CHAPTER XII.

Fourier's integrals.

12.1. Fourier's single integral. Given a function $f(x)$, $-\infty < x < \infty$, consider the expression

$$(1)\quad S_\omega(x) = S_\omega(x; f) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\omega(x-t)}{x-t}\,dt, \quad \omega > 0.$$

This integral exists if $|f(t)|/(1+|t|)$ is integrable over $(-\infty, \infty)$, and so in particular if $f \in L(-\infty, \infty)$, or, using Hölder's inequality, if $f \in L^r(-\infty, \infty)$, $r > 1$. It is an important fact that, if $f(x)$ satisfies conditions ensuring the convergence of Fourier series, then $S_\omega(x) \to f(x)$ as $\omega \to \infty$. This result is known as Fourier's representation of a function by means of a single integral, and is a consequence of the results established in Chapter II and of the following theorem:

Let us fix an arbitrary interval $J_a = (a, a+2\pi)$, and let $f_a(x)$ be the function of period 2π, which is equal to $f(x)$ in J_a. We suppose that $|f(x)|/(1+|x|) \in L(-\infty, \infty)$. Let $s_n(x) = s_n(x; f_a)$ be the partial sums of $\mathfrak{S}[f_a]$. Then, for x belonging to any interval J'_a interior to J_a, the difference $S_\omega(x) - s_{[\omega]}(x)$ tends uniformly to 0 as $\omega \to \infty$ [1]).

12.11. We base the proof of the theorem on a number of lemmas. First of all, it is sufficient to consider, instead of $S_\omega - s_{[\omega]}$, the difference $S_\omega - s^*_{[\omega]}$, where s^*_n are the modified partial sums (§ 2.3). To fix ideas, we assume that $a = 0$, and write $J_0, J'_0, f_0(x)$ instead of $J_a, J'_a, f_a(x)$.

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[1]) See Hobson [3], Pringsheim [2].

(i) *If $g(t) \in L(a, b)$, $-\infty < a < b < \infty$, then $\gamma_\omega = \int_a^b g(t)\,e^{-i\omega t}\,dt \to 0$ as $\omega \to \infty$.*

Transforming the variable of integration we may plainly suppose that $0 < a < b < 2\pi$. Putting $g(t) = 0$ outside (a, b), and applying the device of § 2.21, we obtain, for ω large enough, $2|\gamma_\omega| \leqslant \int_0^{2\pi} |g(t) - g(t + \pi/\omega)|\,dt \to 0$.

(ii) *If $g(t) = f(t)\,h_x(t)$, where $f \in L(a, b)$, and $h_x(t)$, $a \leqslant t \leqslant b$, is a uniformly bounded and uniformly continuous function depending on a parameter x, then $\gamma_\omega \to 0$ uniformly in x.*

Suppose that $0 < \varepsilon < a < b < 2\pi - \varepsilon$, and put $f(t) = 0$ outside (a, b). Let $h_x(t)$ be equal to 0 for $t \leqslant \varepsilon$ and $t \geqslant 2\pi - \varepsilon$, and be linear for $\varepsilon \leqslant t \leqslant a$, $b \leqslant t \leqslant 2\pi - \varepsilon$. The new function $h_x(t)$ is uniformly bounded and uniformly continuous and, since

$$\int_0^{2\pi} |f(t) - f(t + \pi/\omega)|\,dt \to 0, \qquad \operatorname*{Max}_{t,\,x} |h_x(t) - h_x(t + \pi/\omega)| \to 0,$$

the integral majorizing $2|\gamma_\omega|$ tends uniformly to 0 as $\omega \to \infty$. It is plain that *the result holds if $h(t)$ depends on more than one parameter.*

(iii) *Under the conditions of Theorem 12.1, the difference $S_\omega(x) - S_{[\omega]}(x)$ tends uniformly to 0 for $x \in J'_0$.* For, if $[\omega] = n$, $\omega - n = u$, then

$$(1)\quad S_\omega(x) - S_n(x) = \Re\,\frac{e^{i(n+\frac{1}{2}u)x}}{\pi}\int_{-\infty}^{\infty} f(t)\frac{2\sin\frac{1}{2}u(x-t)}{x-t}\,e^{-i(n+\frac{1}{2}u)t}\,dt.$$

To show that the last integral tends uniformly to 0, we break it up into two integrals P and Q, where P is extended over some interval $(-A, A)$, and Q over $(-\infty, -A) + (A, \infty)$. If A is large enough, then $|Q| < \frac{1}{2}\varepsilon$ for $x \in J'_0$. Since the function $h_{x,u}(t) = 2\sin\frac{1}{2}u(x-t)/(x-t)$ is uniformly continuous and uniformly bounded for $0 \leqslant u < 1$ and $x \in J'_0$, an application of (ii) shows that $|P| \to 0$, i. e. $|P + Q| < \varepsilon$ for $\omega > \omega_0$. This proves the lemma.

A moment's consideration shows that Theorem 12.1 is a consequence of (iii) and of the following lemma:

(iv) *Let $f(x) = f'(x) + f''(x)$, where $f'(x) = f(x)$ for $x \in J_0$, $f'(x) = 0$ for $x \,\overline{\in}\, J_0$. Then $\delta_n = S_n(x; f') - s^*_n(x; f_0) \to 0$, $S_n(x; f'') \to 0$ as $n \to \infty$, uniformly in $x \in J'_0$.*

Let $h_x(t)$ be a function of period 2π, equal to $1/(x-t) - \frac{1}{2}\operatorname{ctg}\frac{1}{2}(x-t)$. for $0 \leqslant t < 2\pi$. Since $\mathfrak{M}[h_x(t+\eta) - h_x(t); 0, 2\pi] \to 0$ with η, uniformly in $x \in J'_0$, an argument similar to that of § 2.501 shows that $\delta_n = \Im\,\frac{e^{inx}}{\pi}\int_0^{2\pi} f(t)\,h_x(t)\,e^{-int}\,dt \to 0$ uniformly in $x \in J'_0$. On the other hand, $S_n(x; f'') = U_n + V_n$, where U_n is equal to

$\sum \frac{e^{inx}}{\pi} \int\limits_{2\pi}^{\infty} f(t) \frac{e^{-int}}{x-t} dt$, and V_n is a similar integral formed for the interval $(-\infty, 0)$. To show that $|U_n| + |V_n| \to 0$, we proceed as in the proof of (iii).

This completes the proof of Theorem 12.1.

12.12. *Theorem 12.1 holds if $f(t)$ is integrable over any finite interval and if, moreover, $f(t)/t$ tends to 0 as $t \to \pm\infty$ and is of bounded variation in the neighbourhood of $t = \pm\infty$* [1].

This last condition means that there is a number $B > 0$ such that f is of bounded variation in $(-\infty, -B)$ and in (B, ∞). Without loss of generality we may assume that $f(t)/t$ tends monotonically to 0 as $t \to \pm\infty$, $|t| \geqslant B$, for every f satisfying the conditions of the theorem is a sum of two functions satisfying this more stringent condition.

The proof of the theorem runs close to that of Theorem 12.1, and we need not repeat the whole argument. The proof of the latter theorem was based on Lemmas (iii) and (iv) of § 12.11. Those lemmas hold under new conditions, but in the proofs we must now apply the second mean-value theorem. For example, to prove Lemma 12.11(iii), we break up the right-hand side of 12.11(1) into three integrals extended over $(-\infty, -A)$, $(-A, A)$, and (A, ∞) respectively. The last of them is equal to the limit, for $A' \to \infty$, of the expression

$$(1) \qquad \frac{1}{\pi} \int\limits_{A}^{A'} \frac{f(t)}{t} \cdot \left\{1 + \frac{x}{t-x}\right\} [\sin \omega (t-x) - \sin n (t-x)]\, dt.$$

Applying the second mean-value theorem to the factors $f(t)/t$ and $1/(t-x)$, and observing that $f(t)/t \to 0$ as $t \to \infty$, we see that (1) tends to 0 as $A \to \infty$, $A' \to \infty$. This shows that the integral over (A, ∞) exists *and that it tends to 0 as $A \to \infty$, uniformly in $x \in J_0$ and $\omega \geqslant 1$.* The reader will have no difficulty in completing the proof.

12.2. Fourier's repeated integral.

Suppose that $|f(t)|$ is integrable over $(-\infty, \infty)$. Then the right-hand side of 12.1(1) is equal to

$$(1) \qquad \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(t)\, dt \int\limits_{0}^{\omega} \cos s (x-t)\, ds = \frac{1}{\pi} \int\limits_{0}^{\omega} ds \int\limits_{-\infty}^{\infty} f(t) \cos s (x-t)\, dt,$$

the inversion of the order of integration being clearly justified. Hence $S_\omega(x)$ is a partial integral of the infinite integral

[1] Pringsheim [2]. The condition that $f(t)/t \to 0$ with $1/t$, is necessary, for, if e. g. $f(t) = t$, the integral 12.1(1) diverges.

$$(2) \qquad \frac{1}{\pi} \int\limits_{0}^{\infty} ds \int\limits_{-\infty}^{\infty} f(t) \cos s (x-t)\, dt =$$

$$(3) \qquad = \int\limits_{0}^{\infty} (a_s \cos sx + b_s \sin sx)\, ds = \int\limits_{-\infty}^{\infty} c_s e^{isx}\, ds,$$

where

$$(4) \qquad a_s = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(t) \cos st\, dt, \qquad b_s = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(t) \sin st\, dt,$$

$$(5) \qquad c_s = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt = \tfrac{1}{2}(a_s - ib_s).$$

The expressions a_s, b_s, c_s are analogous to Fourier coefficients; but s is a continuous variable and we obtain a trigonometrical integral of the form (3) instead of a trigonometrical series. Given a function $f(t)$, $-\infty < t < \infty$, such that the integrals (4) have a meaning, we may consider the integral (3) or, what is the same thing, the integral (2), and ask in what sense does it represent $f(x)$. The integral (2) is called *Fourier's repeated integral*. It is plain that if we have (1) for every ω, then the partial integrals $S_\omega(x)$ of (2) are given by the formula 12.1(1), i. e. the problem reduces to that of representing the function by means of Fourier's single integral, a problem which, in the most important cases, is settled by Theorems 12.1 and 12.2. The formula (1), however, is true only under certain conditions bearing on the behaviour of $f(t)$ not at individual points but in the whole interval $(-\infty, \infty)$; more precisely, in the neighbourhood of $t = \pm\infty$. This causes the range of application of Fourier's repeated integral to be more restricted than that of Fourier's single integral [1]. The formula (1) is certainly true when $|f| \in L(-\infty, \infty)$, and so, in view of Theorem 12.1, we have: *If $|f| \in L(-\infty, \infty)$, then $S_\omega(x; f) - s_{[\omega]}(x; f_a) \to 0$, uniformly in $x \in J'_a$, where $S_\omega(x)$ denotes the partial integral of (2), and f_a, J_a, and J'_a have the same meaning as before.*

[1] The range of validity of Fourier's repeated integral can be considerably extended if we suppose that the integrals (4) are *summable* in some sense, e. g. summable (C, k) (§ 12.3). We shall not consider this problem here.

12.21. *The last theorem holds if $f(t)$ is integrable over any finite interval, tends to 0 with $1/t$, and is of bounded variation in the neighbourhood of $t = \pm\infty$.*

Assuming, as we may, that $f(t)$ tends monotonically to 0 as $t \to \pm\infty$, $|t| \geqslant B > 0$, by means of the second mean-value theorem we verify that the inner integral on the right of 12.2(1) converges for every $s > 0$ (but not necessarily for $s = 0$), and that the convergence is uniform over any range $0 < \delta \leqslant s \leqslant \omega$. Hence

$$(1)\quad \frac{1}{\pi}\int_{\delta}^{\omega} ds \int_{-\infty}^{\infty} f(t)\cos s(x-t)\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\omega(x-t)}{x-t}\,dt - \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\delta(x-t)}{x-t}\,dt.$$

We will show that the second integral on the right tends to 0 with δ. For the proof we break up the integral over $(-\infty, \infty)$ into three integrals, extended over $(-\infty, -A)$, $(-A, A)$, and (A, ∞) respectively. Since the integral of $(\sin u)/u$ over any finite interval is bounded, an application of the second mean-value theorem shows that, if $A > B$ is large enough, the first and the third of the three integrals are numerically less than a given $\varepsilon > 0$. Since, for fixed A, the second integral tends to 0 with δ, the last integral on the right of (1) is less than 3ε in absolute value for δ small enough, i. e. it tends to 0. *Thence we obtain* 12.2(1) *(and so also the theorem), where however the outer integral on the right is an improper integral:* $\int_0^{\omega} = \lim_{\delta\to+0}\int_{\delta}^{\omega}$ [1]. That this is essential, and that $g(s) = \int_{-\infty}^{\infty} f(t)\cos s(x-t)\,dt$, considered as a function of s, may be non-integrable (in the Lebesgue sense), in the neighbourhood of $s = 0$, may be seen from the following example. There is a sequence $a_1 \geqslant a_2 \geqslant \ldots \to 0$ such that the sum of the series $\Sigma a_n \cos ns$ is not integrable in the neighbourhood of $s = 0$ (§ 5.121). Let $x = 0$, $f(t) = a_n$ for $n - \frac{1}{2} \leqslant t < n + \frac{1}{2}$, $n = 1, 2, \ldots$, $f(t) = 0$ for $0 \leqslant t < \frac{1}{2}$, $f(-t) = f(t)$. Then $sg(s)/4\sin\frac{1}{2}s = \Sigma a_n \cos ns$, and $g(s)$ is not integrable in the neighbourhood of $s = 0$ (see also § 5.7.4).

This result shows that, *under the hypotheses of the theorem stated at the beginning of the section, the outer integral in Fourier's repeated integral must be understood in the sense* $\lim_{\substack{\omega\to\infty\\ \delta\to 0}}\int_{\delta}^{\omega}$.

12.3. Summability of integrals. So far we applied summability to series only, but a similar theory can be constructed for integrals. We start with the following lemma.

Let $\varphi(x)$ and $\psi(x)$ be two functions defined for $x \geqslant 0$ and integrable over any finite interval $(0, a)$; suppose that $\psi(x) > 0$ for $x > 0$ and let $\Phi(x)$ and $\Psi(x)$ denote respectively the integrals of $\varphi(t)$ and $\psi(t)$ over the interval $(0, x)$. Then, if $\Psi(x) \to \infty$ and $\varphi(x)/\psi(x) \to s$ as $x \to \infty$, we have $\Phi(x)/\Psi(x) \to s$.

For $s = 0$ the lemma was established in § 1.71. If we apply that result to the functions $\varphi_1(x) = \varphi(x) - s\psi(x)$ and $\psi_1(x) = \psi(x)$, we obtain the general result.

[1]) Cf. Tonelli, *Serie trigonometriche*, p. 413.

We write $\Phi_0(x) = \varphi(x)$, and denote by $\Phi_k(x)$, $k = 1, 2, \ldots$, the integral of $\Phi_{k-1}(t)$ over $0 \leqslant t \leqslant x$. Similarly we define $\Psi_k(x)$. It is plain that, if $\Phi_k(x)/\Psi_k(x) \to s$ as $x \to \infty$, then $\Phi_l(x)/\Psi_l(x) \to s$ for $l \geqslant k$. Suppose that $\varphi(x) = 1$; then $\Psi_k(x) = x^k/k!$ We shall say that s is the (C, k) limit of $\varphi(x)$ as $x \to \infty$ and write $(C, k)\,\varphi(x) \to s$, if $\Phi_k(x)\,k!/x^k \to s$, i. e. if

$$(1)\qquad kx^{-k}\int_0^x (x-t)^{k-1}\varphi(t)\,dt \to s \quad \text{as} \quad x \to \infty.$$

Now we may take (1) as the definition of the (C, k) limit for every $k > 0$, integral or fractional. By the $(C, 0)$ limit of the function $\varphi(x)$ as $x \to \infty$, we mean the ordinary limit. Since φ is integrable over any finite interval $(0, a)$, the left-hand side of (1) exists for almost every x (this follows from the results of § 2.11) and is itself integrable over $(0, a)$. If $\varphi(t)$ is bounded over any finite interval—a most frequent case in applications—the left-hand side of (1) exists everywhere.

(i) *If $\alpha \geqslant 0$, $\beta > 0$, and if $(C, \alpha)\,\varphi(x) \to s$, then $(C, \alpha+\beta)\,\varphi(x) \to s$.*

We assume that $\alpha > 0$. In the argument we shall require the formula

$$(2)\qquad \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

a proof of which will be found in most text-books of Analysis [1]). Let us denote the left-hand side of (1) by $k\Phi_k^*(x)/x^k$. We begin by proving that

$$(3)\qquad \Phi^*_{\alpha+\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x \Phi^*_{\alpha}(t)\,(x-t)^{\beta-1}\,dt.$$

For the integral on the right of (3) is equal to

$$\int_0^x (x-t)^{\beta-1}dt\int_0^t \varphi(u)\,(t-u)^{\alpha-1}\,du = \int_0^x \varphi(u)\,du\left\{\int_u^x (x-t)^{\beta-1}(t-u)^{\alpha-1}\,dt\right\}.$$

Thence, transforming the variables in the inner integral on the right, and using (2), we obtain (3).

Now, if $(C, \alpha)\,\varphi(x) \to s$, then $\Phi^*_{\alpha}(t) = st^{\alpha}/\alpha + \varepsilon(t)\,t^{\alpha}$, where $\varepsilon(t) \to 0$ as $t \to \infty$. Since $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$, we obtain from (3)

$$(4)\qquad (\alpha+\beta)\,\Phi^*_{\alpha+\beta}(x)/x^{\alpha+\beta} = s + Cx^{-\alpha-\beta}\int_0^x \varepsilon(t)\,t^{\alpha}(x-t)^{\beta-1}\,dt,$$

where C denotes a constant. Let $\varepsilon > 0$ be an arbitrarily small number and let $|\varepsilon(t)| < \varepsilon$ for $t > x_0$. Breaking up the last integral into two, extended

[1]) The formula (2) can also be obtained from 3.11(3) and the relation $A_n^{\alpha} \simeq n^{\alpha}/\Gamma(\alpha+1)$.

over $(0, x_0)$ and (x_0, x) respectively, the reader will have no difficulty in proving that the right-hand side of (4) tends to s. This completes the proof of the theorem for $\alpha > 0$ [1]). The case $\alpha = 0$ is still simpler.

In the foregoing discussion we supposed that $x \to \infty$, but a similar theory may be developed in the case of x tending to any other limit. For example, the (C, k) limit of $\varphi(x)$ for $x \to +0$ may also be defined by (1), with the difference that in that formula x now tends to $+0$.

Given an integral $J = \int_0^\infty f(t)\, dt$, we shall say that it is summable (C, k), to the value s, if we have (1) with $\varphi(x) = \int_0^x f(t)\, dt$, i. e. if

$$x^{-k} \int_0^x (x-t)^k f(t)\, dt \to s \qquad \text{as } x \to \infty. \tag{5}$$

This definition presupposes that $f(x)$ is integrable over any finite interval. The left-hand side of (5) exists then for almost every x, even if $k > -1$. In view of (i), *summability* (C, α) *implies summability* $(C, \alpha+\beta)$, $\alpha \geqslant 0$, $\beta > 0$, *to the same value* [2]).

Given an arbitrary series $(U)\, u_0 + u_1 + \dots$, let $U_n = u_0 + u_1 + \dots + u_n$, and let $U(x) = U_n$ for $n \leqslant x < n+1$, $n = 0, 1, \dots$ If, for $x \to \infty$, the (C, α) limit of $U(x)$ exists and is equal to s, we shall say that the series U is summable by M. Riesz's method of order α, or summable (R, α), to sum s. M. Riesz has shown that *the methods* (R, α) *and* (C, α) *are equivalent* [3]), *i. e., if a series is summable by one of these methods, it is summable by the other to the same sum.* The proof of the general result is rather complicated, but the special case $\alpha = 1$, which is of independent importance, is fairly easy and may be left to the reader as an exercise.

Since, under the hypothesis of Theorem 12.1, the $(C, 1)$ limit of the difference $S_\omega(x) - s_{[\omega]}(x)$ exists and is equal to 0, and since Fourier series are summable $(C, 1)$ almost everywhere, we obtain:

Under the hypothesis of Theorem 12.1, *the* $(C, 1)$ *limit of* $S_\omega(x)$ *exists almost everywhere and is equal to* $f(x)$. *In particular, this limit exists and is equal to* $\frac{1}{2}[f(x+0) + f(x-0)]$ *at every point of simple discontinuity of* f. *It exists uniformly over any finite interval at all points of which* f *is continuous.*

In the same way we may complete Theorems 12.12 and 12.2. If we assume M. Riesz's equivalence theorem in its general form, we may replace summability $(C, 1)$ by (C, δ), $\delta > 0$. All these results can however be obtained independently of M. Riesz's theorem, by an argument similar to that of § 3.3 [4]).

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1) The result holds for $s = \infty$.
2) The result holds for $\alpha > -1$.
3) A proof will be found in H o b s o n's *Theory of Functions*, 2, p. 90.
4) See e. g. H o b s o n, *loc. cit.* p. 737.

12.4. Fourier transforms [1]). Changing the definition 12.2(5) slightly, we shall write

$$F(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx, \tag{1}$$

where $f(x)$ is now a complex function. When $f(x)$ is represented by Fourier's repeated integral 12.2(2), we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y)\, e^{ixy}\, dy, \tag{2}$$

the integral on the right being defined as $\lim_{\omega \to \infty} \int_{-\omega}^{\omega}$. The function $F(y)$ is called the Fourier *transform* of $f(x)$. It exists for every x if $f \in L(-\infty, \infty)$. We shall now prove that

(i) *If* $f(x) \in L^2(-\infty, \infty)$, *the integral in* (1) *converges, in a certain sense, to a function* $F(y) \in L^2(-\infty, \infty)$. *The function* F *satisfies* (2) *and the relation*

$$\int_{-\infty}^{\infty} |F(y)|^2\, dy = \int_{-\infty}^{\infty} |f(x)|^2\, dx. \tag{3}$$

Let S denote the set of step-functions $f(x)$ which vanish for $|x|$ large. If $f \in S$, we define F by the formula (1); in all other cases we shall define F indirectly. We begin by proving (3) for $f \in S$. Then, for $\omega > 0$,

$$\int_{-\omega}^{\omega} |F(y)|^2\, dy = \frac{1}{2\pi} \int_{-\omega}^{\omega} dy \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx \int_{-\infty}^{\infty} \bar{f}(x')\, e^{ix'y}\, dx' = \tag{4}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \bar{f}(x')\, dx\, dx' \int_{-\omega}^{\omega} e^{iy(x'-x)}\, dy =$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \bar{f}(x') \frac{\sin \omega\, (x - x')}{x - x'}\, dx\, dx' = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x)\, S_\omega(x; \bar{f})\, dx,$$

where S_ω is defined by 12.1(1). The above transformations are perfectly legitimate since the integrals are infinite in appearance

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1) The results of this section are due to P l a n c h e r e l [1], [2]; see also F. R i e s z [9]. Interesting generalizations will be found in W a t s o n [1].

only. Now observe that, in the case considered, $S_\omega(x;\bar{f})$ is uniformly bounded in x and ω, and tends to $\bar{f}(x)$ as $\omega\to\infty$, except, perhaps, at a finite number of points. It is sufficient to consider the case when f is the characteristic function of an interval. But then the result follows (independently of the more difficult Theorem 12.1) from the formula $\int_0^\infty \frac{\sin t}{t}\,dt=\frac{1}{2}\pi$ and from the fact that the partial integrals of the last integral are bounded. Comparing the extreme terms of (4), and making $\omega\to\infty$, we obtain the equation (3), by Lebesgue's theorem on the integration of bounded sequences.

The formula (1) defines an additive operation $F=T[f]$. This operation is actually defined for functions f belonging to a set S, which is everywhere dense in the space $L^2(-\infty,\infty)$ [1]. Hence, in view of the formula (3), valid for $f\in S$, and the remarks of § 9.22 [2], the operation $T[f]$ may be extended, by continuity, to the whole space $L^2(-\infty,\infty)$, and this extension is unique. This operation is of type (2,2) and its modulus is equal to 1. This means that, for every $f\in L^2(-\infty,\infty)$, we have (3) with '$=$' replaced by '$\leqslant$'. To prove that equality actually occurs, let $f\in L^2$, $f_n\in S$, $n=1,2,\ldots$, $\mathfrak{M}_2[f-f_n]\to 0$ [3], $F_n=T[f_n]$. Since $\mathfrak{M}_2[F-F_n]\leqslant\mathfrak{M}_2[f-f_n]\to 0$, Minkowski's inequality gives: $\mathfrak{M}_2[f_n]\to\mathfrak{M}_2[f]$, $\mathfrak{M}_2[F_n]\to\mathfrak{M}_2[F]$. This and the equations $\mathfrak{M}_2[f_n]=\mathfrak{M}_2[F_n]$ imply $\mathfrak{M}_2[f]=\mathfrak{M}_2[F]$, i. e. (3).

It remains to prove (2), which may be writen $f(x)=T^*T[f]$, where T^* denotes the operation we obtain from T by changing the sign of y. Since the operations T and T^* are continuous in the space $L^2(-\infty,\infty)$, it is sufficient to prove the relation $f=T^*T[f]$ when $f\in S$, or, still simpler, when f is the characteristic function of an interval (a,b). Then $F(y)=i\,(e^{-iyb}-e^{-iya})/\sqrt{2\pi}\,y$, and

[1] This is a special case of the more general and difficult Theorem 9.21(i). An independent proof runs as follows. Let S_1 be the set of functions $f(x)\in L^2(-\infty,\infty)$ which vanish for $|x|$ large. S_1 is dense in $L^2(-\infty,\infty)$, and so it is enough to show that S is dense in S_1. Let $f\in S_1$; transforming the variable x, we may suppose that $f(x)$ vanishes outside $(0,2\pi)$. Then there is a continuous function $\sigma(x)$ such that $\mathfrak{M}_2[f-\sigma;\,0,2\pi]<\frac{1}{2}\varepsilon$ (§ 4.21(1)). If $s(x)$ is a step-function vanishing outside $(0,2\pi)$, and such that $\mathfrak{M}_2[\sigma-s;\,0,2\pi]<\frac{1}{2}\varepsilon$, then $\mathfrak{M}_2[f-s;\,-\infty,\infty]<\varepsilon$.

[2] The Stieltjes-Lebesgue integrals considered there reduce to ordinary Lebesgue integrals.

[3] We write $\mathfrak{M}_2[g]$ instead of $\mathfrak{M}_2[g;-\infty,\infty]$.

$$(5)\qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(y)\,e^{ixy}\,dy=\frac{1}{\pi}\int_0^\infty\frac{\sin(x-a)y}{y}\,dy+\frac{1}{\pi}\int_0^\infty\frac{\sin(b-x)y}{y}\,dy,$$

i. e. the left-hand side of (5) is equal to 1 for $a<x<b$, and to 0 for $x<a$ and $x>b$. This completes the proof of (i), if we take for granted the result, which will be established below, that, whenever the integral defining the transform of a function $g\in L^2$ converges almost everywhere, it converges to $T[g]$.

In the previous argument, the operation $T[f]$ was defined directly by (1) only when $f\in S$; to define $T[f]$ for general f we used continuity. We shall now show that, if $g\in L^2(-\infty,\infty)$ vanishes outside some interval $(-A,A)$, then $T[g]$ may still be defined by the formula (1). For let $G(y)=T[g]$ and let $G^*(y)$ be the value of the integral in (1), with f replaced by g. Let $g_n(x)$, $n=1,2,\ldots$ be a sequence of step-functions vanishing outside $(-A,A)$ and such that $\mathfrak{M}_2[g-g_n]\to 0$. If $G_n(y)=T[g_n]$, then $\mathfrak{M}_2[G-G_n]\to 0$ and, à fortiori, $\mathfrak{M}_2[G-G_n;\,-\omega,\omega]\to 0$, for every $\omega>0$. On the other hand, Schwarz's inequality shows that $G_n(y)$ tends uniformly to $G^*(y)$ over any interval, and so $\mathfrak{M}_2[G^*-G_n;\,-\omega,\omega]\to 0$. This and the relation $\mathfrak{M}_2[G-G_n;-\omega,\omega]\to 0$ show that $G^*(y)\equiv G(y)$ for $-\omega<y<\omega$, and so also for $-\infty<y<\infty$.

Let $f\in L^2\,(-\infty,\infty)$ and $\omega>0$. We write

$$(6)\qquad F_\omega(y)=\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}f(x)\,e^{-ixy}\,dx;$$

then $F_\omega(y)=T[f_\omega]$, where $f_\omega(x)$ is equal to $f(x)$ for $|x|<\omega$, and to 0 elsewhere. Since $\mathfrak{M}_2[F_\omega-T[f]]=\mathfrak{M}_2[f_\omega-f]\to 0$ as $\omega\to\infty$, proposition (i) may be restated as follows:

(ii) *For every $f\in L^2(-\infty,\infty)$, the integral in* (1) *converges in mean to a function $F(y)\in L^2(-\infty,\infty)$, that is $\mathfrak{M}_2[F-F_\omega]\to 0$ as $\omega\to\infty$. The integral in* (2) *converges in mean to $f(x)$, and F and f satisfy the Parseval relation* (3).

Since $\mathfrak{M}_2[F-F_\omega]\to 0$, there exists a sequence $\{\omega_k\}$ such that $F_{\omega_k}(y)\to F(y)$ for almost every y (§ 4.2). Therefore, *if the integral in* (1) *converges almost everywhere, it converges to the transform of f.*

It is not difficult to obtain a formula for $F(y)$. Let $\Phi(y)$ and $\Phi_\omega(y)$ denote the integrals of F and F_ω over $(0,y)$. By Schwarz's inequality, $|\Phi(y)-\Phi_\omega(y)|\leqslant y^{1/2}\,\mathfrak{M}_2[F-F_\omega;\,0,y]\to 0$,

i. e. $\Phi(y)=\lim_{\omega\to\infty}\Phi_\omega(y)$. Since $\Phi_\omega(y)=\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega} f(x)\,\frac{e^{-ixy}-1}{-ix}\,dx$, and $\Phi'(y)=F(y)$ for almost every y, we obtain the first of the formulae

$$(7)\quad F(y)=\frac{d}{dy}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{e^{-ixy}-1}{-ix}dx\right\},\quad f(x)=\frac{d}{dx}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(y)\frac{e^{ixy}-1}{iy}dy\right\}.$$

The second formula, which corresponds to (2), may be obtained similarly.

The formula (2) tells us that to every $f(x)\in L^2(-\infty,\infty)$ corresponds a function $g(y)\in L^2(-\infty,\infty)$, whose transform is $f(x)$ (an analogue of the Riesz-Fischer theorem). It suffices to put $g(y)=F(-y)$, where $F(y)$ is the transform of $f(x)$.

12.41. *If $f(x)\in L^p(-\infty,\infty)$, $1<p\leqslant 2$, the integral in* 12.4(1) *converges in mean, with index $p'=p/(p-1)$, to a function $F(y)\in L^{p'}$, which satisfies the equations* 12.4(7) *and the inequality*

$$(1)\qquad \left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|F(y)|^{p'}dy\right\}^{1/p'}\leqslant\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(x)|^{p}dx\right\}^{1/p}.$$ [1]

This is an extension to Fourier integrals of Theorem 9.1(a). We first observe that the formula 12.4(1) defines a functional operation $F=T[f]$, when $f\in L(-\infty,\infty)$ or $f\in L^2(-\infty,\infty)$. Using the notation of § 9.22, we may say that T is of type $(1,\infty)$ and of type $(2,2)$, and that $M_{1,0}=(2\pi)^{-1/2}$, $M_{1/2,1/2}=1$. Hence, by Theorem 9.23, the operation may be extended, so as to become of type (p,p'); and $M_{1/p,1/p'}\leqslant(2\pi)^{1/2-1/p}$. This gives (1), where $F=T[f]$.

Let f_ω have the same meaning as in § 12.4. If $f\in L^p$, then $f_\omega\in L$, and so $F_\omega=T[f_\omega]$ is given by the formula 12.4(6). Since $\mathfrak{M}_{p'}[T[f]-F_\omega]\leqslant M_{1/p,1/p'}\,\mathfrak{M}_p[f-f_\omega]\to 0$, the integral in 12.4(1) converges in mean, with index p', to a function $F(y)\in L^{p'}$. Arguing as in § 12.4, and using Hölder's inequality instead of Schwarz's, we obtain the first formula 12.4(7) (cf. also § 12.5.3).

To prove the second formula 12.4(7), observe that, if $f(x)$ is absolutely integrable over $(-\infty,\infty)$, then the Fourier integral of f may be integrated formally over any finite interval. This follows e. g. from the fact that Fourier series may be integrated

[1]) Titchmarsh [6]; see also M. Riesz [3].

formally and Theorem 12.1. Since $f\in L^p(-\infty,\infty)$, $p>1$, the function equal to $f(x)$ for $|x|<a$ and to 0 elsewhere belongs to $L(-\infty,\infty)$, and so, if $|x|<a$,

$$\int_0^x f(u)\,du=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixy}-1}{iy}\left\{\int_{-a}^{a}f(t)\,e^{-iyt}\,dt\right\}dy=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ixy}-1}{iy}F_a(y)\,dy.$$

Since $\mathfrak{M}_{p'}[F_a-F]\to 0$ as $a\to\infty$, an application of Hölder's inequality shows that we may replace $F_a(y)$ by $F(y)$ in the last integral, and the second formula 12.4(7) follows. This completes the proof of the theorem.

12.42. The result which we obtained is, in one respect, incomplete. Whereas it was proved that the integral in 12.4(1) converges in mean, with index p', the reciprocal relation 12.4(2) was established only in the sense of the second formula 12.4(7). This result was completed by Hille and Tamarkin [3], who showed that *the integral* 12.4(2) *converges in mean, with index p, to $f(x)$.* This theorem is suggested by Theorem 7.3(i), if we observe that the function $F(y)$ is an analogue of a sequence of Fourier coefficients, and the partial integrals of the integral 12.4(2) play the rôle of the partial sums of a Fourier series. The proof is based on the following lemma:

If $f\in L^r(-\infty,\infty)$, $r>1$, the function

$$(1)\qquad g(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}dt=-\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{f(x+t)-f(x-t)}{t}dt$$

exists for almost every x and satisfies an inequality $\mathfrak{M}_r[g]\leqslant A_r\mathfrak{M}_r[f]$, where A_r depends on r only [1]).

Since, in view of Hölder's inequality, the function $f(t)/(t-x)$ is integrable in the neighbourhood of $t=\pm\infty$, the first part of the lemma follows from Theorem 7.1(i). To prove the second part, we put $g_n(x)=\frac{1}{2\pi n}\int_{-\pi n}^{\pi n}f(t)\operatorname{ctg}\frac{x-t}{2n}dt$ and consider the difference $\delta_n=g(x)-g_n(x)$. Then

$$\delta_n=\frac{1}{2\pi n}\int_{-\pi n}^{\pi n}f(t)\left[\frac{2n}{x-t}-\operatorname{ctg}\frac{x-t}{2n}\right]dt+$$

[1]) M. Riesz [4].

$$+\frac{1}{\pi}\int_{-\infty}^{-\pi n}\frac{f(t)}{x-t}\,dt+\frac{1}{\pi}\int_{\pi n}^{\infty}\frac{f(t)}{x-t}\,dt=\alpha_n+\beta_n+\gamma_n.$$

The expressions α_n, β_n, and γ_n tend to 0 as $n\to\infty$ [1]); hence $\delta_n\to 0$, $g_n(x)\to g(x)$, and an application of Fatou's lemma to the inequality $\mathfrak{M}_r[g_n;-\pi n,\pi n]\leqslant A_r\mathfrak{M}_r[f;-\pi n,\pi n]\leqslant A_r\mathfrak{M}_r[f]$ (§ 7.21) shows that $\mathfrak{M}_r[g;-w,w]\leqslant A_r\mathfrak{M}_r[f]$ for every $w>0$, i. e. $\mathfrak{M}_r[g]\leqslant A_r\mathfrak{M}_r[f]$. This completes the proof of the lemma.

If $F_\omega(y)$ is given by 12.4(6), and $w>0$ is any finite number, then

$$\frac{1}{\sqrt{2\pi}}\int_{-w}^{w}F_\omega(y)\,e^{ixy}\,dy=\frac{1}{\pi}\int_{-\omega}^{\omega}f(t)\,\frac{\sin w\,(x-t)}{x-t}\,dt.$$

Since $F_\omega(y)$ tends in mean, with index p', to $F(y)$, we may put $\omega=\infty$ in the last equation, and we obtain

$$(2)\qquad \frac{1}{\sqrt{2\pi}}\int_{-w}^{w}F(y)\,e^{ixy}\,dy=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\,\frac{\sin w\,(x-t)}{x-t}\,dt.$$

Applying the lemma and using the same device as in § 7.3, we obtain that the left-hand side $\Phi_w(x)$ of (2), satisfies the inequality $\mathfrak{M}_p[\Phi_w]\leqslant 2A_p\mathfrak{M}_p[f]$. To show that $\mathfrak{M}_p[\Phi_w-f]\to 0$ as $w\to\infty$, we put $f=f'+f''$, and, correspondingly, $\Phi_w=\Phi_w'+\Phi_w''$, where $f'\in S$ (§ 12.4) and $\mathfrak{M}_p[f'']<\varepsilon$; then

$$\mathfrak{M}_p[f-\Phi_w]\leqslant\mathfrak{M}_p[f'-\Phi_w']+\mathfrak{M}_p[f'']+\mathfrak{M}_p[\Phi_w'']\leqslant\mathfrak{M}_p[f'-\Phi_w']+(2A_p+1)\,\varepsilon,$$

and it is sufficient to show that $\mathfrak{M}_p[f'-\Phi_w']\to 0$. We may restrict ourselves to the case when the function f', which we shall now denote by f again, is the characteristic function of an interval (a,b). Then $F(y)=i\,(e^{-iyb}-e^{-iya})/\sqrt{2\pi}\,y$, and the second mean-value theorem shows that

$$\Phi_w(x)=\frac{1}{\pi}\left[\int_0^{(x-a)w}\frac{\sin y}{y}\,dy+\int_0^{(b-x)w}\frac{\sin y}{y}\,dy\right]=w^{-1}O\left(\frac{1}{|x|}\right)$$

for $|x|$ large. Since $\mathfrak{M}_p[\Phi_w-f;-A,A]$ tends to 0 for any fixed A, and $\mathfrak{M}_p[\Phi_w-f;-\infty,-A]+\mathfrak{M}_p[\Phi_w-f;A,\infty]$ is small for A large, it is easy to see that $\mathfrak{M}_p[\Phi_w-f]\to 0$, and the theorem is established [2]).

[1]) Since $|1/u-\operatorname{ctg}u|\leqslant C<\infty$ for $|u|<\frac{3}{4}\pi$, we obtain that, for fixed x, and n large enough, $|\alpha_n|\leqslant C\,\mathfrak{M}[f;-\pi n,\pi n]/2\pi n\leqslant C\,\mathfrak{M}_r[f](2\pi n)^{-1/r}\to 0$.

[2]) Hence any function $f\in L^p(-\infty,\infty)$, $1<p\leqslant 2$, is the transform of a function $g\in L^{p'}(-\infty,\infty)$.

10.5. Miscellaneous theorems and examples.

1. If $f(x)\in L^2(-\infty,\infty)$, the integral $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{-iyx}\,dx$ is summable $(C,1)$ for almost every y. Plancherel [2].

[Observe that $f(x)$ is the transform of a function of the class $L^2(-\infty,\infty)$].

2. If $f(x)\in L^2(-\infty,\infty)$, then $\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}f(x)\,e^{-iyx}dx=o(\sqrt{\log\omega})$, for almost every y.

[Use the method of § 10.32].

3. If $f(x)\in L^q(-\infty,\infty)$, $q>2$, the function

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,\frac{e^{-ixy}-1}{-ix}\,dx$$

may be almost everywhere non-differentiable.

[Let $\{\alpha_n\}$ be a sequence of real numbers such that $\Sigma|\alpha_n|^q<\infty$, $\Sigma\alpha_n^2=\infty$; put $f(x)=\alpha_n$ for $2^n-\frac{1}{2}<|x|<2^n+\frac{1}{2}$, $n=0,1,\dots$, and $f(x)=0$ elsewhere, and apply Theorem 5.7.7. For a similar result see Titchmarsh [6]].

4. Show that Mellin's inversion formulae

$$\varphi(s)=\int_0^{\infty}x^{s-1}\,\psi(x)\,dx,\quad \psi(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\varphi(s)\,x^{-s}\,ds,$$

may, with suitable conditions, be deduced from the formulae 12.4(1) and 12.4(2).