CHAPTER II.

Carathéodory measure.

§ 1. Preliminary remarks. In the preceding chapter, we supposed given a priori a certain class of sets, together with a measure defined for the sets of this class. A different procedure is usually adopted in theories dealing with special measures. We then begin by determining, as an outer measure, a non-negative function of a set, defined for all sets of the space considered, and it is only a posteriori that we determine a class of measurable sets for which the given outer measure is additive.

An abstract form of these theories, possessing both beauty and generality, is due to C. Carathéodory [I]. The account that we give of it in this chapter, is based on that of H. Hahn [I, Chap. VI], in which the results of Carathéodory are formulated for arbitrary metrical spaces. This account will be preceded by two §§ describing the notions that are fundamental in general metrical spaces.

§ 2. Metrical space. A space $\boldsymbol{M}$ is metrical if to each pair a and b of its points there corresponds a non-negative number $\varrho(a, b)$, called *distance* of the points a and b, that satisfies the following conditions: (i) $\varrho(a, b)=0$ is equivalent to $a=b$, (ii) $\varrho(a, b)=\varrho(b, a)$, (iii) $\varrho(a, b)+\varrho(b, c)\geqslant\varrho(a, c)$. In this chapter, we shall suppose that a metrical space $\boldsymbol{M}$ is fixed, and that all sets of points that arise, are located in $\boldsymbol{M}$.

The notation that we shall use, is as follows. A point a is *limit* of a sequence $\{a_n\}$ of points in $\boldsymbol{M}$, and we write $a=\lim\limits_n a_n$, if $\lim\limits_n \varrho(a, a_n)=0$. Every sequence possessing a limit point is said to be *convergent*. Given a set M, the upper bound of the numbers $\varrho(a,b)$ subject to $a \epsilon M$ and $b \epsilon M$ is called *diameter* of M and is denoted by $\delta(M)$. The set M is *bounded* if $\delta(M)$ is finite. For a class $\mathfrak{M}$ of sets, the upper bound of the numbers $\delta(M)$ subject to $M \epsilon \mathfrak{M}$ is denoted by $\Delta(\mathfrak{M})$ and called *characteristic number* of $\mathfrak{M}$. By the *distance* $\varrho(a, A)$ *of a point a and a set* A, we mean the lower bound of the numbers $\varrho(a,x)$ subject to $x \epsilon A$, and by *the distance* $\varrho(A,B)$ *of two sets* A and B, the lower bound of the numbers $\varrho(x,y)$ for $x \epsilon A$ and $y \epsilon B$.

We call *neighbourhood of a point a with radius* $r>0$, or *open sphere* $S(a;r)$ *of centre a and radius* r, the set of all points x such that $\varrho(a,x)<r$. The set of all points x such that $\varrho(a,x)\leqslant r$ is called *closed sphere of centre a and radius r*, and is denoted by $\bar{S}(a;r)$.

A point a is termed *point of accumulation of a set* A, if every neighbourhood of a contains infinitely many points of A. The set A' of all points of accumulation of A is termed *derived set* of A. The set $A+A'$, that we denote by $\bar{A}$, is termed *closure* of A. If $A=\bar{A}$, the set A is said to be *closed*. The points of a set, other than its points of accumulation, are termed *isolated*. A set is *isolated*, if all its points are isolated. We call *perfect*, any closed set not containing isolated points.

A point a of a set A is said to be an *internal point* of A, if there exists a neighbourhood of a contained in A. The set of all the internal points of a set A is called *interior* of A and denoted by A°. The set $\bar{A}-A^\circ$ is termed *boundary of A*. If $A=A^\circ$, the set A is said to be *open*. Two sets A and B are called *non-overlapping*, if $A\cdot B^\circ = B\cdot A^\circ = 0$.

The class of all open sets will be denoted by $\mathfrak{G}$ and that of all closed sets by $\mathfrak{F}$. In accordance with the convention adopted in § 2 of Chap. I, p. 5, open and closed sets will also be termed sets $(\mathfrak{G})$ and sets $(\mathfrak{F})$ respectively. We see at once that the complement of any set $(\mathfrak{G})$ is a set $(\mathfrak{F})$ and vice-versa.

The sum of a finite number or of an infinity of open sets, as well as the common part of a finite number of such sets, is always an open set. Any common part of a finite number, or of an infinity, of closed sets, and also any sum of a finite number of such sets,

are closed sets. Nevertheless, the sets $(\mathfrak{F}_\sigma)$ and $(\mathfrak{G}_\delta)$ (cf. Chap. I, § 2, p. 5) do not in general coincide with the sets $(\mathfrak{F})$ and $(\mathfrak{G})$, although every set $(\mathfrak{F})$ is clearly a set $(\mathfrak{F}_\sigma)$ and, at the same time, a set $(\mathfrak{G}_\delta)$; for, if F is a closed set and G_n denotes the set of the points x such that $\varrho(x,F)<1/n$, we have $F=\prod_n G_n$, where G_n are open. The corresponding result for the sets $(\mathfrak{G})$ is obtained by passing to the complementary sets. Moreover, it follows that any set expressible as the common part of a set $(\mathfrak{F})$ and a set $(\mathfrak{G})$ is both a set $(\mathfrak{F}_\sigma)$ and a set $(\mathfrak{G}_\delta)$.

We shall denote by $\mathfrak{B}$, the smallest additive class that includes all closed sets (cf. Chap. I, Th. 4.2). This class, clearly, includes also all sets $(\mathfrak{G}_\delta)$ and $(\mathfrak{F}_\sigma)$. The sets $(\mathfrak{B})$ are also termed *measurable* $(\mathfrak{B})$ (in accordance with Chap. I, § 4, p. 7). They are known as *Borel sets*.

We shall also give a few "relative" definitions having reference to a set M. The common part of M with any closed set is *closed in* M; we see at once that, for a set $P\subset M$ to be closed in M, it is necessary and sufficient that $P=M\cdot\bar{P}$, i. e. that the set P contains all its points of accumulation belonging to M. Similarly, any set expressible as the common part of M and an open set is termed *open in* M.

Any set of the form $M\cdot S(a;r)$, where $a\epsilon M$ and $r>0$, is called *portion* of M. If every portion of M contains points of a set A, i. e. if $\bar{A}\supset M$, the set A is said to be *everywhere dense in* M. If a set B is not everywhere dense in any portion of M, i. e. if no portion of M is contained in $\bar{B}$, the set B is said to be *non-dense in* M. In other words, a set B is non-dense in M, if, and only if, each portion of M contains a portion in which there are no points of B. It follows at once that the sum of a finite number of sets non-dense in the set M is itself non-dense in M. The sets expressible as sums of a finite or enumerably infinite number of sets non-dense in M are termed (according to R. Baire [1]) *sets of the first category in* M, and the sets not so expressible are termed *sets of the second category in* M. In all these terms, the expression "in M" is omitted when M coincides with the whole space; thus, by "non-dense sets", we mean sets whose closures contain no sphere and by "sets of the first category", enumerable sums of such sets.

A set M is called *separable*, if it contains an enumerable subset everywhere dense in M.

§ 3. Continuous and semi-continuous functions. If $f(x)$ is a function of a point, defined on a set A containing the point a, we shall denote by $\mathrm{M}_A(f; a; r)$ and $\mathrm{m}_A(f; a; r)$, respectively, the upper and lower bounds of the values assumed by $f(x)$ on the portion $A \cdot \mathrm{S}(a; r)$ of the set A. When r tends to 0, these two bounds converge monotonely towards two limits (finite or infinite) which we shall call respectively *maximum* and *minimum of the function $f(x)$ on the set A at the point a*, and denote by $\mathrm{M}_A(f; a)$ and $\mathrm{m}_A(f; a)$. Their difference $\mathrm{o}_A(f; a) = \mathrm{M}_A(f; a) - \mathrm{m}_A(f; a)$ will be called *oscillation of $f(x)$ on A at a*. We clearly have

(3.1) $$\mathrm{m}_A(f; a) \leqslant f(a) \leqslant \mathrm{M}_A(f; a) \quad \textit{for every point } a \in A.$$

If $f(a) = \mathrm{m}_A(f; a)$, the function $f(x)$ is said to be *lower semi-continuous on the set A at the point a*; similarly, if $f(a) = \mathrm{M}_A(f; a)$, the function $f(x)$ is *upper semi-continuous on A at a*. If both conditions hold together, and if $f(x)$ is finite at the point a, i. e. if $\mathrm{m}_A(f; a) = \mathrm{M}_A(f; a) \neq \infty$, the function $f(x)$ is termed *continuous on A at the point a*. Functions having the appropriate property at all points of the set A, will be termed simply *lower semi-continuous*, or *upper semi-continuous*, or *continuous*, *on A*. In all these terms and symbols, we usually omit all reference to A, when the latter is an open set (in particular, the whole space), or when A is kept fixed, in which case the omission causes no ambiguity.

From these definitions we conclude at once that, if $f(x)$ is upper semi-continuous, the function $-f(x)$ is lower semi-continuous, and vice-versa; and further, that, if two functions are upper (or lower) semi-continuous, so is their sum (supposing, of course, that the functions to be added do not assume at any point infinite values of opposite signs).

(3.2) ***Theorem.*** *For every function $f(x)$ defined on a set A, the set of the points of A at which $f(x)$ is not continuous on A, is the common part of the set A with a set $(\mathfrak{F}_\sigma)$.*

Proof. Let us denote by F_n the set of the points x of A at which either $f(x) = \pm\infty$, or $\mathrm{o}_A(f; x) \geqslant 1/n$. The set $F = \sum_n F_n$ consists of all the points of A at which the function $f(x)$ is not continuous. Now it is easy to see that each of the sets F_n is closed in A, i. e. that $F_n = A \cdot \overline{F}_n$. Therefore F is the common part of A and the set $\sum_n \overline{F}_n$, which is a set $(\mathfrak{F}_\sigma)$.

(3.3) ***Theorem.*** *For a function of a point $f(x)$ to be upper [lower] semi-continuous on a set A, it is necessary and sufficient that, for each number a, the set*

(3.4) $$\mathop{\mathrm{E}}_x [x \in A;\ f(x) \geqslant a] \qquad [\mathop{\mathrm{E}}_x [x \in A;\ f(x) \leqslant a]]$$

be closed in A, i. e. expressible as the common part of A with a set $(\mathfrak{F})$.

Proof. We need only consider the case of upper semi-continuous functions, as the other case follows by change of sign.

Let $f(x)$ be a function upper semi-continuous on A, a an arbitrary number, and $x_0 \in A$ a point of accumulation of the set (3.4). For each $r > 0$, the sphere $\mathrm{S}(x_0; r)$ then contains points of that set, and this requires $\mathrm{M}_A(f; x_0; r) \geqslant a$ and so $\mathrm{M}_A(f; x_0) \geqslant a$. Since by hypothesis $\mathrm{M}_A(f; x_0) = f(x_0)$, we derive $f(x_0) \geqslant a$, so that x_0 belongs to the set (3.4). This set is thus closed in A.

Suppose, conversely, that the set (3.4) is closed in A for each a. Since the relation $\mathrm{M}_A(f; x) = f(x)$ is evident for any x at which $f(x) = +\infty$, let x_0 be a point at which $f(x_0) < +\infty$, and a any number greater than $f(x_0)$. The set (3.4) is closed in A and does not contain x_0, and so, for a sufficiently small value r_a of r, contains no point of the sphere $\mathrm{S}(x_0; r)$. Thus $\mathrm{M}_A(f; x_0) \leqslant \mathrm{M}_A(f; x_0; r_a) \leqslant a$ for every number $a > f(x_0)$, and hence $\mathrm{M}_A(f; x_0) \leqslant f(x_0)$, which, by (3.1), requires $\mathrm{M}_A(f; x_0) = f(x_0)$.

An immediate consequence of Theorem 3.3 (cf. Chap. I, § 7, particularly p. 13) is the following

(3.5) ***Theorem.*** *Every function semi-continuous on a set $(\mathfrak{B})$ is measurable $(\mathfrak{B})$ on this set. More generally, if $\mathfrak{X}$ is any additive class of sets including all closed sets (and so all sets measurable $(\mathfrak{B})$), every function semi-continuous on a set $(\mathfrak{X})$ is measurable $(\mathfrak{X})$ on this set.*

§ 4. Carathéodory measure. A function of a set $\Gamma(X)$, defined and non-negative for all sets of the space $\boldsymbol{M}$, will be called *outer measure in the sense of Carathéodory*, if it fulfills the following conditions:

(C_1) $$\Gamma(X) \leqslant \Gamma(Y) \qquad \textit{whenever} \quad X \subset Y,$$

(C_2) $$\Gamma(\sum_i X_i) \leqslant \sum_i \Gamma(X_i) \qquad \textit{for each sequence } \{X_i\} \textit{ of sets},$$

(C_3) $$\Gamma(X+Y) = \Gamma(X) + \Gamma(Y) \qquad \textit{whenever} \quad \varrho(X, Y) > 0.$$

It should be noted that of these three conditions, the last one, only, has a metrical character. Now in this §, as well as in the §§ 5 and 6, we shall use only properties (C_1) and (C_2) of the Carathéodory measure. Hence all the results of these §§ remain valid in a perfectly arbitrary abstract space.

In order to simplify the wording, we shall suppose, in the rest of this chapter, except in § 8 which is concerned with certain special measures, that an outer Carathéodory measure $\Gamma(X)$ is uniquely determined in the space considered.

A set E will be termed *measurable with respect to the given outer measure* $\Gamma(X)$, if the relation $\Gamma(P+Q)=\Gamma(P)+\Gamma(Q)$ holds for every pair of sets P and Q contained, respectively, in the set E and in its complement CE; or, what amounts to the same, if $\Gamma(X)=\Gamma(X\cdot E)+\Gamma(X\cdot CE)$ holds for every set X. By condition (C_2) this last relation may be replaced by the inequality $\Gamma(X)\geqslant \Gamma(X\cdot E)+\Gamma(X\cdot CE)$.

The class of all the sets that are measurable with respect to Γ, will be denoted by $\mathfrak{L}_\Gamma$. We see at once that this class includes all the sets X for which $\Gamma(X)=0$ (in particular, it includes the empty set). Moreover it is clear that complements of sets $(\mathfrak{L}_\Gamma)$ are also sets $(\mathfrak{L}_\Gamma)$.

The main object of this § is to establish the a d d i t i v i t y of the class $\mathfrak{L}_\Gamma$ (in the sense of Chap. I, § 4) and to prove that the function $\Gamma(X)$ is a m e a s u r e $(\mathfrak{L}_\Gamma)$ in the sense of Chap. I, § 9. This result will constitute Theorems 4.1 and 4.5.

(4.1) ***Theorem.*** *If S is the sum of a sequence $\{X_n\}_{n=1,2,\dots}$ of sets $(\mathfrak{L}_\Gamma)$ no two of which have common points, the set S is again a set $(\mathfrak{L}_\Gamma)$ and $\Gamma(S)=\sum_n \Gamma(X_n)$; more generally, for each set Q*

(4.2) $$\Gamma(Q)=\sum_n \Gamma(Q\cdot X_n)+\Gamma(Q\cdot CS).$$

Proof. Let $S_k=\sum_{n=1}^{k} X_n$. We begin by proving inductively that all the sets S_k are measurable with respect to Γ, and that, for each k and for every set Q,

(4.3) $$\Gamma(Q)\geqslant\sum_{n=1}^{k}\Gamma(Q\cdot X_n)+\Gamma(Q\cdot CS_k).$$

Suppose indeed, that S_p is a set $(\mathfrak{L}_\Gamma)$ and that the inequality (4.3) holds for every set Q, when $k=p$. Since X_{p+1} is, by hypothesis, a set $(\mathfrak{L}_\Gamma)$ and $S_p\cdot X_{p+1}=0$, we then have

$$\begin{aligned}\Gamma(Q)&=\Gamma(Q\cdot X_{p+1})+\Gamma(Q\cdot CX_{p+1})=\\&=\Gamma(Q\cdot X_{p+1})+\Gamma(Q\cdot CX_{p+1}\cdot S_p)+\Gamma(Q\cdot CX_{p+1}\cdot CS_p)=\\&=\Gamma(Q\cdot X_{p+1})+\Gamma(Q\cdot S_p)+\Gamma(Q\cdot CS_{p+1})\geqslant\sum_{n=1}^{p+1}\Gamma(Q\cdot X_n)+\Gamma(Q\cdot CS_{p+1})\end{aligned}$$

and this is (4.3) for $k=p+1$. In view of condition (C_2), p. 43, it follows further that $\Gamma(Q)\geqslant\Gamma(Q\cdot S_{p+1})+\Gamma(Q\cdot CS_{p+1})$, which proves that S_{p+1} is a set $(\mathfrak{L}_\Gamma)$.

Combining the inequality (4.3), thus established, with the inequality $\Gamma(Q\cdot CS_k)\geqslant\Gamma(Q\cdot CS)$, we obtain, by making $k\to\infty$, the inequality $\Gamma(Q)\geqslant\sum_{n=1}^{\infty}\Gamma(Q\cdot X_n)+\Gamma(Q\cdot CS)$, and from this (4.2) follows on account of condition (C_2).

Finally, the same condition enables us to derive from (4.2) that $\Gamma(Q)\geqslant\Gamma(Q\cdot S)+\Gamma(Q\cdot CS)$, and this shows that S is a set $(\mathfrak{L}_\Gamma)$ and completes the proof.

(4.4) ***Lemma.*** *The difference of two sets $(\mathfrak{L}_\Gamma)$ is itself a set $(\mathfrak{L}_\Gamma)$.*

Proof. Let $X\in\mathfrak{L}_\Gamma$ and $Y\in\mathfrak{L}_\Gamma$, and let P and Q be any two sets such that $P\subset X-Y$ and $Q\subset C(X-Y)$. Write $Q_1=Q\cdot Y$ and $Q_2=Q\cdot CY$. Making successive use of the three pairs of inclusions $Q_1\subset Y$, $Q_2\subset CY$; $P\subset X$, $Q_2\subset C(X-Y)\cdot CY\subset CX$; and $Q_1\subset Y$, $P+Q_2\subset CY$, we find $\Gamma(P)+\Gamma(Q)=\Gamma(P)+\Gamma(Q_1)+\Gamma(Q_2)=$ $=\Gamma(P+Q_2)+\Gamma(Q_1)=\Gamma(P+Q)$, which shows that $X-Y$ is a set $(\mathfrak{L}_\Gamma)$.

(4.5) ***Theorem.*** *$\mathfrak{L}_\Gamma$ is an additive class of sets in the space* $\boldsymbol{M}$.

Proof. We have already remarked (p. 44) that the empty set and that complements of sets $(\mathfrak{L}_\Gamma)$ are sets $(\mathfrak{L}_\Gamma)$. To verify the third condition (iii) for additivity (cf. Chap. I, § 4, p. 7), let us observe firstly that, on account of Lemma 4.4 and of the identity $X\cdot Y=X-CY$, the common part of any two sets $(\mathfrak{L}_\Gamma)$ is itself a set $(\mathfrak{L}_\Gamma)$. This result extends by induction to common parts of any finite number of sets $(\mathfrak{L}_\Gamma)$ and, with the help of the identity $\sum_i X_i=C\prod_i CX_i$, we pass to the similar result for finite sums of sets. Finally, if X is the sum of an infinite sequence $\{X_n\}_{n=1,2,\dots}$ of sets $(\mathfrak{L}_\Gamma)$, we have $X=S_1+\sum_{n=1}^{\infty}(S_{n+1}-S_n)$ where $S_n=\sum_{k=1}^{n}X_k$. Now, clearly, of the sets S_1 and $S_{n+1}-S_n$, no two have common points, and, moreover, by the results already proved, they all belong to the class $\mathfrak{L}_\Gamma$. Consequently, to ascertain that X is a set $(\mathfrak{L}_\Gamma)$, we have only to apply Theorem 4.1. The class $\mathfrak{L}_\Gamma$ is thus additive.

Theorem 4.5 connects the considerations of this chapter with those of the preceding one. Thus, in accordance with the conventions adopted in Chap. I, pp. 7 and 16, the sets $(\mathfrak{L}_\Gamma)$ may be termed sets measurable $(\mathfrak{L}_\Gamma)$, and $\Gamma(X)$ may, for $X \in \mathfrak{L}_\Gamma$, be regarded as a measure associated with the class $\mathfrak{L}_\Gamma$. This class, together with the measure Γ, determines further the notions of functions measurable $(\mathfrak{L}_\Gamma)$, of integral $(\mathfrak{L}_\Gamma, \Gamma)$, of additive function of a set $(\mathfrak{L}_\Gamma)$ absolutely continuous $(\mathfrak{L}_\Gamma, \Gamma)$, and the other notions defined generally in Chap. I. Since the outer measure Γ determines already the class $\mathfrak{L}_\Gamma$, we shall omit in the sequel the symbol representing this class, whenever the notation makes explicit reference to the outer measure; thus we shall say "function integrable (Γ)" instead of "function integrable $(\mathfrak{L}_\Gamma, \Gamma)$" and the integral (Γ) of a function $f(x)$ over a set E will be denoted simply by $\int_E f(x)\,d\Gamma(x)$, instead of by $(\mathfrak{L}_\Gamma)\int_E f(x)\,d\Gamma(x)$.

In accordance with Chap. I, § 9, the value taken by $\Gamma(X)$ for a set X measurable $(\mathfrak{L}_\Gamma)$ will be termed *measure* (Γ) of X; when X is quite arbitrary, this value will be called its *outer measure* (Γ).

If E_0 is a subset of a set E such that $\Gamma(E - E_0) = 0$, then for any function $f(x)$ on E the measurability $(\mathfrak{L}_\Gamma)$ of f on E is equivalent to its measurability $(\mathfrak{L}_\Gamma)$ on E_0. This remark and Theorem 11.8, Chap. I, justify the following convention:

If a function $f(x)$ is defined only almost everywhere (Γ) on a set E, then, E_0 denoting the set of the points of E at which $f(x)$ is defined, by *measurability* $(\mathfrak{L}_\Gamma)$, *integrability* (Γ) and *integral* (Γ) *of* f *on the set* E we shall mean those on the set E_0.

Let us note two further theorems.

(4.6) ***Theorem.*** *Given an arbitrary set* E, (i) $\Gamma(E \cdot \sum_n X_n) = \sum_n \Gamma(E \cdot X_n)$ *for every sequence* $\{X_n\}$ *of sets measurable* $(\mathfrak{L}_\Gamma)$ *no two of which have common points,* (ii) $\Gamma(E \cdot \lim_n X_n) = \lim_n \Gamma(E \cdot X_n)$ *for every ascending sequence* $\{X_n\}$ *of sets measurable* $(\mathfrak{L}_\Gamma)$, *and this relation remains valid for descending sequences provided, however, that* $\Gamma(E \cdot X_1) \neq \infty$, (iii) *more generally, for every sequence* $\{X_n\}$ *of sets measurable* $(\mathfrak{L}_\Gamma)$ $\Gamma(E \cdot \liminf_n X_n) \leqslant \liminf_n \Gamma(E \cdot X_n)$, *and, if further* $\Gamma(E \cdot \sum_n X_n) \neq \infty$, *then also* $\Gamma(E \cdot \limsup_n X_n) \geqslant \limsup_n \Gamma(E \cdot X_n)$.

Part (i) of this theorem is contained in Theorem 4.1, and parts (ii) and (iii) follow easily from (i) (cf. Chap. I, the proofs of Theorems 5.1 and 9.1).

A part of Theorem 4.6 will be slightly further generalized. Given a set E, let us denote, for any set X, by $\Gamma^0_E(X)$ the lower bound of the values taken by $\Gamma(E \cdot Y)$ for the sets Y measurable $(\mathfrak{L}_\Gamma)$ that contain X.

(4.7) ***Theorem.*** *Given a set* E, (i) *to every set* X *corresponds a set* $X^0 \supset X$ *measurable* $(\mathfrak{L}_\Gamma)$ *such that* $\Gamma^0_E(X) = \Gamma(E \cdot X^0)$, (ii) $\Gamma(E \cdot \liminf_n X_n) \leqslant \Gamma^0_E(\liminf_n X_n) \leqslant \liminf_n \Gamma^0_E(X_n)$ *for every sequence* $\{X_n\}$ *of sets, and, in particular,* $\Gamma(E \cdot \lim_n X_n) \leqslant \Gamma^0_E(\lim_n X_n) = \lim_n \Gamma^0_E(X_n)$ *for every ascending sequence* $\{X_n\}$.

Proof. *re* (i). For every positive integer n there is a set $Y_n \supset X$, measurable $(\mathfrak{L}_\Gamma)$, such that $\Gamma(E \cdot Y_n) \leqslant \Gamma^0_E(X) + 1/n$. Writing $X^0 = \prod_n Y_n$, we verify at once that the set X^0 has the required properties.

re (ii). Taking (i) into account, let us associate with each set X_n a set $X_n^0 \supset X_n$, measurable $(\mathfrak{L}_\Gamma)$ and such that $\Gamma(E \cdot X_n^0) = \Gamma^0_E(X_n)$. The set $\liminf_n X_n^0 \supset \liminf_n X_n$ is measurable $(\mathfrak{L}_\Gamma)$ and, we therefore have, by Theorem 4.6 (iii)

$$\Gamma^0_E(\liminf_n X_n) \leqslant \Gamma(E \cdot \liminf_n X_n^0) \leqslant \liminf_n \Gamma(E \cdot X_n^0) = \liminf_n \Gamma^0_E(X_n).$$

The second part of (ii) follows at once from the first part.

* **§ 5. The operation (A).** We shall establish here that measurability $(\mathfrak{L}_\Gamma)$ is an invariant of a more general operation than those of addition and multiplication of sets.

We call *determining system*, any class of sets $\mathfrak{A} = \{A_{n_1, n_2, \ldots, n_k}\}$ in which with each finite sequence of positive integers $n_1, n_2, \ldots, n_k$ there is associated a set $A_{n_1, n_2, \ldots, n_k}$. The set

$$(5.1) \qquad \sum_{n_1, n_2, \ldots, n_k, \ldots} A_{n_1} \cdot A_{n_1, n_2} \cdot \ldots \cdot A_{n_1, n_2, \ldots, n_k} \cdot \ldots$$

where the summation extends over all infinite sequences of indices $n_1, n_2, \ldots, n_k, \ldots$, is called *nucleus* of the determining system $\mathfrak{A}$ and denoted by $\mathrm{N}(\mathfrak{A})$. The operation leading from a determining system to its nucleus is often called the *operation* (A).

The operation (A) was first defined by M. Souslin [I] in 1917. When applied to Borel sets, it leads to a wide class of sets (following N. Lusin, we call them *analytic*) and these play an important part in the theory of sets, in the theory of real functions, and even in some problems of classical type. A systematic account of the theory of these sets will be found in the treatises of H. Hahn [II], F. Hausdorff [II], C. Kuratowski [I], N. Lusin [II] and W. Sierpiński [II].

We mentioned at the beginning of this § that the operation (A) includes those of addition and of multiplication of sets. This remark must be understood as follows: *If $\mathfrak{M}$ is a class of sets such that the nucleus of every determining system formed of sets* ($\mathfrak{M}$) *itself belongs to $\mathfrak{M}$, then the sum and the common part of every sequence $\{N_i\}$ of sets* ($\mathfrak{M}$) *are also sets* ($\mathfrak{M}$). In fact, writing $P_{n_1, n_2, \ldots, n_k} = N_{n_1}$ and $Q_{n_1, n_2, \ldots, n_k} = N_k$ we see at once that the nuclei of the determining systems $\{P_{n_1, n_2, \ldots, n_k}\}$ and $\{Q_{n_1, n_2, \ldots, n_k}\}$ coincide respectively with the sum and with the common part of the sequence $\{N_i\}$. Thus, Theorem 5.5, now to be proved, will complete the result contained in Theorem 4.5, and in conjunction with Theorem 7.4, establish measurability ($\mathfrak{L}_\Gamma$) for analytic sets in any metrical space (cf. N. Lusin and W. Sierpiński [1], N. Lusin [3, pp. 25—26], and W. Sierpiński [12; 15]). The proof of this can be simplified if we assume regularity of the outer measure Γ (cf. § 6) (see C. Kuratowski [I, p. 58]).

With every determining system $\mathfrak{A} = \{A_{n_1, n_2, \ldots, n_k}\}$, we shall also associate the following sets. $\mathrm{N}^{h_1, h_2, \ldots, h_s}(\mathfrak{A})$ will denote, for each finite sequence $h_1, h_2, \ldots, h_s$ of positive integers, the sum (5.1) extended over all sequences, $n_1, n_2, \ldots, n_k, \ldots$ such that $n_i \leqslant h_i$ for $i = 1, 2, \ldots, s$. We see at once that the sequence $\{\mathrm{N}^h(\mathfrak{A})\}_{h=1,2,\ldots}$, together with every sequence $\{\mathrm{N}^{h_1, h_2, \ldots, h_k, h}(\mathfrak{A})\}_{h=1,2,\ldots}$ is monotone ascending and that

$$(5.2) \qquad \mathrm{N}(\mathfrak{A}) = \lim_h \mathrm{N}^h(\mathfrak{A}), \qquad \mathrm{N}^{h_1, h_2, \ldots, h_k}(\mathfrak{A}) = \lim_h \mathrm{N}^{h_1, h_2, \ldots, h_k, h}(\mathfrak{A}).$$

Further, for every sequence of positive integers $h_1, h_2, \ldots, h_k, \ldots$, we shall write

$$\mathrm{N}_{h_1}(\mathfrak{A}) = \sum_{n_1 \leqq h_1} A_{n_1}, \quad \mathrm{N}_{h_1, h_2}(\mathfrak{A}) = \sum_{n_1 \leqq h_1,\, n_2 \leqq h_2} A_{n_1} \cdot A_{n_1, n_2}, \quad . \quad . \quad . ,$$

$$\mathrm{N}_{h_1, h_2, \ldots, h_k}(\mathfrak{A}) = \sum_{n_1 \leqq h_1,\, n_2 \leqq h_2, \ldots,\, n_k \leqq h_k} A_{n_1} \cdot A_{n_1, n_2} \cdot \ldots \cdot A_{n_1, n_2, \ldots n_k}, \quad . \quad . \quad . \quad .$$

We see directly that if the sets of the determining system $\mathfrak{A}$ belong to a class of sets $\mathfrak{M}$, the sets $\mathrm{N}_{h_1, h_2, \ldots, h_k}(\mathfrak{A})$ belong to the class $\mathfrak{M}_{\delta\sigma}$.

(5.3) ***Lemma.*** *For every determining system $\mathfrak{A} = \{A_{n_1, n_2, \ldots, n_k}\}$ and for every sequence of positive integers $h_1, h_2, \ldots, h_k, \ldots$*

$$(5.4) \qquad \mathrm{N}_{h_1}(\mathfrak{A}) \cdot \mathrm{N}_{h_1, h_2}(\mathfrak{A}) \cdot \ldots \cdot \mathrm{N}_{h_1, h_2, \ldots, h_k}(\mathfrak{A}) \cdot \ldots \subset \mathrm{N}(\mathfrak{A}).$$

Proof. Let x be any point belonging to the left-hand side of (5.4). We shall show firstly that a positive integer $n_1^0 \leqslant h_1$ can be chosen so that, for each $k \geqslant 2$, the point x belongs to a set $A_{n_1} \cdot A_{n_1, n_2} \cdot \ldots \cdot A_{n_1, n_2, \ldots, n_k}$ for which $n_1 = n_1^0$ and $n_i \leqslant h_i$ for $i = 2, 3, \ldots, k$. Indeed, if there were no such integer n_1^0, we could associate with each index $n \leqslant h_1$, a positive integer k_n, such that x belongs to no product $A_n \cdot A_{n, n_2} \cdot \ldots \cdot A_{n, n_2, \ldots, n_{k_n}}$ for which $n_i \leqslant h_i$ when $i = 2, 3, \ldots, k_n$. Denote by p_1 the greatest of the numbers $k_1, k_2, \ldots, k_{h_1}$. The point x thus belongs to none of the sets $A_{n_1} \cdot A_{n_1, n_2} \cdot \ldots \cdot A_{n_1, n_2, \ldots, n_{p_1}}$ for which $n_i \leqslant h_i$ when $i = 1, 2, \ldots, p_1$, and therefore is not contained in their sum $\mathrm{N}_{h_1, h_2, \ldots, h_{p_1}}(\mathfrak{A})$. This is a contradiction since, by hypothesis, x is an element of the left-hand side of (5.4).

After the index n_1^0, we can determine afresh an index $n_2^0 \leqslant h_2$, so that, for each $k \geqslant 3$, the point x belongs to a set $A_{n_1} \cdot A_{n_1, n_2} \cdot \ldots \cdot A_{n_1, n_2, \ldots, n_k}$ for which $n_1 = n_1^0$, $n_2 = n_2^0$ and $n_i \leqslant h_i$ when $i = 3, 4, \ldots, k$. For, if there were no such index, we could find, as previously, a positive integer $p_2 \geqslant 3$ such that x belongs to no product $A_{n_1} \cdot A_{n_1, n_2} \cdot \ldots \cdot A_{n_1, n_2, \ldots, n_{p_2}}$ for which $n_1 = n_1^0$ and $n_i \leqslant h_i$ when $i = 2, 3, \ldots, p_2$. And this would contradict the definition of the index n_1^0.

Proceeding in this way, we determine an infinite sequence of indices $\{n_i^0\}$ such that $n_i^0 \leqslant h_i$ when $i = 1, 2, \ldots$ and such that $x \in A_{n_1^0} \cdot A_{n_1^0, n_2^0} \cdot \ldots \cdot A_{n_1^0, n_2^0, \ldots, n_k^0} \cdot \ldots$. Thus $x \in \mathrm{N}(\mathfrak{A})$, and this completes the proof.

Lemma 5.3 is due to W. Sierpiński [13]. The proof contains a slightly more precise result than is expressed by the relation (5.4) and shows that the left-hand side of that relation coincides with the sum (5.1), when the latter is extended only to systems of indices $n_1, n_2, \ldots, n_k, \ldots$ restricted to satisfy $n_1 \leqslant h_1, n_2 \leqslant h_2, \ldots, n_k \leqslant h_k, \ldots$.

Let us call *degenerate*, a determining system $\{A_{n_1, n_2, \ldots, n_k}\}$ such that, for some sequence $\{h_k\}$ of positive integers, we have $A_{n_1, n_2, \ldots, n_k} = 0$ whenever $n_k > h_k$. Then, for this sequence $\{h_k\}$, the relation of inclusion (5.4) becomes an identity and we are led to the following theorem:

If a degenerate determining system consists of sets belonging to a class $\mathfrak{M}$, its nucleus is a set ($\mathfrak{M}_{\delta\sigma\delta}$). A similar theorem cannot hold for non-degenerate systems: in fact, as shown by M. Souslin, the operation (A) applied to Borel sets (and even to linear segments) may lead to sets that are not Borel sets (cf. F. Hausdorff [II, p. 182—184]).

(5.5) ***Theorem.*** *The nucleus of any determining system* $\mathfrak{A}=\{A_{n_1,n_2,\ldots,n_k}\}$ *consisting of sets measurable* $(\mathfrak{L}_\Gamma)$ *is itself measurable* $(\mathfrak{L}_\Gamma)$.

Proof. Let us write, for short,

$$N=\mathrm{N}(\mathfrak{A}),\qquad N^{n_1,n_2,\ldots,n_k}=\mathrm{N}^{n_1,n_2,\ldots,n_k}(\mathfrak{A}),\qquad N_{n_1,n_2,\ldots,n_k}=\mathrm{N}_{n_1,n_2,\ldots,n_k}(\mathfrak{A}).$$

We have to show that, for any set E

$$\Gamma(E)\geqslant\Gamma(E\cdot N)+\Gamma(E\cdot CN). \tag{5.6}$$

We may assume that $\Gamma(E)<\infty$, since (5.6) is evidently fulfilled in the opposite case.

Let us denote (as in § 4, p. 47) by $\Gamma_E^0(X)$ the lower bound of the values of $\Gamma(E\cdot Y)$ for sets $Y\supset X$ measurable $(\mathfrak{L}_\Gamma)$ and let ε be an arbitrary positive number. Taking into account (5.2) and Theorem 4.7, we readily define by induction a sequence of positive integers $\{h_k\}$ such that $\Gamma_E^0(N^{h_1})\geqslant\Gamma(E\cdot N)-\varepsilon/2$ and

$$\Gamma_E^0(N^{h_1,h_2,\ldots,h_k})\geqslant\Gamma_E^0(N^{h_1,h_2,\ldots,h_{k-1}})-\varepsilon/2^k\quad for\quad k=2,3,\ldots.$$

Thus the sets $N_{n_1,n_2,\ldots,n_k}\supset N^{n_1,n_2,\ldots,n_k}$ being measurable $(\mathfrak{L}_\Gamma)$ together with the $A_{n_1,n_2,\ldots,n_k}$,

$$\Gamma(E\cdot N_{h_1,h_2,\ldots,h_k})\geqslant\Gamma_E^0(N^{h_1,h_2,\ldots,h_k})\geqslant\Gamma(E\cdot N)-\varepsilon$$

for each k, and therefore

$$\begin{aligned}\Gamma(E)=\Gamma(E\cdot N_{h_1,h_2,\ldots,h_k})+\Gamma(E\cdot CN_{h_1,h_2,\ldots,h_k})\geqslant\\ \geqslant\Gamma(E\cdot N)+\Gamma(E\cdot CN_{h_1,h_2,\ldots,h_k})-\varepsilon.\end{aligned}\tag{5.7}$$

Now the sequence of sets $\{N_{h_1,h_2,\ldots,h_k}\}_{k=1,2,\ldots}$ is descending, and by Lemma 5.3 its limit is a subset of N. The sequence $\{CN_{h_1,h_2,\ldots,h_k}\}_{k=1,2,\ldots}$ is thus ascending and its limit contains the set CN. Hence, making $k\to\infty$ in (5.7), we find, by Theorem 4.6(ii), the inequality $\Gamma(E)\geqslant\Gamma(E\cdot N)+\Gamma(E\cdot CN)-\varepsilon$, and this implies (5.6) since ε is an arbitrary positive number.

§ 6. Regular sets. A set X will be called *regular* (with respect to the outer measure Γ), if there exists a set A measurable $(\mathfrak{L}_\Gamma)$, containing X and such that $\Gamma(A)=\Gamma(X)$. Every measurable set is evidently regular, and so is also every set X whose outer measure (Γ) is infinite, since we then have $\Gamma(X)=\Gamma(\boldsymbol{M})=\infty$. If every set of the space considered is regular with respect to the outer measure Γ, this measure is itself called *regular*; cf. H. Hahn [I, p. 432], C. Carathéodory [1; I, p. 258].

Denoting by $\Gamma^0(X)$ the lower bound of the values of $\Gamma(Y)$ for sets $Y\supset X$ measurable $(\mathfrak{L}_\Gamma)$, we see readily that the relation $\Gamma^0(X)=\Gamma(X)$ expresses a necessary and sufficient condition for the set X to be regular. From Theorem 4.7(ii), taking for the set E the whole space, we derive the following:

(6.1) ***Theorem.*** *For any sequence* $\{X_n\}$ *of regular sets* $\Gamma(\liminf_n X_n)\leqslant\liminf_n\Gamma(X_n)$, *and, if further the sequence* $\{X_n\}$ *is ascending* $\Gamma(\lim_n X_n)=\lim_n\Gamma(X_n)$.

The generality and the importance of this theorem consist in that all outer measures Γ that occur in applications satisfy the condition of regularity, and, for these measures, the last relation of Theorem 6.1 therefore holds for every ascending sequence of sets. Nevertheless, for measures that are not themselves regular, the restriction concerning regularity of the sets X_n is essential for the validity of Theorem 6.1 as is shown by an example of irregular measure due to C. Carathéodory [II, pp. 693—696].

We may observe further that, for any fixed set E, the function of a set $\Gamma_E^0(X)$, defined in § 4, p. 47, is always a regular outer measure, even if the given measure $\Gamma(X)$ is not. Conditions (C_1) and (C_2) together with that of regularity, are at once seen to hold, and (C_3) may be derived from Theorem 7.4, according to which closed sets are measurable $(\mathfrak{L}_\Gamma)$.

§ 7. Borel sets. We shall show in this § that, independently of the choice of the outer measure Γ, the class $\mathfrak{L}_\Gamma$ contains all Borel sets.

(7.1) ***Lemma.*** *If* Q *is any set contained in an open set* G, *and* Q_n *denotes the set of the points* a *of* Q *for which* $\varrho(a,CG)\geqslant 1/n$, *then* $\lim_n\Gamma(Q_n)=\Gamma(Q)$.

Proof. Since the sequence $\{Q_n\}$ is ascending and $Q=\lim_n Q_n$, it suffices to show that $\lim_n\Gamma(Q_n)\geqslant\Gamma(Q)$. For this purpose let us write $D_n=Q_{n+1}-Q_n$. We then have $\varrho(D_{n+1},Q_n)\geqslant 1/n(n+1)>0$, provided that $D_{n+1}\neq 0$ and $Q_n\neq 0$. Hence, taking into account condition (C_3), p. 43, it is readily verified by induction that

$$\Gamma(Q_{2n+1})\geqslant\Gamma\Big(\sum_{k=1}^{n}D_{2k}\Big)=\sum_{k=1}^{n}\Gamma(D_{2k}),\ \Gamma(Q_{2n})\geqslant\Gamma\Big(\sum_{k=1}^{n}D_{2k-1}\Big)=\sum_{k=1}^{n}\Gamma(D_{2k-1}) \tag{7.2}$$

for every positive integer n. Writing, for short, $a_n=\sum_{k=n}^{\infty}\Gamma(D_{2k})$ and

$b_n = \sum_{k=n+1}^{\infty} \Gamma(D_{2k-1})$, we obtain at once, by condition (C_2), p. 43,

$$\Gamma(Q) \leqslant \Gamma(Q_{2n}) + a_n + b_n. \tag{7.3}$$

Now two possibilities arise: either both series, $\sum_{k=1}^{\infty} \Gamma(D_{2k})$ and $\sum_{k=1}^{\infty} \Gamma(D_{2k-1})$ have finite sums, or, one at least has its sum infinite. In the former case $a_n \to 0$ and $b_n \to 0$, so that, the required inequality $\Gamma(Q) \leqslant \lim_n \Gamma(Q_n)$ follows by making $n \to \infty$ in (7.3); while, in the latter case, the inequality is obvious, since by (7.2) we have then $\lim_n \Gamma(Q_n) = \infty$.

(7.4) ***Theorem.*** *Every set measurable* $(\mathfrak{B})$ *is measurable* $(\mathfrak{L}_\Gamma)$.

Proof. Since the class $\mathfrak{L}_\Gamma$ is additive and since $\mathfrak{B}$ is the smallest additive class including the closed sets (cf. § 2, p. 41), it is enough to prove that every closed set is measurable $(\mathfrak{L}_\Gamma)$, i. e., denoting any such a set by F, that

$$\Gamma(P+Q) \geqslant \Gamma(P) + \Gamma(Q) \tag{7.5}$$

holds for every pair of sets $P \subset F$ and $Q \subset CF$. Since the set CF is open, there is, by Lemma 7.1, a sequence $\{Q_n\}$ of sets such that $Q_n \subset Q$, $\varrho(Q_n, F) \geqslant 1/n$ for $n=1, 2, \ldots$, and $\lim_n \Gamma(Q_n) = \Gamma(Q)$. Thus $\varrho(Q_n, P) \geqslant \varrho(Q_n, F) > 0$, and so, on account of condition (C_3), p. 43, we derive $\Gamma(P+Q) \geqslant \Gamma(P+Q_n) = \Gamma(P) + \Gamma(Q_n)$ for each n, and, making $n \to \infty$, we obtain (7.5).

The arguments of this § depend essentially on property (C_3) of outer measure, and on the metrical character of the space $\boldsymbol{M}$, which did not enter into §§ 4—6. It is possible however to give to these arguments a form, independent of condition (C_3), valid for certain topological spaces that are not necessarily metrical (cf. N. Bourbaki [1]).

From the preceding theorem coupled with Theorem 3.5, we derive at once the following

(7.6) ***Theorem.*** (i) *Every function measurable* $(\mathfrak{B})$ *on a set* E *is measurable* $(\mathfrak{L}_\Gamma)$ *on* E. (ii) *Every function that is semi-continuous on a set* $(\mathfrak{L}_\Gamma)$ *is measurable* $(\mathfrak{L}_\Gamma)$ *on this set.*

§ 8. Length of a set. We shall define in this § a class of functions of a set that are outer Carathéodory measures and that play an important part in a number of applications.

Let α be an arbitrary positive number. Given a set X, we shall denote, for each $\varepsilon > 0$, by $\Lambda_\alpha^{(\varepsilon)}(X)$ the lower bound of the sums $\sum_{i=1}^{\infty} [\delta(X_i)]^\alpha$, for which $\{X_i\}_{i=1,2,\ldots}$ is an arbitrary partition of X into a sequence of sets that have diameters less than ε. When $\varepsilon \to 0$, the number $\Lambda_\alpha^{(\varepsilon)}(X)$ tends, in a monotone non-decreasing manner, to a unique limit (finite or infinite) which we shall denote by $\Lambda_\alpha(X)$. The function of a set $\Lambda_\alpha(X)$ thus defined is an outer measure in the sense of Carathéodory. For, when $\varepsilon > 0$, we clearly have (i) $\Lambda_\alpha^{(\varepsilon)}(X) \leqslant \Lambda_\alpha^{(\varepsilon)}(Y)$ if $X \subset Y$, (ii) $\Lambda_\alpha^{(\varepsilon)}(\sum_n X_n) \leqslant \sum_n \Lambda_\alpha^{(\varepsilon)}(X_n)$, if $\{X_n\}$ is any sequence of sets, and (iii) $\Lambda_\alpha^{(\varepsilon)}(X+Y) = \Lambda_\alpha^{(\varepsilon)}(X) + \Lambda_\alpha^{(\varepsilon)}(Y)$, if $\varrho(X, Y) > \varepsilon$. Making $\varepsilon \to 0$, (i), (ii), (iii) become respectively the three conditions (C_1), (C_2), (C_3), p. 43, of Carathéodory for $\Lambda_\alpha(X)$.

We shall prove further that the outer measure Λ_α (for any $\alpha > 0$) is regular in the sense of § 6, i. e. that every set is regular with respect to this measure. We shall even establish a more precise result, namely

(8.1) ***Theorem.*** *For each set* X *there is a set* $H \in \mathfrak{G}_\delta$ *such that* $X \subset H$ *and* $\Lambda_\alpha(H) = \Lambda_\alpha(X)$.

Proof. For each positive integer n, there is a partition of X into a sequence of sets $\{X_i^{(n)}\}_{i=1,2,\ldots}$ such that

$$\delta(X_i^{(n)}) < 1/2n \ \textit{for}\ i=1,2,\ldots,\ \textit{and}\ \sum_{i=1}^{\infty} [\delta(X_i^{(n)})]^\alpha \leqslant \Lambda_\alpha(X) + 1/n. \tag{8.2}$$

We can evidently enclose each set $X_i^{(n)}$ in an open set $G_i^{(n)}$ such that

$$\delta(G_i^{(n)}) \leqslant (1+1/n)\, \delta(X_i^{(n)}). \tag{8.3}$$

Writing $H = \prod_{n=1}^{\infty} \sum_{i=1}^{\infty} G_i^{(n)}$, we see at once that H is a set $(\mathfrak{G}_\delta)$ and that $X \subset H$. Moreover, for each n, $H = \sum_{i=1}^{\infty} H \cdot G_i^{(n)}$ and the relations (8.2) and (8.3) imply that $\delta(H \cdot G_i^{(n)}) < 1/n$ for $i = 1, 2, \ldots$ and that $\Lambda_\alpha^{(1/n)}(H) \leqslant \sum_{i=1}^{\infty} [\delta(H \cdot G_i^{(n)})]^\alpha \leqslant (1+1/n)^\alpha [\Lambda_\alpha(X) + 1/n]$. Making $n \to \infty$, we find in the limit $\Lambda_\alpha(H) \leqslant \Lambda_\alpha(X)$, and, since the converse inequality is obvious, this completes the proof.

In Euclidean n-dimensional space $\boldsymbol{R}_n$ (see Chap. III), the sets whose measure (Λ_n) is zero may be identified with those of measure zero in the Lebesgue sense. By analogy, in any metrical space, sets whose measure (Λ_α) is zero are termed sets having *α-dimensional volume zero*, and in particular, when $\alpha=1, 2, 3$, sets *of zero length, of zero area, of zero volume*, respectively. For the same reason, sets of finite measure (Λ_α) are termed sets *of finite α-dimensional volume* (or *of finite length, finite area, finite volume*, in the cases $\alpha=1, 2, 3$). In particular, in $\boldsymbol{R}_1$, i. e. on the straight line, the outer measure Λ_1 coincides with the Lebesgue measure, and, on this account, we call the number $\Lambda_1(X)$, in general, *outer length* of X, and when X is a set measurable ($\mathfrak{L}_{\Lambda_1}$), simply, *length* of X. For short, we often write Λ instead of Λ_1.

We have mentioned only the more elementary properties of the measures Λ_α, those, namely, that we shall have some further occasion to use. For a deeper study, the reader should consult F. Hausdorff [1]. Among the researches devoted to the notion of length of sets in Euclidean spaces, special mention must be made of the important memoir of A. S. Besicovitch [1]; cf. also W. Sierpiński [1] and J. Gillis [1].

§ 9. Complete space. A metrical space is termed *complete*, if a sequence $\{a_n\}$ of its points converges whenever $\lim\limits_{m,n\to\infty} \varrho(a_m, a_n)=0$. In any metrical space, this is evidently a necessary condition for convergence of the sequence $\{a_n\}$, but, as a rule, not a sufficient one. The following theorem concerns a characteristic property of complete spaces:

(9.1) ***Theorem.*** *In a complete space, when $\{F_n\}$ is a descending sequence of closed and non-empty sets whose diameters tend to zero, the common part $\prod\limits_n F_n$ is not empty.*

Proof. Let a_n be an arbitrarily chosen point of F_n. For $n \geqslant m$, we have $\varrho(a_m, a_n) \leqslant \delta(F_m)$, and hence $\lim\limits_{m,n\to\infty} \varrho(a_m, a_n)=0$. The sequence $\{a_n\}$ is thus convergent. Now the limit point of this sequence clearly belongs to all the sets F_m, since $a_n \epsilon F_n \subset F_m$ whenever $n \geqslant m$, and since the sets F_m are closed by hypothesis.

(9.2) ***Baire's theorem.*** *In a complete space $\boldsymbol{M}$, every non-empty set ($\mathfrak{G}_\delta$) is of the second category on itself, i. e. if H is a set ($\mathfrak{G}_\delta$) in $\boldsymbol{M}$ and $H=\sum\limits_n H_n$, one at least of the sets H_n is everywhere dense in a portion of H.*

Proof. Suppose, accordingly, that $H=\prod\limits_{n=1}^{\infty} G_n$ where G_n are open sets, and further that

$$H=\sum_{n=1}^{\infty} H_n \tag{9.3}$$

where H_n are non-dense in H. The partial sums of the series (9.3) are then also non-dense in H (cf. § 2, p. 41), and it is easy to define inductively a descending sequence of portions S_n of H such that (i) $\overline{S}_n \subset G_n$, (ii) $\overline{S}_n \cdot \sum\limits_{j=1}^{n} H_j=0$, (iii) $\delta(S_n) \leqslant 1/n$. On account of Theorem 9.1 and of (iii), the sets $\overline{S}_n$ have a common point, which by (i), belongs to all the sets G_n, and so to H, while at the same time, by (ii) it belongs to none of the H_n. This contradicts (9.3) and proves the theorem.

The case of Theorem 9.2 that occurs most frequently, is that in which H is a closed set. For closed sets in Euclidean spaces the theorem was established in 1899 by R. Baire [1]. To Baire, we owe also the fundamental applications of the theorem, which have brought out the fruitfulness and the importance of the result for modern real function theory. As regards the theorem by itself however, it was found, almost at the same time and independently by W. F. Osgood [1] in connection with some problems concerning functions of a complex variable (cf. in this connection, the interesting article by W. H. Young [7]). The general form of Theorem 9.2 is due to F. Hausdorff [I, pp. 326—328; II, pp. 138—145].

If a is a non-isolated point (cf. § 2, p. 40) of a set M, the set (a) consisting of the single point a is clearly non-dense in M. It therefore follows from Theorem 9.2 that

(9.4) ***Theorem.*** *In a complete metrical space, every non-empty set ($\mathfrak{G}_\delta$) without isolated points, and in particular every perfect set, is non-enumerable.*

More precisely, by a theorem of W. H. Young [1], every set that fulfills the condition of Theorem 9.4 has the power of the continuum; cf. also F. Hausdorff [II, p. 136].