CHAPTER III.

Functions of bounded variation and the Lebesgue-Stieltjes integral.

§ 1. Euclidean spaces. In this chapter, the notions of measure that we consider undergo a further specialization. Accordingly we introduce for Euclidean spaces, a particular class of outer measures of Carathéodory, determined in a natural way by non-negative additive functions of an interval. These outer measures in their turn determine the corresponding classes of measurable sets and measurable functions, and lead to processes of integration usually known as those of Lebesgue-Stieltjes.

By *Euclidean space of m dimensions* $\boldsymbol{R}_m$, we mean the set of all systems of m real numbers $(x_1, x_2, ..., x_m)$. The number x_k is termed k-th *coordinate* of the point $(x_1, x_2, ..., x_m)$. The point $(0, 0, ..., 0)$ will be denoted by 0.

By *distance* $\varrho(x, y)$ of two points $x=(x_1, x_2, ..., x_m)$ and $y=(y_1, y_2, ..., y_m)$ in the space $\boldsymbol{R}_m$, we mean the non-negative number $[(y_1-x_1)^2+(y_2-x_2)^2+...+(y_m-x_m)^2]^{1/2}$. Distance, thus defined, evidently fulfills the three conditions of Chap. II, p. 40, and hence Euclidean spaces may be regarded as metrical spaces. All the definitions adopted in Chap. II therefore apply in particular to spaces $\boldsymbol{R}_m$. In § 2 we supplement them by some definitions more exclusively restricted to Euclidean spaces.

The space $\boldsymbol{R}_1$ is also termed *straight line* and the space $\boldsymbol{R}_2$, *plane*. Accordingly, the sets in $\boldsymbol{R}_1$ will often be called *linear*, and those in $\boldsymbol{R}_2$ *plane sets*.

§ 2. Intervals and figures. Suppose given a Euclidean space $\boldsymbol{R}_m$.

The set of the points $(x_1, x_2, ..., x_m)$ of $\boldsymbol{R}_m$ that fulfill a linear equation $a_1x_1+a_2x_2+...+a_mx_m=b$, where $b, a_1, a_2, ..., a_m$ are real numbers and, of these, the coefficients $a_1, a_2, ..., a_m$ do not all vanish together, is called *hyperplane* $a_1x_1+a_2x_2+...+a_mx_m=b$. For each fixed $k=1, 2, ..., m$, the hyperplanes $x_k=b$ are said to be *orthogonal to the axis of* x_k. The term *hyperplane* by itself, will be applied exclusively to a hyperplane orthogonal to one of the axes. In $\boldsymbol{R}_1$, hyperplanes coincide with points. In $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ they are respectively *straight lines* and *planes*.

Given two points $a=(a_1, a_2, ..., a_m)$ and $b=(b_1, b_2, ..., b_m)$ such that $a_k \leqslant b_k$ for $k=1, 2, ..., m$, we term *closed interval* $[a_1, b_1; a_2, b_2; ...; a_m, b_m]$ the set of all the points $(x_1, x_2, ..., x_m)$ such that $a_k \leqslant x_k \leqslant b_k$ for $k=1, 2, ..., m$. The points a and b are called *principal vertices* of this interval. If, in the definition of closed interval, we replace successively the inequality $a_k \leqslant x_k \leqslant b_k$ by the inequalities (1⁰) $a_k<x_k<b_k$, (2⁰) $a_k \leqslant x_k<b_k$ and (3⁰) $a_k<x_k \leqslant b_k$, we obtain the definitions (1⁰) of *open interval* $(a_1, b_1; a_2, b_2; ...; a_m, b_m)$, (2⁰) of *interval half open to the right* $[a_1, b_1; a_2, b_2; ...; a_m, b_m)$ and (3⁰) of *interval half open to the left* $(a_1, b_1; a_2, b_2; ...; a_m, b_m]$. If $a_k=b_k$ for at least one index k, all these intervals are said to be *degenerate*. In what follows, an *interval*, by itself, always means either a closed non-degenerate interval or an empty set, unless another meaning is obvious from the context.

We call *face* of the closed interval $I=[a_1, b_1; a_2, b_2; ...; a_m, b_m]$ the common part of I and any one of the $2m$ hyperplanes $x_k=a_k$ and $x_k=b_k$, where $k=1, 2, ..., m$. If J is an open or half open interval we call *faces* of J those of its closure $\bar{J}$. We see at once that the faces of any non-empty interval I are degenerate intervals and that their sum is the boundary of the interval I.

If $b_1-a_1=b_2-a_2=...=b_m-a_m>0$, the interval $[a_1, b_1; a_2, b_2; ...; a_m, b_m]$ is termed *cube* (*square* in $\boldsymbol{R}_2$). We define similarly *open cubes* and *half open cubes* (half open to the right or to the left).

We call *net of closed intervals* in $\boldsymbol{R}_m$ any system of closed non-overlapping intervals that together cover the space $\boldsymbol{R}_m$. Similarly, by *net of half open intervals*, we mean a system of intervals half open on the same side, no two of which have common points, and whose sum covers $\boldsymbol{R}_m$. A sequence of nets $\{\mathfrak{N}_k\}$ (of closed or of half open intervals) is *regular*, if each interval of $\mathfrak{N}_{k+1}$ is contained in an interval of $\mathfrak{N}_k$ and if the characteristic numbers $\Delta(\mathfrak{N}_k)$ (cf. Chap. II, p. 40) tend to 0 as $k\to\infty$. Given a net of half open intervals, we clearly change it into a net of closed intervals by replacing the half open intervals by their closures. The same operation changes any regular sequence of nets of half open intervals into a regular sequence of nets of closed intervals.

(2.1) **Theorem.** *Given an enumerable system of hyperplanes $x_k=a_j$, where $k=1, 2, ..., m$, and $j=1, 2, ...$, we can always construct a regular sequence $\{\mathfrak{N}_k\}$ of nets of cubes (closed or half open) none of which has a face on the given hyperplanes.*

To see this, let b denote a positive number not of the form qa_j/p where p and q are integers and $j=1, 2, ...$ Such a number b certainly exists, since the set of the numbers of the form qa_j/p is at most enumerable. This being so, for each positive integer k let us denote by $\mathfrak{N}_k$ the net consisting of all the cubes half open to the left $(p_1b/2^k, (p_1+1)b/2^k; p_2b/2^k, (p_2+1)b/2^k; ...; p_mb/2^k, (p_m+1)b/2^k]$, where $p_1, p_2, ..., p_m$ are arbitrary integers. The sequence of nets $\{\mathfrak{N}_k\}$ evidently fulfills the required conditions.

Let us observe that, *given a regular sequence $\{\mathfrak{N}_k\}$ of nets of half open [closed] intervals, every open set G is expressible as the sum of an enumerable system of intervals $(\mathfrak{N}_k)$ without common points [non-overlapping]*. To see this, let $\mathfrak{M}_1$ be the set of intervals of $\mathfrak{N}_1$ that lie in G, and let $\mathfrak{M}_{k+1}$, for each $k\geqslant 1$, be the set of intervals $(\mathfrak{N}_{k+1})$ that lie in G but not in any of the intervals $(\mathfrak{M}_k)$. Since $\Delta(\mathfrak{N}_k)\to 0$ as $k\to\infty$, the enumerable system of intervals $\sum_k \mathfrak{M}_k$ covers the set G, and the other conditions required are evidently satisfied also.

On account of Theorem 2.1, we derive at once the following proposition which will often be useful to us in the course of this Chapter:

(2.2) **Theorem.** *Given a sequence of hyperplanes $\{H_i\}$, every open set G is expressible as the sum of a sequence of half open cubes without common points [or of closed non-overlapping cubes] whose faces do not lie on any of the hyperplanes H_i.*

A set expressible as the sum of a finite number of intervals will be termed *elementary figure*, or simply, *figure*. Every sum of a finite number of figures is itself a figure, but this is not in general the case for the common part, or for the difference, of two figures.

We shall therefore define two operations similar to those of multiplication and subtraction of sets, but which differ from the latter in that, when we perform them on figures, the result is again a figure. These operations will be denoted by $\odot$ and $\ominus$ and are defined by the relations

$$A\odot B=\overline{(A\cdot B)^\circ} \qquad \text{and} \qquad A\ominus B=\overline{(A-B)^\circ}.$$

The relation $A\odot B=0$ means that the figures A and B do not overlap (cf. Chap. II, p. 40).

Given an interval $I=[a_1, b_1; a_2, b_2; ...; a_m, b_m]$, the number $(b_1-a_1)\cdot(b_2-a_2)\cdot...\cdot(b_m-a_m)$ will be called *volume* of I (*length* for $m=1$, *area* for $m=2$), and denoted by $\mathrm{L}(I)$ or by $|I|$. If $I=0$, by $\mathrm{L}(I)=|I|$ we mean 0 also. When several spaces $\boldsymbol{R}_m$ are considered simultaneously, we shall, to prevent any ambiguity, denote the volume of an interval I in $\boldsymbol{R}_m$ by $\mathrm{L}_m(I)$. We see at once that every figure R can be subdivided into a finite number of non-overlapping intervals. The sum of the volumes of these intervals is independent of the way in which we make this subdivision; it is termed *volume* (*length*, *area*) of R and denoted, just as in the case of intervals, by $\mathrm{L}(R)$ or by $|R|$.

§ 3. Functions of an interval.

We shall say that $F(I)$ is a *function of an interval* on a figure R [or in an open set G], if $F(I)$ is a finite real number uniquely defined for each interval I contained in R [or in G]. To simplify the wording, we shall usually suppose that functions of an interval are defined in the whole space.

A function of an interval $F(I)$ will be said to be *continuous on a figure* R, if to each $\varepsilon>0$ there corresponds an $\eta>0$ such that $|I|<\eta$ implies $|F(I)|<\varepsilon$ for every interval $I\subset R$. A function of an interval will be said to be *continuous in an open set* G, if it is continuous on every figure $R\subset G$. Finally, functions continuous in the whole space will simply be said to be *continuous*.

The reader will have noticed that we use the terms "on" (or "over") and "in" in slightly different senses. We may express the distinction as follows. Suppose that a certain property (P) of functions of a point, of an interval, or of a set has been defined on figures. We then say that a function has this property *in* an open set G, if it has the property on every figure $R\subset G$. Further, if a function has the property (P) in the whole space, we say simply that *it has the property* (P). Thus, for instance, a function of a set $\Phi(X)$ is *additive* ($\mathfrak{B}$) (cf. Chap. I, § 5 and Chap. II, § 2) in an *open set* G, if it is additive ($\mathfrak{B}$) *on every figure* $R\subset G$; if μ is a measure ($\mathfrak{B}$) (cf. Chap. I, § 9), a function of a point, defined in the whole space, is said to be *integrable* ($\mathfrak{B}$, μ), if it is integrable ($\mathfrak{B}$, μ) on every figure, and so on.

We shall call *oscillation* $\mathrm{O}(F; E)$ of a function of an interval F on a set E, the upper bound of the values $|F(I)|$ for intervals $I\subset E$. If D is an arbitrary set and R a figure, we shall denote by $\mathrm{o}_R(F; D)$ the lower bound of the numbers $\mathrm{O}(F; R\cdot G)$, where G is any open set containing D; the number $\mathrm{o}_R(F; D)$ will be termed *oscillation of F on R at the set D*. Finally we shall say simply *oscillation of F at D*, and use the notation $\mathrm{o}(F; D)$, for the upper bound of the numbers $\mathrm{o}_R(F; D)$ where R denotes any figure (or, what amounts to the same, interval or cube).

In the sequel, D will usually be a hyperplane (a point in $\boldsymbol{R}_1$, a straight line in $\boldsymbol{R}_2$) or else the boundary of a figure. In these cases we shall say that the function F is *continuous*, or *discontinuous*, *at D on R*, according as $\mathrm{o}_R(F; D)=0$ or $\mathrm{o}_R(F; D)>0$. Similarly, we shall say that F is *continuous*, or *discontinuous*, *at D*, according as $\mathrm{o}(F; D)=0$ or $\mathrm{o}(F; D)>0$.

(3.1) ***Theorem.*** *In order that a function F of an interval be continuous on a figure R [or in the whole space], it is necessary and sufficient that $\mathrm{o}_R(F; D)=0$ [or that $\mathrm{o}(F; D)=0$] for every hyperplane D.*

Proof. Since the condition is clearly necessary, let us suppose that the function F is not continuous on R. There is then a number $\varepsilon_0>0$ and a sequence of intervals $\{I^{(n)}=[a_1^{(n)}, b_1^{(n)}; \ldots; a_m^{(n)}, b_m^{(n)}]\}$ contained in R and such that for $n=1, 2, \ldots$, $|F(I^{(n)})|>\varepsilon_0$, and that $|I^{(n)}|\rightarrow 0$. By the second of these conditions, we can extract from the sequence $\{I^{(n)}\}$, a subsequence $\{I^{(n_k)}\}$, in such a manner that $\lim_k (b_i^{(n_k)}-a_i^{(n_k)})=0$ for a positive integer $i=i_0\leqslant m$. The sequences $\{a_{i_0}^{(n_k)}\}_{k=1,2,\ldots}$ and $\{b_{i_0}^{(n_k)}\}_{k=1,2,\ldots}$ then have a common limit point a, and, denoting by D the hyperplane $x_{i_0}=a$, we see that every open set $G\supset D$ contains an infinity of intervals $I^{(n)}$, so that $\mathrm{o}_R(F; D)\geqslant\varepsilon_0>0$.

§ 4. Functions of an interval that are additive and of bounded variation. A function of an interval $F(I)$ is said to be *additive* on a figure R_0 [or in an open set G], if $F(I_1+I_2)=$ $=F(I_1)+F(I_2)$ whenever I_1, I_2 and I_1+I_2 are intervals contained in R_0 [or in G] and I_1, I_2 are non-overlapping. A function additive in the whole space, is for instance the volume $\mathrm{L}(I)=|I|$. Just as in the case of the function $\mathrm{L}(I)$ (cf. § 2, p. 59), every additive function of an interval $F(I)$ on a figure R_0 [or in an open set G] can be continued on all figures in R_0 [or in G] in such a manner that $F(R_1+R_2)=F(R_1)+F(R_2)$ for every pair of figures $R_1\subset R_0$ and $R_2\subset R_0$ that do not overlap. In the sequel, we shall always suppose every additive function of an interval continued in this way on the figures.

If F is an additive function of an interval on a figure R_0, we shall term respectively *upper* and *lower (relative) variations* of F on R_0 the upper and lower bounds of $F(R)$ for figures $R\subset R_0$. We denote these variations by $\overline{\mathrm{W}}(F; R_0)$ and $\underline{\mathrm{W}}(F; R_0)$ respectively. Since every additive function vanishes on the empty set, we have $\overline{\mathrm{W}}(F; R_0)\geqslant 0\geqslant\underline{\mathrm{W}}(F; R_0)$. The number $\overline{\mathrm{W}}(F; R_0)+|\underline{\mathrm{W}}(F; R_0)|$, clearly non-negative, will be called *absolute variation* of F on R_0 and denoted by $\mathrm{W}(F; R_0)$. If $\mathrm{W}(F; R_0)<+\infty$, the function F is said to be *of bounded variation on* R_0. In accordance with the convention of § 3, p. 59, an additive function of an interval in the whole space is of bounded variation, if it is of bounded variation on every figure.

It is obvious that a function of bounded variation on a figure R_0 is equally so on every figure contained in R_0, and also that the sum, the difference, and, more generally, any linear combination of two additive functions of an interval that are of bounded variation on a figure, is itself of bounded variation on the same figure.

An additive function whose values are of constant sign is termed *monotone*. A non-negative monotone additive function is also termed *non-decreasing* (for the same reason as in the case of non-negative additive functions of a set, cf. Chap. I, p. 8). Similarly, non-positive additive functions are also termed *non-increasing*. Every monotone additive function F of an interval on a figure R_0 is clearly of bounded variation on R_0.

If a function is of bounded variation on a figure R_0, its relative variations on R_0 are evidently finite. Conversely, if for an additive function F of an interval on a figure R_0, one or other of the two relative variations is finite, then both are finite, and therefore the absolute variation is finite. For, if $\overline{W}(F; R_0) < +\infty$ say, then as $\underline{W}(F; R_0)$ is the lower bound of the numbers $F(R) = F(R_0) - F(R_0 \ominus R)$, where R is any figure contained in R_0, we find $\underline{W}(F; R_0) \geqslant F(R_0) - \overline{W}(F; R_0) > -\infty$. Moreover, this last inequality may also be written in the form $F(R_0) \leqslant \overline{W}(F; R_0) + \underline{W}(F; R_0)$. Here we replace F by $-F$ to derive the opposite inequality and then finally the equality $F(R_0) = \overline{W}(F; R_0) + \underline{W}(F; R_0)$. Hence *any additive function of bounded variation is the sum of its two relative variations on any figure for which it is defined.* This decomposition is termed the *Jordan decomposition* of an additive function of bounded variation, and is similar to the Jordan decomposition of an additive function of a set (Chap. I, § 6).

If F is an additive function of bounded variation of an interval on a figure R_0, the three monotone functions defined for every figure $R \subset R_0$ by the relations

$$W(R) = \mathrm{W}(F; R), \qquad W_1(R) = \overline{\mathrm{W}}(F; R) \qquad \textit{and} \qquad W_2(R) = \underline{\mathrm{W}}(F; R),$$

are likewise additive on R_0 and are termed, respectively, *absolute, upper,* and *lower variations* of F. The first two are non-negative and the third non-positive. It therefore follows from the Jordan decomposition that every additive function of bounded variation on a figure is, on this figure, the difference of two non-decreasing functions. The converse is obvious.

We shall now prove some elementary theorems concerning continuity properties of functions of bounded variation.

(4.1) ***Theorem.*** *If F is an additive function of an interval, of bounded variation on a figure R_0,* (i) *the series $\sum_n \mathrm{o}_{R_0}(F; D_n)$ converges for every sequence $\{D_n\}$ of hyperplanes distinct from one another, and* (ii) *there is at most an enumerable infinity of hyperplanes D such that $\mathrm{o}_{R_0}(F; D) > 0$.*

Proof. In the proof of (i) we may clearly suppose that the hyperplanes D_n are orthogonal to the same axis. Consider now the first k of these D_n. We can associate with them, k non-overlapping intervals $I_1, I_2, \ldots, I_k$, contained in R_0 and such that $\mathrm{o}_{R_0}(F; D_n) \leqslant |F(I_n)| + 1/k^2$ for $n = 1, 2, \ldots, k$. Hence $\sum_{n=1}^{k} \mathrm{o}_{R_0}(F; D_n) \leqslant \sum_{n=1}^{k} |F(I_n)| + 1/k \leqslant \mathrm{W}(F; R_0) + 1/k$, and, since k is an arbitrary positive integer, $\sum_{n=1}^{\infty} \mathrm{o}_{R_0}(F; D_n) \leqslant \mathrm{W}(F; R_0) < +\infty$.

To establish (ii), suppose that there is a non-enumerable infinity of hyperplanes D such that $\mathrm{o}_{R_0}(F; D) > 0$. There would then be a positive number ε, such that $\mathrm{o}_{R_0}(F; D) > \varepsilon$ for an infinity, (which would even be non-enumerable) of hyperplanes. But this clearly contradicts part (i) which has just been proved.

We obtain at once from Theorem 4.1 the following

(4.2) ***Theorem.*** *For each additive function of an interval of bounded variation, there is at most an enumerable infinity of hyperplanes of discontinuity.*

(4.3) ***Theorem.*** *If F is an additive function of an interval of bounded variation on a figure R_0 and we write $W(I) = \mathrm{W}(F; I)$, the relations $\mathrm{o}_{R_0}(F; D) = 0$ and $\mathrm{o}_{R_0}(W; D) = 0$ are equivalent for every hyperplane D.*

Proof. Suppose, if possible, that

(4.4) $\mathrm{o}_{R_0}(F; D) = 0$ and (4.5) $\mathrm{o}_{R_0}(W; D) > \varepsilon$

for a hyperplane D and a number $\varepsilon > 0$. We shall show that it is then possible to define a sequence of figures $\{R_n\}_{n=1,2,\ldots}$, non-overlapping, contained in R_0, and such that

(4.6) $R_n \cdot D = 0$ and (4.7) $|F(R_n)| > \varepsilon/2$.

To see this, suppose defined k non-overlapping figures $R_1, R_2, \ldots, R_k$ contained in R_0, and let (4.6) and (4.7) hold for $n = 1, 2, \ldots, k$. On account of (4.5) there is then an interval $I \subset R_0$ such that $I \cdot R_n = 0$ for $n = 1, 2, \ldots, k$, and such that $\mathrm{W}(F; I) = W(I) > \varepsilon$. Hence, there exists a figure $R \subset I$ such that $|F(R)| > \mathrm{W}(F; I)/2 > \varepsilon/2$. Moreover since (4.4) asserts that F is continuous on R_0 at D, we may suppose that $R \cdot D = 0$. But if we now choose $R_{k+1} = R$, we see that the figure R_{k+1} does not overlap any of the figures R_n for $1 \leqslant n \leqslant k$, and that (4.6) and (4.7) continue to hold for $n = k+1$.

Having obtained our sequence $\{R_n\}$, we conclude from (4.7) that $\mathrm{W}(F; R_0) \geqslant \sum_n |F(R_n)| = \infty$, and this contradicts our hypotheses. The conditions (4.4) and (4.5) are thus incompatible, i. e. (4.4) implies $\mathrm{o}_{R_0}(W; D) = 0$. And since the converse is obvious, this completes the proof.

From Theorems 4.3 and 3.1, we obtain at once the following

(4.8) ***Theorem.*** *In order that an additive function of bounded variation on a figure R_0 be continuous on R_0, it is necessary and sufficient that its three variations be so.*

§ 5. Lebesgue-Stieltjes integral. Lebesgue integral and measure. We need hardly point out the analogy between additive functions of bounded variation of an interval and additive functions of sets. This analogy will be made clearer and deeper in the present §, by associating a function U^* of a set with each additive function U of bounded variation of an interval. In order to simplify the wording, we shall suppose that the functions of an interval are defined in the whole space.

Suppose given in the first place, a non-negative additive function U of an interval; we then denote for any set E, by $U^*(E)$ the lower bound of the sums $\sum_k U(I_k)$, where $\{I_k\}$ is an arbitrary sequence of intervals such that $E \subset \sum_k I_k^\circ$. For an arbitrary additive function U of bounded variation, with the upper and lower variations W_1 and W_2, we denote by W_1^* and $(-W_2)^*$ the functions of a set that correspond to the non-negative functions W_1 and $-W_2$, and we write, by definition, $U^* = W_1^* - (-W_2)^*$. The function U^* is thus defined for all sets and is finite for bounded sets.

When U is non-negative, U^* is an outer measure in the sense of Carathéodory, i. e. fulfills the three conditions (C_1), (C_2) and (C_3) of Chap. II, § 4. Condition (C_3) is the only one requiring proof, the other two are obvious. Let therefore A and B be any two sets whose distance does not vanish, and let ε be a positive number. There is then a sequence $\{I_n\}$ of intervals such that $A+B \subset \sum_n I_n^\circ$ and $\sum_n U(I_n) \leqslant U^*(A+B)+\varepsilon$. We may clearly suppose that the intervals of the sequence have diameters less than $\varrho(A, B)$, i. e. that none of them contains both points of A and points of B. We then have $U^*(A)+U^*(B) \leqslant \sum_n U(I_n) \leqslant U^*(A+B)+\varepsilon$. This gives the inequality $U^*(A)+U^*(B) \leqslant U^*(A+B)$ and establishes condition (C_3).

The function U^*, determined by a non-negative function of an interval, itself determines, since it is an outer Carathéodory measure (cf. Chap. II, § 4, p. 46), the class $\mathfrak{L}_{U^*}$ of the sets measurable with respect to U^* and the process of integration (U^*). To simplify the notation, we shall omit the asterisk and write simply $\mathfrak{L}_U$ for $\mathfrak{L}_{U^*}$, integral (U) for integral (U^*), measure (U) of a set instead of measure (U^*), $\int_E f\,dU$ instead of $\int_E f\,dU^*$, and so on.

This slight change of notation cannot cause any confusion, since the measure U^* is uniquely determined by the function of an interval U.

When U is a general additive function of an interval, of bounded variation, we shall understand by $\mathfrak{L}_U$ the common part of the classes $\mathfrak{L}_{W_1}$ and $\mathfrak{L}_{-W_2}$, where W_1 and W_2 denote respectively the upper and lower variations of U. A function of a point $f(x)$ will be termed *integrable* (U) on a set E, if $f(x)$ is integrable (W_1) and $(-W_2)$ simultaneously; by its *integral* (U) over E we shall mean the number $\int_E f\,dW_1 - \int_E f\,d(-W_2)$, and we write it $\int_E f\,dU$ as in the case of a non-negative function U. This integration with respect to an additive function of bounded variation of an interval is called *Lebesgue-Stieltjes integration.* In the case of the integration over an interval $I=[a, b]$ in $\boldsymbol{R}_1$, we frequently write $\int_a^b f\,dU$ for $\int_I f\,dU$.

When the function U is continuous, every indefinite integral (U) vanishes, together with the function U^*, on the boundary of any figure. Consequently, an indefinite integral with respect to a continuous function U of bounded variation of an interval is additive not only as function of a set $(\mathfrak{L}_U)$ but also as function of an interval.

The most important case is that in which the given function of an interval U is the special function L (cf. § 2, p. 59) that denotes the volume of an interval. The outer measure (L) is also termed *outer Lebesgue measure*, and the integral (L), *Lebesgue integral*, while functions integrable (L) are often called, as originally, by Lebesgue, *summable*. The class of sets $\mathfrak{L}_{\mathrm{L}}$ will be denoted simply by $\mathfrak{L}$. The outer measure (L) of an arbitrary set E is written $\text{meas}_e E$, and meas E without the suffix when E is measurable $(\mathfrak{L})$. We shall also denote this outer measure by $|E|$ or by $\mathrm{L}(E)$ (or sometimes by $\mathrm{L}_m(E)$ in $\boldsymbol{R}_m$), thus extending to arbitrary sets the notation adopted for all figures R, since for the latter, as we shall see (cf. Th. 6.2), the measure (L) coincides with the values $\mathrm{L}(R)=|R|$. Finally, owing to the special part played by Lebesgue measure in the theory of integration and derivation, the terms "measure of a set", "measurable set", "measurable function", and so on, will, in the sequel, be understood in the Lebesgue sense, whenever another sense has not been explicitly assigned to them.

We also modify slightly the integral notation for a Lebesgue integral; and instead of $\int_E f(x)\,d\mathrm{L}(x)$ we write $\int_E f(x)\,dx$, or else $\int\int_E\ldots\int f(x_1, x_2, \ldots, x_m)\,dx_1\,dx_2\ldots dx_m$, when we wish to indicate the number of dimensions of the space $\boldsymbol{R}_m$ under consideration. This brings us back to the classical notation.

A special part, similar to that of Lebesgue measure in the theory of the integral, is played by Borel sets in the theory of additive classes of sets. In the first place, it follows from Theorem 7.4, Chap. II, that every class $\mathfrak{L}_U$, where U is an additive function of bounded variation of an interval, contains all the sets ($\mathfrak{B}$). In the sequel, we shall agree that additive functions of a set will always mean functions additive ($\mathfrak{B}$), unless there is explicit reference to another additive class of sets. Similarly, additive functions of a set that are absolutely continuous ($\mathfrak{B}$, L) or singular ($\mathfrak{B}$, L), will simply be called *absolutely continuous* or *singular*. In point of fact, Theorem 6.6 below, which asserts that every set measurable ($\mathfrak{L}$) is the sum of a set ($\mathfrak{B}$) and a set of zero measure (L), will show that every additive function of a set, absolutely continuous ($\mathfrak{B}$, L), can be extended in a unique manner to all sets ($\mathfrak{L}$) so as to remain absolutely continuous ($\mathfrak{L}$, L).

The special rôle of the measure (L) and of the sets ($\mathfrak{B}$) showed itself already during the growth of the theory. Lebesgue measure was the starting point for further extensions of the notions of measure and integral, whereas the Borel sets were the origin of general theories of additive classes and functions. The sets ($\mathfrak{B}$) were introduced, with measure (L) defined for them, by E. Borel [I, p. 46—50] in 1898. But it was not until some years later that H. Lebesgue [1; I], by simplifying and extending the definition of measure (L) to all sets ($\mathfrak{L}$), made clear the importance of this measure for the theory of integration and especially for that of derivation of functions. *Vide* E. Borel [1] and H. Lebesgue [6].

We have already seen in § 1 of this book, how, by an apparently very slight modification of the classical definition of Riemann, we obtain the Lebesgue integral. A similar remark may be made with regard to the relationship of Lebesgue measure to the earlier measure of Peano-Jordan. The outer measure of Peano-Jordan for a bounded set E is the lower bound of the numbers $\sum_n |I_n|$ where $\{I_n\}$ is any finite system of intervals covering E. Lebesgue's happy idea was to replace in this definition, the finite systems of intervals by enumerable ones.

We have given in the text a more general form to Lebesgue's definition, relative to an arbitrary non-negative function of an interval. This relativizing of Lebesgue measure is due to J. Radon [1] and to Ch. J. de la Vallée-Poussin [1; I]. The parallel extension of the Lebesgue integral is also due to J. Radon. In the text we have termed it Lebesgue-Stieltjes integral; it is sometimes also termed Lebesgue-Radon integral or Radon integral. For a systematic exposition of the properties of this integral, *vide* H. Lebesgue [II, Chap. XI]. A particularly interesting generalization of the Lebesgue integral, of the Stieltjes type, has been given by N. Bary and D. Menchoff [1]; it differs considerably from the other generalizations of this type. Finally, for an account of the Riemann-Stieltjes integral (which we shall not discuss in this volume) *vide* W. H. Young [2], S. Pollard [1], R. C. Young [1], M. Fréchet [5] and G. Fichtenholz [2].

It was again J. Radon [I, p. 1] who pointed out the importance of the Lebesgue-Stieltjes integral for certain classical parts of Analysis, particularly for potential theory. The modern progress of this theory, which is bound up with the theory of subharmonic functions, has shown up still further the fruitfulness of the Lebesgue-Stieltjes integral in this branch of Analysis (cf. the memoirs of F. Riesz [4] and G. C. Evans [1]).

§ 6. Measure defined by a non-negative additive function of an interval. In this §, U will denote a fixed non-negative additive function of an interval. In the preceding § we made correspond to any such a function U, an outer Carathéodory measure U^*. Besides the properties established in Chap. II for all Carathéodory measures, the function of a set U^* possesses a number of elementary properties of a more special kind which we shall investigate in this §.

(6.1) ***Lemma.*** *If D denotes a hyperplane or a degenerate interval, the relation* $\mathrm{o}(U; D)=0$ *implies* $U^*(D)=0$.

Proof. Since every hyperplane is the sum of a sequence of degenerate intervals (cf. § 2, p. 57), it is enough to prove the lemma in the case in which D is a degenerate interval $[a_1, b_1; a_2, b_2; \ldots; a_m, b_m]$.

Let R be a cube containing D in its interior, G an arbitrary open set such that $D\subset G$, and let

$$D_\varepsilon=[a_1-\varepsilon, b_1+\varepsilon;\ a_2-\varepsilon, b_2+\varepsilon;\ \ldots;\ a_m-\varepsilon, b_m+\varepsilon],$$

where ε is any positive number, sufficiently small to ensure that $D_\varepsilon\subset R\cdot G$. Since D_ε is then an ordinary closed interval containing D in its interior, we find $U^*(D)\leqslant U(D_\varepsilon)\leqslant \mathrm{O}(U; R\cdot G)$, whence $U^*(D)\leqslant \mathrm{o}_R(U; D)=0$.

(6.2) ***Theorem.*** *For every figure R we have*

(6.3) $$U^*(R^\circ) \leqslant U(R) \leqslant U^*(R),$$

and, if the oscillation of U at the boundary of R vanishes,

(6.4) $$U^*(R^\circ) = U(R) = U^*(R).$$

In particular therefore, if U is a continuous function, the equality (6.4) *holds for every figure R.*

Proof. In virtue of Theorems 2.2 and 4.2 the set R° is expressible as the sum of a sequence of non-overlapping cubes $\{I_k\}$ such that the oscillation of U vanishes at all faces of all the I_k. Hence, by the preceding lemma $U^*(R^\circ) = \sum_k U^*(I_k^\circ)$, and since $U(R) \geqslant \sum_{k=1}^{n} U(I_k) \geqslant \sum_{k=1}^{n} U^*(I_k^\circ)$ for each positive integer n, we get $U^*(R^\circ) \leqslant U(R)$.

To establish that $U(R) \leqslant U^*(R)$, it is enough to show that $U(R) \leqslant \sum_k U(I_k)$ for every sequence of intervals $\{I_k\}$ such that $R \subset \sum_k I_k^\circ$. Now, if $\{I_k\}$ is such a sequence, we have, by the well-known covering theorem of Borel-Lebesgue, $R \subset \sum_{k=1}^{N} I_k^\circ$ for some sufficiently large value of N. Hence $U(R) \leqslant \sum_{k=1}^{N} U(R \odot I_k) \leqslant \sum_{k=1}^{N} U(I_k)$.

Finally, denoting by B the boundary of R, let us suppose that $\mathrm{o}(U; B) = 0$. It then follows from Lemma 6.1 that $U^*(B) = 0$, so that $U^*(R) = U^*(R^\circ)$, and the equality (6.4) follows at once from (6.3).

(6.5) ***Theorem.*** *Given an arbitrary set E and any positive ε, there is* (i) *an open set G such that $E \subset G$ and $U^*(G) \leqslant U^*(E) + \varepsilon$,* (ii) *a set $H \in \mathfrak{G}_\delta$ such that $E \subset H$ and $U^*(H) = U^*(E)$.*

Proof. *re* (i). There exists for each $\varepsilon > 0$, a sequence of intervals $\{I_n\}$ such that $E \subset \sum_n I_n^\circ$ and that $\sum_n U(I_n) \leqslant U^*(E) + \varepsilon$. Hence, writing $G = \sum_n I_n^\circ$, we find, on account of condition (C_2) of Carathéodory (Chap. II, p. 43) and Theorem 6.2, that $U^*(G) \leqslant \sum_n U^*(I_n^\circ) \leqslant \sum_n U(I_n) \leqslant$ $\leqslant U^*(E) + \varepsilon$.

re (ii). Let us make correspond to E, for each positive integer n, an open set G_n containing E and such that $U^*(G_n) \leqslant U^*(E) + 1/n$; this is always possible by (i). The set $H = \prod_n G_n$ clearly fulfills our requirements and this completes the proof.

Every set ($\mathfrak{G}_\delta$) is of course measurable ($\mathfrak{L}_U$). Hence it follows at once from Theorem 6.5 that every set is regular (cf. Chap. II, § 6) with respect to the outer measure U^*, and therefore that this measure is itself regular.

(6.6) ***Theorem.*** *Each of the following conditions is necessary and sufficient for a set E to be measurable* ($\mathfrak{L}_U$):

(i) *for every $\varepsilon > 0$ there is an open set $G \supset E$ such that $U^*(G - E) \leqslant \varepsilon$;*

(ii) *there is a set ($\mathfrak{G}_\delta$) containing E and differing from E at most by a set of measure (U) zero;*

(iii) *for every $\varepsilon > 0$ there is a closed set $F \subset E$ such that $U^*(E - F) \leqslant \varepsilon$;*

(iv) *there is a set ($\mathfrak{F}_\sigma$) contained in E and differing from E at most by a set of measure (U) zero.*

Proof. We shall first prove all these conditions necessary. Let E be a set measurable ($\mathfrak{L}_U$) and ε a positive number. We begin by representing E as the sum of a sequence $\{E_n\}_{n=1,2,\dots}$ of sets measurable ($\mathfrak{L}_U$) of finite measure; we may write for instance, $E_n = E \cdot S(0; n)$. This being so, we associate with each set E_n, in accordance with Theorem 6.5, an open set $G_n \supset E_n$ such that $U^*(G_n) \leqslant U^*(E_n) + \varepsilon/2^n$. Hence, the sets E_n being measurable ($\mathfrak{L}_U$), we have $U^*(G_n - E_n) \leqslant \varepsilon/2^n$ for every n, and if we write $G = \sum_n G_n$, we find $E \subset G$ and $U^*(G - U) \leqslant$ $\leqslant \sum_n U^*(G_n - E_n) \leqslant \varepsilon$, and this proves condition (i) necessary.

To prove the necessity of condition (ii), we attach to the given set E measurable ($\mathfrak{L}_U$) a sequence $\{Q_n\}$ of open sets such that $E \subset Q_n$ and $U^*(Q_n - E) \leqslant 1/n$ for each n. Writing $H = \prod_n Q_n$, we see that $H \in \mathfrak{G}_\delta$, $E \subset H$ and $U^*(H - E) = 0$.

Finally, we observe that for any set A, the relation $A \supset CE$ implies $CA \subset E$ and $E - CA = A - CE$; further, if A is a set ($\mathfrak{G}$) or ($\mathfrak{G}_\delta$), the set CA is a set ($\mathfrak{F}$) or ($\mathfrak{F}_\sigma$) respectively. Hence every set E measurable ($\mathfrak{L}_U$) fulfills conditions (iii) and (iv), since, by the results just proved, its complement CE fulfills conditions (i) and (ii).

The sufficiency of conditions (ii) and (iv) is evident, since sets of measure (U) zero, and sets $(\mathfrak{G}_\delta)$ or $(\mathfrak{F}_\sigma)$, are always measurable $(\mathfrak{L}_U)$.

To establish the sufficiency of conditions (i) and (iii), we need only observe that they imply respectively conditions (ii) and (iv). Thus, for instance, if (iii) holds, there is for each positive integer n a closed set $F_n \subset E$ such that $U^*(E - F_n) \leqslant 1/n$. The set $P = \sum_n F_n$ is therefore a set $(\mathfrak{F}_\sigma)$ contained in E and such that $U^*(E-P)=0$.

From Theorem 6.6 it follows in particular that the general form of a set measurable $(\mathfrak{L}_U)$ is $B+N$, where B is a set measurable $(\mathfrak{B})$ and N a set of measure (U) zero. In other words, $\mathfrak{L}_U$ is the smallest additive class containing the Borel sets and the sets of measure (U) zero. It follows that $\mathfrak{L}_U \supset \mathfrak{L}$ whenever the function of an interval U is absolutely continuous (*vide* § 12).

(6.7) ***Theorem.*** *For any set E there is a set $H \epsilon \mathfrak{G}_\delta$ containing E and such that*

(6.8) $U^*(H\cdot X) = U^*(E \cdot X)$ *for every set X measurable $(\mathfrak{L}_U)$.*

Proof. It is enough to show that there is a set $H \supset E$ measurable $(\mathfrak{L}_U)$ for which (6.8) holds. For, by Theorem 6.6 we can always enclose such a set H in a set $(\mathfrak{G}_\delta)$ differing from it by a set of measure (U) zero.

Let us represent E as limit of an ascending sequence $\{E_n\}$ of bounded sets, which are therefore of finite outer measure (U); and let us associate, as we may by Theorem 6.5, with each E_n a set $H_n \epsilon \mathfrak{G}_\delta$ such that $E_n \subset H_n$ and $U^*(H_n) = U^*(E_n)$. Then, for every set X measurable $(\mathfrak{L}_U)$,

$$U^*(H_n\cdot X) = U^*(H_n) - U^*(H_n \cdot CX) \leqslant U^*(E_n(- U^*(E_n\cdot CX) \leqslant U^*(E_n \cdot X);$$

from which, writing $H = \liminf_n H_n$, we deduce by means of Theorem 9.1 of Chap. I, that

$$U^*(H\cdot X) \leqslant \liminf_n U^*(H_n \cdot X) \leqslant \lim_n U^*(E_n \cdot X) \leqslant U^*(E\cdot X),$$

and this implies (6.8), since $H = \liminf_n H_n \supset \lim_n E_n = E$.

In the theorems of this § we have supposed the function U of an interval to be non-negative. But, by slight changes in the wording, the theorems can easily be extended to arbitrary functions of bounded variation. As an example we mention the following theorem which corresponds to Theorem 6.5:

(6.9) ***Theorem.*** *If F is an additive function of bounded variation of an interval, then for any bounded set E and any $\varepsilon > 0$ there is an open set $G \supset E$ such that $|F^*(X) - F^*(E)| \leqslant \varepsilon$ for every bounded set X satisfying the condition $E \subset X \subset G$.*

Proof. Denoting by W_1 and W_2 two functions of an interval that are respectively the upper and the lower variation of F, we can, by Theorem 6.5, enclose E in each of two open sets G_1 and G_2 such that $W_1^*(G_1) \leqslant W_1^*(E) + \varepsilon$ and $W_2^*(G_2) \geqslant W_2^*(E) - \varepsilon$. Therefore, writing $G = G_1 \cdot G_2$ we have $E \subset G$; and for any bounded set X such that $E \subset X \subset G$, we find $0 \leqslant W_1^*(X) - W_1^*(E) \leqslant \varepsilon$ and $0 \leqslant W_2^*(E) - W_2^*(X) \leqslant \varepsilon$, whence by subtraction $|F^*(X) - F^*(E)| \leqslant \varepsilon$.

Let us still prove a theorem which allows us to regard all non-negative additive functions of a set in $\boldsymbol{R}_m$ as determined by non-negative additive functions of an interval. We recall that, according to the conventions of § 3, p. 59 and § 5, p. 66, we always mean by additive functions of a set, functions of a set that are additive $(\mathfrak{B})$ on every figure.

(6.10) ***Theorem.*** *Given any non-negative additive function Φ of a set, there is always a non-negative additive function F of an interval such that $\Phi(X) = F^*(X)$ for every bounded set X measurable $(\mathfrak{B})$.*

Proof. Let us denote for each interval $I = [a_1, b_1; \ldots; a_m, b_m]$, by $\widetilde{I}$ the interval $(a_1, b_1; \ldots; a_m, b_m]$ half open to the left, and let us define the non-negative additive function of an interval by writing $F(I) = \Phi(\widetilde{I})$ for every interval I.

This being so, we observe that any bounded open set G can be expressed (cf. Theorems 2.2 and 4.2) as the sum of a sequence $\{I_n\}$ of non-overlapping intervals at whose faces the oscillation of F vanishes; and therefore by Theorem 6.2, $\Phi(G) = \sum_n \Phi(\widetilde{I}_n) = \sum_n F(I_n) = \sum_n F^*(\widetilde{I}_n) = F^*(G)$. Thus the equation $\Phi(X) = F^*(X)$ holds whenever X is a bounded open set, and therefore also whenever X is a bounded set $(\mathfrak{G}_\delta)$, since the latter is expressible as the limit of a descending sequence of bounded open sets. It follows further that $\Phi(X) = F^*(X) = 0$ for every bounded set X of measure (F) zero, since, by Theorem 6.6, such a set X can be enclosed in a bounded set $(\mathfrak{G}_\delta)$ of measure (F) zero. This completes the proof, since every bounded set $(\mathfrak{B})$ is, by Theorem 6.6, the difference of a bounded set $(\mathfrak{G}_\delta)$ and of a set of measure (F) zero.

The proof of Theorem 6.10 could also be attached to the following general theorem concerning functions additive $(\mathfrak{B})$ defined on any metrical space $\boldsymbol{M}$: *if two such functions coincide for every open set, they are identical for all sets* $(\mathfrak{B})$. This theorem is easily proved.

§ 7. Theorems of Lusin and Vitali-Carathéodory.

We shall establish in this § two theorems concerning the approximation to measurable functions by continuous functions and by semi-continuous functions. As in the preceding §, U will stand for a non-negative additive function of an interval, fixed in any manner for the space $\boldsymbol{R}_m$.

(7.1) ***Lusin's Theorem.*** *In order that a function $f(x)$, finite on a set E, be measurable $(\mathfrak{L}_U)$ on E, it is necessary and sufficient that for every $\varepsilon>0$, there exists a closed set $F\subset E$ such that $U^*(E-F)<\varepsilon$, and on which $f(x)$ is continuous.*

Proof. To show the condition necessary, we suppose $f(x)$ finite and measurable $(\mathfrak{L}_U)$ on E, and we deal first with two particular cases:

(i) $f(x)$ is a simple function on E. The set E is, in this case, the sum of a finite sequence $E_1, E_2, \ldots, E_n$ of sets measurable $(\mathfrak{L}_U)$ no two of which have common points, such that $f(x)$ is constant on each of these sets. By Theorem 6.6 there exists, for each set E_i, a closed set $F_i\subset E_i$ such that $U^*(E_i-F_i)<\varepsilon/n$. Writing $F=\sum_{i=1}^{n}F_i$, we then have $F\subset E$ and $U^*(E-F)<\varepsilon$, and moreover the function $f(x)$ is clearly continuous on F.

(ii) E is a set of finite measure (U). In this case, by Theorem 7.4, Chap. I, (applied separately to the non-negative and to the non-positive parts of $f(x)$), there is a sequence $\{f_n(x)\}_{n=1,2,\ldots}$ of simple functions, finite and measurable $(\mathfrak{L}_U)$, that converges on E to $f(x)$. By Egoroff's Theorem (Chap. I, Th. 9.6), this sequence converges uniformly on a set $P\subset E$, measurable $(\mathfrak{L}_U)$ and such that $U^*(E-P)<\varepsilon/2$. This set P may further be supposed to be closed, on account of Theorem 6.6. Finally, by (i) we can attach to each function $f_n(x)$, a closed set $P_n\subset E$ such that $U^*(E-P_n)<\varepsilon/2^{n+1}$,

and on which $f_n(x)$ is continuous. Hence, writing $F=P\cdot\prod_n P_n$, we get $U^*(E-F)\leqslant U^*(E-P)+\sum_n U^*(E-P_n)\leqslant\varepsilon$; and moreover, all the $f_n(x)$, and therefore also the function $f(x)=\lim_n f_n(x)$, are continuous on F, a set which is evidently closed.

We now come to the general case where E is any set measurable $(\mathfrak{L}_U)$. Let $E_n=E\cdot(S_n-S_{n-1})$, where $S_0=0$ and $S_n=S(0;n)$ for $n\geqslant1$. By (ii), there exists, for each $n\geqslant1$, a closed set $Q_n\subset E_n$ such that $U^*(E_n-Q_n)<\varepsilon/2^n$, and on which $f(x)$ is continuous. Writing $F=\sum_{n=1}^{\infty}Q_n$, the set F is closed, we have $U^*(E-F)\leqslant\leqslant\sum_n U^*(E_n-Q_n)<\varepsilon$, and $f(x)$ is continuous on F.

The proof of the necessity of the condition is thus complete. Let us now suppose, conversely, that the condition is satisfied. The set E is then expressible as the sum of a set N of measure (U) zero and of a sequence $\{F_n\}$ of closed sets on each of which $f(x)$ is continuous. The function f is thus measurable $(\mathfrak{L}_U)$ on N and on each of the sets F_n (cf. Chap. II, Th. 7.6), and therefore on the whole set E.

For the various proofs of Lusin's theorem, *vide* N. Lusin [1], W. Sierpiński [6] and L. W. Cohen [1].

(7.2) ***Lemma.*** *Given a function $f(x)$, measurable $(\mathfrak{L}_U)$ and non-negative in the space $\boldsymbol{R}_m$, there exists, for each $\varepsilon>0$, a lower semi-continuous function $h(x)$ such that*

$$h(x)\geqslant f(x) \quad \textit{at each point } x, \tag{7.3}$$

and

$$\int_{\boldsymbol{R}_m}[h(x)-f(x)]\,dU(x)\leqslant\varepsilon \tag{7.4}$$

(where, in accordance with the convention of Chap. I, p. 6, the difference $h(x)-f(x)$ is to be understood to vanish at any point x for which $h(x)=f(x)=+\infty$).

Proof. (i) First suppose that $f(x)$ is bounded and vanishes outside a bounded set E measurable $(\mathfrak{L}_U)$. Let $\eta=\varepsilon/[1+U^*(E)]$. We write $E_k=\underset{x}{\mathrm{E}}[x\in E;\ (k-1)\eta\leqslant f(x)<k\eta]$ for $k=1,2,\ldots$ and we associate with each set E_k an open set $G_k\supset E_k$ such that

(7.5) $$U^*(G_k - E_k) \leqslant 1/k \cdot 2^k.$$

Further, denoting by $c_k(x)$ the characteristic function of G_k, we write $h(x) = \sum_{k=1}^{\infty} k\eta \cdot c_k(x)$. Since each function $c_k(x)$ is evidently lower semi-continuous, the function $h(x)$ is so too. We also observe that $h(x)$ fulfills condition (7.3). On the other hand

$$\int_{\boldsymbol{R}_m} h(x)\,dU(x) = \sum_{k=1}^{\infty}{}' k\eta \int_{\boldsymbol{R}_m} c_k(x)\,dU(x) = \sum_{k=1}^{\infty}{}' k\eta \cdot U^*(G_k) =$$
$$= \sum_{k=1}^{\infty}{}' (k-1)\eta \cdot U^*(E_k) + \sum_{k=1}^{\infty}{}' \eta \cdot U^*(E_k) + \sum_{k=1}^{\infty}{}' k\eta \cdot U^*(G_k - E_k),$$

whence, by (7.5), we obtain

$$\int_{\boldsymbol{R}_m} h(x)\,dU(x) \leqslant \int_{\boldsymbol{R}_m} f(x)\,dU(x) + \eta \cdot U^*(E) + \eta \leqslant \int_{\boldsymbol{R}_m} f(x)\,dU(x) + \varepsilon.$$

From this, remembering that $f(x)$ is integrable on $\boldsymbol{R}_m$, (7.4) follows at once.

(ii) We now pass to the general case and represent firstly $f(x)$ in the form $f(x) = \sum_{n=1}^{\infty} f_n(x)$, where the $f_n(x)$ are bounded non-negative functions measurable $(\mathfrak{L}_U)$, each of which vanishes outside a bounded set. We may do this, for instance, by writing $f_n(x) = s_n(x) - s_{n-1}(x)$ where

$$s_n(x) = \begin{cases} f(x) & \textit{for} \quad \varrho(0,x) < n \quad \textit{and} \quad f(x) \leqslant n, \\ n & \textit{for} \quad \varrho(0,x) < n \quad \textit{and} \quad f(x) > n, \\ 0 & \textit{for} \quad \varrho(0,x) \geqslant n \end{cases} \qquad n = 0, 1, 2, \ldots$$

By what has been proved in (i), there exists for each function $f_n(x)$ a lower semi-continuous function $h_n(x)$ such that $h_n(x) \geqslant f_n(x)$ at every point x, and that $\int_{\boldsymbol{R}_m} [h_n(x) - f_n(x)]\,dU(x) < \varepsilon/2^n$. The function $h(x) = \sum_{n=1}^{\infty} h_n(x)$ is then evidently lower semi-continuous and fulfills condition (7.3). Finally $\int_{\boldsymbol{R}_m} [h(x) - f(x)]\,dU(x) \leqslant \sum_{n=1}^{\infty}{}' \int_{\boldsymbol{R}_m} [h_n(x) - f_n(x)]\,dU(x) \leqslant \varepsilon$, and this completes the proof.

(7.6) ***Theorem of Vitali-Carathéodory.*** *Given a function $f(x)$ measurable $(\mathfrak{L}_U)$ in the space $\boldsymbol{R}_m$, there exist two monotone sequences of functions $\{l_n(x)\}$ and $\{u_n(x)\}$ for which the following conditions are satisfied:*

(i) *the functions l_n are lower semi-continuous and the functions u_n are upper semi-continuous,*

(ii) *each of the functions l_n is bounded below and each of the functions u_n is bounded above,*

(iii) *the sequence $\{l_n\}$ is non-increasing and the sequence $\{u_n\}$ is non-decreasing,*

(iv) $l_n(x) \geqslant f(x) \geqslant u_n(x)$ *for every x,*

(v) $\lim_n l_n(x) = f(x) = \lim_n u_n(x)$ *almost everywhere (U),*

(vi) *on every set E on which $f(x)$ is integrable (U), so are the functions $l_n(x)$ and $u_n(x)$ and we have*

$$\lim_n \int_E l_n(x)\,dU(x) = \lim_n \int_E u_n(x)\,dU(x) = \int_E f(x)\,dU(x).$$

Proof. By expressing the function $f(x)$ as the sum of its non-negative and non-positive parts $\overset{\circ}{f}(x)$ and $\underset{\circ}{f}(x)$ (Chap. I, § 7), we may suppose that $f(x)$ is of constant sign, say non-negative. By the preceding lemma, we can associate with $f(x)$ a sequence of lower semi-continuous functions $\{h_n(x)\}_{n=1,2,\ldots}$ such that $h_n(x) \geqslant f(x)$ for every x and

(7.7) $$\lim_n \int_{\boldsymbol{R}} [h_n(x) - f(x)]\,dU(x) = 0.$$

Writing $l_n(x) = \min[h_1(x), h_2(x), \ldots, h_n(x)]$ we therefore obtain a non-increasing sequence of lower semi-continuous functions $\{l_n(x)\}$ that evidently fulfills conditions (i), (ii), (iii) and (iv); moreover, it follows from (7.7) that $\lim_n \int_{\boldsymbol{R}_m} [l_n(x) - f(x)]\,dU(x) = 0$, and hence that the functions $l_n(x)$ fulfill also conditions (v) and (vi).

In order now to define the sequence $\{u_n(x)\}$, we attach to the function $1/f(x)$ a non-increasing sequence of lower semi-continuous functions $\{g_n(x)\}$ such that $\lim_n g_n(x) = 1/f(x)$ almost everywhere (U). Such a sequence certainly exists by what has just been proved. The functions $1/g_n(x)$ then form a non-decreasing sequence of upper semi-continuous functions, that converges almost everywhere (U) to $f(x)$. If we now write $u_n(x) = 1/g_n(x)$ when $1/g_n(x) \leqslant n$, and

$u_n(x)=n$ when $1/g_n(x)>n$, we obtain a sequence $\{u_n(x)\}$ of bounded functions with the same properties, which therefore satisfies conditions (i—v). Finally, since the functions $u_n(x)$ are non-negative, we can apply Lebesgue's Theorem (Chap. I, Th. 12.6) to derive from (iii) and (v) that $\lim_n \int_E u_n(x)\,dU(x)=\int_E f(x)\,dU(x)$ on every set E measurable ($\mathfrak{L}_U$), and this implies (vi).

Conditions (i) and (v) of Theorem 7.6 imply that every function measurable ($\mathfrak{L}_U$) is almost everywhere (U) the limit of a convergent sequence (with finite or infinite limit) of semi-continuous functions, and thus coincides almost everywhere (U) with a function of the second class of Baire. This result, due to G. Vitali [2] (cf. also W. Sierpiński [6]) was completed by C. Carathéodory [I, p. 406], who established for every measurable function $f(x)$ the existence of two sequences of functions fulfilling conditions (i)—(v). Condition (vi), which includes, as we shall see later, the theorem of de la Vallée Poussin and Perron on the existence, for summable functions, of majorant and minorant functions, has been added here because its proof is naturally related to those of conditions (i—v).

There is an obvious analogy between the property of measurable functions expressed by the theorem of Vitali-Carathéodory, and the properties of measurable sets stated in conditions (i) and (iii) of Theorem 6.6. By taking into account the geometrical definition of the integral (cf. below § 10), we might even base the proof of Theorem 7.6 directly on Theorem 6.6 (*vide* the first ed. of this book, pp. 88—91).

§ 8. Theorem of Fubini. Given two Euclidean spaces $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, if $x=(a_1, a_2, ..., a_p)$ and $y=(a_{p+1}, a_{p+2}, ..., a_{p+q})$ are two points situated respectively in these two spaces, we shall denote by (x, y) the point $(a_1, a_2, ..., a_{p+q})$ in the space $\boldsymbol{R}_{p+q}$. If X and Y are two sets situated respectively in the spaces $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, we shall denote by $X\times Y$ the set of all points (x, y) in $\boldsymbol{R}_{p+q}$ such that $x\in X$ and $y\in Y$. In particular, if X and Y are two intervals — closed, open, or half open on the same side — $X\times Y$ also is an interval in $\boldsymbol{R}_{p+q}$, which is closed, open, or half open on the same side as X and Y. Every interval $I=[a_1, b_1; ...; a_{p+q}, b_{p+q}]$ can evidently be expressed — and in a unique manner — in the form $I_1\times I_2$ where I_1 and I_2 are intervals in $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$ respectively; we merely have to write $I_1=[a_1, b_1; ...; a_p, b_p]$ and $I_2=[a_{p+1}, b_{p+1}; ...; a_{p+q}, b_{p+q}]$.

Given two additive functions of an interval, U and V, in the spaces $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$ respectively, we determine a function of an interval T in $\boldsymbol{R}_{p+q}$ by writing $T(I_1\times I_2)=U(I_1)\cdot V(I_2)$ for each pair of intervals $I_1\subset \boldsymbol{R}_p$ and $I_2\subset \boldsymbol{R}_q$. The function T thus defined, clearly additive when U and V are, will be denoted by UV. In particular,

we see easily that $\mathrm{L}_{p+q}=\mathrm{L}_p\mathrm{L}_q$, where L_p, L_q and L_{p+q} denote the volume in the spaces $\boldsymbol{R}_p$, $\boldsymbol{R}_q$ and $\boldsymbol{R}_{p+q}$ respectively (cf. § 2, p. 59).

It is known since Cauchy that, if I_1 and I_2 are respectively two intervals in the spaces $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, integration of any continuous function over the interval $I_1\times I_2\subset \boldsymbol{R}_{p+q}$ may be reduced to two successive integrations over the intervals I_1 and I_2. By repeating the process, any integral of a continuous function on an m-dimensional interval may be reduced to m successive integrations on linear intervals in $\boldsymbol{R}_1$. This classical theorem was extended by H. Lebesgue [1] to functions measurable ($\mathfrak{L}$) that are bounded, and then by G. Fubini [1] (cf. also L. Tonelli [2]) to all functions integrable (L), whether bounded or not. We shall state this result in the following form:

(8.1) ***Fubini's Theorem.*** *Suppose given two non-negative additive functions U and V of an interval in the spaces $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$ respectively, and let $f(x, y)$ be a non-negative function measurable ($\mathfrak{L}_{UV}$) in $\boldsymbol{R}_{p+q}$. Then*

(i_1) *$f(x, y)$ is a function of x, measurable ($\mathfrak{L}_U$) in $\boldsymbol{R}_p$ for every $y\in \boldsymbol{R}_q$, except at most a set of measure (V) zero,*

(i_2) *$f(x, y)$ is a function of y, measurable ($\mathfrak{L}_V$) in $\boldsymbol{R}_q$ for every $x\in \boldsymbol{R}_p$, except at most a set of measure (U) zero,*

(ii)
$$\int_{\boldsymbol{R}_{p+q}} f(x, y)\,dUV(x, y)=\int_{\boldsymbol{R}_q}\Big[\int_{\boldsymbol{R}_p} f(x, y)\,dU(x)\Big]\,dV(y)=$$
$$=\int_{\boldsymbol{R}_p}\Big[\int_{\boldsymbol{R}_q} f(x, y)\,dV(y)\Big]\,dU(x).$$

Proof. Let us write for short, $T=UV$. By symmetry, it is enough to show that every non-negative function $f(x, y)$ measurable ($\mathfrak{L}_T$) in $\boldsymbol{R}_{p+q}$ fulfills condition (i_1) and also the relation

$$\int_{\boldsymbol{R}_{p+q}} f(x, y)\,dT(x, y)=\int_{\boldsymbol{R}_q}\Big[\int_{\boldsymbol{R}_p} f(x, y)\,dU(x)\Big]\,dV(y). \tag{8.2}$$

For brevity, we shall say that a function $f(x, y)$ in $\boldsymbol{R}_{p+q}$ has the property (F), if it is non-negative and measurable ($\mathfrak{L}_T$) in $\boldsymbol{R}_{p+q}$, and if it fulfills condition (i_1) and the relation (8.2). For the sake of clearness, the reasoning that follows is divided into a several auxiliary propositions.

(8.3) *The sum of two functions with the property* (F), *and the limit of any non-decreasing sequence of such functions, have the property* (F). *Also, the difference of two functions with the property* (F), *has the property* (F), *provided that it is non-negative and that one at least of the given functions is finite and integrable* (T) *on the space* $\boldsymbol{R}_{p+q}$.

For the sum, and for the difference, of two functions, the statement is obvious. Let therefore $\{h_n(x, y)\}$ be a non-decreasing sequence of functions in $\boldsymbol{R}_{p+q}$ having the property (F), and let $h(x, y) = \lim_n h_n(x, y)$. The definite integrals $\int_{\boldsymbol{R}_p} h_n(x, y)\, dU(x)$ exist, and constitute a non-decreasing sequence, for every $y \in \boldsymbol{R}_q$ except at most those of a set of measure (V) zero. Consequently, by Lebesgue's theorem on integration of monotone sequences of functions:

$$\int_{\boldsymbol{R}_{p+q}} h(x,y)\, dT(x,y) = \lim_n \int_{\boldsymbol{R}_{p+q}} h_n(x, y)\, dT(x, y) = \lim_n \int_{\boldsymbol{R}_q} \Big[\int_{\boldsymbol{R}_p} h_n(x,y)\, dU(x)\Big]\, dV(y) =$$

$$= \int_{\boldsymbol{R}_q} \Big[\lim_n \int_{\boldsymbol{R}_p} h_n(x, y)\, dU(x)\Big]\, dV(y) = \int_{\boldsymbol{R}_q} \Big[\int_{\boldsymbol{R}_p} h(x, y)\, dU(x)\Big]\, dV(y),$$

and this establishes the property (F) for the function $h(x, y)$.

(8.4) *The characteristic function of any set* $E \subset \boldsymbol{R}_{p+q}$ *measurable* ($\mathfrak{L}_T$) *has the property* (F).

We shall establish this, first for very special sets E and then, by successive stages, for general measurable sets. Suppose in the first place that

1° $E = A \times B$, where A and B are intervals half open to the left, situated respectively in $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, and such that the oscillations of U and of V vanish at the boundaries of A and B respectively (cf. § 3, p. 60). The oscillation of the function $T = UV$ therefore vanishes at the boundary of the interval $E = A \times B$, and we find by Theorem 6.2

$$T^*(E) = T(\overline{E}) = U(\overline{A}) \cdot V(\overline{B}) = U^*(A) \cdot V^*(B). \tag{8.5}$$

On the other hand, for every $y \in \boldsymbol{R}_q$ the function $c_E(x, y)$ is in x the characteristic function either of the half open interval A, or of the empty set, according as $y \in B$ or $y \in \boldsymbol{R}_q - B$. This function is therefore measurable ($\mathfrak{L}_U$), and indeed measurable ($\mathfrak{B}$), for every $y \in \boldsymbol{R}_q$, and, by (8.5)

$$\int_{\boldsymbol{R}_{p+q}} c_E(x, y)\, dT(x, y) = T^*(E) = U^*(A) \cdot V^*(B) = \int_{\boldsymbol{R}_q} \Big[\int_{\boldsymbol{R}_p} c_E(x, y)\, dU(x)\Big]\, dV(y).$$

2° E is an open set. We shall begin by showing that, in this case, E is the sum of a sequence of half open intervals $\{I_n\}$ no two of which have common points, these intervals I_n being of the form $A_n \times B_n$ where (a) A_n and B_n are intervals, half open to the left, situated in $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$ respectively, and (b) the oscillations of U and V vanish at the boundaries of A_n and B_n respectively.

To see this, let $\{\mathfrak{U}^{(k)}\}$ be a regular sequence of nets in $\boldsymbol{R}_p$ formed of intervals half open to the left and such that the oscillation of U vanishes at the boundary of each of these intervals; by Theorems 4.2 and 2.1, such a sequence certainly exists. And let $\{\mathfrak{B}^{(k)}\}$ be a sequence of nets similarly constructed for the space $\boldsymbol{R}_q$ and for the function V. We denote, for each k, by $\mathfrak{T}^{(k)}$ the system of all half open intervals in $\boldsymbol{R}_{p+q}$ which are of the form $A \times B$ where $A \in \mathfrak{U}^{(k)}$ and $B \in \mathfrak{B}^{(k)}$. The systems of intervals $\mathfrak{T}^{(k)}$, thus defined, form a regular sequence of nets of half open intervals in the space $\boldsymbol{R}_{p+q}$. The set E being open, we can therefore express it (cf. § 2, p. 58) as the sum of a sequence of half open intervals $\{I_n\}$ taken from the nets $\mathfrak{T}^{(k)}$ and without points in common to any two. We see at once that each interval I_n of this sequence is of the form $A_n \times B_n$ where A_n and B_n satisfy conditions (a) and (b).

This being so, we have $c_E(x, y) = \sum_n c_{I_n}(x, y)$, where on account of the result established for the case 1°, each of the characteristic functions $c_{I_n}(x, y)$ has the property (F). Therefore, to verify that the function $c_E(x, y)$ also has this property, we need only apply (8.3).

3° E is a set ($\mathfrak{G}_\delta$). First suppose that, besides, the set E is bounded. E is then the limit of a descending sequence of bounded open sets $\{G_n\}$. The functions $c_{G_1}(x, y) - c_{G_n}(x, y)$ constitute a non-decreasing sequence of non-negative functions which have, by 2° and (8.3), the property (F). Consequently, again on account of (8.3), the limit function of this sequence $h(x, y) = c_{G_1}(x, y) - c_E(x, y)$ itself has the property (F) and the same is therefore true of the function $c_E(x, y) = c_{G_1}(x, y) - h(x, y)$.

Now if E is an arbitrary set ($\mathfrak{G}_\delta$), we can express it as the limit of an ascending sequence $\{H_n\}$ of bounded sets ($\mathfrak{G}_\delta$). By what has just been proved, the characteristic functions of the sets H_n have the property (F) and, consequently, the function $c_E(x,y)=\lim_n c_{H_n}(x,y)$ itself has the property (F).

4^0 E is a set of measure (T) zero. There is then, by Theorem 6.5, a set $H\in\mathfrak{G}_\delta$ containing E and of measure (T) zero. By the result established for sets $(\mathfrak{G}_\delta)$, the function $c_H(x,y)$ has the property (F), and therefore $\int\limits_{R_q}\Big[\int\limits_{R_p} c_H(x,y)\,dU(x)\Big]dV(y)=$ $=\int\limits_{R_{p+q}} c_H(x,y)\,dT(x,y)=T^*(H)=0$. Hence, for every $y\in R_q$, except at most a set Y of measure (V) zero, $\int\limits_{R_p} c_H(x,y)\,dU(x)=0$, i. e. $c_H(x,y)$, as function of x, vanishes almost everywhere (U) in R_p. Hence, *a fortiori*, $c_E(x,y)\leqslant c_H(x,y)$ as function of x, vanishes almost everywhere (U), and is consequently measurable $(\mathfrak{L}_U)$, for all $y\in R_q$, except at most for those of the set Y of measure (V) zero. Finally, we clearly have $\int\limits_{R_q}\Big[\int\limits_{R_p} c_E(x,y)\,dU(x)\Big]dV(y)=0=T^*(E)=\int\limits_{R_{p+q}} c_E(x,y)\,dT(x,y)$. The function $c_E(x,y)$ thus has the property (F).

On account of Theorem 6.6 every set E measurable $(\mathfrak{L}_T)$ is expressible in the form $E=H-Q$, where H is a set $(\mathfrak{G}_\delta)$ and Q is a set of measure (T) zero contained in H. We thus have $c_E(x,y)=$ $=c_H(x,y)-c_Q(x,y)$, and by (8.3) the proposition (8.4) reduces to the special cases 3^0 and 4^0 already treated.

The proposition (8.4) being thus established, let $f(x,y)$ be any non-negative function measurable $(\mathfrak{L}_T)$ in the space R_{p+q}. By Theorem 7.4, Chap. I, the function f is the limit of a non-decreasing sequence of simple functions, finite, non-negative, and measurable $(\mathfrak{L}_T)$. Now each of these simple functions is a linear combination, with positive coefficients, of a finite number of characteristic functions of sets measurable $(\mathfrak{L}_T)$, and therefore has the property (F) on account of (8.4). Thus the function f is the limit of a non-decreasing sequence of functions with the property (F), and so, by (8.3), f itself has the property (F). This completes the proof of Theorem 8.1.

Let us make special mention of the particular case of the theorem in which $f(x,y)$ is the characteristic function of a measurable set:

(8.6) ***Theorem.*** *If U and V are two non-negative additive functions of an interval in the spaces R_p and R_q respectively, and if Q is a set measurable $(\mathfrak{L}_{UV})$ in the space R_{p+q}, then*

(i$_1$) *the set $\mathop{E}\limits_x[(x,y)\in Q]$ is measurable $(\mathfrak{L}_U)$ for every $y\in R_q$, except at most a set of measure (V) zero,*

(i$_2$) *the set $\mathop{E}\limits_y[(x,y)\in Q]$ is measurable $(\mathfrak{L}_V)$ for every $x\in R_p$, except at most a set of measure (U) zero, and*

(ii) *the measure (UV) of Q is equal to*

$$\int\limits_{R_q} U^*\{\mathop{E}\limits_x[(x,y)\in Q]\}\,dV(y)=\int\limits_{R_p} V^*\{\mathop{E}\limits_y[(x,y)\in Q]\}\,dU(x).$$

Fubini's theorem is frequently stated in the following form:

(8.7) ***Theorem.*** *Let U and V be two additive functions of bounded variation of an interval in the spaces R_p and R_q respectively. Then for every function $f(x,y)$ integrable (UV) on R_{p+q}, the relation* (ii) *of theorem* 8.1 *holds good and the function $f(x,y)$ is integrable (U) in x on R_p for every $y\in R_q$, except at most a set of measure (V) zero, and integrable (V) in y on R_q for every $x\in R_p$, except at most a set of measure (U) zero.*

We reduce this statement at once to that of Theorem 8.1 by expressing the function f as the sum of its non-negative and non-positive parts, and by applying to the functions of an interval U and V the Jordan decomposition (§ 4, p. 62).

Further generalizations of Fubini's theorem for the Lebesgue-Stieltjes integration (in particular including the theorems analogous to Theorem 15.1 of Chap. I) were studied by L. C. Young in his Fellowship Dissertation (Cambridge 1931, unpublished). An account of these researches will be given in the book *The theory of Stieltjes integrals and distribution-functions* by L. C. Young (Oxford, Clarendon Press).

It follows in particular from Theorem 8.6 that for any set Q measurable in the sense of Lebesgue in the space R_{p+q}, its measure (L_{p+q}) is given by the definite integrals $\int\limits_{R_q} L_p\{\mathop{E}\limits_x[(x,y)\in Q]\}\,dL_q(y)=\int\limits_{R_p} L_q\{\mathop{E}\limits_y[(x,y)\in Q]\}\,dL_p(x)$. It is nevertheless to be remarked that the existence of these two integrals does not in general enable us to draw any conclusion as to measurability $(\mathfrak{L})$ of the set Q. W. Sierpiński [5] has in fact constructed in the plane a set non-measurable $(\mathfrak{L})$ having exactly one point in common with every parallel to the axes. This construction depends, needless to say, on the axiom of selection of Zermelo.

For an interesting discussion of Fubini's theorem for Lebesgue integration of functions of variable sign, *vide* G. Fichtenholz [1].

We complete the theorems of this § by the following

(8.8) ***Theorem.*** *If Q is a set measurable $(\mathfrak{B})$ in the space $\boldsymbol{R}_{p+q}$, the set $\underset{x}{\mathrm{E}}[(x,y)\in Q]$ is measurable $(\mathfrak{B})$ in the space $\boldsymbol{R}_p$ for every $y\in\boldsymbol{R}_q$, and the set $\underset{y}{\mathrm{E}}[(x,y)\in Q]$ is measurable $(\mathfrak{B})$ in $\boldsymbol{R}_q$ for every $x\in\boldsymbol{R}_p$.*

Similarly, if a function $f(x,y)$ is measurable $(\mathfrak{B})$ in the space $\boldsymbol{R}_{p+q}$, then in $\boldsymbol{R}_p$ the function $f(x,y)$ is measurable $(\mathfrak{B})$ in x for every $y\in\boldsymbol{R}_q$, and in $\boldsymbol{R}_q$ the function $f(x,y)$ is measurable $(\mathfrak{B})$ in y for every $x\in\boldsymbol{R}_p$.

Proof. It will be enough to prove the first half of the theorem, since the second half obviously follows from the first. Let us denote by $\mathfrak{B}_0$ the class of all sets Q in $\boldsymbol{R}_{p+q}$ such that the sets $\underset{x}{\mathrm{E}}[(x,y)\in Q]$, for every $y\in\boldsymbol{R}_q$, and the sets $\underset{y}{\mathrm{E}}[(x,y)\in Q]$, for every $x\in\boldsymbol{R}_p$, are measurable $(\mathfrak{B})$ in the spaces $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$ respectively. If a set $Q\subset\boldsymbol{R}_{p+q}$ is closed, so are the sets $\underset{x}{\mathrm{E}}[(x,y)\in Q]$ and $\underset{y}{\mathrm{E}}[(x,y)\in Q]$. The class $\mathfrak{B}_0$ thus contains all closed sets of the space $\boldsymbol{R}_{p+q}$, and on the other hand we see at once that $\mathfrak{B}_0$ is additive. It follows that $\mathfrak{B}_0$ includes all Borel sets in the space $\boldsymbol{R}_{p+q}$ (cf. the definition, Chap. II, p. 41), and this completes the proof.

* § 9. Fubini's theorem in abstract spaces.

We shall return in this § to the abstract considerations of Chap. I and show that for abstract spaces, theorems similar to those of the preceding § hold good.

Given any two sets X and Y, we shall denote by $X\times Y$ the set of all pairs of elements (x,y) for which $x\in X$ and $y\in Y$. The set $X\times Y$ is often called *combinatory product* or *Cartesian product* (cf. C. Kuratowski [I, p. 7]) of the sets X and Y. The following identities are obvious

(9.1) $$(X_1\times Y_1)\cdot(X_2\times Y_2)=(X_1\cdot X_2)\times(Y_1\cdot Y_2),$$

(9.2) $$(X_2\times Y_2)-(X_1\times Y_1)=[(X_2-X_1)\times Y_2]+[(X_2\cdot X_1)\times(Y_2-Y_1)],$$

the sets X_1, X_2, Y_1, Y_2 being quite arbitrary.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are additive classes of sets in the spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively, $\mathfrak{X}\mathfrak{Y}$ will denote the smallest additive class of sets in the space $\boldsymbol{X}\times\boldsymbol{Y}$, containing all product-sets of the form $X\times Y$, where $X\in\mathfrak{X}$ and $Y\in\mathfrak{Y}$.

For auxiliary purposes, we shall make use in this § of the following definition: a class $\mathfrak{N}$ of sets will be termed *normal*, if (i) the sum of every sequence of sets $(\mathfrak{N})$ no two of which have common points is itself a set $(\mathfrak{N})$ and (ii) the limit of every descending sequence of sets $(\mathfrak{N})$ is a set $(\mathfrak{N})$.

We shall begin by proving the following analogue of Theorem 8.8:

(9.3) ***Theorem.*** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two additive classes of sets in the spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. Then, if Q is a set measurable $(\mathfrak{X}\mathfrak{Y})$ in the space $\boldsymbol{X}\times\boldsymbol{Y}$, the set $\underset{x}{\mathrm{E}}[(x,y)\in Q]$ is measurable $(\mathfrak{X})$ for every $y\in\boldsymbol{Y}$, and the set $\underset{y}{\mathrm{E}}[(x,y)\in Q]$ is measurable $(\mathfrak{Y})$ for every $x\in\boldsymbol{X}$.*

In the same way a function $f(x,y)$ which is measurable $(\mathfrak{X}\mathfrak{Y})$ in the space $\boldsymbol{X}\times\boldsymbol{Y}$, is measurable $(\mathfrak{X})$ in x for every $y\in\boldsymbol{Y}$ and measurable $(\mathfrak{Y})$ in y for every $x\in\boldsymbol{X}$.

Proof. It is enough to prove only the first part concerning sets. To do this, we denote by $\mathfrak{M}$ the class of all sets Q in $\boldsymbol{X}\times\boldsymbol{Y}$ such that the set $\underset{x}{\mathrm{E}}[(x,y)\in Q]$ is measurable $(\mathfrak{X})$ for every $y\in\boldsymbol{Y}$, and that the set $\underset{y}{\mathrm{E}}[(x,y)\in Q]$ is measurable $(\mathfrak{Y})$ for every $x\in\boldsymbol{X}$. We see at once that the class $\mathfrak{M}$ is additive in the space $\boldsymbol{X}\times\boldsymbol{Y}$ and that, besides, it includes all sets $X\times Y$ for which $X\in\mathfrak{X}$ and $Y\in\mathfrak{Y}$. Hence $\mathfrak{X}\mathfrak{Y}\subset\mathfrak{M}$, and this completes the proof.

Before proceeding further we shall establish the following lemma:

(9.4) ***Lemma.*** *If $\mathfrak{X}$ and $\mathfrak{Y}$ are two additive classes of sets in the spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively, the class $\mathfrak{X}\mathfrak{Y}$ coincides with the smallest normal class that includes the sets $X\times Y$ for which $X\in\mathfrak{X}$ and $Y\in\mathfrak{Y}$.*

Proof. For brevity let us term elementary any set $X\times Y$ for which $X\in\mathfrak{X}$ and $Y\in\mathfrak{Y}$, and let $\mathfrak{N}_0$ denote the smallest normal class which includes all elementary sets (i. e. the common part of all the normal classes that include these sets). Clearly $\mathfrak{N}_0\subset\mathfrak{X}\mathfrak{Y}$ since $\mathfrak{X}\mathfrak{Y}$ is also a normal class. In order to establish the opposite inclusion, it is enough to prove that the class $\mathfrak{N}_0$ is additive, and this will be an immediate consequence of the following two properties of the class $\mathfrak{N}_0$:

(9.5) *The common part of any sequence of sets* $(\mathfrak{N}_0)$ *is itself a set* $(\mathfrak{N}_0)$.

(9.6) *The complement (with respect to the space* $X\times Y$*) of any set* $(\mathfrak{N}_0)$ *is again a set* $(\mathfrak{N}_0)$.

To prove (9.5), it is enough, since the class $\mathfrak{N}_0$ is normal, to show that the common part of two sets $(\mathfrak{N}_0)$ is a set $(\mathfrak{N}_0)$.

For this purpose, let $\mathfrak{N}_1$ be the class of all the sets $(\mathfrak{N}_0)$ whose common parts with every elementary set belong to $\mathfrak{N}_0$. From the identity (9.1), it follows that the common part of two elementary sets is an elementary set, and hence that $\mathfrak{N}_1$ includes all the elementary sets. On the other hand, we verify at once that $\mathfrak{N}_1$ is a normal class. This gives $\mathfrak{N}_0\subset\mathfrak{N}_1$, and since by definition $\mathfrak{N}_1\subset\mathfrak{N}_0$, we obtain $\mathfrak{N}_1=\mathfrak{N}_0$.

Let now $\mathfrak{N}_2$ be the class of all the sets $(\mathfrak{N}_0)$ whose common parts with every set $(\mathfrak{N}_0)$ belong to $\mathfrak{N}_0$. Since $\mathfrak{N}_1=\mathfrak{N}_0$, the class $\mathfrak{N}_2$ includes all elementary sets. Furthermore, $\mathfrak{N}_2$ is clearly a normal class. We therefore have $\mathfrak{N}_2=\mathfrak{N}_0$, and this proves (9.5).

To establish (9.6), let $\mathfrak{N}_3$ be the class of all the sets $(\mathfrak{N}_0)$ whose complements are also sets $(\mathfrak{N}_0)$. On account of the identity (9.2) the complement of any elementary set is the sum of two elementary sets without common points, and so, a set $(\mathfrak{N}_0)$. Therefore the class $\mathfrak{N}_3$ includes all elementary sets and, to conclude that $\mathfrak{N}_3=\mathfrak{N}_0$, it suffices to show that the class $\mathfrak{N}_3$ is normal.

Let therefore $\{X_n\}$ be any sequence of sets $(\mathfrak{N}_3)$ without common points to any two of them, and let X be the sum of the sequence. The set X clearly belongs to the class $\mathfrak{N}_0$. On the other hand, the sets CX_n are, by hypothesis, sets $(\mathfrak{N}_0)$; so that, by (9.5), the same is true of their product $CX=\prod_n CX_n$. Thus we have at the same time, $X\in\mathfrak{N}_0$ and $CX\in\mathfrak{N}_0$, and therefore $X\in\mathfrak{N}_3$.

Again, let $\{Y_n\}_{n=1,2,\dots}$ be a descending sequence of sets $(\mathfrak{N}_3)$, and Y its limit. The set Y clearly belongs to the class $\mathfrak{N}_0$. On the other hand, consider the identity

$$CY=\sum_{n=1}^{\infty}CY_n=CY_1+\sum_{n=1}^{\infty}Y_n\cdot CY_{n+1},$$

and observe that no two of the sets CY_1 and $Y_n\cdot CY_{n+1}$ for $n=1,2,\dots$ have common points. Since these sets belong, by (9.5), to the class $\mathfrak{N}_0$, so does the set CY. Thus we have both $Y\in\mathfrak{N}_0$ and $CY\in\mathfrak{N}_0$, whence $Y\in\mathfrak{N}_3$. The class $\mathfrak{N}_3$ is therefore normal, and this establishes (9.6) and completes the proof of Lemma 9.4.

We can restate Lemma 9.4 in the following more general form:

(9.7) *Given in an abstract space* T *a class* $\mathfrak{Q}$ *of sets additive in the weak sense, then the smallest class that is additive (in the complete sense) and contains* $\mathfrak{Q}$*, coincides with the smallest normal class containing* $\mathfrak{Q}$.

The proof is the same as for Lemma 9.4.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are additive classes in the spaces X and Y respectively, the finite sums of the sets $X\times Y$ for which $X\in\mathfrak{X}$ and $Y\in\mathfrak{Y}$, constitute, according to formulae (9.1) and (9.2), a class that is additive in the weak sense (*vide* Chap. I, p. 7) in the space $X\times Y$. Another example of a class of sets additive in the weak sense consists of the class of all the sets of an arbitrary metrical space M, that are both sets $(\mathfrak{G}_\delta)$ and $(\mathfrak{F}_\sigma)$. The smallest class that is additive (in the complete sense) and contains these sets is clearly the class of Borel sets in M.

The assertion of (9.7) enables us to prove easily the following theorem due to H. Hahn [2, p. 437] and in some respect analogous to Theorem 6.6:

Let $\mathfrak{Q}$ *be a class of sets, additive in the weak sense in a space* T*, and let* $\mathfrak{T}$ *be the smallest class of sets that is additive in the complete sense and contains* $\mathfrak{Q}$*. Suppose further that* τ *is a measure* $(\mathfrak{T})$ *such that the space* T *either has finite measure* (τ)*, or, more generally, is expressible as the sum of a sequence of sets of finite measure* (τ)*. Then (i) for every set* E *measurable* $(\mathfrak{T})$ *and for every* $\varepsilon>0$*, there exists a set* $F\in\mathfrak{Q}_\delta$*, and a set* $G\in\mathfrak{Q}_\sigma$*, such that* $F\subset E\subset G$ *and that* $\tau(E-F)<\varepsilon$ *and* $\tau(G-E)<\varepsilon$*; (ii) for every set* E *measurable* $(\mathfrak{T})$ *there exist a set* $(\mathfrak{Q}_{\delta\sigma})$ *contained in* E*, and a set* $(\mathfrak{Q}_{\sigma\delta})$ *containing* E*, which differ from* E *at most by sets of measure* (τ) *zero.*

(9.8) **Theorem.** *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be additive classes of sets in the spaces* X *and* Y *respectively, and let* μ *and* ν *be measures defined respectively for these classes. Suppose that* $\mu(X)<\infty$ *and* $\nu(Y)<\infty$*, or, more generally, that*

(9.9)
$$X=\sum_n X_n,\qquad Y=\sum_n Y_n$$
where $X_n\in\mathfrak{X}$, $Y_n\in\mathfrak{Y}$, $\mu(X_n)<\infty$ *and* $\nu(Y_n)<\infty$ *for* $n=1,2,\dots$

Then, for every set $Q\subset X\times Y$ *measurable* $(\mathfrak{X}\mathfrak{Y})$, (i) $\mu\{\underset{x}{\mathrm{E}}[(x,y)\in Q]\}$, *as function of* y*, is measurable* $(\mathfrak{Y})$ *in the space* Y *and* $\nu\{\underset{y}{\mathrm{E}}[(x,y)\in Q]\}$*, as function of* x*, is measurable* $(\mathfrak{X})$ *in the space* X*; furthermore*

(ii)
$$(\mathfrak{Y})\int_Y \mu\{\underset{x}{\mathrm{E}}[(x,y)\in Q]\}\,d\nu(y)=(\mathfrak{X})\int_X \nu\{\underset{y}{\mathrm{E}}[(x,y)\in Q]\}\,d\mu(x).$$

Proof. We may clearly suppose that no two sets X_n, and also no two sets Y_n, have common points. The same will then be true of the sets $X_n\times Y_m$ in the space $X\times Y$.

Let us denote by $\mathfrak{N}$ the class of all the sets P measurable $(\mathfrak{X}\mathfrak{Y})$ in the space $X\times Y$, such that conditions (i) and (ii) of the theorem hold good for every set $Q=P\cdot(X_n\times Y_m)$ where n and m are arbitrary positive integers. Since $Q=\sum_{n,m}Q\cdot(X_n\times Y_m)$ for every set $Q\subset\boldsymbol{X}\times\boldsymbol{Y}$, and since no two of the sets $X_n\times Y_m$ have common points, it follows easily from Lebesgue's Theorem 12.3, Chap. I, that every set Q belonging to the class $\mathfrak{N}$ fulfills conditions (i) and (ii). We have to prove that this class includes all sets measurable $(\mathfrak{X}\mathfrak{Y})$.

To do this, we observe that it follows at once from the identity (9.1) that every set $X\times Y$ for which $X\in\mathfrak{X}$ and $Y\in\mathfrak{Y}$, belongs to $\mathfrak{N}$. On the other hand, since by hypothesis $\mu(X_n)<\infty$ and $\nu(Y_n)<\infty$ for every n, we easily deduce from Lebesgue's Theorems 12.3 and 12.11 Chap. I, that the sum of any sequence of sets $(\mathfrak{N})$ no two of which have common points, and the limit of any descending sequence of sets $(\mathfrak{N})$, are themselves sets $(\mathfrak{N})$. The class $\mathfrak{N}$ is therefore normal, and by Lemma 9.4, contains all sets $(\mathfrak{X}\mathfrak{Y})$. This proves the theorem.

If we suppose the hypotheses of Theorem 9.8 satisfied, a measure can be defined for the class $\mathfrak{X}\mathfrak{Y}$ so as to correspond naturally to the measures μ and ν that are given for the classes $\mathfrak{X}$ and $\mathfrak{Y}$. We do this by calling *measure* $(\mu\nu)$ of a set Q measurable $(\mathfrak{X}\mathfrak{Y})$ the common value of the integrals (ii) of Theorem 9.8. It is immediate that we then have $\mu\nu(X\times Y)=\mu(X)\cdot\nu(Y)$ for every pair of sets $X\in\mathfrak{X}$ and $Y\in\mathfrak{Y}$.

This definition enables us to state Theorem 9.8 in a manner analogous to Theorem 8.6. But the analogy would be incomplete if we neglected to extend at the same time the class $\mathfrak{X}\mathfrak{Y}$. Thus, for instance if $\mathfrak{X}$ and $\mathfrak{Y}$ denote respectively the classes of sets measurable in the Lebesgue sense in Euclidean spaces $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, the class $\mathfrak{X}\mathfrak{Y}$ does not coincide with that of the sets measurable $(\mathfrak{L})$ in $\boldsymbol{R}_{p+q}$, although it is evidently included in the latter. The extension of the class $\mathfrak{X}\mathfrak{Y}$, that we require in the general case, will be defined as follows.

Given an additive class of sets $\mathfrak{T}$ and a measure τ associated with this class, we shall call the class $\mathfrak{T}$ *complete with respect to the measure* τ if it includes all subsets of sets $(\mathfrak{T})$ of measure (τ) zero. Thus for instance, if Γ denotes any measure of Carathéodory, the class $\mathfrak{L}_\Gamma$ is complete with respect to Γ (cf. Chap. II, p. 44), and in particular, the class $\mathfrak{L}$ in a Euclidean space is complete with respect to Lebesgue measure; whereas the class of sets measurable $(\mathfrak{B})$ is not complete with respect to that measure.

Every additive class of sets $\mathfrak{T}$ may be completed with respect to any measure τ defined for the class, i. e. there is always an additive class $\mathfrak{S}\supset\mathfrak{T}$ such that the function of a set τ can be continued as a measure on all sets $(\mathfrak{S})$ and such that $\mathfrak{S}$ is complete with respect to the measure τ thus continued. Among the classes $\mathfrak{S}$ of this kind, there is a smallest one that we shall denote by $\overline{\mathfrak{T}}^{\tau}$. As is seen directly, this class consists of all sets of the form $T-N_1+N_2$, where $T\in\mathfrak{T}$ and N_1, N_2 are arbitrary subsets of sets $(\mathfrak{T})$ of measure (τ) zero. The extension of the measure τ to all sets of this form is evident.

We can now state the following theorem which corresponds to Fubini's Theorem 8.1:

(9.10) ***Theorem.*** *Under the hypotheses of Theorem* 9.8, *if* $f(x,y)$ *is a non-negative function measurable* $(\overline{\mathfrak{X}\mathfrak{Y}}^{\mu\nu})$ *in the space* $\boldsymbol{X}\times\boldsymbol{Y}$,

(i_1) $f(x,y)$ *as function of* x *is measurable* $(\overline{\mathfrak{X}}^{\mu})$ *in* $\boldsymbol{X}$ *for every* $y\in\boldsymbol{Y}$, *except at most a set of measure* (ν) *zero;*

(i_2) $f(x,y)$ *as function of* y *is measurable* $(\overline{\mathfrak{Y}}^{\nu})$ *in* $\boldsymbol{Y}$ *for every* $x\in\boldsymbol{X}$, *except at most a set of measure* (μ) *zero;*

(ii) $$\int\limits_{\boldsymbol{X}\times\boldsymbol{Y}} f(x,y)\,d\mu\nu(x,y)=\int\limits_{\boldsymbol{Y}}\Big[\int\limits_{\boldsymbol{X}} f(x,y)\,d\mu(x)\Big]\,d\nu(y)=\int\limits_{\boldsymbol{X}}\Big[\int\limits_{\boldsymbol{Y}} f(x,y)\,d\nu(y)\Big]\,d\mu(x).$$

Proof. In the special case in which f is the characteristic function of a set measurable $(\mathfrak{X}\mathfrak{Y})$, the theorem is an immediate consequence of Theorem 9.8. The same is true when f is the characteristic function of a set measurable $(\overline{\mathfrak{X}\mathfrak{Y}}^{\mu\nu})$ of measure $(\mu\nu)$ zero, and in consequence Theorem 9.8 remains true when f is the characteristic function of any set $(\overline{\mathfrak{X}\mathfrak{Y}}^{\mu\nu})$.

This being so, we pass as usual to the case in which f is a finite function, simple and measurable $(\overline{\mathfrak{X}\mathfrak{Y}}^{\mu\nu})$ and finally, with the help of Theorems 7.4 and 12.6, Chap. I, to the general case in which f is any non-negative function measurable $(\overline{\mathfrak{X}\mathfrak{Y}}^{\mu\nu})$.

Condition (9.9) is essential to the validity of Theorems 9.8 and 9.10. To see this, let us consider some examples for which the condition is not fulfilled. Put $\boldsymbol{X}=\boldsymbol{Y}=\boldsymbol{R}_1$, and let $\mathfrak{X}=\mathfrak{Y}$ be the class of all sets in $\boldsymbol{R}_1$ that are measurable in the sense of Lebesgue. We choose for μ the ordinary Lebesgue measure, and we define the measure ν by making $\nu(Y)$ equal to the number of elements of Y (so that $\nu(Y)=\infty$ if Y is an infinite set). Finally, let Q be the set of all the points (x,x) in $\boldsymbol{R}_2=\boldsymbol{X}\times\boldsymbol{Y}$ such that $0\leqslant x\leqslant 1$. The integrals occuring in condition (ii) of Theorem 9.8 are then respectively 0 and 1 so that condition (ii) does not hold. (We could also, by a suitable modification of the set Q, choose $\boldsymbol{Y}=\boldsymbol{R}_2$ and take as measure ν the length Λ_1; cf. Chap. II, § 8.)

Another example showing the importance of condition (9.9) is due to A. Lindenbaum. Put $X=Y=R_1$, let $\mathfrak{X}=\mathfrak{Y}$ be the class of all Borel sets in R_1, and let $\mu(X)=\nu(X)$ denote for every set X the number of its elements. By a theorem of the theory of analytic sets (cf., for instance, C. Kuratowski [I, p. 261]) there exists in the plane $R_2=X\times Y$ a Borel set Q such that the set of $x\in R_1$ for which $\underset{y}{\mathrm{E}}[(x,y)\in Q]$ reduces to a single point, is not measurable in the sense of Borel. In other words, the set of $x\in X$ for which $\nu\{\underset{y}{\mathrm{E}}[(x,y)\in Q]\}=1$ is not measurable ($\mathfrak{X}$). Thus condition (i) of Theorem 9.8 does not hold.

For the results of this §, *vide* H. Hahn [2]; cf. also S. Ulam [2], Z. Łomnicki and S. Ulam [1], and W. Feller [1]. For a discussion of Fubini's theorem applied to functions whose values belong to an abstract vector space, *vide* also S. Bochner [2]. Finally we observe that certain theorems, analogous to those established in this § for measurable sets and sets of measure zero, can be stated for the property of Baire and the Baire categories. Cf. on this point C. Kuratowski and S. Ulam [1].

§ 10. Geometrical definition of the Lebesgue-Stieltjes integral. The geometrical definition of an integral is inspired by the older and more natural idea of regarding the integral as the measure of an "area", or of a "volume", attached to the function in a certain way that is well known.

Let us begin by fixing our notation. Given a function $f(x)$ defined on a set $Q\subset R_m$, we call *graph* of $f(x)$ on Q, and we denote by $\mathrm{B}(f;Q)$, the set of all points (x,y) of R_{m+1} for which $x\in Q$ and $y=f(x)\neq\infty$. If $f(x)$ is non-negative on Q, the set of all the points (x,y) of R_{m+1} such that $x\in Q$ and $0\leqslant y\leqslant f(x)$ is termed, according to C. Carathéodory, *ordinate-set* of f on Q and will be denoted by $\mathrm{A}(f;Q)$.

As in §§ 3—7 we shall suppose the space R_m fixed and a non-negative additive function U of an interval given in R_m. And in accordance with § 2, p. 59 and § 5, p. 65, L_1 denotes the Lebesgue measure in R_1.

(10.1) ***Lemma.*** *If $Q\subset R_m$ is a set measurable ($\mathfrak{L}_U$), the set $\underset{(x,y)}{\mathrm{E}}[x\in Q;\, a\leqslant y\leqslant b]$ in R_{m+1} is, for every pair of real numbers a and $b\geqslant a$, measurable ($\mathfrak{L}_{U\mathrm{L}_1}$), and its measure ($U\mathrm{L}_1$) is $(b-a)\cdot U^*(Q)$.*

Proof. Let us write for short, $Q_{a,b}=\underset{(x,y)}{\mathrm{E}}[x\in Q;\, a\leqslant y\leqslant b]$ and $T=U\mathrm{L}_1$. We shall begin by showing that if Q has measure (U) zero, the set $Q_{a,b}$ is of measure (T) zero, and so is certainly measurable ($\mathfrak{L}_T$).

To see this, observe that there is then, for any $\varepsilon>0$, a sequence of intervals $\{I_n\}$ in R_m such that $Q\subset\sum_n I_n^\circ$ and $\sum_n U(I_n)\leqslant\varepsilon$. Writing $J_n=I_n\times[a-\varepsilon,\, b+\varepsilon]$, we obtain a sequence of intervals $\{J_n\}$ in R_{m+1}, such that $Q_{a,b}\subset\sum_n J_n^\circ$ and $\sum_n T(J_n)=\sum_n U(I_n)\cdot(b-a+2\varepsilon)\leqslant(b-a+2\varepsilon)\cdot\varepsilon$. Thus $T^*(Q_{a,b})=0$.

Let now Q be any set measurable ($\mathfrak{L}_U$). By Theorem 6.6, Q is the sum of a sequence of closed sets and a set of measure (U) zero. Therefore, by the above, the set $Q_{a,b}$ is also the sum of a sequence of closed sets and of a set of measure (T) zero, and is thus measurable ($\mathfrak{L}_T$). Finally, for every real number y, we have $\underset{x}{\mathrm{E}}[(x,y)\in Q_{a,b}]=Q$ if $a\leqslant y\leqslant b$, and $\underset{x}{\mathrm{E}}[(x,y)\in Q_{a,b}]=0$ if y is outside the interval $[a,b]$. Hence, by Theorem 8.6, we have $T^*(Q_{a,b})=\int_a^b U^*(Q)\,dy=$ $=(b-a)\cdot U^*(Q)$, which completes the proof.

(10.2) ***Theorem.*** *If $f(x)$ is a function measurable ($\mathfrak{L}_U$) on a set $Q\subset R_m$, its graph on Q is of measure ($U\mathrm{L}_1$) zero.*

Proof. Since any set measurable ($\mathfrak{L}_U$) can be expressed as the sum of a sequence of bounded measurable sets, we can restrict ourselves to the case in which Q is bounded.

Let us fix an $\varepsilon>0$ and write $Q_n=\underset{x}{\mathrm{E}}[x\in Q;\, n\varepsilon\leqslant f(x)<(n+1)\varepsilon]$ for every integer n. By Lemma 10.1 the measure ($U\mathrm{L}_1$) of the graph of $f(x)$ on Q_n does not exceed $\varepsilon\cdot U^*(Q_n)$; therefore, on the whole set Q, it does not exceed $\varepsilon\cdot U^*(Q)$, and so vanishes.

We can now prove the following theorem which includes the geometrical definition of the Lebesgue integral:

(10.3) ***Theorem.*** *In order that a function $f(x)$ defined and non-negative on a set $Q\subset R_m$ measurable ($\mathfrak{L}_U$) be measurable ($\mathfrak{L}_U$) on Q, it is necessary and sufficient that its ordinate-set $\mathrm{A}(f;Q)$ on Q be measurable ($\mathfrak{L}_{U\mathrm{L}_1}$). When this condition is fulfilled, the definite integral (U) of f on Q is equal to the measure ($U\mathrm{L}_1$) of the set $\mathrm{A}(f;Q)$.*

Proof. Write, for short, $T=U\mathrm{L}_1$ and suppose first that $f(x)$ is a simple function, finite, non-negative, and measurable ($\mathfrak{L}_U$) on Q, i. e. that $f=\{v_1,Q_1;\, v_2,Q_2;\, \ldots;\, v_n,Q_n\}$ where Q_i are sets measurable ($\mathfrak{L}_U$)

no two of which have common points. By Lemma 10.1, all the sets $A(f; Q_i)$ are measurable $(\mathfrak{L}_T)$ and $T^*[A(f; Q_i)] = v_i \cdot U^*(Q_i)$ for $i = 1, 2, ..., n$. Hence, the set $A(f; Q) = \sum_i A(f; Q_i)$ is itself measurable $(\mathfrak{L}_T)$, and its measure (T) is equal to $\sum_i T^*[A(f; Q_i)] = \sum_i v_i \cdot U^*(Q_i) = \int_Q f(x)\, dU(x)$.

Let now f be any non-negative function measurable $(\mathfrak{L}_T)$ on Q. There is a non-decreasing sequence $\{h_n(x)\}$ of simple functions, finite, non-negative, and measurable $(\mathfrak{L}_U)$ on Q such that $f(x) = \lim_n h_n(x)$.

We then have

$$A(f; Q) = \lim_n A(h_n; Q) + B(f; Q). \tag{10.4}$$

Now, by the above, all the sets $A(h_n; Q)$ are measurable $(\mathfrak{L}_T)$ and $T^*[A(h_n; Q)] = \int_Q h_n\, dU$ for $n = 1, 2, ...$. On the other hand, by Theorem 10.2, the set $B(f; Q)$ has measure (T) zero. It therefore follows at once from (10.4) that the set $A(f; Q)$ is itself measurable $(\mathfrak{L}_T)$ and that

$$T^*[A(f; Q)] = \lim_n \int_Q h_n\, dU = \int_Q f\, dU.$$

It remains to prove that, if the set $A(f; Q)$ is measurable $(\mathfrak{L}_T)$, the function f is measurable $(\mathfrak{L}_U)$. To do this, write for short $A = A(f; Q)$, and observe that, for every non-negative number y, the set $\underset{x}{E}[x \in Q; f(x) \geqslant y]$ coincides with the set $\underset{x}{E}[(x, y) \in A]$. Thus, by Theorem 8.6, if A is measurable $(\mathfrak{L}_T)$, the set $\underset{x}{E}[x \in Q; f(x) \geqslant y]$ is measurable $(\mathfrak{L}_U)$ for all numbers y except at most those of a set of measure (L_1) zero. But this suffices for the measurability of f on Q (cf. Chap. I, (7.2)) and so completes the proof.

* **§ 11. Translations of sets.** As an application of Theorem 8.6, we shall prove in this § a theorem on parallel translations of sets. As a matter of course, in what follows, translations could be replaced by rotations, or by certain other transformations constituting continuous groups and preserving Lebesgue measure.

Given two points $x = (x_1, x_2, ..., x_m)$ and $y = (y_1, y_2, ..., y_m)$ in the space $\boldsymbol{R}_m$, we shall denote by $x + y$ the point $(x_1 + y_1, x_2 + y_2, ..., x_m + y_m)$ and by $|x|$ the number $(x_1^2 + x_2^2 + ... + x_m^2)^{1/2}$. We shall write $x \to 0$ when $|x| \to 0$. If Q is a set in the space $\boldsymbol{R}_m$ and a any point of this space, $Q^{(a)}$ will denote the set of all points $x + a$ where $x \in Q$. The set $Q^{(a)}$ is termed *translation of Q by the vector a*. If Φ is an additive function of a set in $\boldsymbol{R}_m$ and $a \in \boldsymbol{R}_m$, we shall write $\Phi^{(a)}(X) = \Phi(X^{(a)})$ for every set X bounded and measurable $(\mathfrak{B})$.

(11.1) ***Theorem.*** *If Q is a bounded set $(\mathfrak{B})$ of measure* (L) *zero in the space $\boldsymbol{R}_m$ and Φ is an additive function of a set $(\mathfrak{B})$ in $\boldsymbol{R}_m$, the function Φ vanishes for almost all translations of Q, i. e. $\Phi(Q^{(a)}) = \Phi^{(a)}(Q) = 0$ for almost all points a of $\boldsymbol{R}_m$.*

Proof. We may clearly assume Φ to be a non-negative function, and Q to be a bounded set $(\mathfrak{G}_\delta)$. Hence, by Theorem 6.10, there is a non-negative additive function U of an interval, such that $\Phi(X) = U^*(X)$ for every set X bounded and measurable $(\mathfrak{B})$.

Denote by $\widetilde{M}$, for any set $M \subset \boldsymbol{R}_m$, the set of all points (x, y) of the space $\boldsymbol{R}_{2m}$ which are such that $x \in \boldsymbol{R}_m$, $y \in \boldsymbol{R}_m$ and $x + y \in M$. The set $\widetilde{M}$ is clearly open whenever the set M is open. It follows at once that if M is a set $(\mathfrak{G}_\delta)$, so is the set $\widetilde{M}$. Finally, observe that for every point $z \in \boldsymbol{R}_m$ we have $\underset{x}{E}[(x, z) \in \widetilde{M}] = \underset{y}{E}[(z, y) \in \widetilde{M}] = M^{(-z)}$.

Since the given set Q is, by hypothesis, a set $(\mathfrak{G}_\delta)$, so is the set $\widetilde{Q}$, and by Theorem 8.6, $\int_{\boldsymbol{R}_m} U^*(Q^{(-z)})\, dL_m(z) = \int_{\boldsymbol{R}_m} L_m(Q^{(-z)})\, dU(z) = 0$, because all translations $Q^{(-z)}$ of the set Q are of measure (L_m) zero. Hence $\Phi(Q^{(-z)}) = U^*(Q^{(-z)}) = 0$ for every $z \in \boldsymbol{R}_m$, except at most a set of measure (L_m) zero. Replacing $-z$ by a, we obtain the required statement.

(11.2) ***Theorem.*** *Given an additive function of a set Φ, each of the following three conditions is both necessary and sufficient for the function Φ to be absolutely continuous*:

1^0 $\lim_{a \to 0} \Phi(Q^{(a)}) = \lim_{a \to 0} \Phi^{(a)}(Q) = \Phi(Q)$ *for every bounded set Q measurable $(\mathfrak{B})$ and of measure* (L) *zero*;

2^0 $\lim_{a \to 0} \Phi(Q^{(a)}) = \lim_{a \to 0} \Phi^{(a)}(Q) = \Phi(Q)$ *for every bounded set Q measurable $(\mathfrak{B})$*;

3^0 $\lim_{a \to 0} W[\Phi^{(a)} - \Phi; I] = 0$ *for every interval I.*

Proof. It is evidently sufficient to establish the necessity of condition 3⁰ and the sufficiency of condition 1⁰.

Suppose first that Φ is an absolutely continuous additive function of a set. In virtue of Theorem 14.11, Chap. I, Φ is thus the indefinite integral of a function f measurable ($\mathfrak{B}$). Let $I=[a_1, b_1; \ldots; a_m, b_m]$ be an interval in the space considered and let J be an interval containing I in its interior, for instance the interval $[a_1-1, b_1+1; \ldots; a_m-1, b_m+1]$.

Let ε be any positive number. Since the function $f(x)$ is integrable over J, there exists a number $\eta>0$ such that $\int_X |f(x)|\,dx<\varepsilon/3$ for every set $X\subset J$ measurable ($\mathfrak{B}$) and of measure (L) less than η. Therefore

$$(11.3)\qquad \int_X |f(x)|\,dx<\varepsilon/3 \quad and \quad \int_X |f(x+u)|\,dx=\int_{X^{(u)}} |f(x)|\,dx<\varepsilon/3$$
$$if \quad X\in\mathfrak{B}, \quad X\subset I, \quad |X|<\eta \quad and \quad |u|<1.$$

On the other hand, by Lusin's Theorem 7.1, there exists a closed set $F\subset I^\circ$ such that the function f is continuous on F and such that $|I-F|<\eta/2$. Let $\sigma<1$ be a positive number such that $F^{(u)}\subset I$ whenever $|u|<\sigma$, and such that

$$(11.4)\qquad |f(x+u)-f(x)|<\frac{\varepsilon}{3\cdot|I|} \quad whenever \quad x\in F,\ x+u\in F,\ and\ |u|<\sigma.$$

Let now a be any point of $\boldsymbol{R}_m$ such that $|a|<\sigma$. By (11.4)

$$(11.5)\qquad \int_{F\cdot F^{(-a)}} |f(x+a)-f(x)|\,dx\leqslant|I|\cdot(\varepsilon/3|I|)=\varepsilon/3.$$

On the other hand, $|I-F\cdot F^{(-a)}|\leqslant|I-F|+|I-F^{(-a)}|\leqslant 2\cdot|I-F|<\eta$, and therefore, by (11.3),

$$\int_{I-F\cdot F^{(-a)}} |f(x+a)-f(x)|\,dx<2\varepsilon/3.$$

If we add this inequality to (11.5) we obtain $\int_I |f(x+a)-f(x)|\,dx<\varepsilon$, i. e. the variation $\mathrm{W}[\Phi^{(a)}-\Phi; I]=\int_I |f(x+a)-f(x)|\,dx$ tends to 0 with $|a|$. The function Φ therefore fulfills condition 3⁰.

It remains to prove the sufficiency of condition 1⁰. Now, if the function Φ fulfills this condition, Φ vanishes by Theorem 11.1 for every bounded set measurable ($\mathfrak{B}$) of measure (L) zero, and so is absolutely continuous.

It was long known that every absolutely continuous function Φ fulfills conditions 1⁰, 2⁰ and 3⁰ of Theorem 11.2. The converse, however, (i. e. the sufficiency of these conditions, in order that the function Φ of a set be absolutely continuous) was established more recently. The sufficiency of condition 3⁰ was first proved by A. Plessner [1] (with the help of trigonometric series and for functions of a real variable). As regards the other conditions (1⁰ and 2⁰), and as regards Theorem 11.1, *vide* H. Milicer-Grużewska [1], and N. Wiener and R. C. Young [1]. In the text we have followed the method used by the latter authors.

§ 12. Absolutely continuous functions of an interval.

An additive function F of an interval will be termed *absolutely continuous on a figure* R_0, if to each $\varepsilon>0$ there corresponds a number $\eta>0$ such that for every figure $R\subset R_0$ the inequality $|R|<\eta$ implies $|F(R)|<\varepsilon$. In conformity with § 3, p. 59, we shall understand by absolute continuity in an open set G, absolute continuity on every figure $R\subset G$, and by absolute continuity, absolute continuity in the whole space.

Every additive function of an interval, absolutely continuous on a figure R_0, *is of bounded variation on* R_0. For, if F is a function that is absolutely continuous on R_0, there exists a number $\eta>0$ such that, for every figure $R\subset R_0$, the inequality $|R|<\eta$ implies $|F(R)|<1$. Therefore, if we subdivide R_0 into a finite number of intervals $I_1, I_2, \ldots, I_n$ of measure less than η, we obtain $\mathrm{W}(F; R_0)\leqslant\leqslant\sum_{k=1}^{n}\mathrm{W}(F; I_k)\leqslant 2n$.

An additive function of an interval F, of bounded variation on a figure R_0, will be termed *singular on* R_0, if for each $\varepsilon>0$ there exists a figure $R\subset R_0$ such that $|R|<\varepsilon$ and $\mathrm{W}(F; R_0\ominus R)<\varepsilon$.

The reader will observe the analogy between the above definitions and the criteria given in Theorems 13.2 and 13.3, Chap. I, in order that an additive function of a set should be absolutely continuous or singular. This analogy could be pushed further by introducing the notions of absolutely continuous function, and of singular function, with respect to a non-negative additive function of an interval. But this "relativization", although useful in certain cases, would not play an essential part in the remainder of this book.

The following theorem is, almost word for word, a duplicate of Theorem 13.1 of Chapter I.

(12.1) ***Theorem.*** *1° In order that an additive function of an interval be absolutely continuous [singular] on a figure R_0, it is necessary and sufficient that its two variations, the upper and the lower, should both be so. 2° Every linear combination, with constant coefficients, of two additive functions of an interval which are absolutely continuous [singular] on a figure R_0 is itself absolutely continuous [singular] on R_0. 3° The limit of a bounded monotone sequence of additive functions of an interval that are absolutely continuous [singular] on a figure R_0 is also absolutely continuous [singular] on R_0. 4° If an additive function of an interval is absolutely continuous [singular] on a figure R_0, the function is so on every figure $R \subset R_0$. 5° If an additive function of an interval is absolutely continuous [singular] on each of the figures R_1 and R_2, the function is so on the figure R_1+R_2. 6° An additive function of an interval cannot be both absolutely continuous and singular on a figure R_0, without vanishing identically on R_0.*

Part 3°, at most, perhaps requires a proof. (It differs slightly from the corresponding part of Theorem 13.1, Chap. I.) Let therefore F be the limit of a bounded monotone sequence $\{F_n\}$ of additive functions of an interval on a figure R_0. Let ε be any positive number. Since the functions $F-F_n$ are monotone on R_0, there exists a positive integer n_0 such that

(12.2) $|F(R)-F_{n_0}(R)| \leqslant |F(R_0)-F_{n_0}(R_0)| < \varepsilon/2$ *for every figure* $R \subset R_0$.

This being so, let us suppose that the functions F_n are absolutely continuous on R_0. There is then an $\eta>0$ such that, for every figure $R \subset R_0$, $|R|<\eta$ implies the inequality $|F_{n_0}(R)|<\varepsilon/2$ and therefore, by (12.2), the inequality $|F(R)|<\varepsilon$. The function F is thus absolutely continuous on R_0.

Suppose next that the functions F_n are singular on R_0. There is then a figure $R_1 \subset R_0$ such that $|R_1|<\varepsilon$ and $\mathrm{W}[F_{n_0}; R_0 \ominus R_1]<\varepsilon/2$. Hence, by (12.2), $\mathrm{W}[F; R_0 \ominus R_1]<\varepsilon$, which shows that the function F is singular. This completes the proof.

We shall now establish two simple theorems that show explicitly the connection between the absolutely continuous or singular functions of an interval and those of a set. To avoid misunderstanding, we draw the reader's attention to the abbreviations adopted in § 5, p. 66, in the terminology of functions of a set.

(12.3) ***Theorem.*** *In order that a non-negative additive function F of an interval be absolutely continuous, it is necessary and sufficient that the corresponding function of a set F^* should be so.*

Proof. Suppose that the function F is absolutely continuous. In order to prove that the function F^* is so too, it is enough to show that F^* vanishes on every bounded set of measure (L) zero. Let therefore E be such a set, and let J be an interval that contains E in its interior. For any $\varepsilon>0$, let η be a positive number such that

(12.4) $|R|<\eta$ *implies* $|F(R)|<\varepsilon$ *for every figure* $R \subset J$.

Since $|E|=0$, there exists a sequence of intervals $\{I_n\}$ in J such that

(12.5) $E \subset \sum_n I_n^\circ$ *and* $\sum_n |I_n| < \eta$.

Denote by R_k the sum of the k first intervals of this sequence. By Theorem 4.6 (or 6.1) of Chap. II, and Theorem 6.2, the relations (12.4) and (12.5) give $F^*(E) \leqslant \lim_k F^*(R_k^\circ) \leqslant \lim_k F(R_k) \leqslant \varepsilon$, from which it follows that $F^*(E)=0$.

Conversely, if F^* is an absolutely continuous function of a set, the absolute continuity of F follows at once from the inequality $F(R) \leqslant F^*(R)$ which holds by Theorem 6.2 for every figure R.

(12.6) ***Theorem.*** *In order that a non-negative additive function of an interval F be singular, it is necessary and sufficient that the corresponding function of a set F^* should be so.*

Proof. Suppose that the function of an interval F is singular, and let J be any interval. Given any number $\varepsilon>0$, there is then a figure $R \subset J$ such that $|R|<\varepsilon$ and $F(J \ominus R)<\varepsilon$. Consequently, by Theorem 6.2, we have $F^*(J^\circ - R) \leqslant F(J \ominus R)<\varepsilon$, which shows on account of Theorem 13.3, Chap. I, that the non-negative function of a set F^* is singular in the interior of every interval J, and therefore in the whole space.

Suppose, conversely, that the function of a set F^* is singular, and let ε be any positive number. Given any interval I there is then a set $E \subset I^\circ$ such that $|E|=0$ and $F^*(I^\circ - E)=0$. Consequently, there is a sequence of intervals $\{I_n\}$ in I such that

(12.7) $I^\circ - E \subset \sum_n I_n$ and (12.8) $\sum_n F(I_n) < \varepsilon$.

Denote by R_k the sum of the k first intervals of this sequence. Since $|E|=0$, we obtain from (12.7) that $|R_{k_0}|>|I|-\varepsilon$ for a sufficiently large k_0, and writing $P=I \ominus R_{k_0}$, this gives $|P|<\varepsilon$. Again, by (12.8), $F(I \ominus P)<\varepsilon$, which proves that the function F is singular.

§ 13. Functions of a real variable. The most important of the notions and theorems of this chapter were originally given a rather different form: they were made to refer, not to additive functions of an interval, but to functions of a real variable. It is, however, easy to establish between functions of a real variable and additive functions of a linear interval, a correspondence rendering it immaterial which of these two kinds of functions is considered.

To do this, let $f(x)$ be an arbitrary finite function of a real variable on an interval I_0. Let us term *increment* of $f(x)$ over any interval $I=[a, b]$ contained in I_0, the difference $f(b)-f(a)$. Thus defined the increment is an additive function of a linear interval $I\subset I_0$, and corresponds in a unique manner to the function $f(x)$. Conversely, if we are given any additive function $F(I)$ of a linear interval I, this in itself defines, except for an additive constant, a finite function of a real variable $f(x)$ whose increments on the intervals I coincide with the corresponding values of the function $F(I)$.

We shall understand by *upper*, *lower* and *absolute*, *variations* of a function of a real variable $f(x)$ on an interval I, the upper, lower, and absolute, variations of the increment of $f(x)$ over I. To denote these numbers, we shall use symbols similar, to those adopted for additive functions of an interval, i. e.: $\overline{W}(f; I)$, $\underline{W}(f; I)$, and $W(f; I)$.

A finite function will be termed *of bounded variation* on an interval I_0, if its increment is a function of an interval of bounded variation on I_0. Similarly the function is *absolutely continuous*, or *singular*, if its increment is absolutely continuous, or singular. As we see immediately, in order that a function $f(x)$ be of bounded variation on an interval I_0, it is necessary and sufficient that there exists a finite number M such that $\sum_i |f(b_i)-f(a_i)|<M$ for every sequence of non-overlapping intervals $\{[a_i, b_i]\}$ contained in I_0. Similarly, in order that $f(x)$ be absolutely continuous, it is necessary and sufficient that to each $\varepsilon>0$ there corresponds an $\eta>0$ such that $\sum_i |f(b_i)-f(a_i)|<\varepsilon$ for every sequence of non-overlapping intervals $\{[a_i, b_i]\}$ contained in I_0 and for which $\sum_i |b_i-a_i|<\eta$.

If $f(x)$ and $g(x)$ are two bounded functions on an interval I_0, and M denotes the upper bound of the absolute values of $f(x)$ and $g(x)$ on I_0, we have

$$|f(b)\,g(b)-f(a)\,g(a)|\leqslant M[|f(b)-f(a)|+|g(b)-g(a)|]$$

for every interval $[a, b]\subset I_0$. It follows at once that

(13.1) *The product of two functions of bounded variation [absolutely continuous] on an interval is itself of bounded variation [absolutely continuous] on this interval.*

Finally we see that if a function of an interval F corresponds to a finite function of a point f (i. e. is the increment of f), we have $o_I(f; a)=o_I(F; a)$ for any interval I and any point $a\in I$ (cf. Chap. II, § 3, p. 42, and the present Chapter, § 3, p. 60). Thus, in particular, in order that the function f be continuous at a point a according to the definition of § 3, Chap. II, it is necessary and sufficient that the function of an interval F that corresponds to f should be so according to the definition of § 3 of the present Chapter.

If at a point a a function of a real variable f has a unique, limit on the right, this limit will be denoted by $f(a+)$; similarly, $f(a-)$ will stand for a unique limit on the left. If the function f is defined in a neighbourhood of a point a and both limits $f(a+)$, $f(a-)$ exist, then the oscillation $o(f; a)$ (*vide* Chap. II, p. 42) is equal to the largest of the three numbers $|f(a+)-f(a-)|$, $|f(a+)-f(a)|$, and $|f(a-)-f(a)|$.

If both limits $f(a+)$ and $f(a-)$ exist and are finite, and $f(a)=\frac{1}{2}[f(a+)+f(a-)]$ the function $f(x)$ is termed *regular at the point a*. It is *regular* if it is regular at every point.

Let f be any function of a real variable, of bounded variation, and $\{a_n\}$ a sequence of points. Let us put $s(a)=0$ and

$$s(x)=\begin{cases} f(a+)-f(a)+\sum_n^{(a,x)}[f(a_n+)-f(a_n-)]+f(x)-f(x-) & \text{for } x>a\\ f(a-)-f(a)+\sum_n^{(a,x)}[f(a_n-)-f(a_n+)]+f(x)-f(x+) & \text{for } x<a,\end{cases}$$

where the summation $\sum_n^{(a,x)}$ is extended to all indices n such that $a<a_n<x$, when $x>a$, and $a>a_n>x$, when $x<a$. The function s thus defined is termed the *saltus-function of f corresponding to the sequence $\{a_n\}$ of points*. It is continuous everywhere except, perhaps,

at the points a_n, and by subtracting it from f we obtain a function of bounded variation, continuous at all points of continuity of f and, besides, at all the points a_n. If $\{a_n\}$ is the sequence of all points of discontinuity of f, the corresponding function s is called simply the *saltus-function of f*. By varying the fixed point a we get the various saltus-functions of f which can obviously differ only by constants. A function of bounded variation which is its own saltus-function, is called a *saltus-function.*

The functions of a real variable whose increments over each interval I coincide respectively with the variations $\overline{W}(f; I)$, $\underline{W}(f; I)$ and $W(f; I)$ of a function f, are also termed (*upper, lower,* and *absolute*) *variations of f*. By applying the Jordan decomposition (§ 4, p. 62), we can express any function of a real variable f of bounded variation as the sum of two functions that are respectively its upper and lower variations. Thus any function of bounded variation is the difference of two monotone non-decreasing functions, and consequently is measurable ($\mathfrak{B}$) and has at every point the two unilateral limits, on the right and left. Moreover, the set of its points of discontinuity is at most enumerable, since the sum of its oscillations at the points of discontinuity lying in any finite interval is always finite (this is actually the special case of Theorem 4.1).

In various cases it is more convenient to operate on functions of a real variable than on additive functions of an interval in R_1. The difference is, of course, only formal, and all the definitions adopted for functions of an interval can be stated, with obvious modifications, in terms of functions of a real variable. We need not state them here explicitly. If F is a function of a real variable, of bounded variation, the meaning of expressions such as Lebesgue-Stieltjes integral with respect to F, integral (F), sets ($\mathfrak{L}_F$), and so on, may be regarded as absolutely clear, in view of the definitions of § 5. If F is a continuous function and g a function integrable (F), the integral $\int_I g\,dF$, where I is a variable interval, is an additive continuous function of an interval I (*vide* § 5, p. 65). There is, consequently, a continuous function of a real variable whose increment on any interval I coincides with the definite integral (F) of g over I. This function, which is determined uniquely except for an additive constant, is also termed *indefinite integral (F) of g*.

When there is no ambiguity, the additive function of an interval that is determined by a finite function of a real variable F, will be denoted by the same letter F, i. e. $F(I)$ will stand for the increment of $F(x)$ on an interval I. By means of the corresponding function of an interval, any function of a real variable F of bounded variation determines an additive function of a set which we denote by F^* (cf. § 5). We see at once that $F^*(X)=F(b+)-F(a-)$ when $X=[a, b]$, and that $F^*(X)=F(a+)-F(a-)$ when $X=(a)$, i. e. when X is the set consisting of a single point a.

If $W(x)$ is the absolute variation of a function $F(x)$ of bounded variation, we clearly have $W(F^*; X)\leqslant W^*(X)$ for every set X bounded and measurable ($\mathfrak{B}$). The opposite inequality does not hold in general. If, for instance, X is a set consisting of one point only, and F is the characteristic function of X, then $W(F^*; X)=$ $=F^*(X)=0$, while $W^*(X)=2$. We can, however, state the following theorem:

(13.2) ***Theorem.*** *If $F(x)$ is a function of a real variable of bounded variation, and $W(x)$ is the absolute variation of $F(x)$, then $W(F^*; X)=$ $=W^*(X)$ for every set X bounded and measurable ($\mathfrak{B}$) at all points of which $F(x)$ is continuous.*

Proof. Suppose first that the set X is contained in an open interval J_0 in which the function $F(x)$, and consequently the function $W(x)$ also, is continuous. Let $G\subset J_0$ be an arbitrary open set such that $X\subset G$. Then, expressing G as the sum of a sequence of closed non-overlapping intervals $\{I_n\}$, we get $W^*(X)\leqslant W^*(G)=\sum_n W^*(I_n)=$ $=\sum_n W(I_n)\leqslant\sum_n W(F^*; I_n)=W(F^*; G)$; whence $W^*(X)\leqslant W(F^*; X)$, and since the opposite inequality is obvious, $W^*(X)=W(F^*; X)$.

Let us pass now to the general case. Let I_0 be an interval containing X in its interior, and let $\varepsilon>0$. Denote by $\{a_n\}$ the sequence of points of discontinuity of $F(x)$ interior to I_0, and by $S_N(x)$ the saltus-function of $F(x)$ corresponding to the points a_n for $n>N$. Let us put $G(x)=F(x)-S_N(x)$, where N is a positive integer sufficiently large in order that $W(S_N; I_0)\leqslant\varepsilon$. The points $a_1, a_2, \ldots, a_N$, none of which belongs to X, divide I_0 into a finite number of subintervals $J_0, J_1, \ldots, J_N$ in the interior of which the function $G(x)$ is continuous. Hence, denoting by $V(x)$ the absolute variation of

$G(x)$, there follows, by what has already been proved, $V^*(X\cdot J_k)=$ $=\mathrm{W}(G^*; X\cdot J_k)$ for $k=0,1,\ldots,N$, whence $V^*(X)=\mathrm{W}(G^*; X)$. On the other hand, $|W^*(X)-V^*(X)|$ and $|\mathrm{W}(F^*; X)-\mathrm{W}(G^*; X)|$ are both at most equal to $\mathrm{W}(S_N; I_0)\leqslant\varepsilon$. Thus $|W^*(X)-\mathrm{W}(F^*; X)|\leqslant 2\varepsilon$, and finally $W^*(X)=\mathrm{W}(F^*; X)$.

If $F(x)$ is a finite function of a real variable and E an arbitrary set in $\boldsymbol{R}_1$, the set of the values of $F(x)$ for $x\in E$ will be denoted by $F[E]$.

(13.3) ***Theorem.*** *If $F(x)$ is a function of a real variable of bounded variation and $W(x)$ is the absolute variation of $F(x)$, then $|F[E]|\leqslant W^*(E)$ for every set E in $\boldsymbol{R}_1$; and if further the function $F(x)$ is non-decreasing, and continuous at all points of E, then $|F[E]|=F^*(E)$.*

Proof. Let ε be a positive number and $\{I_n\}$ a sequence of intervals such that $E\subset\sum_n I_n$ and $W^*(E)+\varepsilon\geqslant\sum_n W(I_n)$. Then, if m_n and M_n denote the lower and upper bounds, respectively, of $F(x)$ on I_n, the sequence of intervals $\{[m_n, M_n]\}$ covers the set $F[E]$, and consequently $|F[E]|\leqslant\sum_n(M_n-m_n)\leqslant\sum_n W(I_n)\leqslant W^*(E)+\varepsilon$. Hence, $|F[E]|\leqslant W^*(E)$.

Suppose now $F(x)$ continuous at the points of E and non-decreasing. By what has already been proved, $|F[E]|\leqslant F^*(E)$. To establish the opposite inequality, let η be an arbitrary positive number, and $\{J_n\}$ a sequence of intervals subject to the conditions $F[E]\subset\sum_n J_n$ and $|F[E]|+\eta\geqslant\sum_n|J_n|$. Let E_n denote the set of the points $x\in E$ such that $F(x)\in J_n$. Then $F^*(E_n)\leqslant|J_n|$ for each n; and consequently $F^*(E)\leqslant\sum_n|J_n|\leqslant|F[E]|+\eta$, whence $F^*(E)\leqslant|F[E]|$.

The characteristic function of a set consisting of a single point provides the simplest example of a singular function of a real variable, that does not vanish identically. This function is however discontinuous. It is easy to give examples of functions of an interval that are additive, singular, continuous, and not identically zero, in the spaces $\boldsymbol{R}_m$ for $m\geqslant 2$. For simplicity, consider the plane, and denote, for any interval I, by $F(I)$ the length of the segment of the line $y=x$ contained in I; the function of an interval $F(I)$ will evidently have the desired properties. A similar example for $\boldsymbol{R}_1$ is less trivial. We shall therefore conclude this § with a short description of an elementary method for the construction of continuous singular functions of a real variable.

We shall begin with the following remark which frequently proves useful.

(13.4) *Let E be a linear, bounded, perfect and non-dense set, and α and $\beta>\alpha$ two arbitrary numbers. Then, if a and b denote the lower and upper bounds of E, a function $F(x)$ may be defined on the interval $J_0=[a,b]$ so as to satisfy the following conditions:* (i) *$F(a)=\alpha$, $F(b)=\beta$,* (ii) *$F(x)$ is constant on each interval contiguous to the set E, and* (iii) *$F(x)$ is continuous and non-decreasing on the interval J_0 and strictly increasing on the set E.*

To see this, let $\{I_n\}$ be the sequence of intervals contiguous to E, and let us agree to write $I_n\lessdot I_m$ whenever the interval I_n is situated on the left of I_m. By induction (cf. e. g. F. Hausdorff [II, p. 50]) we can easily establish a one-to-one correspondence between the intervals I_n and the rational numbers of the open interval (α,β) so that, denoting by $u(I_n)$ the number which corresponds to the interval I_n, the relation $I_n\lessdot I_m$ implies $u(I_n)<u(I_m)$. Let us now put $F(x)=u(I_n)$ for $x\in I_n$ where $n=1,2,\ldots,$ and then extend $F(x)$ by continuity to the whole of the interval J_0. We see at once that the function $F(x)$ thus obtained satisfies all the required conditions (i), (ii) and (iii) of (13.4).

Now let us choose for the set E in (13.4) a set of measure zero. Then if $\{I_n\}$ is the sequence of the intervals contiguous to E, we have

$$\mathrm{W}(F;\sum_{k=1}^{n} I_k)=\sum_{k=1}^{n}\mathrm{W}(F; I_k)=0$$

for each positive integer n; and since $|J_0-\sum_{k=1}^{n}I_k|\to 0$ as $n\to\infty$, the function $F(x)$ is evidently singular on the interval J_0.

The singular function obtained by the foregoing construction is continuous and monotone non-decreasing; the function is not constant on the whole interval J_0, but is so on certain partial intervals. Now, by the method of condensation of singularities, it is easy to derive from it a singular continuous function that increases everywhere.

To do this, suppose in (13.4), $|E|=0$, $a=\alpha=0$, $b=\beta=1$, and extend the function $F(x)$ on to the whole axis $\boldsymbol{R}_1$ by stipulating $F(x+1)=1+F(x)$. Write

$$H(x)=\sum_{n=1}^{\infty}\frac{F(nx)}{2^n}. \tag{13.5}$$

This series is a uniformly convergent series of singular functions, since $F(nx)$ is clearly singular with $F(x)$. Now the functions $F(nx)$ are monotone non-decreasing. By Theorem 12.1 (3^0), the function $H(x)$ is thus singular. This function is also continuous, as the limit of a uniformly convergent series. To prove that $H(x)$ is strictly increasing, let x_1 and $x_2>x_1$ denote an arbitrary pair of points in $[0,1]$. For $n>1/(x_2-x_1)$, we have $nx_2-nx_1>1$, and consequently $F(nx_2)>F(nx_1)$; while for every n, $F(nx_2)\geqslant F(nx_1)$, whence by (13.5), $H(x_2)>H(x_1)$ as asserted.

Various examples of this kind have been constructed by A. Denjoy [1], W. Sierpiński [3], H. Hahn [I, p. 538], L. C. Young [1] and G. Vitali [4]; cf. also O. D. Kellog [1], and E. Hille and J. D. Tamarkin [1].

§ 14. Integration by parts. As in the preceding §, we shall deal only in this § with functions of a real variable. For the latter, we shall establish two classical theorems, of importance on account of their many applications to various branches of Analysis. We shall first prove them for the Lebesgue-Stieltjes integral and then specialize them for the ordinary Lebesgue integral.

(14.1) ***Theorem on integration by parts.*** *If $U(x)$ and $V(x)$ are two functions of bounded variation, we have for every interval $I_0=[a,b]$*

$$\int_a^b U\,dV+\int_a^b V\,dU=U(b+)V(b+)-U(a-)V(a-),$$

provided that at each point of I_0 either one at least of the functions U and V is continuous, or both are regular.

Proof. In order to simplify the notation assume $a=0$ and $b=1$, and consider the triangle $Q=\underset{(x,y)}{\mathrm{E}}[0\leqslant x\leqslant 1; y\leqslant x]$ on the plane $\boldsymbol{R}_2$. The set $\underset{x}{\mathrm{E}}[(x,y)\,\varepsilon\, Q]$ is then the interval $[y,1]$ or the empty set, according as y belongs, or does not belong, to the interval $[0,1]$. Similarly, $\underset{y}{\mathrm{E}}[(x,y)\,\varepsilon\, Q]$ is the interval $[0,x]$ or the empty set, according as we have, or do not have, $0\leqslant x\leqslant 1$. Hence, by Fubini's theorem in the form (8.6),

$$\int_0^1[U(1+)-U(y-)]\,dV(y)=\int_0^1[V(x+)-V(0-)]\,dU(x),$$

i. e.

$$\int_0^1 U(x-)\,dV(x)+\int_0^1 V(x+)\,dU(x)=U(1+)V(1+)-U(0-)V(0-). \tag{14.2}$$

Interchanging U and V and adding the corresponding equation to (14.2), we get, on dividing by 2,

$$\int_0^1 \tfrac{1}{2}[U(x+)+U(x-)]\,dV(x)+\int_0^1 \tfrac{1}{2}[V(x+)+V(x-)]\,dU(x)= \\ =U(1+)V(1+)-U(0-)V(0-). \tag{14.3}$$

Let M be the set of the points in $I_0=[0,1]$ at which the function $U(x)$ is regular. Then

$$\int_M \tfrac{1}{2}[U(x+)+U(x-)]\,dV(x)=\int_M U(x)\,dV(x). \tag{14.4}$$

On the other hand, the set I_0-M is at most enumerable and, by hypothesis, the function $V(x)$, and consequently both its relative variations, are continuous at each point of I_0-M. Thus, the definite integral (V) of any function over the set I_0-M is zero, and it follows from (14.4) that $\int_0^1 \frac{1}{2}[U(x+)+U(x-)]\,dV(x)=\int_0^1 U(x)\,dV(x)$. Similarly, the second member on the left-hand side of the relation (14.3) is equal to $\int_0^1 V(x)\,dU(x)$, and this relation may be written

$$\int_0^1 U\,dV+\int_0^1 V\,dU=U(1+)V(1+)-U(0-)V(0-),$$

which proves the theorem.

The theorem may be also proved independently of Fubini's theorem, but then the proof is slightly longer. The proof given above was communicated to the author by L. C. Young.

(14.5) ***Second Mean Value Theorem.*** *If $U(x)$ and $V(x)$ are two non-decreasing functions and the function $V(x)$ is continuous, then in any interval $[a,b]$ there exists a point ξ such that*

$$\int_a^b U\,dV=U(a)\cdot[V(\xi)-V(a)]+U(b)\cdot[V(b)-V(\xi)]. \tag{14.6}$$

Proof. Since the values of $U(x)$ outside the interval $[a,b]$ do not affect (14.6), we may suppose that $U(a-)=U(a)$ and, $U(b+)=U(b)$. Therefore, making use of Theorem 14.1 and of the first mean value theorem (Chap. I, Th. 11.13), we obtain

$$\int_a^b U\,dV=U(b)V(b)-U(a)V(a)-\int_a^b V\,dU= \\ =U(b)V(b)-U(a)V(a)-\mu\cdot[U(b)-U(a)], \tag{14.7}$$

where μ is a number lying between the bounds of the function $V(x)$ on $[a,b]$. But, since this function is by hypothesis continuous, there exists in $[a,b]$ a point ξ such that $\mu=V(\xi)$. Substituting this value for μ in (14.7) we obtain the relation (14.6).

As a special case of Theorem 14.1 we have the following theorem on integration by parts for the Lebesgue integral:

(14.8) ***Theorem.*** *If* $u(x)$ *and* $v(x)$ *are two summable functions on an interval* $[a, b]$ *and* $U(x)$ *and* $V(x)$ *are their indefinite integrals* (L), *then*

$$(14.9) \qquad \int_b^b U(x)\,v(x)\,dx + \int_a^b V(x)\,u(x)\,dx = U(b)\cdot V(b) - U(a)\cdot V(a).$$

Proof. Observe first that by writing for instance $u(x)=0$ and $v(x)=0$ outside the interval $[a, b]$, we may suppose that the functions $u(x)$ and $v(x)$, and their indefinite integrals $U(x)$ and $V(x)$, are defined on the whole straight line $\boldsymbol{R}_1$. Also, by altering, if necessary, the values of the functions $u(x)$ and $v(x)$ on a set of measure (L) zero, which does not affect the values of the integrals in (14.9), we may suppose that these functions, together with the functions $U(x)\,v(x)$ and $V(x)\,u(x)$, are measurable ($\mathfrak{B}$) (cf. Theorem 7.6 of Vitali-Carathéodory or else Lusin's Theorem 7.1). We may, therefore write, according to Theorem 15.1, Chap. I,

$$\int_a^b U(x)\,v(x)\,dx = \int_a^b U(x)\,dV(x) \qquad \text{and} \qquad \int_a^b V(x)\,u(x)\,dx = \int_a^b V(x)\,dU(x),$$

and (14.9) follows at once from Theorem 14.1.

Similarly, we derive at once from Theorem 14.5 the *second mean value theorem for the Lebesgue integral*:

(14.10) ***Theorem.*** *If* $U(x)$ *is a non-decreasing function on an interval* $[a, b]$ *and* $v(x)$ *is a summable function on this interval, then*

$$\int_a^b U(x)\,v(x)\,dx = U(a)\int_a^\xi v(x)\,dx + U(b)\int_\xi^b v(x)\,dx,$$

where ξ *is a point of* $[a, b]$.