CHAPTER V.

Area of a surface $z = F(x, y)$.

§ 1. Preliminary remarks. We saw (cf. Chap. IV, § 8) that the Lebesgue theory enables us to solve completely the elementary problems concerning the length of a curved line and the expression of this length by an integral. However, similar problems concerning curved surfaces involve difficulties of a much more serious kind. Certain classical treatises on the differential and integral calculus, even in the second half of the XIX-th century, contain an inaccurate definition of the area of a surface. By analogy with the definition of length of a curve, the authors attempted to define the area of a surface as the limit of the areas of polyhedra inscribed in the surface and tending to it. H. A. Schwarz [I, p. 309] (cf. also M. Fréchet [3]) was the first to remark that such a limit may not exist and that it is possible to choose a sequence of inscribed polyhedra whose areas tend to any number not less than the actual area of the surface. About the same time Peano and Hermite subjected the old definition to similar criticisms and proposed new definitions based on quite different ideas. It was H. Lebesgue who first returned in his Thesis [1] to the old method, in a modified form that may be roughly described as follows: the area of a surface is the lower limit of the areas of polyhedra tending uniformly to the surface in question (without, however, being necessarily inscribed in the latter).

Nevertheless, in the more general case in which the surface is given parametrically, this definition requires various additional notions and considerations (cf. T. Radò [I; 1; 4]) and the results obtained are far from being as complete as those available for curves. The difficulties that arise belong to Geometry and

Topology rather than to the Theory of functions of a real variable. (For the special case in which the functions $x=X(u,v)$, $y=Y(u,v)$ and $z=Z(u,v)$ which define the surface parametrically fulfil the Lipschitz condition *vide* T. Radò [4] and H. Rademacher [4]).

We shall therefore restrict ourselves to the case of continuous surfaces of the form $z=F(x,y)$. The most elegant and the most complete results concerning these surfaces are due to L. Tonelli [5; 6; 7]; they will be given in § 8 and are the principal object of this chapter.

Tonelli's theory is based on the definition of area proposed by Lebesgue. As regards the modern work on area of surfaces based on other definitions, we should mention: W. H. Young [4], J. C. Burkill [3], S. Banach [5], A. Kolmogoroff [3] and J. Schauder [1].

T. Radò [1, pp. 154—169; 2] has developed further the methods of Tonelli by means of older ideas due to de Geöcze and with the help of certain functionals introduced by the latter. The principal result of Radò (*vide* Theorem 7.3), applications of which will be discussed below, enables us to define the area of a surface as the limit of certain simple expressions, whereas the Lebesgue definition only enables us to obtain it as a lower limit. Another expression is due to L. C. Young (*vide* below § 8) and constitutes a direct generalization of the classical formula for the area of a surface.

Except where the contrary is expressly stated, the reasoning of this chapter will be formulated for functions of two real variables. The extension to spaces of any number of dimensions offers no difficulty.

§ 2. Area of a surface. By a *continuous surface* on a plane interval I_0, we shall mean any equation of the form $z=F(x,y)$, where F is a continuous function on I_0.

A continuous surface $z=P(x,y)$ on an interval I_0 is termed *polyhedron* if there exists a decomposition of I_0 into a finite number of non-overlapping triangles $T_1, T_2, \ldots, T_n$ such that the function P is linear on each of these triangles, i. e. such that $P(x,y)=a_ix+b_iy+c_i$ for $(x,y)\in T_i$, where $i=1,2,\ldots,n$ and a_i, b_i, c_i are constant coefficients. We shall call, respectively, *faces* and *vertices* of the polyhedron $z=P(x,y)$, the parts and the points of the graph (cf. Chap. III, § 10) of the function P, which correspond to the triangles T_i, and to the vertices of the T_i. The sum of the areas of the faces in the sense of elementary Geometry, i.e. the number $\sum_i |T_i|\cdot(a_i^2+b_i^2+1)^{\frac{1}{2}} = \int\int_{I_0} [(\partial P/\partial x)^2+(\partial P/\partial y)^2+1]^{\frac{1}{2}}\,dx\,dy$, will be called *elementary area* of the polyhedron $z=P(x,y)$ on I_0 and denoted by $\mathrm{S}_0(P; I_0)$.

Given any continuous surface $z=F(x,y)$ on an interval I_0, we shall term its *area* on I_0, and denote by $\mathrm{S}(F; I_0)$, the lower limit of the elementary areas of polyhedra tending uniformly to this surface, i.e. the lower bound of all the numbers s for each of which there exists, given any $\varepsilon>0$, a polyhedron $z=P(x,y)$ on I_0 such that 1⁰ $|P(x,y)-F(x,y)|<\varepsilon$ at every point $(x,y)\in I_0$ and 2⁰ $\mathrm{S}_0(P; I_0)\leqslant s$.

We might verify here that for polyhedra the elementary area agrees with the area according to the general definition just given. As, however, this is an easy consequence of the theorems given further on (*vide* p. 181), a special proof is unnecessary at this point. It should be remarked that, in accordance with the definition adopted, the area of a surface may be either finite or infinite.

The following theorem is an immediate consequence of the definition.

(2.1) ***Theorem.*** *For any sequence of continuous functions* $\{F_n\}$ *which converges uniformly on an interval* I_0 *to a function* F, *we have* $\liminf_n \mathrm{S}(F_n; I_0) \geqslant \mathrm{S}(F; I_0)$.

§ 3. The Burkill integral. Instead of treating the theory of area of surfaces by itself, it is more convenient to associate it with certain differential properties of functions of an interval. However, the functions of an interval occurring in the theory of area are not in general additive, and in consequence we shall have to complete in some minor points the theory of functions of an interval, developed in the two preceding chapters.

We shall begin with some subsidiary definitions. To simplify the wording we shall understand in the sequel by *subdivision* of a figure R_0 any finite system of non-overlapping intervals $I_1, I_2, \ldots, I_n$ such that $R_0=\sum_k I_k$. Given any function of an interval U and given a finite system of intervals $\mathfrak{J}=\{I_k\}$, we shall write, for brevity, $U(\mathfrak{J})$ in place of $\sum_k U(I_k)$. In particular therefore, $\mathrm{L}(\mathfrak{J})$ will denote the sum of the areas of the intervals belonging to the system $\mathfrak{J}$.

We call *upper* and *lower integral* in the sense of Burkill of a function of an interval $U(I)$ over a figure R_0, and we denote by $\overline{\int_{R_0}} U$ and $\underline{\int_{R_0}} U$ respectively, the upper and the lower limit of the numbers $U(\mathfrak{I})$ for arbitrary subdivisions $\mathfrak{I}$ of R_0, whose characteristic numbers $\Delta(\mathfrak{I})$ tend to zero (cf. Chap. II, p. 40). When these integrals are equal, their common value is called the *Burkill definite integral* (or simply the *integral*) of the function U over R_0 and is denoted by $\int_{R_0} U$. If this integral exists and is finite, the function U is said to be *integrable* on R_0 (in the sense of Burkill). If the function U is integrable on every figure R (in the whole plane or in a figure R_0) its integral $\int_R U$ considered as a function of the figure R is called *indefinite integral* of U (in the whole plane or on R_0).

(3.1) ***Theorem*** 1^0 *If U is a function of an interval and R_1, R_2 are non-overlapping figures, we have*

$$\overline{\int_{R_1+R_2}} U \geqslant \overline{\int_{R_1}} U + \overline{\int_{R_2}} U \quad \text{and} \quad \underline{\int_{R_1+R_2}} U \leqslant \underline{\int_{R_1}} U + \underline{\int_{R_2}} U, \tag{3.2}$$

provided that both integrals of U over R_1+R_2 are finite.

2^0 *Any function of an interval U which is integrable on a figure R_0, is equally so on every figure $R \subset R_0$ and its indefinite integral on R_0 is an additive function of a figure.*

Proof. Part 1^0 of the theorem is a direct consequence of the definition of the Burkill integrals, and part 2^0 follows at once from the formulae (3.2) when we subtract the second of these formulae from the first.

If U is a function of an interval on a figure R, we shall call *variation of U on R at a set D* the upper limit of $|U(\mathfrak{I})|$ as $\Delta(\mathfrak{I}) \to 0$, where $\mathfrak{I}$ denotes any finite system of non-overlapping intervals contained in R and possessing common points with D. The following analogue of Theorem 4.1, Chap. III, may be noted.

(3.3) ***Theorem.*** *Given on a figure R_0 a function of an interval U such that $\overline{\int_{R_0}} |U| < +\infty$, there can be at most an enumerable infinity of straight lines D, which are parallel to the coordinate axes and at which the variation of U on R_0 is not zero.*

In fact, the number of straight lines which are parallel to the axis of x or of y, and at which the variation of U on R_0 exceeds a positive number ε, cannot be greater than $2\varepsilon^{-1} \cdot \overline{\int_{R_0}} |U| < +\infty$.

(3.4) ***Lemma.*** *Given a function of an interval U integrable on a figure R_0, there exists, for each $\varepsilon > 0$, an $\eta > 0$ such that for every system $\mathfrak{I} = \{I_1, I_2, \ldots, I_p\}$ of non-overlapping intervals situated in R_0, the inequality $\Delta(\mathfrak{I}) < \eta$ implies the inequality*

$$\left| \sum_{k=1}^{p} \left[U(I_k) - \int_{I_k} U \right] \right| < \varepsilon. \tag{3.5}$$

Proof. Let $\eta > 0$ be a number such that, for every subdivision $\mathfrak{T}$ of R_0, $\Delta(\mathfrak{T}) < \eta$ implies $|U(\mathfrak{T}) - \int_{R_0} U| < \varepsilon/2$, and let $\mathfrak{I} = \{I_1, I_2, \ldots, I_p\}$ be any finite system of non-overlapping intervals situated in R_0, such that $\Delta(\mathfrak{I}) < \eta$. Let $R_1 = R_0 \ominus \sum_{k=1}^{p} I_k$. By Theorem 3.1, the function U is integrable on R_1. It follows that there exists a subdivision $\mathfrak{I}_1$ of R_1 such that

$$\Delta(\mathfrak{I}_1) < \eta \quad \text{and} \quad \left| U(\mathfrak{I}_1) - \int_{R_1} U \right| < \varepsilon/2. \tag{3.6}$$

Now $\mathfrak{I} + \mathfrak{I}_1$ clearly constitutes a subdivision of R_0 such that $\Delta(\mathfrak{I} + \mathfrak{I}_1) < \eta$. We therefore have $|U(\mathfrak{I} + \mathfrak{I}_1) - \int_{R_0} U| < \varepsilon/2$, and we need only subtract the second of the relations (3.6) from it to obtain (3.5).

If R_0 is a fixed figure, then to any $\eta > 0$ there corresponds a positive integer p such that every interval $I \subset R_0$ may be subdivided in p subintervals of diameter less than η. Hence applying Lemma 3.4, we obtain at once the following

(3.7) ***Theorem.*** *If a function of an interval $U(I)$ which is integrable on a figure R_0, is continuous, then the same is true of its indefinite integral $B(R) = \int_R U$.*

(3.8) ***Theorem.*** *If U is a function of an interval which is integrable on a figure R_0 and if B is its indefinite integral, then $\overline{B}(x,y) = \overline{U}(x,y)$ and $\underline{B}(x,y) = \underline{U}(x,y)$ at almost all points $(x,y) \in R_0$.*

In particular therefore, if one of the functions U and B is almost everywhere derivable in R_0, the same is true of the other and the derivatives of U and of B are almost everywhere equal.

Proof. Suppose that the set of the points (x,y) at which $\overline{U}(x,y)>\overline{B}(x,y)$ has positive measure. We could then determine a set $E\subset R_0$ of positive outer measure and a number $\alpha>0$, such that $\overline{U}(x,y)-\overline{B}(x,y)>\alpha$ at each point (x,y) of E. Therefore, on account of Vitali's Covering Theorem (Chap. IV, Theorem 3.1), we could determine in R_0, for any $\eta>0$, a finite system of non-overlapping intervals $\mathfrak{J}=\{I_k\}_{k=1,2,\dots,n}$ such that $\Delta(\mathfrak{J})<\eta$, $\mathrm{L}(\mathfrak{J})>|E|/2$, and $U(I_k)-B(I_k)>\alpha\cdot|I_k|$ for $k=1,2,\dots,n$. Now it follows from the last two relations that $U(\mathfrak{J})-B(\mathfrak{J})>\alpha\cdot|E|/2$, which contradicts Lemma 3.4. Hence, $\overline{U}(x,y)\leqslant\overline{B}(x,y)$ almost everywhere in R_0. In the same way we prove that the opposite inequality holds also almost everywhere in R_0, and this completes the proof.

(3.9) ***Theorem.*** *Suppose that U is a continuous function of an interval on a figure R_0 and that* (i) $\overline{\int\limits_{R_0}}|U|<+\infty$ *and* (ii) $U(I)\leqslant U(\mathfrak{S})$ *for every interval $I\subset R_0$ and every subdivision $\mathfrak{S}$ of I. Then the function U is integrable on R_0.*

Proof. Given a number $\varepsilon>0$, let $\mathfrak{T}=\{J_i\}_{i=1,2,\dots,p}$ be a subdivision of R_0 such that

$$U(\mathfrak{T})>\overline{\int\limits_{R_0}}U-\varepsilon. \tag{3.10}$$

Let us denote by $D_1,D_2,\dots,D_r$ the sides of the intervals $(\mathfrak{T})$ which do not belong to the boundary of R_0. By Theorem 3.3 it may be assumed, in view of the continuity of the function U and of condition (ii), that the variation of U on R_0 vanishes at each side D_l. It follows that there exists an $\eta>0$ such that, given any finite system $\mathfrak{S}$ of non-overlapping intervals situated in R_0 and having points in common with the sides D_l, the inequality $\Delta(\mathfrak{S})<\eta$ implies $|U(\mathfrak{S})|<\varepsilon$. We can clearly assume that η does not exceed the length of any side of the intervals $(\mathfrak{T})$.

This being so, consider an arbitrary subdivision $\mathfrak{J}=\{I_1,I_2,\dots,I_n\}$ of R_0 such that $\Delta(\mathfrak{J})<\eta$. By numbering the intervals of $\mathfrak{J}$ suitably, we may evidently suppose that $I_1,I_2,\dots,I_q$ are those having points in common with the sides D_l, while the remaining intervals of $\mathfrak{J}$ (if any) have none. Finally, let us agree to write $U(J_i\odot I_k)=0$ when $J_i\odot I_k=0$. Then $|\sum_{k=1}^{q}U(I_k)|<\varepsilon$ and $|\sum_{i=1}^{p}\sum_{k=1}^{q}U(J_i\odot I_k)|<\varepsilon$, so that, by (3.10) and by condition (ii) of the theorem, we have $\overline{\int\limits_{R_0}}U-\varepsilon<U(\mathfrak{T})\leqslant U(\mathfrak{J})-\sum_{k=1}^{q}U(I_k)+\sum_{i=1}^{p}\sum_{k=1}^{q}U(J_i\odot I_k)<U(\mathfrak{J})+2\varepsilon$. It follows that $\overline{\int\limits_{R_0}}U\leqslant\underline{\int\limits_{R_0}}U+3\varepsilon$, and so, that $\overline{\int\limits_{R_0}}U=\underline{\int\limits_{R_0}}U$.

In connection with this §, *vide* J. C. Burkill [2; 3; 4], R. C. Young [2] and F. Riesz [6; 7].

§ 4. Bounded variation and absolute continuity for functions of two variables. Given a function $F(x,y)$ continuous on an interval $I=[a_1,b_1;a_2,b_2]$, let us denote for any value x subject to $a_1\leqslant x\leqslant b_1$, by $\mathrm{W}_1(F;x;a_2,b_2)$ the absolute variation of the function $F(x,y)$ with respect to the variable y on the interval $[a_2,b_2]$, and for any value y subject to $a_2\leqslant y\leqslant b_2$, by $\mathrm{W}_2(F;y;a_1,b_1)$ that of the function $F(x,y)$ with respect to x on $[a_1,b_1]$. Denoting by J_1 and J_2 respectively the linear intervals $[a_1,b_1]$ and $[a_2,b_2]$ we shall also write $\mathrm{W}_1(F;x;J_2)$ for $\mathrm{W}_1(F;x;a_2,b_2)$ and $\mathrm{W}_2(F;y;J_1)$ for $\mathrm{W}_2(F;y;a_1,b_1)$.

By continuity of the function F, the non-negative expressions $\mathrm{W}_1(F;x;J_2)$ and $\mathrm{W}_2(F;y;J_1)$ are, as is easily seen, lower semi-continuous functions of the variables x and y respectively. When the integrals $\int_{a_1}^{b_1}\mathrm{W}_1(F;x;J_2)\,dx$ and $\int_{a_2}^{b_2}\mathrm{W}_2(F;y;J_1)\,dy$ are both finite, the function F is said to be *of bounded variation* on I in the Tonelli sense. It follows at once that any function of bounded variation of two variables x,y is of bounded variation with respect to x for almost every value of y and with respect to y for almost every value of x.

A continuous function $F(x,y)$ will be termed *absolutely continuous* on an interval $I=[a_1,b_1;a_2,b_2]$ in the Tonelli sense, if it is of bounded variation on I and moreover, absolutely continuous with respect to x for almost every value of y in $[a_2,b_2]$, and absolutely continuous with respect to y for almost every value of x in $[a_1,b_1]$.

We say that a function $F(x,y)$ fulfils the *Lipschitz condition* on I, if there exists a finite constant N such that $|F(x',y')-F(x'',y'')|\leqslant N\cdot(|x'-x''|+|y'-y''|)$ for every pair of points (x',y') and (x'',y'') of I.

Any function which fulfils the Lipschitz condition on an interval I is evidently absolutely continuous on I. In particular polyhedra and also functions of two variables with continuous partial derivatives, are always absolutely continuous functions.

A function F which is continuous and of bounded variation [absolutely continuous, or subject to the Lipschitz condition] on an interval $I_0=[a_1,b_1;\ a_2,b_2]$ can easily be continued, even so as to be periodic, over the whole plane in such a manner as to remain continuous and of bounded variation [absolutely continuous, or subject to the Lipschitz condition] on every interval. In fact, denoting by I_1 one of the intervals congruent to I_0 with a common side parallel to the axis of x, let us continue the function F from the interval I_0 on to the interval I_1 by symmetry relative to the common side of these intervals. Let us further continue similarly the function F from the interval I_0+I_1 on to an interval I_2 congruent to I_0+I_1 which has with the latter a common side parallel to the axis of y. The function F is then defined on the interval $I_0+I_1+I_2$ whose sides are respectively of lengths $2\cdot(b_1-a_1)$ and $2\cdot(b_2-a_2)$. Writing $u=2\cdot(b_1-a_1)$ and $v=2\cdot(b_2-a_2)$, and continuing the function F from the interval $I_0+I_1+I_2$ on to the rest of the plane by the periodicity condition $F(x+u,y)=F(x,y+v)=F(x,y)$, we see easily that the continuation obtained for the function F has the properties required.

Besides the definition of Tonelli several other definitions have been given of conditions under which a function of two variables is said to be of bounded variation. For a discussion of these definitions see C. R. Adams and J. A. Clarkson [1; 2]. Throughout this chapter, use is made of Tonelli's concept only.

We shall subsequently make use of the following theorem concerning the partial derivates of any continuous function:

(4.1) ***Theorem.*** *Given a continuous function $F(x,y)$, its partial Dini derivates, $\overline{F}_x^+, \overline{F}_x^-, \underline{F}_x^+, \underline{F}_x^-$ and $\overline{F}_y^+, \overline{F}_y^-, \underline{F}_y^+, \underline{F}_y^-$, are functions measurable* ($\mathfrak{B}$).

Proof. It will suffice to prove this for any one of these derivates, say $\overline{F}_x^+$.

Given an arbitrary real number a, consider the set

$$E=\underset{(x,y)}{\mathrm{E}}\,[\overline{F}_x^+(x,y)<a],$$

and denote by E_n the set of all the points (x,y) such that for every h the inequality $0<h\leqslant 1/n$ implies $[F(x+h,y)-F(x,y)]/h\leqslant a-1/n$.

We find that $E=\sum_n E_n$ and, since by continuity of the function F each of the sets E_n is closed, E is a set ($\mathfrak{F}_\sigma$), so that the derivate $\overline{F}_x^+$ is a function measurable ($\mathfrak{B}$).

Theorem 4.1 may be compared with Theorems 4.2 and 4.3 of Chap. IV concerning measurability of the derivates of functions of one real variable. Nevertheless it is to be remarked that contrary to what occurs for functions of one variable, the partial Dini derivates of a function measurable ($\mathfrak{B}$) need not in general be measurable ($\mathfrak{B}$), although they are still measurable ($\mathfrak{L}$) (the proof of this requires, however, the theory of analytic sets; *vide* F. Hausdorff [II, p. 274], M. Neubauer [1] and A. E. Currier [1]). On the other hand, a function of two variables may be measurable ($\mathfrak{L}$) without its partial Dini derivates being so.

§ 5. The expressions of de Geöcze.

We shall make correspond to any function $F(x,y)$ which is continuous on an interval $I=[a_1,b_1;\ a_2,b_2]$, the following expressions introduced by Z. de Geöcze [1] into the theory of areas of surfaces:

$$\mathrm{G}_1(F;I)=\int_{a_1}^{b_1}|F(x,b_2)-F(x,a_2)|\,dx,\quad \mathrm{G}_2(F;I)=\int_{a_2}^{b_2}|F(b_1,y)-F(a_1,y)|\,dy,$$

$$\mathrm{G}(F;I)=\{[\mathrm{G}_1(F;I)]^2+[\mathrm{G}_2(F;I)]^2+|I|^2\}^{1/2}.$$

While studying the fundamental properties of these expressions, we shall often find the following two inequalities useful:

$$(5.1)\qquad [(\sum_{i=1}^n x_i)^2+(\sum_{i=1}^n y_i)^2+(\sum_{i=1}^n z_i)^2]^{1/2}\leqslant\sum_{i=1}^n (x_i^2+y_i^2+z_i^2)^{1/2}$$

for any three sequences $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$ of real numbers;

$$(5.2)\qquad \Big[\Big(\int_E x\,dt\Big)^2+\Big(\int_E y\,dt\Big)^2+\Big(\int_E z\,dt\Big)^2\Big]^{1/2}\leqslant\int_E (x^2+y^2+z^2)^{1/2}\,dt$$

for any measurable set E in a space $\boldsymbol{R}_m$ and any three non-negative functions $x(t)$, $y(t)$ and $z(t)$, measurable on E.

The inequality (5.1) is easily deduced by induction from the case $n=2$ which can be verified directly. The inequality (5.2), in the special case in which the functions $x(t)$, $y(t)$, $z(t)$ are simple, is an obvious consequence of (5.1); and we pass at once to the general case with the help of Theorems 7.4 and 12.6 of Chapter I.

(5.3) ***Theorem.*** *The expressions of de Geöcze* $G_1(I)=G_1(F; I)$, $G_2(I)=G_2(F; I)$ *and* $G(I)=G(F; I)$, *associated with a continuous function* $F(x, y)$, *are continuous functions of the interval* I *and their integrals over any interval exist (finite or infinite); these integrals over any interval* $I_0=[a_1, b_1; a_2, b_2]$ *fulfil the following relations:*

(5.4) $$\int_{I_0} G_1=\int_{a_1}^{b_1} W_1(F; x; a_2, b_2)\,dx \quad \textit{and} \quad \int_{I_0} G_2=\int_{a_2}^{b_2} W_2(F; y; a_1, b_1)\,dy$$

(5.5) $$G_1(I_0)\leqslant\int_{I_0} G_1,\quad G_2(I_0)\leqslant\int_{I_0} G_2 \quad \textit{and} \quad G(I_0)\leqslant\int_{I_0} G,$$

(5.6) $$\int_{I_0} G_i\leqslant\int_{I_0} G\leqslant\int_{I_0} G_1+\int_{I_0} G_2+|I_0| \quad \textit{where } i=1, 2.$$

Proof. Given an arbitrary $\varepsilon>0$, let $\eta<\varepsilon$ be a positive number such that, for any pair of the points (x, y_1) and (x, y_2) in I_0,

(5.7) $$|y_2-y_1|<\eta \quad \textit{implies} \quad |F(x, y_2)-F(x, y_1)|<\varepsilon.$$

Let M denote the upper bound of $F(x, y)$ on I_0 and consider in I_0 an interval $I=[\alpha_1, \beta_1; \alpha_2, \beta_2]$ such that $|I|<\eta^2$. We then have either $\beta_1-\alpha_1<\eta$ or $\beta_2-\alpha_2<\eta$. In the former case, we find $G_1(I)\leqslant(\beta_1-\alpha_1)\cdot 2M<2M\eta\leqslant 2M\varepsilon$, and in the latter we derive from (5.7), $G_1(I)\leqslant(\beta_1-\alpha_1)\cdot\varepsilon\leqslant(b_1-a_1)\cdot\varepsilon$, so that in both cases $G_1(I)\leqslant(2M+b_1-a_1)\cdot\varepsilon$. The function $G_1(I)$ is therefore continuous. The same is of course true of $G_2(I)$ and the continuity of these two functions at once implies that of $G(I)$.

This being so, we shall show that the functions G_1 and G_2 are integrable and, at the same time, we shall deduce the formulae (5.4).

Let $\{\mathfrak{I}_n\}$ be a sequence of subdivisions of I_0 such that $\lim_n \Delta(\mathfrak{I}_n)=0$ and $\lim_n G_1(\mathfrak{I}_n)=\int_{I_0} G_1$; and denote, for any positive integer n and any $\xi\,\epsilon\,[a_1, b_1]$, by $V_n(\xi)$ the sum of the absolute increments of the function $F(\xi, y)$ on the linear intervals cut off on the line $x=\xi$ by the rectangles of the subdivision $\mathfrak{I}_n$. We then have, on the one hand,

(5.8) $$G_1(\mathfrak{I}_n)=\int_{a_1}^{b_1} V_n(\xi)\,d\xi \quad \textit{for} \quad n=1, 2, \ldots,$$

and on the other hand, on account of continuity of the function F, $\lim_n V_n(\xi)=W_1(F; \xi; a_2, b_2)$ for any $\xi\,\epsilon\,[a_1, b_1]$. Therefore, in virtue of

Fatou's Lemma (Chap. I, Theorem 12.10) and by (5.8), we obtain $\underline{\int_{I_0}} G_1\geqslant\int_{a_1}^{b_1} W_1(F; \xi; a_2, b_2)\,d\xi$. But since $G_1(\mathfrak{I})\leqslant\int_{a_1}^{b_1} W_1(F; \xi; a_2, b_2)\,d\xi$ for every subdivision $\mathfrak{I}$ of I_0, we have also $\overline{\int_{I_0}} G_1\leqslant\int_{a_1}^{b_1} W_1(F; \xi; a_2, b_2)\,d\xi$. Therefore the function G_1 has a unique integral over I_0 and this integral fulfils the first of the relations (5.4). The existence of the integral $\int_{I_0} G_2$ and the validity of the second of these relations are deduced by symmetry.

Let us pass on now to the function G. We first remark that the integral $\int_{I_0} G$ clearly exists in the case in which one at least of the integrals $\int_{I_0} G_1$ and $\int_{I_0} G_2$ is infinite, and is then also infinite on account of the relations

(5.9) $$G_1(I)\leqslant G(I) \quad \textit{and} \quad G_2(I)\leqslant G(I) \quad \textit{for any interval } I.$$

In the remaining case, the two integrals in question being finite, the evident inequality $G(I)\leqslant G_1(I)+G_2(I)+|I|$ yields

(5.10) $$\overline{\int_{I_0}} G\leqslant\int_{I_0} G_1+\int_{I_0} G_2+|I_0|<+\infty;$$

and on the other hand, for every subdivision $\mathfrak{I}$ of any interval I the equally obvious relations

(5.11) $$G_1(I)\leqslant G_1(\mathfrak{I}) \quad \textit{and} \quad G_2(I)\leqslant G_2(\mathfrak{I})$$

lead, in view of the inequality (5.1), to

(5.12) $$G(I)\leqslant G(\mathfrak{I}).$$

Now, continuity of the function G being already established, the formulae (5.10) and (5.12) imply, by Theorem 3.9, that this function has over I_0 a unique integral.

To complete the proof we need only remark that the formulae (5.11) and (5.12) imply at once the formulae (5.5) and finally that formula (5.6) follows directly from (5.9) and (5.10).

As a corollary of Theorem 5.3, and more particularly as a consequence of the formulae (5.4) and (5.6), we have:

(5.13) ***Theorem.*** *In order that the function of an interval $G(I)=\mathrm{G}(F;I)$, associated with a continuous function $F(x,y)$, be integrable on an interval I_0 (i. e. in order that $\int_{I_0} G<+\infty$), it is necessary and sufficient that the function $F(x,y)$ be of bounded variation on I_0.*

§ 6. Integrals of the expressions of de Geöcze. Given a continuous function $F(x,y)$, we shall denote, for any interval I_0, by $\mathrm{H}_1(F;I_0)$, $\mathrm{H}_2(F;I_0)$ and $\mathrm{H}(F;I_0)$ respectively, the integrals of the functions of an interval $G_1(I)=\mathrm{G}_1(F;I)$, $G_2(I)=\mathrm{G}_2(F;I)$ and $G(I)=\mathrm{G}(F;I)$ over the interval I_0. All these integrals exist on account of Theorem 5.3 and their importance in the theory of area of surfaces is due to the fact that, as will be shown in the next §, the number $\mathrm{H}(F;I_0)$ coincides with the area of the surface $z=F(x,y)$ on I_0.

(6.1) ***Theorem.*** *For any function $F(x,y)$ which is continuous and of bounded variation, the expressions $H_1(I)=\mathrm{H}_1(F;I)$, $H_2(I)=\mathrm{H}_2(F;I)$ and $H(I)=\mathrm{H}(F;I)$ are additive, continuous, and non-negative functions of the interval I, and we have at almost all points (x,y) of the plane*

$$H_1'(x,y)=|F_y'(x,y)|,\qquad H_2'(x,y)=|F_x'(x,y)|,\tag{6.2}$$
$$H'(x,y)=\{[F_x'(x,y)]^2+[F_y'(x,y)]^2+1\}^{\frac{1}{2}}.$$

Proof. Additivity and continuity of the functions in question follow at once from Theorems 3.1 and 3.7 on account of Theorem 5.3. We have therefore only to establish the relations (6.2). Now, for any interval $I=[a_1,b_1;a_2,b_2]$ we have according to Theorem 5.3 and Theorem 7.4 of Chap. IV, the following relation (in which the transformation is effected in accordance with Fubini's Theorem 8.1, Chap. III, rendered applicable to the partial derivates of the function $F(x,y)$ by Theorem 4.1):

$$H_1(I)=\int_{a_1}^{b_1}\mathrm{W}_1(F;x;a_2,b_2)\,dx\geqslant\int_{a_1}^{b_1}\Big[\int_{a_2}^{b_2}|F_y'(x,y)|\,dy\Big]dx=\int\!\!\int_I|F_y'(x,y)|\,dx\,dy;$$

whence

$$H_1'(x,y)\geqslant|F_y'(x,y)|\quad\textit{for almost every point } (x,y).\tag{6.3}$$

Let us now denote by $\{J_n\}$ the sequence of the linear intervals with rational extremities. In view of Theorem 7.4, Chap. IV, we

have for $n=1,2,\ldots$ and for every linear interval J,

$$\int_J\mathrm{W}_1(F;x;J_n)\,dx=H_1(J\times J_n)\geqslant\int\!\!\int_{J\times J_n}H_1'(x,y)\,dx\,dy=\int_J\Big[\int_{J_n}H_1'(x,y)\,dy\Big]dx,$$

and consequently, for each positive integer n, the inequality $\mathrm{W}_1(F;x;J_n)\geqslant\int_{J_n}H_1'(x,y)\,dy$ holds at every point x, except at most those of a set E_n of linear measure zero. Therefore, writing $E=\sum_n E_n$, we obtain the inequality $\mathrm{W}_1(F;x;J)\geqslant\int_J H_1'(x,y)\,dy$, whenever J has rational extremities and x lies outside the set E of linear measure zero. If we now regard the two sides of this inequality, for a given value of x outside the set E, as functions of the linear interval J, we obtain by derivation with respect to this interval (on account of Theorem 7.9, Chap. IV) for almost all y, the inequality

$$|F_y'(x,y)|\geqslant H_1'(x,y).\tag{6.4}$$

Therefore, since the derivatives $H_1'(x,y)$ and $F_y'(x,y)$ are measurable (cf. Theorem 4.1), it follows from Fubini's theorem (in the form (8.6), Chap. III) that the set of the points (x,y) at which the relation (6.4) is not fulfilled, is of plane measure zero. By (6.3) we therefore have almost everywhere the first of the relations (6.2).

The proof of the second relation now follows by symmetry, and that of the third from the remark that if we write $G_1(I)=\mathrm{G}_1(F;I)$, $G_2(I)=\mathrm{G}_2(F;I)$ and $G(I)=\mathrm{G}(F;I)$, we have by Theorem 3.8,

$$H'(x,y)=G'(x,y)=\{[G_1'(x,y)]^2+[G_2'(x,y)]^2+1\}^{\frac{1}{2}}=$$
$$=\{[H_1'(x,y)]^2+[H_2'(x,y)]^2+1\}^{\frac{1}{2}}=\{[F_y'(x,y)]^2+[F_x'(x,y)]^2+1\}^{\frac{1}{2}}$$

at almost every point (x,y) of the plane. This completes the proof.

(6.5) ***Theorem.*** *In order that the function of an interval $H(I)=\mathrm{H}(F;I)$, corresponding to a continuous function $F(x,y)$ of bounded variation on an interval $I_0=[a_1,b_1;a_2,b_2]$, be absolutely continuous on this interval, it is necessary and sufficient that the function $F(x,y)$ itself be absolutely continuous; and when this is the case, we have*

$$H(I_0)=\int\!\!\int_{I_0}\{[F_x'(x,y)]^2+[F_y'(x,y)]^2+1\}^{1/2}\,dx\,dy.\tag{6.6}$$

Proof. By Theorem 5.3, absolute continuity of the function $H(I)$ is equivalent to absolute continuity of the functions $H_1(I)$ and $H_2(I)$ together.

Therefore if the function H is absolutely continuous on I_0, we have, by Theorem 6.1, for any interval $I_\xi=[a_1, \xi; a_2, b_2]$, where $a_1 \leqslant \xi \leqslant b_1$, the relation

$$\int_{a_1}^{\xi} \mathrm{W}_1(F; x; a_2, b_2)\, dx = H_1(I_\xi) = \iint_{I_\xi} |F'_y(x,y)|\, dx\, dy = \int_{a_1}^{\xi}\Big[\int_{a_2}^{b_2} |F'_y(x,y)|\, dy\Big]\, dx,$$

and, taking the derivative with respect to ξ, we obtain for almost every value of x,

$$\mathrm{W}_1(F; x; a_2, b_2) = \int_{a_2}^{b_2} |F'_y(x,y)|\, dy. \tag{6.7}$$

Now, for any given value of x (for which $F(x,\eta)$ is of bounded variation in η) the difference $\mathrm{W}_1(F; x; a_2, \eta) - \int_{a_2}^{\eta} |F'_y(x,y)|\, dy$ is a non-negative and non-decreasing function of the variable η (cf. Theorem 7.4, Chap. IV). It therefore follows from (6.7) that we have for almost every value of x, and for any $\eta \in [a_2, b_2]$,

$$\mathrm{W}_1(F; x; a_2, \eta) = \int_{a_2}^{\eta} |F'_y(x,y)|\, dy,$$

i. e. that the function $\mathrm{W}_1(F; x; a_2, \eta)$, and consequently also $F(x,\eta)$, is absolutely continuous with respect to η on $[a_2, b_2]$ for almost every value of x. By the symmetry of the variables, we conclude also that the function $F(\xi, y)$ is at the same time absolutely continuous with respect to ξ on $[a_1, b_1]$ for almost every value of y. The function F, which is by hypothesis of bounded variation on I_0, is therefore absolutely continuous in the Tonelli sense on this interval.

Conversely, if the function F is absolutely continuous on I_0, we have by Theorems 7.8 and 7.9, Chap. IV, for every subinterval $I=[\alpha_1, \beta_1; \alpha_2, \beta_2]$ of I_0, the relations:

$$H_1(I) = \int_{\alpha_1}^{\beta_1} \mathrm{W}_1(F; x; \alpha_2, \beta_2)\, dx = \iint_I |F'_y(x,y)|\, dx\, dy,$$

$$H_2(I) = \int_{\alpha_2}^{\beta_2} \mathrm{W}_2(F; y; \alpha_1, \beta_1)\, dy = \iint_I |F'_x(x,y)|\, dx\, dy,$$

so that the two functions of an interval H_1 and H_2, and therefore also H, are absolutely continuous.

Finally, since the function H is absolutely continuous, the formula (6.6) is a direct consequence of the third of the relations (6.2), the latter being valid almost everywhere on account of Theorem 6.1.

Up to the present we have regarded the expression $\mathrm{H}(F; I)$ as a function of an interval I. If we treat this expression as a functional depending on the function F, we obtain the following theorem, whose geometrical interpretation will appear in § 8 when the theorem appears to be a generalization of Theorem 2.1.

(6.8) ***Theorem.*** *Given any sequence of continuous functions $\{F_n\}$ which converges to a continuous function F, we have for every interval I*

$$\liminf_n \mathrm{H}(F_n; I) \geqslant \mathrm{H}(F; I). \tag{6.9}$$

Proof. Denoting by $\mathfrak{Z}_p$ the subdivision of I into p^2 equal intervals, similar to I, we have by Theorem 5.3 for any pair of integers p and n, $\mathrm{H}(F_n; I) \geqslant \mathrm{G}(F_n; \mathfrak{Z}_p)$, and by Fatou's Lemma (Chap. I, Theorem 12.10), for every integer p, $\liminf_n \mathrm{G}(F_n; \mathfrak{Z}_p) \geqslant \mathrm{G}(F; \mathfrak{Z}_p)$. We therefore have $\liminf_n \mathrm{H}(F_n; I) \geqslant \mathrm{G}(F; \mathfrak{Z}_p)$, and this leads to (6.9) when $p \to \infty$.

§ 7. Radò's Theorem. Before passing to the proof of the result of Radò, according to which the area of any surface $z=F(x,y)$ on an interval I is equal to $\mathrm{H}(F; I)$, we shall prove the following

(7.1) ***Theorem.*** *If a continuous function $F(x,y)$ has on an interval $I_0=[a_1, b_1; a_2, b_2]$ continuous partial derivatives, there exists a sequence of polyhedra $\{z=P_n(x,y)\}$ inscribed in the surface $z=F(x,y)$, such that the sequence converges uniformly to this surface and such that*

$$\lim_n \mathrm{S}_0(P_n; I_0) = \iint_{I_0} \{[F'_x(x,y)]^2 + [F'_y(x,y)]^2 + 1\}^{\frac{1}{2}}\, dx\, dy = \mathrm{H}(F; I). \tag{7.2}$$

Proof. Let $\mathfrak{Z}_n = \{I_{n,1}, I_{n,2}, \ldots, I_{n,n^2}\}$ denote the subdivision of I_0 into n^2 equal intervals similar to I_0, and $(x_{n,i}, y_{n,i})$, where $i=1, 2, \ldots, n^2$, the lower left-hand vertex of $I_{n,i}$. Let us divide any interval $I_{n,i}$ into two right-angled triangles $T'_{n,i}$ and $T''_{n,i}$ by a diagonal, in such a way that the vertex $(x_{n,i}, y_{n,i})$ is that of the right angle of $T'_{n,i}$. Consider for any n the polyhedron $z=P_n(x,y)$ inscribed in the surface $z=F(x,y)$ and corresponding to the net formed on I_0 by the $2n^2$ triangles $T'_{n,i}$ and $T''_{n,i}$ where $i=1, 2, \ldots, n^2$.

For brevity let $h_n=(b_1-a_1)/n$ and $k_n=(b_2-a_2)/n$; and let μ_n denote the upper bound of the differences $|F'_x(x'',y'')-F'_x(x',y')|$ and $|F'_y(x'',y'')-F'_y(x',y')|$ for all points (x',y') and (x'',y'') of I_0 such that $|x''-x'|\leqslant h_n$ and $|y''-y'|\leqslant k_n$.

Now if $s'_{n,i}$ and $s''_{n,i}$ denote respectively the elementary areas of the faces of the polyhedron $z=P_n(x,y)$ which correspond to the triangles $T'_{n,i}$ and $T''_{n,i}$, we notice at once that the areas of the projections of the former of these faces on the planes xz and yz are respectively equal to

$$\tfrac{1}{2}h_n\cdot|F(x_{n,i},y_{n,i}+k_n)-F(x_{n,i},y_{n,i})|=\tfrac{1}{2}h_nk_n\cdot|F'_y(x_{n,i},y'_{n,i})|$$

and

$$\tfrac{1}{2}k_n\cdot|F(x_{n,i}+h_n,y_{n,i})-F(x_{n,i},y_{n,i})|=\tfrac{1}{2}h_nk_n\cdot|F'_x(x'_{n,i},y_{n,i})|$$

where $x_{n,i}\leqslant x'_{n,i}\leqslant x_{n,i}+h_n$ and $y_{n,i}\leqslant y'_{n,i}\leqslant y_{n,i}+k_n$.

We therefore have $s'_{n,i}=\frac{1}{2}\{[F'_x(x'_{n,i},y_{n,i})]^2+[F'_y(x_{n,i},y'_{n,i})]^2+1\}^{1/2}\cdot|I_{n,i}|$, and so, by the inequality (5.1), p. 171,

$$\sum_{i=1}^{n^2}s'_{n,i}-\frac{1}{2}\sum_{i=1}^{n^2}\{[F'_x(x_{n,i},y_{n,i})]^2+[F'_y(x_{n,i},y_{n,i})]^2+1\}^{1/2}\cdot|I_{n,i}|\leqslant\frac{1}{2}\cdot\sqrt{2}\,\mu_n\cdot|I_0|.$$

Since the partial derivatives F'_x and F'_y are by hypothesis continuous, it follows by making $n\to\infty$ that

$$\lim_n\sum_{i=1}^{n^2}s'_{n,i}=\frac{1}{2}\int\int_{I_0}\{[F'_x(x,y)]^2+[F'_y(x,y)]^2+1\}^{1/2}dx\,dy,$$

and the same limit is clearly obtained for the sum of the $s''_{n,i}$. By addition, together with an appeal to Theorem 6.5, we now derive the formula (7.2) and this completes the proof.

In what follows we shall apply the method of mean value integrals. Given in the plane a summable function $F(x,y)$, the sequence of functions $F_n(x,y)=n^2\int_0^{1/n}\int_0^{1/n}F(x+u,y+v)\,du\,dv$ where $n=1,2,\ldots$, will be called *sequence of mean value integrals* of the function $F(x,y)$. It is clear that *if the function F is continuous,* (i) *the sequence of its mean value integrals $\{F_n(x,y)\}$ converges to $F(x,y)$ at every point (x,y) of the plane, and uniformly on any interval, and* (ii) *the partial derivatives $\partial F_n/\partial x$ and $\partial F_n/\partial y$ exist everywhere and are continuous.* In fact, at any point (x,y) a direct calculation gives

$$\partial F_n(x,y)/\partial x=n^2\int_0^{1/n}[F(x+1/n,y+v)-F(x,y+v)]\,dv$$

and

$$\partial F_n(x,y)/\partial y=n^2\int_0^{1/n}[F(x+u,y+1/n)-F(x+u,y)]\,du.$$

It was T. Radò [2] who first applied in the theory of area of surfaces the method of mean value integrals. The rôle of these mean values is due to the fact that in the case in which the given function F is continuous, the sequence of areas of the surfaces $z=F_n(x,y)$ on any interval tends to the area of the surface $z=F(x,y)$ (*vide*, below, Theorem 7.3).

In the definition given above, the functions F_n are defined at each point (x,y) as "mean values" of the function F over squares of which (x,y) is a vertex; it goes without saying that we could also make use of mean values taken over squares, or circles, having (x,y) as their centres. These mean values over circles are used for instance in potential theory (cf. F. Riesz [4] and G. C. Evans [1]).

(7.3) ***Radò's Theorem.*** *If $F(x,y)$ is a continuous function and $\{F_n(x,y)\}$ is the sequence of mean value integrals of $F(x,y)$, then*

(7.4) $$\mathrm{H}(F;I_0)=\mathrm{S}(F;I_0)=\lim_n \mathrm{S}(F_n;I_0)$$

for every interval I_0.

Proof. Let $\{z=P_n(x,y)\}$ be a sequence of polyhedra converging uniformly to the surface $z=F(x,y)$, such that

(7.5) $$\lim_n \mathrm{S}_0(P_n;I_0)=\mathrm{S}(F;I_0).$$

Since the functions $P_n(x,y)$ are absolutely continuous, it follows from Theorem 6.5 (cf. also § 2, p. 165) that $\mathrm{S}_0(P_n;I_0)=\mathrm{H}(P_n;I_0)$ for every n. Consequently, since the sequences of functions $\{F_n\}$ and $\{P_n\}$ converge uniformly to the function F, it follows by using successively Theorem 2.1, the formula (7.5) and Theorem 6.8, that

(7.6) $$\liminf_n \mathrm{S}(F_n;I_0)\geqslant\mathrm{S}(F;I_0)=\lim_n\mathrm{H}(P_n;I_0)\geqslant\mathrm{H}(F;I_0).$$

Now if the function F is not of bounded variation on I_0, it follows from Theorem 5.13 that $\mathrm{H}(F;I_0)=+\infty$ and consequently the formula (7.4) follows at once from (7.6). We may therefore assume that the function F is of bounded variation on I_0, and further (cf. § 4, p. 170) that F is continuous and of bounded variation on each interval of the plane.

Let us agree to denote, for any set E in the plane, by $E^{(u,v)}$ the parallel translation of E by the vector (u, v) (cf. Chap. III, § 11); similarly, for a family of sets $\mathfrak{E}$ in the plane, $\mathfrak{E}^{(u,v)}$ will denote the family of all the sets obtained from sets $(\mathfrak{E})$ by subjecting them to this translation. For any subinterval $I=[a_1, b_1; a_2, b_2]$ we then obtain

$$\mathrm{G}_1(F_n; I)=\int_{a_1}^{b_1}|F_n(x, b_2)-F_n(x, a_2)|\,dx \leqslant$$

$$\leqslant n^2\int_{a_1}^{b_1}\Big[\int_0^{1/n}\int_0^{1/n}|F(x+u, b_2+v)-F(x+u, a_2+v)|\,du\,dv\Big]\,dx=$$

$$=n^2\int_0^{1/n}\int_0^{1/n}\mathrm{G}_1(F; I^{(u,v)})\,du\,dv,$$

and a similar formula for G_2. Hence by the inequality (5.2), p. 171,

$$\mathrm{G}(F_n; I)=\{[\mathrm{G}_1(F_n; I)]^2+[\mathrm{G}_2(F_n; I)]^2+|I|^2\}^{1/2}\leqslant$$

$$\leqslant n^2\int_0^{1/n}\int_0^{1/n}\{[\mathrm{G}_1(F; I^{(u,v)})]^2+[\mathrm{G}_2(F; I^{(u,v)})]^2+|I^{(u,v)}|^2\}^{1/2}\,du\,dv=$$

$$=n^2\int_0^{1/n}\int_0^{1/n}\mathrm{G}(F; I^{(u,v)})\,du\,dv.$$

Denoting by $\mathfrak{J}_p$ the subdivision of I_0 into p^2 equal intervals similar to I_0, we obtain therefore, for every p,

$$\mathrm{G}(F_n; \mathfrak{J}_p)\leqslant n^2\int_0^{1/n}\int_0^{1/n}\mathrm{G}(F; \mathfrak{J}_p^{(u,v)})\,du\,dv;$$

and since by Lemma 3.4, $\mathrm{G}(F; \mathfrak{J}_p^{(u,v)})$ tends to $\mathrm{H}(F; I_0^{(u,v)})$ as $p\to\infty$, uniformly in u and v, we obtain in the limit

$$\mathrm{H}(F_n; I_0)\leqslant n^2\int_0^{1/n}\int_0^{1/n}\mathrm{H}(F; I_0^{(u,v)})\,du\,dv. \tag{7.7}$$

Finally the areas of the figures $I_0^{(u,v)}\ominus I_0$ and $I_0\ominus I_0^{(u,v)}$ tend to 0 with u and v and each of these figures is a sum of two intervals. Hence since the expression $\mathrm{H}(F; I)$ is by Theorem 6.1 a continuous function of the interval I, we have $\lim_{u\to 0,\, v\to 0}\mathrm{H}(F; I_0^{(u,v)})=\mathrm{H}(F; I_0)$. On the other hand, since the functions $F_n(x, y)$ have continuous partial derivatives, we have, by Theorem 7.1, $\mathrm{S}(F_n; I_0)\leqslant\mathrm{H}(F_n; I_0)$ for each n. Therefore making $n\to\infty$ in (7.7), we find $\limsup_n \mathrm{S}(F_n; I_0)\leqslant\mathrm{H}(F; I_0)$, which in conjunction with (7.6) gives the required relation (7.4).

§ 8. Tonelli's Theorem. The theorem of Radò just established, enables us to replace in all the theorems of this chapter the expression $\mathrm{H}(F; I)$ by the surface area $\mathrm{S}(F; I)$.

Thus for instance, Theorem 6.5 (formula (6.6)) expresses the fact that *the elementary area of a polyhedron coincides with its area according to the general definition of area of a surface.*

Theorem 6.8 contains a generalization of Theorem 2.1; it enables us to replace in its statement uniform convergence by ordinary convergence: we thus obtain a theorem similar to Lemma of Fatou (Chap. I, Theorem 12.10). It follows that the uniform convergence of the inscribed polygons, required in the definition of area, may be replaced by the ordinary convergence, so that the area of a continuous surface $z=F(x, y)$ is the lower limit of the areas of polyhedra tending to this surface. Further, by Theorem 7.1, *if a function $F(x, y)$ has continuous partial derivatives, there exists a sequence of polyhedra inscribed in the surface $z=F(x, y)$, tending uniformly to the latter and having areas which converge to the area of this surface.* (For further generalizations *vide* S. Kempisty [1]. Cf. also on this subject H. Rademacher [3], W. H. Young [5], M. Fréchet [2] and T. Radó [5].) Finally, we obtain the following theorem, which sums up the most essential considerations of this chapter:

(8.1) ***Tonelli's Theorem.*** a) *In order that a continuous surface $z=F(x,y)$ have a finite area on an interval I_0, it is necessary and sufficient that the function $F(x, y)$ be of bounded variation on I_0.*

b) *When this is the case, we have*

$$\mathrm{S}(F; I_0)\geqslant\int\int_{I_0}\Big[\Big(\frac{\partial F}{\partial x}\Big)^2+\Big(\frac{\partial F}{\partial y}\Big)^2+1\Big]^{1/2}dx\,dy;$$

the expression $S(I)=\mathrm{S}(F; I)$ is then an additive continuous function of the interval $I\subset I_0$ and we have for almost every point $(x, y)\in I_0$

$$S'(x, y)=\Big[\Big(\frac{\partial F}{\partial x}\Big)^2+\Big(\frac{\partial F}{\partial y}\Big)^2+1\Big]^{1/2}.$$

c) *In order that we should have*

$$\mathrm{S}(F; I_0)=\int\int_{I_0}\Big[\Big(\frac{\partial F}{\partial x}\Big)^2+\Big(\frac{\partial F}{\partial y}\Big)^2+1\Big]^{1/2}dx\,dy, \tag{8.2}$$

it is necessary and sufficient that the function $F(x, y)$ be absolutely continuous on I_0; and in order that this be the case it is necessary and sufficient that the area $\mathrm{S}(F; I)$ be an absolutely continuous function of the interval $I\subset I_0$.

Proof. The assertion a) follows directly from Theorem 5.13; b) and c) follows from Theorems 6.1 and 6.5 on account of Theorem 7.4, Chap. IV.

With regard to Theorem 8.1 *vide* L. Tonelli [5; 6; 7]. The necessity of condition a) was established a little earlier by G. Lampariello [1].

According to Tonelli's theorem, the relation of equality (8.2) can hold for a continuous surface $z=F(x,y)$ only in the case in which the function F is absolutely continuous. Nevertheless, as proved by L. C. Young, this relation will remain valid for arbitrary continuous surfaces, as soon as we replace on the right-hand side the partial derivatives by ratios of finite differences and transpose the passage to the limit outside the integral sign. In fact:

(8.3) ***Theorem.*** *For any continuous surface $z=F(x,y)$ and any interval I_0 we have*

$$(8.4)\qquad S(F;I_0)=\lim_{\alpha,\beta\to 0}\iint_{I_0}\left\{\left[\frac{F(x+\alpha,y)-F(x,y)}{\alpha}\right]^2+\left[\frac{F(x,y+\beta)-F(x,y)}{\beta}\right]^2+1\right\}^{1/2}dx\,dy;$$

and in order that the function F be of bounded variation on I_0, it is necessary and sufficient that

$$(8.5)\qquad \limsup_{\alpha,\beta\to 0}\frac{1}{|\alpha|+|\beta|}\iint_{I_0}|F(x+\alpha,y+\beta)-F(x,y)|\,dx\,dy<+\infty.$$

Proof. Let $\{F_n\}$ be the sequence of mean value integrals (cf. § 7, p. 178) of the function F. Denote, for brevity, by $R(x,y;\alpha,\beta)$ the expression under the integral sign on the right-hand side of (8.4), and by $R_n(x,y;\alpha,\beta)$, for each positive integer n, the expression obtained from $R(x,y;\alpha,\beta)$ by replacing F by F_n. Finally let us write for $n=1,2,\ldots$

$$R_n(x,y)=\lim_{\alpha,\beta\to 0}R_n(x,y;\alpha,\beta)=\{[\partial F_n(x,y)/\partial x]^2+[\partial F_n(x,y)/\partial y]^2+1\}^{1/2}.$$

In order to establish the identity (8.4), it evidently suffices to show that

$$(8.6)\qquad S(F;I_0)\leqslant\liminf_{\alpha,\beta\to 0}\iint_{I_0}R(x,y;\alpha,\beta)\,dx\,dy$$

and

$$(8.7)\qquad S(F;I_0)\geqslant\limsup_{\alpha,\beta\to 0}\iint_{I_0}R(x,y;\alpha,\beta)\,dx\,dy.$$

For this purpose, let I be any interval in the interior of I_0. By means of the inequality (5.2), p. 171, we easily find that

$$(8.8)\qquad R_n(x,y;\alpha,\beta)\leqslant n^2\int_0^{1/n}\int_0^{1/n}R(x+u,y+v;\alpha,\beta)\,du\,dv.$$

Now let n be a positive integer, sufficiently large in order that $(x,y)\in I$, $|u|<1/n$ and $|v|<1/n$ should imply $(x+u,y+v)\in I_0$. We then have $\iint_I R(x+u,y+v;\alpha,\beta)\,dx\,dy\leqslant\iint_{I_0}R(x,y;\alpha,\beta)\,dx\,dy$ and consequently, by (8.8), $\iint_I R_n(x,y;\alpha,\beta)\,dx\,dy\leqslant\iint_{I_0}R(x,y;\alpha,\beta)\,dx\,dy$. Making $\alpha\to 0$ and $\beta\to 0$, we obtain in the limit $S(F_n;I)\leqslant\liminf_{\alpha,\beta\to 0}\iint_{I_0}R(x,y;\alpha,\beta)\,dx\,dy$. This relation being thus established for each interval $I\subset I_0^\circ$, we may replace, on its left-hand side, I by I_0, and making still $n\to\infty$ we obtain the relation (8.6).

In order to prove the relation (8.7), let us first observe that the latter is obvious in the case in which $S(F;I_0)=+\infty$. We may therefore assume that the function $F(x,y)$ is of bounded variation on I_0 and moreover (cf. § 4, p. 170) of bounded variation on every interval in the plane and periodic with respect to each variable. We can therefore determine an interval $J_0=[a_1,b_1;a_2,b_2]$ containing I_0 in its interior, such that its sides b_1-a_1 and b_2-a_2 are the periods of $F(x,y)$ with respect to x and y respectively.

This being so, we find easily, on account of the inequality (5.1), p. 171, that, for any pair of positive integers n and k,

$$R_n(x,y;\alpha,\beta)\leqslant\frac{1}{k}\sum_{j=0}^{k-1}R_n(x+j\alpha/k,\,y+j\beta/k;\,\alpha/k,\beta/k).$$

By integrating the two sides of this inequality over J_0, and taking account of the periodicity of the function F, we obtain

$$\iint_{J_0}R_n(x,y;\alpha,\beta)\,dx\,dy\leqslant\iint_{J_0}R_n(x,y;\alpha/k,\beta/k)\,dx\,dy;$$

and hence, passing to the limit, making first $k\to\infty$, and then $n\to\infty$, we find by Radò's Theorem 7.3,

$$\iint_{J_0} R(x, y;\, \alpha, \beta)\, dx\, dy \leqslant \lim_n \iint_{J_0} R_n(x, y)\, dx\, dy = \lim_n \mathrm{S}(F_n; J_0) = \mathrm{S}(F; J_0),$$

and so

$$\limsup_{\alpha,\beta\to 0} \iint_{J_0} R(x, y;\, \alpha, \beta)\, dx\, dy \leqslant \mathrm{S}(F; J_0). \tag{8.9}$$

Now, by the result already established in the inequality (8.6), we have $\liminf\limits_{\alpha,\beta\to 0} \iint_I R(x, y;\, \alpha, \beta)\, dx\, dy \geqslant \mathrm{S}(F; I)$ for every interval I. It follows at once that (8.9) remains valid when we replace the interval J_0 by any subinterval of J_0, and in particular by the interval I_0. We thus obtain the relation (8.7).

Finally let us remark that on account of the relation (8.4), in order that the area of the surface $z=F(x, y)$ on I_0 be finite, it is necessary and sufficient that

$$\limsup_{\alpha\to 0} \frac{1}{|\alpha|} \iint_{I_0} |F(x+\alpha, y) - F(x, y)|\, dx\, dy < +\infty$$

and

$$\limsup_{\beta\to 0} \frac{1}{|\beta|} \iint_{I_0} |F(x, y+\beta) - F(x, y)|\, dx\, dy < +\infty.$$

Now this pair of relations is easily seen to be equivalent to the relation (8.5) which therefore expresses a condition necessary and sufficient in order that the function F should be of bounded variation on I_0. This completes the proof.

A statement analogous to Theorem 8.3 can be made for curves (cf. Chap. IV, § 8). *If C is a continuous curve defined by the equations $x=X(t)$, $y=Y(t)$, its length on any interval $I_0=[a, b]$ is given by the formula*

$$\mathrm{S}(C; I_0) = \lim_{h\to 0} \int_a^b \left\{ \left[\frac{X(t+h) - X(t)}{h}\right]^2 + \left[\frac{Y(t+h) - Y(t)}{h}\right]^2 \right\}^{1/2} dt. \tag{8.10}$$

In particular therefore, *in order that a continuous functions $G(t)$ be of bounded variation on an interval $[a, b]$, it is necessary and sufficient that*

$$\limsup_{h\to 0} \frac{1}{|h|} \int_a^b |G(t+h) - G(t)|\, dt < +\infty. \tag{8.11}$$

This assertion can be proved by the method of mean value integrals in a manner quite similar to that we made use of in the theory of areas of surfaces $z=F(x, y)$, but for curves this method can be very much simplified. Let us observe further that the relation (8.11) may be interpreted in a more general sense. In fact, given any summable function $G(t)$, the relation (8.11) is the necessary and sufficient condition in order that the function G be almost everywhere on $[a, b]$ equal to a function of bounded variation (*vide* G. H. Hardy and J. E. Littlewood [1]; cf. also A. Zygmund [I, p. 106]).

Finally the relation (8.10) holds for any rectifiable curve given by the equations $x=X(t)$, $y=Y(t)$, where the functions $X(t)$ and $Y(t)$ are not necessarily continuous, provided however that for each t the point $(X(t), Y(t))$ lies on the segment joining the points $(X(t-), Y(t-))$ and $(X(t+), Y(t+))$.