $\mathfrak{Q}_k$ for $k\geqslant n$. The sets $R_{i,n}$ are evidently measurable $(\mathfrak{B})$ (they are actually sets $(\mathfrak{G}_\delta)$). Moreover, since the function G fulfils the condition (1_α^+), the sum $\sum_n R_{i,n}$ must, for each integer i, cover the whole space $\boldsymbol{R}_m$. Hence, writing $E_{i,n}=E_i\cdot(R_{i,n}-R_{i,n-1})$ for $n>1$, and $E_{i,1}=E_i\cdot R_{i,1}$, we find that

$$\Lambda_\alpha(E_i)=\sum_n \Lambda_\alpha(E_{i,n}) \quad \textit{for} \quad i=1,2,.... \tag{4.7}$$

This being so, it follows from Lemma 4.1 that for each pair of positive integers i and n, we can determine a sequence $\{Q_{i,n}^{(j)}\}_{j=1,2,...}$ of cubes which belong to the nets $\mathfrak{Q}_k$ for $k\geqslant n$, and fulfil the following conditions:

$$\sum_j [\delta(Q_{i,n}^{(j)})]^\alpha \leqslant (4m)^m\cdot[\Lambda_\alpha(E_{i,n})+1/2^n]; \tag{4.8}$$

(4.9) *to each point $x\in E_{i,n}$ there corresponds an integer $k\geqslant k_0$ such that each cube of the net $\mathfrak{Q}_k$, containing x, belongs to the sequence $\{Q_{i,n}^{(j)}\}_{j=1,2,...}$;*

(4.10) *each cube $Q_{i,n}^{(j)}$ has points in common with the set $E_{i,n}$ and therefore fulfils the inequality $G(Q_{i,n}^{(j)})>-\varepsilon\cdot[\delta(Q_{i,n}^{(j)})]^\alpha/2^i\cdot[\Lambda_\alpha(E_i)+1]$.*

For brevity, let us agree to say that an interval has the property (A), when it is representable as the sum of a finite number of non-overlapping intervals I each of which either fulfils the inequality $G(I)>0$, or else coincides with one of the cubes $Q_{i,n}^{(j)}$. We remark that on account of (4.10), (4.8) and (4.7), the inequality

$$G(R)\geqslant-\varepsilon\cdot\sum_{i,n,j}[\delta(Q_{i,n}^{(j)})]^\alpha/2^i[\Lambda_\alpha(E_i)+1]\geqslant$$
$$\geqslant-(4m)^m\,\varepsilon\cdot\sum_{i,n}[\Lambda_\alpha(E_{i,n})+1/2^n]/2^i[\Lambda_\alpha(E_i)+1]=-(4m)^m\cdot\varepsilon$$

is valid whenever R is a figure consisting of any finite number of non-overlapping cubes $Q_{i,n}^{(j)}$, and therefore that the inequality $G(I)\geqslant-(4m)^m\cdot\varepsilon$ must hold for every interval I having the property (A).

We shall now show that the interval I_0 itself has the property (A), so that $G(I_0)\geqslant-(4m)^m\varepsilon$. Let us suppose the contrary. We could then, starting with I_0, construct a decreasing sequence $\{I_p\}$ of cubes belonging to the nets $\mathfrak{Q}_k$ and none of which has the property (A). Let x_0 be the common point of these cubes. Then either $x_0\in E$, and consequently, by (4.9), the sequence contains cubes $Q_{i,n}^{(j)}$; or $x_0\in CE$, so that $(\mathfrak{Q})\underline{G}(x_0)>0$, and therefore $G(I_p)>0$ for each sufficiently large p. Thus in both cases, the sequence $\{I_p\}$ would contain intervals with the property (A) and we arrive at a contradiction. It follows that $G(I_0)\geqslant-(4m)^m\varepsilon$, and therefore, by (4.6), that

$$F(I_0)=G(I_0)+V(I_0)-\varepsilon\cdot|I_0|\geqslant\int_{I_0}g(x)\,dx-[1+(4m)^m+|I_0|]\varepsilon;$$

since ε is an arbitrary positive number, this gives the relation (4.5).

* § 5. Applications to functions of a complex variable.

We now interpret the points of the plane $\boldsymbol{R}_2$ as complex numbers and, as usual, we call *complex function of a complex variable* every function of the form $u+iv$ where u and v are real functions in the whole plane, or in an open set. The functions u and v are termed *real part* and *imaginary part* of the function f. A complex function is said to be *continuous* (at a point, or in an open set), if its real and imaginary parts are both continuous.

Given a complex function f, continuous in an open set G, and having the real and imaginary parts u and v respectively, we shall write for every interval $I=[a_1,b_1;a_2,b_2]$ contained in G:

$$\begin{aligned} &\mathrm{J}_1(f;I)=-\int_{a_1}^{b_1}[u(x,b_2)-u(x,a_2)]\,dx-\int_{a_2}^{b_2}[v(b_1,y)-v(a_1,y)]\,dy,\\ &\mathrm{J}_2(f;I)=\int_{a_2}^{b_2}[u(b_1,y)-u(a_1,y)]\,dy-\int_{a_1}^{b_1}[v(x,b_2)-v(x,a_2)]\,dx,\end{aligned} \tag{5.1}$$

and

$$\mathrm{J}(f;I)=\mathrm{J}_1(f;I)+i\mathrm{J}_2(f;I).$$

The expression $\mathrm{J}(f;I)$, which will also be denoted by $\int_{(I)} f\,dz$, will be called curvilinear integral of the function f along the boundary of the interval I. The function f will be termed *holomorphic* in an open set G, if $\mathrm{J}(f;I)=0$ for every interval $I\subset G$. (The equivalence of this definition of the term „holomorphic" — used here in place of terms such as "regular", "analytic", etc. — with the more familiar definitions of the theory of complex functions, follows from the well-known theorem of Morera [1].) We verify at once that this relation holds when $f(z)=az+b$ where a and b are any complex constants.

If f is a complex function, continuous in an open set G, the expressions $J_1(f;I)$ and $J_2(f;I)$ are continuous additive functions of the interval I in G. Moreover

$$|J_1(f;I)|\leqslant|J(f;I)| \quad \textit{and} \quad |J_2(f;I)|\leqslant|J(f;I)|$$

for each interval I in G. On account of Theorem 2.3, we therefore obtain at once the following theorem due to J. Wolff [1] (cf. also H. Looman [2] and J. Ridder [1; 2]):

(5.2) ***Theorem.*** *A complex function f, continuous in an open set G, is holomorphic in G if at almost all points z of G,*

$$\liminf_{\delta(Q)\to 0}\frac{1}{|Q|}\left|\int_{(Q)} f(z)\,dz\right|=0,$$

and if at all points z of G, except at most those of an enumerable set,

$$\limsup_{\delta(Q)\to 0}\frac{1}{|Q|}\left|\int_{(Q)} f(z)\,dz\right|<+\infty,$$

where Q denotes any square containing z.

A complex function is called *derivable* at a point z_0, if the ratio $[f(z)-f(z_0)]/(z-z_0)$ tends to a finite limit when z tends to z_0 in any manner. This limit is called *derivative* of f at z_0 and is denoted by $f'(z_0)$.

Let f be any complex function, defined in the neighbourhood of a point z_0. If we have $\limsup_{h\to 0}|[f(z_0+h)-f(z_0)]/h|<+\infty$, we can write $f(z)=f(z_0)+M(z)\cdot(z-z_0)$, where $M(z)$ is a function of z which is bounded in the neighbourhood of z_0; and we then easily find that the ratio $|J(f;Q)|/|Q|$, and *a fortiori* the ratios $|J_1(f;Q)|/|Q|$ and $|J_2(f;Q)|/|Q|$, must remain bounded when Q denotes any sufficiently small square containing z_0. If, further, the function f is derivable at z_0, we have $f(z)=f(z_0)+f'(z_0)\cdot(z-z_0)+\varepsilon(z)\cdot(z-z_0)$, where $|\varepsilon(z)|\to 0$ as $z\to z_0$, and the ratios in question tend to zero as $\delta(Q)\to 0$. Finally, let us observe that if the function f is continuous, the expressions $J_1(f;I)$ and $J_2(f;I)$, considered as functions of the interval I, both fulfil the condition (l_1) of § 4. Therefore, if we apply Theorem 4.4, we obtain the following theorem due to A. S. Besicovitch [3] (cf. also S. Saks and A. Zygmund [2]):

(5.3) ***Theorem.*** *A complex function f, continuous in an open set G, is holomorphic in G if it is derivable at almost all the points of G and if further $\limsup_{h\to 0}|[f(z+h)-f(z)]/h|<+\infty$ at each point z of G except at most those of a set which is the sum of a sequence of sets of finite length.*

The theorem of Besicovitch may be regarded as a generalization of the classical theorem of E. Goursat [1]: *A complex function f, continuous in an open set G, is holomorphic in G if it is everywhere derivable in G.* T. Pompeiù [1] showed that it is enough to suppose f derivable almost everywhere, provided that we restrict the expression $\limsup_{h\to 0}|[f(z+h)-f(z)]/h|$ to be bounded in G. Finally, H. Looman [3] (cf. also J. Ridder [2]) replaced the condition that the expression $\limsup_{h\to 0}|[f(z+h)-f(z)]/h|$ is bounded by the condition that this expression is finite at each point of G. Theorem 5.3 evidently includes all these generalizations.

The theorems of Morera and of Goursat, and their generalizations furnished by Theorems 5.2 and 5.3, contain criteria for holomorphism which are based on the notion of curvilinear integral and of derivation in the complex domain. The classical theorem of Cauchy is an instance of a criterion of a different kind, expressible in terms of real variable conditions on the real and imaginary parts of a complex function; we have in fact, according to this theorem: in order that a continuous function of a complex variable $f(z)=u(x,y)+iv(x,y)$ be holomorphic in an open set G, it is necessary and sufficient that the partial derivatives u'_x, u'_y, v'_x, v'_y should all exist in G and be continuous, and that they everywhere fulfil the Cauchy-Riemann equations $u'_x=v'_y$, and $u'_y=-v'_x$.

A series of researches begun by P. Montel [1] has been devoted to the reduction of these conditions, particularly that of the continuity of the partial derivatives. The problem was finally solved by H. Looman [2] and D. Menchoff (*vide* the first ed. of this book, p. 243, and D. Menchoff [I]) who succeeded in removing completely the condition in question without replacing it by any other. It is remarkable that a classical problem of such an elementary aspect should only have been solved by a quite essential use of methods of the theory of real functions.

(5.4) ***Lemma.*** *Let w be a real function of one variable, derivable almost everywhere in an interval $[a, b]$; let F be a closed non-empty subset of this interval, and let N be a finite constant such that*

$$|w(x_2)-w(x_1)|\leqslant N\cdot|x_2-x_1|$$

whenever $x_1 \epsilon F$ and $x_2 \epsilon [a,b]$. Then

$$(5.5)\qquad \left|w(b)-w(a)-\int_F w'(x)\,dx\right|\leqslant N\cdot(b-a-|F|).$$

Proof. Let us denote by F_1 the set obtained by adding the points a and b to the set F. The function $\tilde{w}$, equal to w on F_1 and linear on the intervals contiguous to F_1, is evidently absolutely continuous on $[a,b]$ (and even fulfils the Lipschitz condition). Hence

$$(5.6)\qquad w(b)-w(a)=\tilde{w}(b)-\tilde{w}(a)=\int_a^b \tilde{w}'(x)\,dx.$$

Now $\tilde{w}'(x)=w'(x)$ at almost all the points x of F and $|\tilde{w}'(x)|\leqslant N$ at each point x outside F. The relation (5.5) therefore follows at once from (5.6).

(5.7) ***Lemma.*** *Let $w(x,y)$ be a real function whose partial derivatives with respect to the two variables x and y exist at every point of a square Q, except at most at the points of an enumerable set; and let F be a closed non-empty subset of Q, and N a finite constant such that*

$$|w(x_2,y_1)-w(x_1,y_1)|\leqslant N\cdot|x_2-x_1| \quad \text{and} \quad |w(x_1,y_2)-w(x_1,y_1)|\leqslant N\cdot|y_2-y_1|$$

whenever $(x_1,y_1)\epsilon F$, $(x_2,y_1)\epsilon Q$, and $(x_1,y_2)\epsilon Q$.

Then if $[a_1,b_1; a_2,b_2]$ denotes the smallest interval (which may be degenerate) containing F, we have

$$(5.8)\qquad \begin{aligned} &\left|\int_{a_1}^{b_1}[w(x,b_2)-w(x,a_2)]\,dx-\iint_F w'_y(x,y)\,dx\,dy\right|\leqslant 5N\cdot|Q-F|\\ &\left|\int_{a_2}^{b_2}[w(b_1,y)-w(a_1,y)]\,dy-\iint_F w'_x(x,y)\,dx\,dy\right|\leqslant 5N\cdot|Q-F|. \end{aligned}$$

Proof. Let us choose arbitrarily two points (x',a_2) and (x'',b_2) belonging to the set F and situated on the two sides of the interval $[a_1,b_1;a_2,b_2]$ parallel to the x-axis. For any point ξ of $[a_1,b_1]$ we have $|w(\xi,b_2)-w(\xi,a_2)|\leqslant|w(\xi,b_2)-w(x'',b_2)|+|w(x'',b_2)-w(x',b_2)|+$ $+|w(x',b_2)-w(x',a_2)|+|w(x',a_2)-w(\xi,a_2)|$; and hence, denoting by l

the length of the side of the square Q, we obtain

$$(5.9)\qquad \begin{aligned} &|w(\xi,b_2)-w(\xi,a_2)|\leqslant\\ &\leqslant N\cdot[|x''-\xi|+|x'-x''|+|a_2-b_2|+|\xi-x'|]\leqslant 4Nl. \end{aligned}$$

We now denote for any point ξ of $[a_1,b_1]$, by F_ξ the set of all the points y of $[a_2,b_2]$ such that $(\xi,y)\epsilon F$. Let A be the set of the points ξ of the interval $[a_1,b_1]$ for each of which $F_\xi\neq 0$, and let B denote the set of the remaining points of $[a_1,b_1]$. On account of Lemma 5.4 we have

$$\left|w(\xi,b_2)-w(\xi,a_2)-\int_{F_\xi} w_y(\xi,y)\,dy\right|\leqslant N\cdot(b_2-a_2-|F_\xi|)$$

whenever $\xi\epsilon A$, and if we integrate the two sides of this inequality with respect to ξ on the set A, we find

$$(5.10)\qquad \begin{aligned} &\left|\int_A [w(\xi,b_2)-w(\xi,a_2)]\,d\xi-\iint_F w'_y(\xi,y)\,dy\,d\xi\right|\leqslant\\ &\leqslant N\cdot[(b_1-a_1)(b_2-a_2)-|F|]\leqslant N\cdot|Q-F|. \end{aligned}$$

On the other hand if we integrate (5.9) with respect to ξ on the set B, we obtain $\left|\int_B [w(\xi,b_2)-w(\xi,a_2)]\,d\xi\right|\leqslant 4Nl\cdot|B|\leqslant 4N\cdot|Q-F|$, and by adding this to (5.10) we obtain the first of the inequalities (5.8). The second inequality follows by symmetry.

(5.11) ***Theorem of Looman-Menchoff.*** *If the functions $u(x,y)$ and $v(x,y)$, continuous in an open set G, are derivable with respect to x and with respect to y at each point of G except at most at the points of an enumerable set, and if $u'_x(x,y)=v'_y(x,y)$ and $u'_y(x,y)=-v'_x(x,y)$ at almost all the points (x,y) of G, then the complex function $f=u+iv$ is holomorphic in G.*

Proof. Let us denote by F the set of the points (x,y) of G such that the function f is not holomorphic in any neighbourhood of (x,y). The set F is evidently closed in G and the function f is holomorphic in $G-F$. It thus has to be proved that F is empty.

Suppose therefore, if possible, that $F\neq 0$ and let F_n denote, for each positive integer n, the set of the points (x,y) of G such that, whenever $|h|\leqslant 1/n$, none of the four differences $u(x+h,y)-u(x,y)$ $u(x,y+h)-u(x,y)$, $v(x+h,y)-v(x,y)$, $v(x,y+h)-v(x,y)$ exceeds $|nh|$ in absolute value. By continuity of the functions u and v, each of the sets F_n is closed in G. On the other hand, the sets F_n cover the whole set G, except at most an enumerable set consisting of the points at which the functions u and v are not both derivable with respect to x and with respect to y simultaneously. Therefore, on account of Baire's Theorem (Chap. II, Theorem 9.1), the set $F \subset G$ contains a portion which either reduces to a single point, or else is contained entirely in one of the F_n. The former possibility is ruled out, since, as we easily see on account of the continuity of f, the set F cannot contain any isolated points. There must therefore exist a positive integer N and an open sphere S such that $0 \neq F \cdot S \subset F_N$.

Let Q be any square contained in S, such that $\delta(Q) \leqslant 1/N$ and $Q \cdot F \neq 0$. We denote by $I = [a_1, b_1; a_2, b_2]$ the smallest interval containing $Q \cdot F$. By applying the evaluations of Lemma 5.7 to the integrals on the left-hand sides of the formulae (5.1) and by taking into account the relations $u'_x(x,y) = v'_y(x,y)$ and $u'_y(x,y) = -v'_x(x,y)$ which are, by hypothesis, fulfilled almost everywhere, we find $|\mathrm{J}_1(f;I)| \leqslant 10N \cdot |Q - F|$ and $|\mathrm{J}_2(f;I)| \leqslant 10N \cdot |Q - F|$, and therefore $|\mathrm{J}(f;I)| \leqslant 20N \cdot |Q - F|$. This last inequality may also be written $|\mathrm{J}(f;Q)| \leqslant 20N \cdot |Q - F|$, since the figure $Q \ominus I$ contains no points of F in its interior, and since therefore $\mathrm{J}(f;R) = 0$ for each interval R contained in $Q \ominus I$.

Now let $z_0 = (x_0, y_0)$ be any point of S, and let Q be any square containing z_0. By what has just been shown, if $z_0 \in F$ we have $|\mathrm{J}(f;Q)|/|Q| \leqslant 20N \cdot |Q - F|/|Q|$ as soon as $\delta(Q) \leqslant 1/N$; the ratio $\mathrm{J}(f;Q)/|Q|$ therefore remains bounded as $\delta(Q) \to 0$ and tends to zero whenever z_0 is a point of density of F. Further $|\mathrm{J}(f;Q)|/|Q| \to 0$ as $\delta(Q) \to 0$, whenever $z_0 \in S - F$, since $\mathrm{J}(f;Q) = 0$ for every square Q which does not contain points of F. Therefore by Theorem 2.3, the function f must be holomorphic in S. This is, however, excluded since $S \cdot F \neq 0$. We thus arrive at a contradiction and this completes the proof.

Theorem 5.11 was stated (even in a more general form) by P. Montel [2] as early as 1913, but without proof. The proof supplied by H. Looman [2] in 1923 was found to contain a serious gap which was only finally filled in by D. Menchoff (cf. D. Menchoff [I] and the first edition of this book, p. 243).

By making use of general theorems on derivates (*vide*, below, Chap. IX) it is possible to weaken slightly the hypotheses of the theorem. Thus instead of assuming partial derivability of the function u and v, it is sufficient to suppose that at each point of G (except at most those of an enumerable set) these functions have with respect to each variable, x and y, their partial Dini derivates finite. This condition implies (cf. Chap. VII, § 10, p. 236, or Chap. IX, § 4) partial derivability of the functions u and v with respect to each variable at almost all points of G (this generalization of the theorem of Looman-Menchoff does not require any alteration of the proof; for other and much deeper generalizations, *vide* the memoirs of D. Menchoff [1; 2]).

The extension of Theorem 5.11 which we have indicated, includes in particular the theorem of Looman mentioned above, p. 197, but not however the theorem of Besicovitch (5.3). It would be interesting to establish a theorem which would include both the theorem of Besicovitch and that of Looman-Menchoff.

§ 6. The Perron integral. For functions of one real variable, as announced in § 1, the method of major and minor functions leads to an important generalization of the Lebesgue integral.

A function of a real variable, f, is termed *integrable in the sense of Perron*, or *$\mathcal{P}$-integrable*, on a figure R_0 in $\boldsymbol{R}_1$, if 1° f has both major and minor functions on R_0, and if 2° the lower bound of the numbers $U(R_0)$, where U is any major function of f on R_0, and the upper bound of the numbers $V(R_0)$, where V is any minor function of f, are equal. The common value of the two bounds is then called *definite Perron integral*, or *definite $\mathcal{P}$-integral*, of f on R_0, and denoted by $(\mathcal{P})\int_{R_0} f(x)\,dx$. It is evident that *for $\mathcal{P}$-integrability of a function f on a figure R_0 it is necessary and sufficient that for each $\varepsilon > 0$ there should exist a major function U and a minor function V of f on R_0 such that $U(R_0) - V(R_0) < \varepsilon$.*

Since (cf. § 3, p. 190) the function $U - V$ is monotone non-decreasing for every major function U and every minor function V of f, it follows that *every function which is $\mathcal{P}$-integrable on a figure R_0, is so also on every figure $R \subset R_0$.* The function of an interval $P(I) = (\mathcal{P})\int_I f(x)\,dx$, thus defined for every interval $I \subset R_0$, is called *indefinite Perron integral*, or *indefinite $\mathcal{P}$-integral*, of f on R_0. As we see at once, $P(I)$ is an additive function of the interval I. Moreover, given any positive number ε, there exist always a major function U and a minor function V of f, such that $0 \leqslant U(I) - P(I) \leqslant \varepsilon$ and $0 \leqslant P(I) - V(I) \leqslant \varepsilon$ for every interval $I \subset R_0$; and since $\underline{U}(x) > -\infty$ and $\overline{V}(x) < +\infty$ at each point x of R_0, it follows at once that the function P is continuous. Just as in the case of the Lebesgue integral, a function of a real variable is termed *indefinite $\mathcal{P}$-integral* [*major function, minor function*] of a function f, if this is the case for the function of an interval determined by it (cf. Chap. III, § 13).

As we see at once from Theorem 3.2, *every function which is integrable in the sense of Lebesgue on a figure R_0, is so in the sense of Perron, and its definite Lebesgue and Perron integrals over R_0 are equal.* On the other hand, if F is the primitive of Newton (cf. § 1) of a function f, the function F is at the same time a major and a minor function of f, and therefore is the indefinite $\mathcal{P}$-integral of f. It follows that *Perron's process of integration includes both that of Lebesgue and that of Newton.*

We shall establish some fundamental properties of the Perron integral.

(6.1) ***Theorem.*** *Every $\mathcal{P}$-integrable function is measurable, and is almost everywhere finite and equal to the derivative of its indefinite integral.*

Proof. Let f be a function of a real variable, $\mathcal{P}$-integrable on an interval I_0, and let P be its indefinite $\mathcal{P}$-integral on I_0. It has to be proved that the function P has at almost all points x, a finite derivative equal to $f(x)$.

For this purpose, let ε be any positive number and U a major function of f such that

$$U(I_0)-P(I_0)<\varepsilon^2. \tag{6.2}$$

Let us write $H=U-P$. The function H, as monotone non-decreasing, is almost everywhere derivable, and if we denote by E the set of the points x of I_0 at which $H'(x)\geqslant\varepsilon$, we find, by (6.2) and Theorem 7.4, Chap. IV, that $|E|<\varepsilon$.

Now at each point $x\in I_0$ where the function H is derivable, $\underline{U}(x)=H'(x)+\underline{P}(x)$; hence $\underline{P}(x)>-\infty$ and $\underline{P}(x)\geqslant\underline{U}(x)-\varepsilon\geqslant f(x)-\varepsilon$ at almost all the points x of I_0-E. Therefore, since $|E|\leqslant\varepsilon$, ε being an arbitrary positive number, it follows that $-\infty\neq\underline{P}(x)\geqslant f(x)$ at almost all the points x of I_0. By symmetry this gives also $+\infty\neq\overline{P}(x)\leqslant f(x)$, and finally $\infty\neq P'(x)=f(x)$ almost everywhere in I_0.

(6.3) ***Theorem.*** *If two functions f and g are almost everywhere equal on a figure R_0 and one of them is $\mathcal{P}$-integrable on R_0, so is the other and the definite $\mathcal{P}$-integrals of f and g over R_0 are equal.*

Proof. Suppose that the function f is $\mathcal{P}$-integrable and denote by A the value of its definite integral over R_0. Let ε be any positive number and let U and V be two functions of an interval, which are respectively a major and a minor function of f on R_0 and which fulfil the inequalities

$$U(R_0)\geqslant A\geqslant V(R_0) \quad and \quad U(R_0)-V(R_0)\leqslant\varepsilon/3. \tag{6.4}$$

Let us denote by E the set of the points x at which $f(x)\neq g(x)$. The function equal to $+\infty$ at all the points of E and to 0 everywhere else is therefore almost everywhere zero, and by Theorem 3.2 has a major function G such that $0\leqslant G(R_0)\leqslant\varepsilon/3$. We have $\underline{G}(x)=+\infty$ at each point $x\in E$ and writing $U_1=U+G$, $V_1=V-G$, we see that the functions of an interval U_1 and V_1 thus defined are respectively a major and a minor function of the function g on R_0. Moreover by (6.4), $U_1(R_0)\geqslant A\geqslant V_1(R_0)$ and $U_1(R_0)-V_1(R_0)\leqslant\varepsilon$. Therefore the function g is $\mathcal{P}$-integrable on R_0 and $A=(\mathcal{P})\int_{R_0} g(x)\,dx$, which completes the proof.

(6.5) ***Theorem.*** *Every function f which is $\mathcal{P}$-integrable and almost everywhere non-negative on a figure R_0, is summable on this figure.*

Proof. We may assume, by Theorem 6.3, that the function f is everywhere non-negative on R_0. Therefore if U is any major function of f, we have $\underline{U}(x)\geqslant f(x)\geqslant 0$ at every point $x\in R_0$, and consequently, by Theorem 3.1, the function U is monotone non-decreasing. Its derivative $U'(x)$ is therefore summable on R_0, and, since $U'(x)\geqslant f(x)\geqslant 0$ almost everywhere, the function f is also summable on R_0.

Theorem 6.5 shows that, although Perron integration is more general than Lebesgue integration, the two processes are completely equivalent in the case of integration of functions of constant sign.

§ 7. Derivates of functions of a real variable. Certain of the theorems of §§ 2 and 3 can be given a more complete statement when we deal with functions of one real variable. We shall begin with the following proposition which is due to Zygmund:

(7.1) ***Theorem.*** *If $F(x)$ is a finite function of a real variable such that* (i) $\limsup_{h\to 0+} F(x-h)\leqslant F(x)\leqslant\limsup_{h\to 0+} F(x+h)$ *at every point x, and* (ii) *the set of the values assumed by $F(x)$ at the points x where $\overline{F}^+(x)\leqslant 0$ contains no non-degenerate interval, then the function F is monotone non-decreasing.*

Proof. Suppose, if possible, that there exist two points a and b such that $a<b$ and that $F(b)<F(a)$. Then, denoting by E the set of the points x at which $\overline{F}^+(x)\leqslant 0$, we can determine a value y_0 not belonging to the set $F[E]$ and such that $F(b)<y_0<F(a)$. Let

x_0 be the upper bound of the points x of $[a,b]$, for which $F(x) \geqslant y_0$. We shall obviously have $a < x_0 < b$, $F(x_0) = y_0$, and $F(x) \leqslant y_0$ for each point x of the interval $[x_0, b]$. Therefore $\overline{F}^+(x_0) \leqslant 0$, although x_0 does not belong to E. This is in contradiction with the definition of the set E.

Let us mention the following consequence of Theorem 7.1:

Dini's Theorem. *Given on an interval $I=[a,b]$ a continuous function $F(x)$, the upper and lower bounds of each of its four Dini derivates are respectively equal to the upper and lower bounds of the ratio $\dfrac{F(x_2)-F(x_1)}{x_2-x_1}$, where x_1 and x_2 are any points of I.*

Let, for instance, m be the lower bound of the derivate $\overline{F}^+(x)$ on the interval I, and suppose first that $m > -\infty$. Then, if m' denotes any finite number less than m, the function $F(x) - m' \cdot x$ has everywhere on $[a, b]$ its upper right-hand derivate positive; and so by Theorem 7.1, $F(x_2) - F(x_1) \geqslant m' \cdot (x_2 - x_1)$, and therefore also $[F(x_2) - F(x_1)]/(x_2 - x_1) \geqslant m$, for every pair of points x_1 and x_2 of I such that $x_1 < x_2$. Since the inequality just obtained is trivial in the case $m = -\infty$, the theorem follows.

An immediate consequence is the following theorem:

If any one of the four Dini derivates of a continuous function is continuous at a point, so are the three others, and all four derivates in question are equal, so that the function considered is derivable at this point.

These two propositions were proved by U. Dini [I] in 1878.

(7.2) ***Theorem.*** *If H is a finite function of one variable such that* (i) $\limsup_{h\to 0+} H(x-h) \leqslant H(x) \leqslant \limsup_{h\to 0+} H(x+h)$ *at every point x, and* (ii) $\overline{H}^+(x) \geqslant 0$ *at every point x except at most at those of an enumerable set, then the function H is monotone non-decreasing.*

Proof. Let ε be a positive number and write $F(x) = H(x) + \varepsilon x$. We have $\overline{F}^+(x) \geqslant \varepsilon > 0$ at each point x except at most at those of a finite or enumerable set E. The set $F[E]$ being, with E, at most enumerable, it follows from Theorem 7.1 that the function $F(x) = H(x) + \varepsilon x$ is non-decreasing for each $\varepsilon > 0$; and by making $\varepsilon \to 0$ we obtain the assertion of the theorem.

(7.3) ***Theorem.*** *Suppose that F is a continuous function and g a $\mathscr{P}$-integrable function of a real variable, and that, further, we have* (i) $\overline{F}^+(x) \geqslant g(x)$ *at almost all points x and* (ii) $\overline{F}^+(x) > -\infty$ *at every point x, except at most at those of an enumerable set; then*

(7.4) $$F(b) - F(a) \geqslant (\mathscr{P})\int_a^b g(x)\,dx$$

for every pair of points a and b such that $a < b$.

If, in addition, (i_1) $\overline{F}^+(x) \geqslant g(x) \geqslant \underline{F}^+(x)$ *at almost all points x and* (ii_1) $\overline{F}^+(x) > -\infty$ *and* $\underline{F}^+(x) < +\infty$ *at every point x except at most at those of an enumerable set, then the function F is an indefinite $\mathscr{P}$-integral of g.*

Proof. We may obviously assume that $\overline{F}^+(x) \geqslant g(x)$ at every point x. Therefore, denoting by V any minor function of g, and writing $H = F - V$, we shall have $\overline{H}^+(x) \geqslant \overline{F}^+(x) - \overline{V}^+(x) \geqslant 0$ at every point x, except at most at those of a finite or enumerable set where $\overline{F}^+(x) = -\infty$. Further, since the function F is continuous, the inequality $\overline{V}(x) < +\infty$, which holds at every point x, implies that the function H satisfies the condition (i) of Theorem 7.2. Consequently, by Theorem 7.2, $H(b) - H(a) \geqslant 0$, i. e. $F(b) - F(a) \geqslant V(b) - V(a)$, and since V is any minor function of g, we obtain the inequality (7.4).

The second part of the assertion is an immediate consequence of the first part.

As we easily see, the condition of continuity of the function F in the first part of Theorem 7.3 may be replaced by the condition (i) of Theorem 7.1.

Theorem 7.3 constitutes, on account of Theorem 7.4, Chap. IV, a generalization of the following theorem of Lebesgue [I, p. 122; 2; 3; 4; II, p. 183]: *in order that one of the derivates of a continuous function, supposed finite, be summable, it is necessary and sufficient that this function be of bounded variation; its absolute variation is the integral of the absolute value of the derivate in question.* Let us add that in the case in which the function F is assumed to be of bounded variation, Theorem 7.3 is included in Theorem 9.6 of Chap. IV.

The condition (ii) of the first part of Theorem 7.3, as well as the condition (ii_1) of the second, is quite essential for the validity of the theorem. It is possible, in fact, to give an example of a continuous function whose derivative exists everywhere and is summable, without the function being the indefinite integral of its derivative, and this because the latter assumes infinite values. To see this, we shall first show that given any closed set E of measure zero in an interval $J_0 = [a, b]$, there exists a function G, absolutely continuous and increasing in J_0, which has a derivative everywhere in J_0 and fulfils the conditions

(7.5) $$G'(x) = +\infty \quad \text{for} \quad x \in E \quad \text{and} \quad G'(x) \neq +\infty \quad \text{for} \quad x \in J_0 - E.$$

Let us suppose for simplicity that E contains the end-points a and b of J_0 and let us denote by $\{[a_n, b_n]\}$ the sequence of the intervals contiguous to E. Let $\{h_n\}$ be a sequence of positive numbers such that

(7.6) $$\lim_n h_n/(b_n - a_n) = +\infty \quad \text{and} \quad (7.7) \quad \sum_{n=1}^{\infty} h_n = 1$$

(it suffices to write, for instance, $h_n = \sqrt{r_n} - \sqrt{r_{n+1}}$, where $r_n = \dfrac{1}{b-a}\sum_{i=n}^{\infty}(b_i - a_i)$).

Let us write

(7.8) $$g(x) = \begin{cases} h_n/(x-a_n)^{1/2}(b_n-x)^{1/2}, & \text{when} \quad a_n < x < b_n, \\ +\infty, & \text{when} \quad x \in E. \end{cases}$$

Thus defined the function $g(x)$ is non-negative on J_0 and summable on J_0, since $\int_{a_n}^{b_n} g(x)dx=\pi h_n$, so that by (7.7) we have $\int_a^b g(x)dx=\pi$. Let G be the indefinite integral of g on J_0. In order to verify that the function G fulfils the conditions (7.5), we observe that the function $g(x)$ is continuous for every $x\in J_0-E$; on the other hand, if we denote by m_n the lower bound of $g(x)$ in $[a_n, b_n]$ we derive from (7.6) that $\lim_n m_n=\lim_n 2h_n/(b_n-a_n)=+\infty$, from which it follows that $\lim_{x\to x_0} g(x)=+\infty=g(x_0)$ for every $x_0\in E$. Consequently $G'(x)=g(x)$ for every x, and therefore, by (7.8), the conditions (7.5) hold.

Now let (cf. Chap. III, (13.4)), $H(x)$ be a continuous non-decreasing singular function on J_0, which is constant on each interval contiguous to the set E, and such that $H(a)\neq H(b)$. Let us put $F=G+H$. As we verify easily from (7.5), we have $F'(x)=G'(x)=g(x)$ at every point x of J_0. The function F therefore has everywhere a derivative which is summable on J_0. But, since H is the function of singularities of the function F, the latter is certainly not absolutely continuous, let alone the indefinite integral of its derivative. (The functions G and F provide at the same time an example of two functions whose derivatives, finite or infinite, exist and are everywhere equal, without the difference $G-F$ being a constant; cf. H. Hahn [1] and S. Ruziewicz [1].)

In connection with these examples, it may be interesting to mention the following theorem (*vide* G. Goldowsky [1] and L. Tonelli [8]):

(7.9) ***Theorem.*** *If a continuous function F has a (finite or infinite) derivative at each point of $\boldsymbol{R}_1$ except perhaps at the points of an enumerable set, and if this derivative is almost everywhere non-negative, the function F is monotone non-decreasing.*

Proof. Let E be the set of the points x such that the function F is not monotone in any neighbourhood of x. The set E is evidently closed, and the function F is non-decreasing on every interval contained in CE. It therefore has to be proved that the set E is empty.

Suppose, if possible, that $E\neq 0$, and denote for every positive integer n by P_n the set of the points x for which the inequality $0<x'-x<1/n$ implies $F(x')-F(x)\leqslant-(x'-x)$ however we choose x'. Similarly let Q_n be the set of the points x for which the same inequality implies $F(x')-F(x)\geqslant -2(x'-x)$. We see easily that the sets P_n and Q_n are closed, and that they cover the whole straight line $\boldsymbol{R}_1$ except at most the finite or enumerable set of the points at which the function F is without a derivative. Consequently, by Baire's Theorem (Chap. II, Theorem 9.2) the set E must contain a portion which either 1º reduces to a single point, or else 2º is contained in one of the sets P_n, or finally 3º is contained in one of the sets Q_n. The first case is obviously impossible, since the set E has no isolated points. Let us therefore consider case 2º, and suppose that there exists a positive integer n_0 and an open interval I such that $0\neq E\cdot I\subset P_{n_0}$. We may clearly suppose that $\delta(I)<1/n_0$. Since by hypothesis, $F'(x)\geqslant 0$ almost everywhere, the set P_{n_0} is certainly non-dense. Let $[a,b]$ denote any interval contiguous to $E\cdot I$. The function F is then non-decreasing on $[a,b]$ and this contradicts the fact that, since a and b belong to P_{n_0} and $b-a<1/n_0$, we have

$$F(b)-F(a)\leqslant-(b-a)<0.$$

There now remains only case 3º. In this case there exists an open interval I such that the set $E\cdot I$ is non-empty and is contained in one of the sets Q_n. But then $\overline{F}^{+}(x)\geqslant -2$ everywhere in I, and $F'(x)\geqslant 0$ almost everywhere, in I. Therefore, by Theorem 7.3, the function F is non-decreasing in I, and this again is impossible since the interval I contains points of E in its interior.

We thus arrive at a contradiction in each of the three cases, and this proves our assertion.

Let us mention a corollary of Theorem 7.9:

If F is a continuous function having a derivative at every point, except perhaps at those of an enumerable set, and if there exists a finite constant M such that $|F'(x)|\leqslant M$ at almost all points x, then the function F is the indefinite integral of its derivative.

***§ 8. The Perron-Stieltjes integral.** Among the various generalizations of the Stieltjes type for the Perron integral (*vide* for instance R. L. Jeffery [2; 3], J. Ridder [9] and A. J. Ward [3]), that due to Ward has the advantage of including the others and of defining the process of Stieltjes integration with respect to any finite function whatsoever. In this § we shall give the fundamental definitions and results of the theory of Ward. For a deeper analysis, in the case in which the function with respect to which we integrate is of generalized bounded variation in the restricted sense (*vide* below, Chap. VII) the reader should consult the memoir of Ward referred to.

As in the two preceding §§ we shall consider only functions defined in $\boldsymbol{R}_1$, i. e. functions of a linear interval or of a real variable. We shall, moreover, restrict ourselves to integration of finite functions. This restriction is essential for the methods which we shall employ.

Given two finite functions f and G, an additive function of an interval U will be termed *major function* of f *with respect to* G on an interval I_0, if to each point x there corresponds a number $\varepsilon>0$ such that $U(I)\geqslant f(x)G(I)$ for every interval I containing x and of length less than ε. The definition of *minor function with respect to* G is symmetrical, and by following the method of § 6, p. 201, with the help of the notions of major and minor functions with respect to G, we define *Perron-Stieltjes integration*, or *$\mathcal{PS}$-integration* with respect to any finite function G whatever. The $\mathcal{PS}$-integral of a function f with respect to a function G on an interval $I_0=[a,b]$ will be denoted by $(\mathcal{PS})\int_{I_0} f(x)dG(x)$, or by $(\mathcal{PS})\int_a^b f(x)dG(x)$.

If U and V are respectively a major and a minor function of the same function f with respect to the same function G, their difference $U-V$ is evidently monotone non-decreasing. The criterion for $\mathcal{PS}$-integrability of a function is entirely similar to that for $\mathcal{P}$-integrability given in § 6, p. 201, and it follows that every function which is $\mathcal{PS}$-integrable on an interval I_0, is so equally on each subinterval of I_0. We are thus led to the notion of *indefinite* $\mathcal{PS}$*-integral* with respect to any finite function G. This indefinite integral is an additive function of an interval, and is continuous at each point of continuity of the function G. Finally we observe that the $\mathcal{PS}$-integral possesses the distributive property which we may express as follows: *If each of the two finite functions* f_1 *and* f_2 *is* $\mathcal{PS}$*-integrable on an interval* I_0 *with respect to each of the two functions* G_1 *and* G_2, *then each linear combination of the functions* f_1 *and* f_2 *is* $\mathcal{PS}$*-integrable with respect to each linear combination of the functions* G_1 *and* G_2, *and we have*

$$(\mathcal{PS})\int_{I_0}(a_1f_1+a_2f_2)\,d(b_1G_1+b_2G_2)=\sum_{i,k=1,2}a_ib_k\cdot(\mathcal{PS})\int_{I_0}f_i\,dG_k$$

for all numbers a_1, a_2, b_1 *and* b_2.

If $G(x)=x$ for every point x (or, what amounts practically to the same, if $G(I)=|I|$ for each interval I) $\mathcal{PS}$-integration with respect to G coincides with $\mathcal{P}$-integration. In fact, if f is any finite function, each major [minor] function of f with respect to the function $G(x)=x$ in the sense of Ward, is at the same time a major [minor] function of f in the sense of the definition of § 3; the converse is not true in general, but we see at once that if U is a major function of f in the sense of § 3, the function $U(x)+\varepsilon x$ is for each $\varepsilon>0$ a major function of f with respect to $G(x)=x$. Thus the Perron-Stieltjes integral includes the ordinary Perron integral, at any rate as regards integration of finite functions. On the other hand, the Perron-Stieltjes integral includes also the Lebesgue-Stieltjes integral. We have in fact

(8.1) ***Theorem.*** *A finite function f integrable in the Lebesgue-Stieltjes sense on an interval* $I_0=[a_0,b_0]$ *with respect to a function of bounded variation* G, *is so also in the Perron-Stieltjes sense and we have*

$$(8.2)\qquad (\mathcal{PS})\int_{a_0}^{b_0}f\,dG=\int_{a_0}^{b_0}f\,dG-\{f(a_0)[G(a_0)-G(a_0-)]+f(b_0)[G(b_0+)-G(b_0)]\}.$$

Proof. Let us denote for brevity, by A the right-hand side of the relation (8.2). We may evidently assume that the function f is non-negative and it is enough to consider only the following two cases:

1° G is a continuous non-decreasing function. The proof is then just as in Theorem 3.2. Let ε be any positive number. Since the function f is finite, there exists by the theorem of Vitali-Carathéodory (Chap. III, § 7) a lower semi-continuous function g, integrable (G) in the Lebesgue-Stieltjes sense, such that $g(x)>f(x)$ at each point x and such that $\int_{I_0}[g(x)-f(x)]\,dG(x)<\varepsilon$. Denoting by U the indefinite integral (G) of the function g, and taking account of the lower semi-continuity of g, we see easily that U is a major function of f with respect to G on I_0. Moreover, the function G being continuous by hypothesis, the number A is equal to the integral $\int_{a_0}^{b_0}f\,dG$ and we find $0\leqslant U(I_0)-A<\varepsilon$. By symmetry we determine also a minor function V of f with respect to G on I_0 in such a manner that $0\leqslant A-V(I_0)<\varepsilon$, and this establishes $\mathcal{PS}$-integrability of f on I_0 and at the same time the validity of the formula (8.2).

2° G is a non-decreasing saltus-function. Let us denote by $\{x_n\}_{n=1,2,\dots}$ the sequence of the points of discontinuity of G which are in the interior of the interval I_0; and let ε be any positive number and $\{k_n\}_{n=1,2,\dots}$ a sequence of positive numbers such that

$$(8.3)\qquad \sum_n k_n\cdot[G(x_n+)-G(x_n-)]<\varepsilon \quad\textit{and}\quad \lim_n k_n=+\infty.$$

Let us define a function h in $\boldsymbol{R}_1$, by writing: $h(x)=f(x)$ for all the points x of I_0 which are distinct from the points x_n; $h(x_n)=f(x_n)+k_n$ for $n=1,2,\dots$; and $h(x)=f(a_0)$ for $x<a_0$, and $h(x)=f(b_0)$ for $x>b_0$. Finally let us write, for each interval $I=[a,b]$,

$$U(I)=\int_a^b h(x)\,dG(x)-\{h(a)[G(a)-G(a-)]+h(b)[G(b+)-G(b)]\}.$$

The function of an interval U thus defined is evidently additive, and as we easily verify, is a major function of f with respect to G on I_0. Moreover, it follows at once from (8.3) that $0\leqslant U(I_0)-A\leqslant\varepsilon$. Similarly we determine a minor function V of f with respect to G so as to have $0\leqslant A-V(I_0)\leqslant\varepsilon$; hence $A=(\mathcal{PS})\int_{I_0}f\,dG$, and this completes the proof.

Formula (8.2) brings out the fact that the definite Perron-Stieltjes and Lebesgue-Stieltjes integrals are not always equal, even for a function f integrable in both senses. This is due to the fact that the indefinite integral of Lebesgue-Stieltjes is not in general an additive function of an interval. We could, of course, modify the definition of this integral so as to ensure its additivity as a function of an interval. The term in brackets $\{\ \}$ would then disappear from the formula (8.2), but it would then be necessary to give up the additivity of the indefinite Lebesgue-Stieltjes integral considered as a function of a set (cf. Chap. VIII, § 2).

Let us mention further the following generalization of Theorem 6.1 on derivation of the indefinite Perron integral:

(8.4) ***Theorem.*** *If P is an indefinite $\mathcal{PS}$-integral of a finite function f with respect to a function G, then, at almost all points x, the ratio*

$$(8.5)\qquad [P(I)-f(x)G(I)]/|I|$$

tends to 0 as $\delta(I)\to 0$, where I denotes any interval containing x.

Hence at almost all points x, $\overline{P}(x)=f(x)\overline{G}(x)$ and $\underline{P}(x)=f(x)\underline{G}(x)$ or else $\overline{P}(x)=f(x)\underline{G}(x)$ and $\underline{P}(x)=f(x)\overline{G}(x)$ according as $f(x)\geqslant 0$ or $f(x)\leqslant 0$; in particular $P'(x)=0$ at almost all points x where $f(x)=0$.

Proof. The proof is quite similar to that of Theorem 6.1. Let I_0 be an interval, ε a positive number, and U a major function of f with respect to G on I_0 such that $U(I_0)-P(I_0)<\varepsilon^2$. We write $H=U-P$. The function H is monotone non-decreasing, and we have $H'(x)<\varepsilon$ at every point $x\,\epsilon\, I_0$ except at most those of a set E of measure less than ε. Now, since $U(I)-f(x)G(I)\geqslant 0$ for every point x and for every sufficiently small interval I containing x, the lower limit of the ratio (8.5), as $\delta(I)\to 0$, exceeds $-\varepsilon$ at each point x except at most at those of E. Therefore, ε being any positive number, this limit is non-negative for almost all points x. Combining this with the symmetrical result for the upper limit of the same ratio, we complete the proof.

Another generalization of Theorem 6.1, also due to Ward, uses the following definition of relative derivation, which is slightly different from that given in Chap. IV, § 2 (cf. A. J. Ward [3] and A. Roussel [1]).

Given two finite functions of a real variable F and G, we shall say that a number a is the *Roussel derivative* of the function F with respect to G at a point x_0, if when I denotes any interval containing x_0, we have (i) $F(I)-a\cdot G(I)\to 0$ and (ii) $|F(I)-a\cdot G(I)|/\mathrm{O}(G;I)\to 0$, as $\delta(I)\to 0$ (the ratio in (ii) is to be interpreted to mean 0 whenever its numerator and denominator vanish together; $\mathrm{O}(G;I)$ denotes, in accordance with Chap. III, p. 60, the oscillation of G on I).

When the oscillation of the function G at x_0 is finite, the condition (ii) evidently implies (i); however, when $\mathrm{o}(G;x_0)=+\infty$, the condition (i) plays an essential part, whereas (ii) is then satisfied independently of F and of a.

It is also to be observed that when $\mathrm{o}(G;x_0)<+\infty$, and when F is a function which has the relative derivative $F'_G(x_0)$ (cf. Chap. IV, § 2, p. 109), the latter is also the Roussel derivative of F with respect to G. Finally, in the case of derivation with respect to monotone functions, the two methods are completely equivalent. In particular therefore, when $G(x)=x$, Roussel derivation with respect to G it equivalent to ordinary derivation.

The proof of the theorem on Roussel derivability of the indefinite $\mathcal{PS}$-integral is much the same as that of Theorems 6.1 and 8.4; it depends, however, on the following lemma which may be regarded as a generalization of a result of W. Sierpiński [4].

(8.6) ***Lemma.*** *Let G be a finite function of a real variable, E a bounded set in $\boldsymbol{R}_1$, and $\mathfrak{J}$ a system of intervals such that each point of E is a (right- or left-hand) end-point of an interval ($\mathfrak{J}$) of arbitrarily small length.*

Then, given any number $\mu<|G[E]|$, we can select from $\mathfrak{J}$ a finite system $\{I_k\}$ of non-overlapping intervals such that

$$\sum_k |G[I_k]|\geqslant \frac{1}{2}\mu.$$

Proof. Suppose, for simplicity, that the set E lies in the open interval (0,1). For each positive integer n, let A_n and B_n denote respectively the sets of the points of E each of which is respectively a left- or right-hand end-point of an interval ($\mathfrak{J}$) contained in (0,1) and of length exceeding $1/n$. We evidently have $E=\lim\limits_n(A_n+B_n)$ and there therefore exists a positive integer n_0 such that $|G[A_{n_0}+B_{n_0}]|>\mu$. Suppose, for definiteness, that $|G[A_{n_0}]|>\frac{1}{2}\mu$.

Now, it is easily seen that, if $|G[A_{n_0}]|=+\infty$, there exists a point x_0 such that $|G[A_{n_0}\cdot J]|=+\infty$ for any interval J containing x_0 in its interior. Hence, from the family of intervals ($\mathfrak{J}$) whose left-hand end-points belong to A_{n_0} and whose lengths exceed $1/n_0$, we can obviously select an interval I so as to have $|G[I]|\geqslant|G[A_{n_0}\cdot I]|>\frac{1}{2}\mu$.

Suppose now that $|G[A_{n_0}]|<+\infty$. Then, by induction, we can extract from $\mathfrak{J}$ a finite sequence of intervals $\{I_k=(a_k,b_k)\}_{k=1,2,\ldots,p}$ in such a manner that, writing for symmetry $b_0=0$ and $a_{p+1}=1$, we have: (i) $b_k-a_k>1/n_0$ for $k=1,2,\ldots,p$, (ii) $b_{k-1}<a_k$ and $|G[A_{n_0}\cdot(b_{k-1},a_k)]|\leqslant(|G[A_{n_0}]|-\frac{1}{2}\mu)/n_0$ for $k=1,2,\ldots,p$, and (iii) the

interval $(b_p, a_{p+1})=(b_p, 1)$ contains no points of A_{n_0}. Since, on account of (i), we certainly have $p<n_0$, it follows from (ii) and (iii) that $\sum_k |G[I_k]| \geqslant |G[A_{n_0}]| - p\cdot(|G[A_{n_0}]|-\frac{1}{2}\mu)/n_0 \geqslant \frac{1}{2}\mu$, i. e. that the system of intervals $\{I_k\}$ fulfils the required conditions.

(8.7) ***Theorem.*** *Every finite function f which is $\mathscr{PS}$-integrable with respect to a function G on an interval I_0, is the Roussel derivative with respect to G of its indefinite $\mathscr{PS}$-integral at each point x of I_0 except at most those of a set E such that $|G[E]|=0$.*

Proof. Let ε be any positive number and U a major function of f with respect to G such that $U(I_0)-P(I_0)<\varepsilon^2$, where P denotes the indefinite $\mathscr{PS}$-integral of f. Let us write $H=U-P$, and denote by E_ε the set of the points x of I_0 for which there exist intervals I of arbitrarily small lengths, such that $x\in I$ and that $H(I)\geqslant\varepsilon\cdot|G[I]|$. It follows that each point of E_ε is an end-point of intervals I, as small as we please, which fulfil the inequality $H(I)\geqslant\frac{1}{2}\varepsilon\cdot|G[I]|$. Therefore, denoting by μ any number less than $|G[E_\varepsilon]|$ and applying Lemma 8.6, we can determine in I_0 a finite system of non-overlapping intervals $\{I_k\}$ such that $H(I_k)\geqslant\frac{1}{2}\varepsilon\cdot|G[I_k]|$ for $k=1, 2, \ldots, p$ and that $\sum_k |G[I_k]| \geqslant \frac{1}{2}\mu$. Consequently, since H is non-decreasing, $\varepsilon^2>H(I_0)\geqslant\varepsilon\mu/4$; and therefore $\mu<4\varepsilon$, and hence $|G[E_\varepsilon]|\leqslant 4\varepsilon$.

Now let x be any point of I_0. We have for every sufficiently small interval I containing x,

$$P(I)-f(x)\,G(I)=U(I)-f(x)\,G(I)-H(I)\geqslant -H(I)>-\varepsilon^2;$$

and, unless x belongs to the set E_ε, we also have

$$P(I)-f(x)\,G(I)\geqslant -H(I)\geqslant -\varepsilon\cdot|G[I]|\geqslant -\varepsilon\cdot \mathrm{O}(G; I).$$

Combining this with the similar upper evaluations of $P(I)-f(x)\,G(I)$ obtained by symmetry, we see, since ε is an arbitrary positive number, that f is the Roussel derivative of the function P with respect to G, at every point x of I_0 except at most those of a set E such that $|G[E]|=0$.