CHAPTER XI.

Riemann's theory of trigonometrical series.

11.1. In the previous chapters we have, almost exclusively, considered the behaviour of Fourier series. Now we shall prove a number of theorems concerning the properties of trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

with coefficients tending to 0, but otherwise quite arbitrary. The fundamental results in this field are due to Riemann, and these results, with their subsequent extensions, constitute what is now called the Riemann theory of trigonometrical series. The chief results of the Riemann theory concern the problems of *uniqueness* and of *localization* for trigonometrical series.

In what follows we shall suppose, unless otherwise stated, that the coefficients of the trigonometrical series considered tend to 0.

11.11. The Cantor-Lebesgue theorem. In the sequel we shall frequently use the following notation:

$$\tfrac{1}{2}a_0 = A_0(x), \quad a_n \cos nx + b_n \sin nx = A_n(x),$$

$$b_n \cos nx - a_n \sin nx = B_n(x), \quad n > 0,$$

$$A_n(x) = \rho_n \cos(nx + \alpha_n), \text{ where } \rho_n^2 = a_n^2 + b_n^2, \quad \rho_n \geqslant 0.$$

The following theorem is called the Cantor-Lebesgue theorem:

(i) *If* $A_n(x)$ *tends to* 0, *as* $n \to \infty$, *for every* x *belonging to a set* E *of positive measure, then* $a_n \to 0$, $b_n \to 0$.

For, if ρ_n does not tend to 0, there exists a sequence $n_1 < n_2 < \ldots$ of indices, and an $\varepsilon > 0$ such that $\rho_{n_k} > \varepsilon$, $k = 1, 2, \ldots$ From this, and the relation $\rho_n \cos(nx + \alpha_n) \to 0$, we obtain that $\cos(n_k x + \alpha_{n_k}) \to 0$ and, à fortiori, $\cos^2(n_k x + \alpha_{n_k}) \to 0$ for $x \in E$. The terms of the last sequence do not exceed 1, and so, by Lebesgue's theorem on the integration of bounded sequences, the expression

$$(1) \qquad \int_E \cos^2(n_k x + \alpha_{n_k})\, dx = \tfrac{1}{2} \int_E dx + \tfrac{1}{2} \int_E \cos 2\,(n_k x + \alpha_{n_k})\, dx$$

tends to 0. Since the numbers $\dfrac{1}{\pi} \int_E \cos 2n_k x\, dx$, $\dfrac{1}{\pi} \int_E \sin 2n_k x\, dx$ are Fourier coefficients of the characteristic function of E, they tend to 0, and the right-hand side of the equation (1) tends to $\frac{1}{2}|E| > 0$. This contradiction proves the theorem. As corollaries we obtain the following propositions, the second of which contains the first as a special case.

(ii) *If the series* 11.1(1) *converges in a set E of positive measure, then* $a_n \to 0$, $b_n \to 0$.

(iii) *If the series* 11.1(1) *is summable* (C, k), $k > -1$, *in a set E of positive measure, then* $a_n = o(n^k)$, $b_n = o(n^k)$.

To prove (iii), we observe that $a_n n^{-k} \cos nx + b_n n^{-k} \sin nx \to 0$ for $x \in E$ (§ 3.13) and apply (i). From (iii) and Theorem 2.221 we infer that, in the general case, the method (C, k), $k < 1$, is too weak to sum Fourier-Denjoy series.

11.12. A generalization of the previous theorem. Given any sequence of real numbers $\alpha_1, \alpha_2, \ldots$, and a number $-1 < \beta < 1$, we shall denote by E_n the set of points in the interval $(0, 2\pi)$ for which $\cos(nx + \alpha_n) \geqslant \beta$. We have $|E_n| = 2\pi\theta$, where the positive number θ is equal to $(\arccos \beta)/\pi$, and so $|E_n|$ depends on β only.

For any infinite sequence $n_1 < n_2 < \ldots$, and fixed β, the product $E = E_{n_1} E_{n_2} \ldots$ is of measure 0. Clearly we may omit as many factors in the product as we please, since this only extends E. In the first place, we observe that, if S is any finite system of intervals, then $|SE_n| \to \theta |S|$ as $n \to \infty$. Now let $\theta < \theta_1 < 1$, $m_1 = n_1$, and suppose that we have already defined $m_1, m_2, \ldots, m_{k-1}$. If $S_{k-1} = E_{m_1} E_{m_2} \ldots E_{m_{k-1}}$, we can find a number $m_k > m_{k-1}$ belonging to $\{n_l\}$ and such that $|S_{k-1} E_{m_k}| \leqslant \theta_1 |S_{k-1}|$. Hence, putting

$S_k = E_{m_1} E_{m_2} \ldots E_{m_k}$, we have $|S_k| \leqslant 2\pi\theta_1^k$. Therefore $|E_{m_1} E_{m_2} \ldots| = 0$ and, à fortiori, $|E| = 0$.

Sets such as the set E which we have just considered, will be called H-sets [1]). Every H-set is defined by the sequences $n_1, n_2, \ldots, \alpha_{n_1}, \alpha_{n_2}, \ldots$ (the second of which we may denote by $\alpha_1, \alpha_2, \ldots$ simply) and the number β. If $n_k = 3^k$, $\alpha_k = 0$, $k = 1, 2, \ldots$, and $\beta = -\frac{1}{2}$, we obtain Cantor's ternary set constructed on $(0, 2\pi)$.

We shall say that a set is a H_σ-set if it is a sum of a finite or enumerable sequence of H-sets. Since every H-set is closed and of measure 0, it is non-dense. Therefore sets of type H_σ are of the first category and of measure 0.

We shall require the following lemma.

If $\{\alpha_k\}$ is an arbitrary sequence of real numbers and $n_1 < n_2 < \ldots$ an arbitrary sequence of integers, then, except perhaps for x belonging to a set E of type H_σ, we have $\overline{\lim}\, |\cos(n_k x + \alpha_k)| = 1$.

If $0 < \gamma < 1$ and if $|\cos(n_k x + \alpha_k)| \leqslant \gamma$, then, à fortiori $\cos(n_k x + \alpha_k) \geqslant -\gamma$. Let $G_i^{(\gamma)}$ denote the set of x such that $|\cos(n_k x + \alpha_k)| \leqslant \gamma$ for $k \geqslant i$. From what we have just said it follows that $G_i^{(\gamma)} \subset F_i^{(\gamma)}$, where $F_i^{(\gamma)}$ is an H-set. Therefore $G^{(\gamma)} = G_1^{(\gamma)} + G_2^{(\gamma)} + \ldots$ is contained in an H_σ-set, and the same is true for the set $E = G^{(1/2)} + G^{(2/3)} + G^{(3/4)} + \ldots$, outside which we have $\overline{\lim}\, |\cos(n_k x + \alpha_k)| = 1$.

Now we are in a position to prove the following theorem due to Steinhaus.

Except perhaps in a set E of measure 0 *and of type* H_σ,

$$\overline{\lim_{n\to\infty}}\, |a_n \cos nx + b_n \sin nx| = \overline{\lim_{n\to\infty}} \sqrt{a_n^2 + b_n^2}$$ [2]).

Let $A_n(x) = \rho_n \cos(nx + \alpha_n)$, and let $\{n_k\}$ be a sequence such that $\lim \rho_{n_k} = \overline{\lim}\, \rho_n$. If E is the set E of the lemma, then outside E we have

$$\overline{\lim}\, |A_n(x)| \geqslant \overline{\lim}\, |A_{n_k}(x)| = \lim \rho_{n_k} = \overline{\lim}\, \rho_n,$$

i. e. $\overline{\lim}\, |A_n(x)| \geqslant \overline{\lim}\, \rho_n$. Since the inverse inequality is satisfied for every x, the theorem follows.

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[1]) These sets were introduced by Rajchman [1].

[2]) Steinhaus [8] proved that $|E| = 0$; that E is of type H_σ was shown by Rajchman [1].

It is plain that the Cantor-Lebesgue theorem is a consequence of Steinhaus's. Since H_σ-sets are of the first category, we obtain, in particular, that, *if $A_n(x)$ tends to 0 in a set of the second category, then $a_n \to 0$, $b_n \to 0$* [1]).

11.2. Riemann's theorems on the formal integration of trigonometrical series [2]). Given the series 11.1(1) with $a_n, b_n \to 0$, consider the function

$$(1) \qquad F(x) = \tfrac{1}{4} a_0 x^2 - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}.$$

The series on the right, which is obtained by integrating 11.1(1) formally twice, converges absolutely and uniformly, and so $F(x)$ is continuous. It will be readily seen that

$$(2) \qquad \frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2} = A_0 + \sum_{n=1}^{\infty} A_n(x)\left(\frac{\sin nh}{nh}\right)^2.$$

The numerator of the ratio on the left will be denoted by $\Delta^2 F(x, 2h)$. The upper and lower limits of indetermination of $\Delta^2 F(x,h)/h^2$, as $h \to 0$, will be denoted by $\overline{D}^2 F(x)$ and $\underline{D}^2 F(x)$ respectively. The common value of $\overline{D}^2 F(x)$ and $\underline{D}^2 F(x)$, if it exists, will be denoted by $D^2 F(x)$ and called the *generalized second derivative* of F at the point x. If $D^2 F(x_0)$ exists and is finite, we shall say that the series 11.1(1) is, at the point x_0, summable by Riemann's method of summation, or summable R, to the value $D^2 F(x)$.

(i) *If 11.1(1), where $a_n, b_n \to 0$, converges at a point x to sum s, it is also summable R to the same sum.*

It is sufficient to show that $\Delta^2 F(x, 2h_i)/4h_i^2$ tends to s for every sequence $\{h_i\}$ of positive numbers tending to 0. Let us put $A_0+A_1+\ldots+A_n = s_n$, $(\sin^2 h)/h^2 = u(h)$. Applying Abel's transformation, we see that the right-hand side of (2), for $h = h_i$, is equal to

$$(3) \qquad \sum_{n=0}^{\infty} s_n \cdot \{u(nh_i) - u((n+1)h_i)\}.$$

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[1]) Young [14].

[2]) Riemann [1]. Proposition (i) of this section is a special case of Theorem 10.42, but we prefer not to use that result.

Here we have a linear transformation of the sequence $s_n \to s$, and, to prove that (3) tends to s, it is sufficient to show that the Toeplitz conditions of § 3.1 are satisfied. Conditions (i) and (ii) are obviously satisfied. To verify (iii) we observe that

$$(4) \qquad \sum_{n=0}^{\infty} |u(nh_i) - u((n+1)h_i)| \leqslant \sum_{n=0}^{\infty} \int_{nh_i}^{(n+1)h_i} |u'(t)|\, dt = \int_0^{\infty} |u'(t)|\, dt,$$

and that the last integral is finite.

Theorem (i) may be generalized as follows.

(i') *If the series 11.1(1) has partial sums $s_n(x)$ bounded at x, and if $\underline{s}(x) = \underline{\lim}\, s_n(x)$, $\overline{s}(x) = \overline{\lim}\, s_n(x)$, then the numbers $\underline{D}^2 F(x)$ and $\overline{D}^2 F(x)$ are both contained in the interval $(s - k\delta,\ s + k\delta)$, where $2s = \underline{s}(x) + \overline{s}(x)$, $2\delta = \overline{s}(x) - \underline{s}(x)$, and k is an absolute constant.*

This follows from § 3.101, if for k we take the upper bound, for all $\{h_i\}$, of the sums on the left of (4).

(ii) *If a_n and b_n tend to 0, then*

$$(5) \qquad \frac{F(x+2h)+F(x-2h)-2F(x)}{4h} = A_0 h + \sum_{n=1}^{\infty} A_n \frac{\sin^2 nh}{n^2 h} \to 0$$

as $h \to 0$.

It is again sufficient to prove (5) for any sequence $\{h_i\}$ of positive numbers tending to 0. The series in (5) is a linear transformation of the sequence $A_n \to 0$, and so it is sufficient to verify Toeplitz's conditions (i) and (iii) (condition (ii) need not be tested). The first of them is obviously satisfied. To prove (iii) we observe that

$$(6) \qquad h_i + \sum_{n=1}^{\infty} \frac{\sin^2 nh_i}{n^2 h_i} \leqslant h_i + \sum_{n=1}^{N} \frac{n^2 h_i^2}{n^2 h_i} + \sum_{n=N+1}^{\infty} \frac{1}{n^2 h_i} < (N+1) h_i + 1/Nh_i.$$

If we put $N = [1/h_i] + 1$, then $1/h_i < N \leqslant 1/h_i + 1$ and the right-hand side of (6) is less than 4 for $|h_i| \leqslant 1$. This completes the proof.

It is plain that (5) is satisfied uniformly in x.

The relation (5) is satisfied at every point x, irrespectively of the convergence or divergence of the series 11.1(1). If $G(x)$ is the sum of an arbitrary trigonometrical series with coefficients $o(n^{-2})$, then $\Delta^2 G(x, 2h)/4h \to 0$ for every x, and $h \to 0$; for G may be considered as the function F corresponding to a trigonometrical series with coefficients tending to 0.

If for a function $F(x)$ we have $\Delta^2 F(x_0, h)/h \to 0$, then F will be said to be *smooth* at the point x_0. For, writing $\Delta^2 F(x_0, h)/h$ in the form $\{F(x_0+h)-F(x_0)\}/h - \{F(x_0)-F(x_0-h)\}/h$, we see that F cannot have an angular point at x_0: if the right-hand and the left-hand derivatives at x_0 exist, they must be equal.

11.21. Fatou's theorems. Instead of the function $F(x)$ defined by 11.2(1), we may consider the function

$$L(x) = \tfrac{1}{2} a_0 x + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n} \tag{1}$$

obtained from 11.1(1) by a single integration. Then

$$\frac{L(x+h)-L(x-h)}{2h} = A_0 + \sum_{n=1}^{\infty} A_n(x) \left(\frac{\sin nh}{nh}\right).$$

The trouble is that, in the general case, the series in (1) need not converge everywhere, even if 11.1(1) converges for every x (a simple example is provided by the series $\Sigma (\sin nx)/\log n$), and this makes the function $L(x)$ much less convenient in applications.

If $L(x)$ exists in a neighbourhood of a point x_0 and if the ratio $\{L(x_0+h)-L(x_0-h)\}/2h$ tends to a limit s as $h \to 0$, we shall say that the series 11.1(1) is summable by Lebesgue's method of summation, or summable L, to the value s, at the point x_0.

(i) *If a_n and b_n are $o(1/n)$, a necessary and sufficient condition that the series 11.1(1) should converge, at a point x, to sum s, is that it should be summable L to s* [1]).

In view of the conditions imposed upon a_n and b_n, $F(x)$ exists for every x. Let $s_n(x) = A_0 + A_1 + \dots + A_n$, $N = [1/h]$; then

$$\frac{L(x+h)-L(x-h)}{2h} - s_N = \tag{2}$$
$$= \sum_{n=1}^{N} A_n \left(\frac{\sin nh}{nh} - 1\right) + \sum_{n=N+1}^{\infty} A_n \frac{\sin nh}{nh} = P + Q.$$

The terms of Q are $o(n^{-2}h^{-1})$, and so $Q = o(N^{-1}h^{-1}) = o(1)$. Since

[1]) Fatou [1]. In this proposition, as well as in (ii) below, the number s may be infinite.

$(\sin u)/u - 1 = O(u^2) = O(u)$ for $|u| \leqslant 1$, the terms of P are $h \cdot o(1)$, and $P = o(Nh) = o(1)$. Therefore $P + Q = o(1)$, and, in fact, uniformly in x, and the theorem follows.

By the Riesz-Fischer theorem, trigonometrical series with coefficients $o(1/n)$ are Fourier series.

(ii) *If a_n and b_n are $o(1/n)$, and if 11.1(1) is the Fourier series of a function f such that $f(x) \to s$ as $x \to x_0 + 0$, then the series converges at the point x_0 to the value s.*

(iii) *If a_n and b_n are $o(1/n)$ and if 11.1(1) is $\mathfrak{S}[f]$, where f is continuous in an interval $a \leqslant x \leqslant b$, then the series converges uniformly in that interval.*

To prove (ii) we observe that, at the point x_0, the function $L(x)$ has a right-hand derivative equal to s. Since $L(x)$ is a smooth function (§ 11.2), the left-hand derivative at x_0 exists and is also equal to s. Hence $\{L(x_0+h)-L(x_0-h)\}/2h \to s$, and so, by (i), $s_n(x_0) \to s$.

To prove (iii) we notice that, if $h \to +0$, then $\{L(x+h)-L(x)\}/h$ tends to $f(x)$, uniformly in the interval $(I)\ a \leqslant x \leqslant a + \frac{1}{2}(b-a)$. Since $\Delta^2 L(x, h)/h \to 0$ uniformly in x (§ 11.2), we obtain that $\{L(x)-L(x-h)\}/h \to f(x)$, and so also $\{L(x+h)-L(x-h)\}/2h \to f(x)$, uniformly in I. Similarly we prove the last relation in the remaining part of (a, b), and it is sufficient to observe that the left-hand side of (2) tends to 0 uniformly in x.

11.3. Uniqueness of trigonometrical series. In previous chapters we have learnt to associate with every integrable and periodic function $f(x)$ a special trigonometrical series — the Fourier series of $f(x)$ — which, as we have shown, represents $f(x)$ in various ways. It is natural to inquire whether functions can be represented by trigonometrical series other than Fourier series. This problem has many aspects, according to the sense which we assign to the word 'represent'. The problem of the convergence, or summability, in mean was discussed in Chapter IV. In this paragraph we shall consider the representation of functions by means of trigonometrical series which are everywhere convergent. The following results are fundamental for the theory of trigonometrical series.

(i) *If a trigonometrical series converges everywhere to* 0, *the series vanishes identically, i. e. all the coefficients are equal to* 0.

(ii) *If two trigonometrical series converge to the same sum in the interval* $(0, 2\pi)$, *the series are identical, i. e. corresponding coefficients in the two series are equal.*

(iii) *If a trigonometrical series converges in the interval* $(0, 2\pi)$ *to an integrable function* $f(x)$, *the series is* $\mathfrak{S}[f]$.

Of these theorems, (ii) follows from (i), and the latter is, in turn, a consequence of (iii). Theorem (i) is due to Cantor; (iii) was established, in the case of f bounded and integrable in the Riemann sense, by Du Bois-Reymond, and in the general case by de la Vallée Poussin [1]).

The most important step in the proof of (iii) will have been achieved when we have shown that the function $F(x)$ defined by 11.2(1) satisfies an equation

$$(1) \quad F(x) = \int_a^x dy \int_a^y f(t)\, dt + Ax + B, \qquad (A, B \text{ constants})$$

i. e. that the formal integration of the series 11.1(1) corresponds to the integration of $f(x)$. For let $F_1(x) = F(x) - {}^1/_4\, a_0 x^2$; it is clear that the series 11.2(1) without the quadratic term is $\mathfrak{S}[F_1]$. The function $F_1(x)$ is a second integral and, as may be seen from 11.2(1), a periodic function. Let us put $2c_n = a_n - ib_n$ and write $\mathfrak{S}[F_1]$ in the complex form. Integrating by parts twice and observing that $F_1(x)$ and $F_1'(x)$ are periodic, we have, for $n \neq 0$,

$$-\frac{c_n}{n^2} = \frac{1}{2\pi}\int_0^{2\pi} F_1 e^{-inx}\, dx = -\frac{1}{2\pi n^2}\int_0^{2\pi} F_1'' e^{-inx}\, dx =$$

$$= -\frac{1}{2\pi n^2}\int_0^{2\pi} [f - \tfrac{1}{2} a_0]\, e^{-inx}\, dx = -\frac{1}{2\pi n^2}\int_0^{2\pi} f e^{-inx}\, dx,$$

i. e.

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f e^{-inx}\, dx.$$

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1) Cantor [1], Du Bois-Reymond [3], de la Vallée-Poussin [3]; Denjoy [4] showed that, with a suitable definition of an integral, more general that that of Lebesgue, every trigonometrical series convergent to a finite sum is a Fourier series.

To find the same formula for $c_0 = \frac{1}{2} a_0$, it is sufficient to observe that the function $F_1'(x) = F'(x) - \frac{1}{2} a_0 x$ is periodic, and so the integral of $F_1''(x) = F''(x) - \frac{1}{2} a_0 = f(x) - \frac{1}{2} a_0$ over the interval $(0, 2\pi)$ is equal to 0.

11.31. We shall now prove a number of lemmas, which give a little more than we actually require.

(i) *If a continuous function* $F(x)$, $a < x < b$, *satisfies the inequality* $\overline{D}^2 F(x) \geqslant 0$, *except perhaps at an enumerable set* E, *where however* F *is smooth, then* F *is convex.*

It is sufficient to consider the case $\overline{D}^2 F > 0$ for, if we put $F_n(x) = F(x) + x^2/n$, then $\overline{D}^2 F_n(x) > 0$, $F_n(x) \to F(x)$, and the limit of a sequence of convex functions is convex. If $F(x)$ were not convex, there would exist two points α and β, and a linear function $l(x) = mx + n$, such that $\rho(x) = F(x) - l(x)$ would vanish for $x = \alpha$, $x = \beta$, and would assume positive values somewhere in (α, β). It is important to observe that, if we replace m by m_1, where $m_1 > m$ and $m_1 - m$ is sufficiently small, we shall still have the same situation. Let x_0 be a point in (α, β) where $\rho(x)$ attains its maximum; hence $\Delta^2\rho(x_0, h) \leqslant 0$ for h positive and sufficiently small. It follows that $\overline{D}^2\rho(x_0) = \overline{D}^2 F(x_0) \leqslant 0$, which contradicts our hypothesis, and so proves the lemma, unless $x_0 \in E$.

Suppose now that x_0 belongs to E, and divide the inequality $\Delta^2\rho(x_0, h) = \rho(x_0 + h) - \rho(x_0) + \rho(x_0 - h) - \rho(x_0) \leqslant 0$ by $h \to +0$. The function $\rho(x)$ is smooth at x_0, and so, taking into account that $\rho(x_0 + h) - \rho(x_0) \leqslant 0$, $\rho(x_0 - h) - \rho(x_0) \leqslant 0$ for h small enough, we obtain that the right-hand and the left-hand derivatives of $\rho(x)$ at x_0 exist and are equal to 0, i. e. $\rho'(x_0) = F'(x_0) - m = 0$; in particular $F'(x_0)$ exists. Therefore if, instead of m, we take a number $m_1 > m$ sufficiently near to m, and such that $m_1 \neq F'(\xi)$ for every $\xi \in E$, the point x_0 does not belong to E, and in this case the lemma has already been established.

(ii) *If a function* $F(x)$, $a < x < b$, *has a continuous derivative* $F'(x)$ *and if, at a point* x_0, *all the derivates of* $F'(x)$ *are contained between* m *and* M, *then* $m \leqslant \underline{D}^2 F(x_0) \leqslant \overline{D}^2 F(x_0) \leqslant M$.

By the mean-value theorem, the ratio $\Delta^2 F(x_0, h)/h^2$ is equal to $[F(x_0 + h_1) - F(x_0 - h_1)]/2h_1$, $0 < h_1 < h$; and since the last ratio is the arithmetic mean of the expressions $[F(x_0 \pm h_1) - F(x_0)]/h_1$, it is contained between m and M.

(iii) *Let $f(x)$, $a \leqslant x \leqslant b$, be an integrable function, $f_1(x)$ the indefinite integral of $f(x)$, and $\varepsilon > 0$ an arbitrary number. Then there exist two functions $\varphi(x)$ and $\psi(x)$ such that* (a) $|f_1(x) - \varphi(x)| < \varepsilon$, $|f_1(x) - \psi(x)| < \varepsilon$, (b) *at every point where $f(x) \neq +\infty$ all the derivates of ψ exceed $f(x)$, and at every point where $f(x) \neq -\infty$ all the derivates of $\varphi(x)$ are less than $f(x)$.*

For the proof we refer the reader to any of the standard treatises on the Lebesgue integral[1]).

(iv) *Let $f(x)$, $a \leqslant x \leqslant b$, be an integrable function, finite except perhaps at an enumerable set E. Let $F(x)$, $a \leqslant x \leqslant b$, be a continuous function such that $\underline{D}^2 F(x) \leqslant f(x) \leqslant \overline{D}^2 F(x)$, except perhaps in E, where however F is smooth. Then F is of the form* 11.3(1).

Let $\varphi_n(x)$ and $\psi_n(x)$ be a pair of functions φ and ψ from (iii) corresponding to $\varepsilon = 1/n$, $n = 1, 2, \ldots$ Let $J[g; a, x]$ denote the integral of any function $g(t)$ over (a, x). Let $f_1(x) = J[f; a, x]$, $f_2(x) = J[f_1; a, x]$, $\Phi_n(x) = J[\varphi_n; a, x]$, $\Psi_n(x) = J[\psi_n; a, x]$. From (ii) it follows that $\underline{D}^2 \Psi_n(x) > f(x) \geqslant \underline{D}^2 F(x)$, $\overline{D}^2 \Phi_n(x) < f(x) \leqslant \overline{D}^2 F(x)$ for $x \bar{\in} E$. From this, and from the obvious inequalities

$$\underline{D}^2 \Psi_n \leqslant \overline{D}^2(\Psi_n - F) + \underline{D}^2 F, \quad \overline{D}^2 \Phi_n \geqslant \underline{D}^2(\Phi_n - F) + \overline{D}^2 F,$$

we obtain $\overline{D}^2\{\Psi_n(x) - F(x)\} \geqslant 0$ and $\underline{D}^2\{\Phi_n(x) - F(x)\} \leqslant 0$, i. e. $\overline{D}^2\{F(x) - \Phi_n(x)\} \geqslant 0$, for $x \bar{\in} E$. Using (i), we see that $\Psi_n - F$ and $F - \Phi_n$ are convex functions. Since $\varphi_n(x) \to f_1(x)$, $\psi_n(x) \to f_1(x)$, and so $\Phi_n(x) \to f_2(x)$, $\Psi_n(x) \to f_2(x)$ as $n \to \infty$, we obtain that $f_2(x) - F(x)$ and $F(x) - f_2(x)$ are convex functions. Hence $F(x) - f_2(x)$ is a linear function and the lemma follows. Incidentally, in view of (ii), the result shows that $\underline{D}^2 F(x) = \overline{D}^2 F(x) = f(x)$ almost everywhere.

(v) *If $F(x)$ is convex in an interval (a, b), then $\overline{D}^2 F(x)$ exists for almost every x and is integrable over any interval $(a + \varepsilon, b - \varepsilon)$, $\varepsilon > 0$.*

Since $F(x)$ is the indefinite integral of a non-decreasing function $\xi(x)$ (§ 4.141), we have

$$(1) \quad \frac{F(x+h) + F(x-h) - 2F(x)}{h^2} = \frac{1}{h^2} \int_0^h [\xi(x+t) - \xi(x-t)]\, dt.$$

[1]) See e. g. de la Vallée Poussin, *Intégrales de Lebesgue*, Saks, *Théorie de l'intégrale.*

By Lebesgue's classical theorem, $\xi'(x)$ exists almost everywhere and is integrable over $(a + \varepsilon, b - \varepsilon)$. At every point x where $\xi'(x)$ exists, we have $\xi(x+t) - \xi(x-t) = 2t\,\xi'(x) + o(t)$, and so the right-hand side of (1) tends to $\xi'(x)$. This proves the lemma.

11.32. We are now in a position to prove Theorem 11.3 (iii), and even the following more general result, in which the upper and lower sums of a series with partial sums s_n mean the numbers $\overline{\lim}\, s_n$ and $\underline{\lim}\, s_n$ respectively.

If the upper sum $f^*(x)$ and the lower sum $f_*(x)$ of the series 11.1(1), *where $a_n \to 0$, $b_n \to 0$, are both integrable, and finite outside an enumerable set E of points, the series is $\mathfrak{S}[f]$, where $f(x) = \overline{D}^2 F(x)$ (or $f(x) = \underline{D}^2 F(x)$) and F is given by* 11.3(1).

For from Theorem 11.2(i') it follows that $\overline{D}^2 F(x)$ and $\underline{D}^2 F(x)$ are both integrable, and are finite for $x \bar{\in} E$. The function F is smooth (§ 11.2(ii)); hence, if we put $f(x) = \overline{D}^2 F(x)$, the conditions of the last lemma but one of § 11.31 are satisfied, F is of the form 11.3(1), and this, as we know, proves the theorem.

The following remark, which requires no proof, will be useful later: *If the conditions of the last theorem are satisfied in an interval (a, b), the function $F(x)$ satisfies the equation* 11.3(1) *for $a < x < b$.*

The proof of Theorem 11.3(iii) which we have given is not very simple; it is therefore of interest to observe that Theorem 11.3(i), which is very important in itself, is much easier. For, under the hypothesis of that theorem, the function $F(x)$ satisfies the condition $D^2 F(x) = 0$, and so, using Lemma 11.31(i) in its simplest form ($\underline{D}^2 F = \overline{D}^2 F = D^2 F = 0$), we obtain that the functions F and $-F$ are convex; hence F is a linear function $Ax + B$. The equation

$$(1) \qquad {}^1/_4\, a_0 x^2 - Ax - B - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} = 0$$

holds for all x, and so, making $x \to \infty$ and observing that the sum on the left represents a bounded function, we obtain $A = 0$, $a_0 = 0$. Now the left-hand side of (1) is a trigonometrical series converging uniformly to 0; hence $B = a_1 = b_1 = a_2 = \ldots = 0$ and the theorem follows.

11.33. Theorem 11.32 may be generalized as follows:

If $f_*(x)$ and $f^*(x)$ are finite outside an enumerable set E, and if $f_*(x) \geqslant g(x)$, where $g(x)$ is integrable (in particular, if $f_*(x)$ is integrable), the series is a Fourier series.

In this paragraph we shall only prove the theorem in the special case $f_*(x) = f^*(x) = f(x)$ [1]). The general result is a corollary of a theorem which we shall prove in § 11.6.

Let $g_1, g_2, \varphi_n, \Phi_n$ be functions which have a similar meaning to that in the proof of Theorem 9.31(iv), but correspond to the function g. It follows that, outside E, $\overline{D}^2(F-\Phi_n) \geqslant \overline{D}^2 F - \overline{D}^2 \Phi_n = = D^2 F - \overline{D}^2 \Phi_n \geqslant f - g \geqslant 0$. Thus $F - \Phi_n$ is convex, and, making $n \to \infty$, we obtain that $F - g_2$ is also convex. Hence $D^2(F-g_2) = f-g$ exists almost everywhere and is integrable over any finite interval (§ 11.31(v)). Thence we deduce that f is integrable, and the theorem considered follows from Theorem 11.32.

11.4. The principle of localization. It was proved in § 2.5 that the behaviour of $\mathfrak{S}[f]$ at a point x_0 depends only on the values of f in an arbitrarily small neighbourhood of x_0. This is a special case of the following more general theorem, due to Riemann, which involves arbitrary trigonometrical series with coefficients tending to 0: *The behaviour of the series* 11.1(1) *at a point x_0 depends only on the values of the function $F(x)$, defined by* 11.2(1), *in an arbitrarily small neighbourhood of x_0.* More precisely:

Let $F_1(x)$ and $F_2(x)$ be the functions F corresponding to two trigonometrical series; if $F_1(x) = F_2(x)$ in an interval $(x_0-\varepsilon, x_0+\varepsilon)$, or, more generally, if $F_1(x) - F_2(x)$ is equal to a linear function in $(x_0-\varepsilon, x_0+\varepsilon)$, the series considered are equiconvergent at the point x_0.

If two integrable functions $f_1(x)$ and $f_2(x)$ are equal in an interval $(x_0-\varepsilon, x_0+\varepsilon)$, then, since Fourier series may be integrated term by term, the functions F_1 and F_2, corresponding to $\mathfrak{S}[f_1]$ and $\mathfrak{S}[f_2]$, differ by a linear function in $(x_0-\varepsilon, x_0+\varepsilon)$; this shows that the principle of localization for Fourier series is actually a special case of the theorem just stated.

[1]) This result has been obtained by Banach (as a generalization of an earlier result of Steinhaus [2] for the case $g(x)=0$) but never published.

11.41. Rajchman's theory of formal multiplication of trigonometrical series. A new approach to problems of localization is due to Rajchman, who developed the theory of formal multiplication of trigonometrical series with coefficients tending to 0 [1]). Not only does this theory enable us to obtain Riemann's results, but it can also be applied to problems where Riemann's classical method would not work.

We shall write trigonometrical series in the complex form (§ 1.43). Given two trigonometrical series

$$(1) \qquad \text{a)} \sum_{n=-\infty}^{+\infty} c_n e^{inx}, \qquad \text{b)} \sum_{n=-\infty}^{+\infty} \gamma_n e^{inx},$$

we shall call the series

$$(2) \qquad \sum_{n=-\infty}^{+\infty} C_n e^{inx}, \quad \text{where} \quad C_n = \sum_{p=-\infty}^{+\infty} c_p \gamma_{n-p},$$

their formal product, provided that the series defining C_n converge. This is certainly the case if the first of the series (1) has coefficients tending to 0 and the second converges absolutely. We shall assume for simplicity that the series considered are real, i. e. that $c_{-n} = \bar{c}_n$, $\gamma_{-n} = \bar{\gamma}_n$. It is plain that also $C_{-n} = \bar{C}_n$.

We require the following lemma, in which we suppose, as an exception to this rule, that c_n and γ_n are arbitrary complex numbers.

If $c_n \to 0$ as $n \to \pm\infty$, and if $\sum |\gamma_n|$ converges, then $C_n = \sum_{p=-\infty}^{\infty} c_p \gamma_{n-p}$ tends to 0 as $n \to \pm\infty$.

For let $M = \text{Max}\,|c_n|$; then, as $n \to +\infty$,

$$|C_n| \leqslant M \sum_{p=-\infty}^{[n/2]} |\gamma_{n-p}| + \underset{p>n/2}{\text{Max}}\,|c_p| \sum_{p=[n/2]+1}^{\infty} |\gamma_{n-p}| \leqslant$$

$$\leqslant M \sum_{q=n-[n/2]}^{\infty} |\gamma_q| + \underset{p>n/2}{\text{Max}}\,|c_p| \sum_{q=-\infty}^{+\infty} |\gamma_q| \to 0.$$

As regards the case $n \to -\infty$, we observe that $C_{-m} = \sum_{p=-\infty}^{\infty} c'_p \gamma'_{m-p}$, where $c'_p = c_{-p}$, $\gamma'_p = \gamma_{-p}$.

If c_n and γ_n depend on a parameter, and the conditions imposed upon c_n and γ_n are satisfied uniformly, then $C_n \to 0$ uniformly.

[1]) Rajchman [2], [3], Zygmund [11]. In the last paper a discussion of the case of coefficients not tending to 0 is given.

11.42. We shall say that the series 11.41(1b) is *rapidly convergent* to sum s, if the series converges to s and if, moreover, $\Gamma_0+\Gamma_1+...+\Gamma_n+...<\infty$, where $\Gamma_n=|\gamma_n|+|\gamma_{n+1}|+...$ We certainly have rapid convergence if, for example, $\gamma_n=O(n^{-3})$, $n>0$. The following theorem is fundamental for the whole theory.

(i) *Suppose that $c_n\to 0$ and that the series* 11.41(1b) *converges rapidly to* 0 *for x belonging to a set E. Then the product* 11.41(2) *converges uniformly to* 0 *in the set E.*

Let $R_k(x)$ denote the sum of the terms $\gamma_n e^{inx}$ with $n\geqslant k$. If $x_0\in E$, $k>0$, then $|R_{-k}(x_0)|=|R_{k+1}(x_0)|\leqslant\Gamma_{k+1}$, and so the series $\sum\limits_{-\infty}^{+\infty}|R_k(x_0)|$ is uniformly convergent in E. Now

$$S_m(x_0)=\sum_{n=-m}^{m}C_n e^{inx_0}=\sum_{n=-m}^{m}e^{inx_0}\sum_{p=-\infty}^{\infty}c_p\gamma_{n-p}=$$
$$=\sum_{p=-\infty}^{\infty}c_p e^{ipx_0}\sum_{n=-m}^{m}\gamma_{n-p}e^{i(n-p)x_0}=$$
$$=\sum_{p=-\infty}^{\infty}c_p e^{ipx_0}R_{-m-p}(x_0)-\sum_{p=-\infty}^{\infty}c_p e^{ipx_0}R_{m-p+1}(x_0).$$

Applying the lemma of the last section (with $c_p e^{ipx_0}$ and $R_{n-p}(x_0)$ instead of c_p and γ_{n-p}) we see that $S_m(x_0)$ tends uniformly to 0 for $x_0\in E$, $m\to\infty$. This proves (i).

The reader will observe that the above theorem remains true even if the coefficients c_n and γ_n of the series 11.41(1) depend themselves on the variable x, provided that the formal product is defined by 11.41(2). This is not surprising since proposition (i), as well as (ii) below, are nothing but theorems on the Laurent multiplication of arbitrary series [1]).

(ii) *If $c_n\to 0$, and if the series* 11.41(1b) *converges rapidly to sum $\lambda(x)$, the series*

$$\sum_{n=-\infty}^{\infty}C_n e^{inx}\quad \textit{and}\quad \lambda(x)\sum_{n=-\infty}^{\infty}c_n e^{inx}$$

are uniformly equiconvergent in the interval $(0,2\pi)$.

[1]) Similar theorems can be established for other rules of multiplication, in particular for Cauchy's rule.

Let us write $\gamma_0^*=\gamma_0-\lambda(x)$, $\gamma_n^*=\gamma_n$ for $n\neq 0$, and consider the formal product $\sum C_n^* e^{inx}$ of the series $\sum c_n e^{inx}$ and $\sum\gamma_n^* e^{inx}$. In view of (i) and the additional remark, the formal product converges to 0 uniformly in the interval $(0,2\pi)$, and it is sufficient to notice that $C_n^*=C_n-\lambda(x)\,c_n$.

Now we shall state a number of corollaries which, although very simple, have important applications.

(a) *If $\lambda(x_0)\neq 0$, a necessary and sufficient condition that* 11.41(2) *should converge at the point x_0, is that* 11.41(1a) *should converge there.*

Let $\mathfrak{A}$ be any Toeplitz method of summation (§ 3.1). Observing that, if $\sum C_n^* e^{inx_0}$ converges to 0, it is summable $\mathfrak{A}$ to 0, we obtain:

(b) *If $\lambda(x_0)\neq 0$, a necessary and sufficient condition that* 11.41(2) *should be summable $\mathfrak{A}$ at the point x_0 is that* 11.41(a) *should be summable $\mathfrak{A}$ at that point. If the sum of the latter series is s, the sum of the former is $\lambda(x_0)\cdot s$.*

(c) *If the series* 11.41(a) *is uniformly convergent, or summable $\mathfrak{A}$, over a set $\mathcal{E}$, so is the series* 11.41(2). *The converse is also true if $|\lambda(x)|\geqslant\varepsilon>0$ for $x\in\mathcal{E}$.*

Proposition (b) may be completed by considering limits of indetermination. Restricting ourselves to the case of ordinary convergence (the reader will have no difficulty in stating the general result) we have:

(d) *If the upper and lower sums of* 11.41(1a) *at the point x_0 are $\bar{s}$ and $\underline{s}$ respectively, the upper and lower sums of* 11.41(2) *are $\lambda(x_0)\cdot\bar{s}$ and $\lambda(x_0)\cdot\underline{s}$ if $\lambda(x_0)>0$, and $\lambda(x_0)\cdot\underline{s}$ and $\lambda(x_0)\cdot\bar{s}$ if $\lambda(x_0)<0$.*

11.43. Now we shall prove certain theorems about the series conjugate to formal products. It will be recalled that the series conjugate to 11.41(1a) may be obtained from the latter by replacing c_n by $c_n\varepsilon_n$, where $\varepsilon_n=-i\operatorname{sign}n$ (§ 1.13).

(i) *Under the hypotheses of Theorem* 11.42(i), *the series conjugate to the formal product converges uniformly over E.*

(ii) *Under the hypotheses of Theorem* 11.42(ii), *the series*

$$(1)\quad \text{a)}\ \sum_{n=-\infty}^{\infty}C_n\varepsilon_n e^{inx}\quad \textit{and}\quad \text{b)}\ \lambda(x)\sum_{n=-\infty}^{\infty}c_n\varepsilon_n e^{inx}\quad (\varepsilon_n=-i\operatorname{sign}n)$$

are uniformly equiconvergent in the wider sense.

Let $\overline{S}_n(x)$ denote the partial sums of the series (1a). Writing $c'_n = c_n e^{inx_0}$, and similarly defining C'_n and γ'_n, we have

$$\overline{S}_m(x_0) = \sum_{n=-m}^{m} \varepsilon_n C'_n = \sum_{n=-m}^{m} \varepsilon_n \sum_{p=-\infty}^{\infty} c'_p \gamma'_{n-p} = \sum_{p=-\infty}^{\infty} c'_p \sum_{n=-m}^{m} \gamma'_{n-p} \varepsilon_n =$$

$$= -i \sum_{p=-\infty}^{\infty} c'_p \sum_{n=1}^{m} (\gamma'_{n-p} - \gamma'_{-n-p}) =$$

$$= -i \sum_{p=-\infty}^{\infty} c'_p \{R_{1-p}(x_0) - R_{m-p+1}(x_0) - R_{-m-p}(x_0) + R_{-p}(x_0)\}$$

and, in view of Lemma 11.41, if $x_0 \in E$ and $m \to \infty$, the last expression tends uniformly to $-i \sum_{p=-\infty}^{\infty} c'_p \{R_{1-p}(x_0) + R_{-p}(x_0)\}$. This proves (i). To prove (ii) we use the same device as in the case of Theorem 11.42(ii). We consider the formal product $\Sigma C^*_n e^{inx}$ of the series $\Sigma c_n e^{inx}$ and $\Sigma \gamma^*_n e^{inx}$. The coefficients C^*_n depend on x, but if we define the series 'conjugate' to the product as $\Sigma \varepsilon_n C^*_n e^{inx}$, the latter series will, as the proof of (i) shows, be uniformly convergent. Since $C^*_n = c_n - \lambda(x)\gamma_n$, the theorem is established.

The following is one of the corollaries of (ii):

(a) *If the series* $\sum_{n=-\infty}^{\infty} c_n \varepsilon_n e^{inx}$ *is uniformly summable* $\mathfrak{A}$ *over a set* $\mathcal{E}$, *so is* (1a). *The converse is also true if* $|\lambda(x)| \geqslant \varepsilon > 0$ *over* $\mathcal{E}$.

A characteristic feature of the theorems on formal multiplication which we have proved is that we suppose next to nothing about one of the factors, whereas upon the second we impose rather stringent conditions. However, if the first series is a Fourier series, the conditions imposed upon the second series may be relaxed slightly. The reader will observe that Theorems 2.53 and 2.531 may be considered as theorems on the formal multiplication of trigonometrical series in the case when the first factor is a Fourier series.

We shall now give a number of applications of the theory of formal multiplication.

11.44. As a first application we shall show that, *given an arbitrary closed set* $E \subset (0, 2\pi)$, *there is a trigonometrical series with coefficients tending to* 0 *which converges in* E *and diverges outside* E[1]).

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[1]) Rajchman [2]. It is plain that, if E contains one of the points $0, 2\pi$, it must contain the other.

We start with the fact that there is a trigonometrical series

$$(1) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (a_n \to 0,\ b_n \to 0)$$

which diverges everywhere (§ 8.5). Let $\lambda(x)$ be a function, with Fourier coefficients $O(n^{-3})$, which is equal to 0 in E and different from 0 elsewhere[1]). The formal product of (1) by $\mathfrak{S}[\lambda]$ gives the required example, for, in view of Theorems 11.42, this product converges to 0 in E and diverges outside E.

Since, in view of Theorem 11.43(i), the series conjugate to the product considered converges in E, we obtain at once: *for every closed set E situated on the circumference of the unit circle there is a power series with coefficients tending to* 0 *which converges in E and diverges in the remaining points of the circumference*[2]).

11.441. The only example which we so far know of an everywhere divergent series is Kolmogoroff's example considered in § 8.5. Since that example is a Fourier series, the theory of formal multiplication was not indispensable in the argument of the previous section, and we could use Theorem 2.53 instead. Moreover, Kolmogoroff's series is fairly complicated, and it is therefore desirable to have a simpler example. Following Steinhaus, we shall show that *the series*

$$(1) \qquad \sum_{k=3}^{\infty} \frac{\cos k(x - \log\log k)}{\log k}$$

diverges for every x[3]).

Let $l_k = [\log k]$, $v_k = \log\log k$, and

$$G_n(x) = \sum_{k=n+1}^{n+l_n} \frac{\cos k(x - v_k)}{\log k}, \qquad G_n = \sum_{k=n+1}^{n+l_n} \frac{1}{\log k}, \qquad I_n = (v_n, v_{n+1}).$$

---

[1]) Let $\{(\alpha_n, \beta_n)\}$ be the sequence of intervals contiguous to E, and let $\lambda_n(x)$ be equal to $(x - \alpha_n)^4 (\beta_n - x)^4$ in (α_n, β_n) and to 0 elsewhere. If $\gamma_n > 0$, $\Sigma \gamma_n < \infty$, we may put $\lambda(x) = \Sigma \gamma_n \lambda_n(x)$, for $\lambda'''(x)$ exists and is continuous.

[2]) For a more complete result see Mazurkiewicz [1].

[3]) Steinhaus [10]. The first example of an everywhere divergent trigonometrical series with coefficients tending to 0 was given by Steinhaus [9]. Other examples will be found in Hardy and Littlewood [9], [18]. See also Wilton [1].

Since $G_n \geqslant l_n/\log(n+l_n) \to 1$, we have $G_n > 0.9$ for $n > n_0$. The inequality $|\sin u| \leqslant |u|$ gives

(2) $$0 \leqslant G_n - G_n(x) \leqslant \frac{1}{2\log n} \sum_{k=n+1}^{n+l_n} k^2 (x - v_k)^2.$$

If $n < k \leqslant n + l_n$, then $v_n < v_k \leqslant v_{n+l_n}$. Hence, if x belongs to the interval (v_n, v_{n+1}), $n \geqslant 3$, then $|x - v_k| \leqslant v_{n+l_n} - v_n$; applying the mean-value theorem we obtain $|x - v_k| \leqslant l_n/n \log n \leqslant 1/n$, and the right-hand side of (2) is less than $(n+l_n)^2\, l_n/2\, n^2 \log n < 0.6$ for $n > n_1$. Collecting the results, we see that

$$G_n(x) = G_n - (G_n - G_n(x)) > 0.9 - 0.6 = 0.3, \quad x \in I_n, \quad n \geqslant \operatorname{Max}(n_0, n_1).$$

Since every point x belongs (mod 2π) to an infinite number of the intervals I_n, the series (1) diverges for every x.

11.45. Fatou's theorem on power series. If the series

(1) $$\sum_{n=0}^{\infty} \alpha_n z^n = F(z)$$

converges at a point of the unit circle, then $\alpha_n \to 0$. The converse is false (the power series whose real part for $z = e^{ix}$ is the series 11.441(1), diverges at every point of the unit circle), but

If $\alpha_n \to 0$, the series (1) converges at every point of the unit circle where the function $f(x)$ is regular. The convergence is uniform on every closed arc of regularity.

This theorem, due substantially to Fatou [1]), is a consequence of more general results which will be established later. In view however of its importance, we shall prove it separately. Considering the real and imaginary parts of (1) for $z = e^{ix}$, we see that the theorem will be established when we have shown that, *if the series 11.44(1) is uniformly summable A, for $a \leqslant x \leqslant b$, to a function $g(x)$ which together with its first and second derivatives is continuous, the series is uniformly convergent in every interval (a', b') interior to (a, b).*

Let $\lambda(x)$ be a function equal to 1 in (a', b'), equal to 0 outside (a, b), and such that $\lambda'''(x)$ exists and is continuous. Since

[1]) Fatou [1], M. Riesz [1], [5], [6]. The part concerning uniform convergence, was first stated by M. Riesz.

the coefficients of $\mathfrak{S}[\lambda]$ are $O(n^{-3})$, the formal product of 11.44(1) by $\mathfrak{S}[\lambda]$ converges uniformly to 0 outside (a, b). By Theorem 11.42(ii), this product is uniformly summable A for $a \leqslant x \leqslant b$, to the value $\lambda(x)\, g(x)$. Hence it is uniformly summable A in the whole interval $0 \leqslant x \leqslant 2\pi$, to a sum $\varphi(x)$ which has a continuous second derivative. It follows that the product is $\mathfrak{S}[\varphi]$; for if α_n, β_n are the coefficients of the product, and $\varphi(r, x)$ the corresponding harmonic function, then

$$\alpha_n r^n = \frac{1}{\pi}\int_0^{2\pi} \varphi(r, x) \cos nx\, dx, \qquad \beta_n r^n = \frac{1}{\pi}\int_0^{2\pi} \varphi(r, x) \sin nx\, dx$$

and, making $r \to 1$, we see that α_n and β_n are Fourier coefficients of φ. Since $\varphi''(x)$ exists and is continuous, the numbers α_n, β_n are $O(n^{-2})$, and so $\mathfrak{S}[\varphi]$ converges uniformly. Observing that $\lambda(x) = 1$ for $a' \leqslant x \leqslant b'$, and applying Theorem 11.42(ii), we see that 11.44(1) converges uniformly over (a', b'), and the theorem is established.

The reader will notice that the condition concerning g'' was not indispensable. We only used it as a simple test ensuring the convergence of $\mathfrak{S}[\varphi]$. It would also be sufficient to assume that g satisfies the Dini-Lipschitz condition, or is continuous *and* of bounded variation.

11.46. Proof of the principle of localization. Let $\mathfrak{A}$ be a linear method of summation. We shall say that $\mathfrak{A}$ is of type U, if every trigonometrical series with coefficients tending to 0, and summable $\mathfrak{A}$ to a finite and integrable function $f(x)$, is $\mathfrak{S}[f]$. In § 11.3 we showed that ordinary convergence is of type U. It is important to notice that the method R is also of type U; this was implicitely proved in § 11.3, for the essence of the Riemann method in problems of uniqueness just consists in treating convergent series as series summable R. In § 11.6 we shall prove that Abel's method of summation is of type U.

In what follows we shall frequently consider formal products of trigonometrical series by the Fourier series of functions λ. To avoid repetition we shall tacitly assume that $\lambda''(x)$ exists and is of bounded variation. Then the Fourier coefficients of f are $O(n^{-3})$ and the theorems on formal multiplication can be applied. It will be also convenient to suppose that, if of two functions $\varphi(x)$ and $\psi(x)$ one is equal to 0 in an interval (α, β), the product $\varphi\psi$ exists and is equal to 0 in (α, β) even if the second factor is not defined in that interval.

(i) *Let* $\mathfrak{A}$ *be any method of summation of type U. If, for* $a<x<b$, *the series* 11.44(1) *is summable* $\mathfrak{A}$ *to a finite and integrable function* $f(x)$, *then, for* $a' \leqslant x \leqslant b'$, *the series is uniformly equiconvergent with* $\mathfrak{S}[\lambda f]$, *where* $\lambda(x)$ *is equal to* 1 *for* $a' \leqslant x \leqslant b'$, $a<a'<b'<b$, *and to* 0 *outside* (a, b) (mod 2π). *The series conjugate to* 11.44(1), *and* $\tilde{\mathfrak{S}}[\lambda f]$, *are uniformly equiconvergent in the wider sense in the interval* (a', b') [1].

To prove the first part of the theorem we observe that the product of 11.44(1) by $\mathfrak{S}[\lambda]$ converges to 0 outside (a, b), and is summable $\mathfrak{A}$ to λf in (a, b). Hence this product is summable $\mathfrak{A}$ in the whole interval $(0, 2\pi)$ to sum $\lambda(x) f(x)$. This sum is integrable; hence the product is $\mathfrak{S}[\lambda f]$ and it remains to apply Theorem 11.42(ii). To obtain the second part of the theorem we apply Theorem 11.43(ii).

Now we are in a position to prove the Riemann principle of localization which will be established in the following general form (we preserve the notation of § 11.4):

(ii) *Let* S_1 *and* S_2 *be two trigonometrical series with coefficients tending to* 0, *and let* $F_1(x)$ *and* $F_2(x)$ *denote the sums of the series* S_1 *and* S_2 *integrated formally twice. If the difference* $F_1(x)-F_2(x)$ *is a linear function in an interval* $a \leqslant x \leqslant b$, *the series* S_1 *and* S_2 *are uniformly equiconvergent in every interval* (a', b') *interior to* (a, b). *The series conjugate to* S_1 *and* S_2 *are uniformly equiconvergent in the wider sense in the interval* (a', b') [2].

Let 11.44(1) be the difference of S_1 and S_2. We have to show that this series, as well as its conjugate, are uniformly convergent over (a', b'), the sum of the former being 0. Integrating 11.41(1) twice, we obtain a function $F(x)=F_1(x)-F_2(x)$ which is linear over (a, b). Since $\Delta^2 F(x, h)/h^2 = 0$ for any x interior to (a, b), and h sufficiently small, the series 11.44(1) is summable R to 0 for $a<x<b$, and it suffices to apply proposition (i).

As a special case we obtain the following theorem.

---

[1] Rajchman [2], Zygmund [11].

[2] Riemann [1], Rajchman [2], Neder [2], Zygmund [11].

(iii) *Suppose that the sum* $F(x)$ *of the series* 11.44(1) *integrated twice satisfies an equation*

$$(1)\qquad F(x) = Ax + B + \int_a^x dy \int_a^y f(t)\,dt, \qquad a \leqslant x \leqslant b,$$

where A *and* B *are constants, and* $f(t)$ *is a function integrable over the interval* (a, b). *Let* $f^*(x)$ *be the function equal to* $f(x)$ *in* (a, b) *and to* 0 *elsewhere* (mod 2π). *Then the series* 11.44(1) *and* $\mathfrak{S}[f^*]$ *are uniformly equiconvergent in every interval* (a', b') *interior to* (a, b). *The series conjugate to* 11.44(1), *and* $\tilde{\mathfrak{S}}[f^*]$, *are uniformly equiconvergent in the wider sense in the interval* (a', b').

For the proof we notice that Fourier series may be integrated term by term; hence, if $F_1(x)$ is the sum of $\mathfrak{S}[f^*]$ integrated twice, $F_1(x)$ satisfies an equation similar to (1), and so $F(x)-F_1(x)$ is linear over (a, b).

A special case of (iii), which was already used in the proof of (ii), deserves a separate statement:

(iv) *If the sum* $F(x)$ *of the series* 11.44(1) *integrated twice is linear in an interval* (a, b), *the series* 11.44(1) *as well as its conjugate are uniformly convergent in every interval interior to* (a, b), *the sum of the former series being* 0.

11.47. Theorem 11.46(iii) states that, if $F(x)$ satisfies the equation 11.46(1), the series 11.44(1) and $\mathfrak{S}[f^*]$ are uniformly equiconvergent over (a', b'). From this and from the fact that Fourier series may be integrated term by term we deduce

Under the conditions of Theorem 11.46(iii), *the series* 11.44(1) *may be integrated formally over any interval* (a', b') *interior to* (a, b); *the series*

$$(1)\qquad \tfrac{1}{2}a_0 x + C + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n} \qquad (C \text{ const.})$$

converges uniformly over (a', b').

It is only the second part of this theorem which needs a proof, and the result will follow when we have shown that (1) converges at some point interior to (a, b). To show this we observe that the periodic part of (1) is a Fourier series with coefficients $o(1/n)$, and so it is sufficient to apply Theorem 11.21(i).

The theorem which we have just obtained may be slightly generalized, viz., *under the same conditions as above, the series* (1) *converges uniformly, and so represents the indefinite integral of f, in the whole interval* $a \leqslant x \leqslant b$. This is an immediate corollary of Theorem 11.21(iii). In particular,

If the series 11.44(1) *converges in the interval* $a \leqslant x \leqslant b$, *except perhaps at an at most enumerable set E of points, to an integrable function* $f(x)$, *the series* (1) *converges uniformly over* (a, b) *to the integral of* f [1]).

11.48 [2]). Following Young, the series 11.44(1) is called a *restricted Fourier series*, associated with an interval (a, b) and a function $f(x) \in L(a, b)$, if this series is a formally differentiated Fourier series of a function $\Phi(x)$ which is the indefinite integral of $f(x)$ for $a < x < b$.

If 11.44(1) *is a restricted Fourier series associated with an interval* (a, b) *and a function* $f(x)$, *and if* $f^*(x)$ *has the same meaning as in* § 11.46, *the series* 11.44(1) *and* $\mathfrak{S}[f^*]$ *are uniformly equiconvergent over any interval* (a', b') *interior to* (a, b). *The series conjugate to* 11.44(1) *and* $\bar{\mathfrak{S}}[f^*]$ *are uniformly equiconvergent for* $a' \leqslant x \leqslant b'$, *but in the wider sense.*

The theorem is a corollary of Theorem 11.46(iii) if we observe that the function $F(x)$ corresponding to 11.44(1) is of the form 11.46(1).

11.49. Riemann's formulae. Riemann deduced his principle of localization from an important formula which we shall now prove, in a slightly more general form.

Let $a < a' < b' < b$, and let $\lambda(x)$ be a function equal to 1 in (a', b'), vanishing outside (a, b) (mod 2π) and having Fourier coefficients $O(n^{-3})$.

If $F(x)$ *is the sum of the series* 11.44(1) *integrated twice, the sequences*

$$(1)\quad \tfrac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) - \frac{1}{\pi}\int_a^b F(t)\,\lambda(t)\,\frac{d^2}{dt^2}D_n(t-x)\,dt$$

$$(2)\quad \sum_{k=1}^{n}(a_k \sin kx - b_k \cos kx) - \frac{1}{\pi}\int_a^b F(t)\,\lambda(t)\,\frac{d^2}{dt^2}\bar{D}_n(t-x)\,dt$$

tend uniformly to limits in the interval (a', b'). *In the case of the sequence* (1) *the limit is* 0 [1]).

In this theorem, D_n and $\bar{D}_n$ denote the Dirichlet kernel and the conjugate Dirichlet kernel respectively. Since the expressions (1) and (2) depend only on the values of $F(x)$ within the interval (a, b), the above theorem contains the principle of localization.

To grasp the meaning of the theorem suppose that $a_0 = 0$, and denote the series 11.44(1) by S; F is then a periodic function with coefficients $o(n^{-2})$. Assume for a while that the formal product of $\mathfrak{S}[F]$ and $\mathfrak{S}[\lambda]$ has coefficients $o(n^{-2})$ (which is easy to prove but is not required for the proof of the theorem). Then $F\lambda$ may be considered as the function $F_1(x)$ corresponding to a trigonometrical series S_1. Since $F(x) = F_1(x)$ in (a', b'), the series $S - S_1$ converges uniformly to 0 in every interval $(a' + \delta, b' - \delta)$, $\delta > 0$ (§ 11.46(ii)), and it suffices to observe that (1) is the difference of the n-th partial sums of the series S and S_1. Similarly we prove the part of the theorem concerning the sequence (2). In other words, Riemann's formulae are, in a degree, consequences of the principle of localization. The only defect of the above argument is that it gives convergence in the interval $(a' + \delta, b' - \delta)$ and not in (a', b'). Although this point is of minor importance, we shall prove our theorem in its complete form, first for aesthetic reasons and second since in the original paper of Riemann the interval (a', b') reduces to a point, and so the above argument could not be applied to that case [2]). We require the following lemma:

If V and W are trigonometrical series, then we have the equation $(VW)'' = V''W + 2V'W' + VW''$, *where products are formal products and dashes denote formal differentiation.*

For if c_n, γ_n, C_n denote the complex coefficients of V, W, VW respectively, the n-th coefficient of $(VW)''$ is

[1]) Lusin [2], Hobson [2]. It is sufficient to assume that the upper and lower sums of the series 11.44(1) are finite for $a \leqslant x \leqslant b$, $x \,\bar{\in}\, E$, and that one of them is integrable over (a, b).

[2]) Young [15], [16]; see also Hobson's *Theory of functions*, 2, p. 686.

[1]) Riemann [1], Neder [2], Zygmund [11].

[2]) On the other hand, this argument imposes less stringent conditions upon λ, for, as can easily be verified, it suffices to suppose that the Fourier coefficients of λ are $o(n^{-2})$.

$$-\sum_{p=-\infty}^{\infty} c_p \gamma_{n-p}\cdot n^2$$

and it is enough to notice that $-n^2 = -(n-p)^2 + 2i(n-p)\,ip - p^2$.

Suppose now that $a_0 = 0$, and let S denote the series 11.44(1). The expression (1) is the n-th partial sum of the series

$$S - \mathfrak{S}''[F\lambda] = S - \{\mathfrak{S}[F]\,\mathfrak{S}[\lambda]\}'' =$$
$$= (S - \mathfrak{S}''[F]\,\mathfrak{S}[\lambda]) - 2\,\mathfrak{S}'[F]\,\mathfrak{S}'[\lambda] - \mathfrak{S}[F]\,\mathfrak{S}''[\lambda].$$

Since $\mathfrak{S}''[F] = S$ and $S - S\,\mathfrak{S}[\lambda] = S(1 - \mathfrak{S}[\lambda]) = S\,\mathfrak{S}[1-\lambda]$, we obtain the equation

$$(3)\quad S - \mathfrak{S}''[F\lambda] = S\,\mathfrak{S}[1-\lambda] - 2\,\mathfrak{S}'[F]\,\mathfrak{S}'[\lambda] - \mathfrak{S}[F]\,\mathfrak{S}''[\lambda].$$

Observing that S, $\mathfrak{S}'[F]$, $\mathfrak{S}[F]$ have coefficients tending to 0, and $\mathfrak{S}[1-\lambda]$, $\mathfrak{S}'[\lambda]$, $\mathfrak{S}''[\lambda]$ have coefficients $O(n^{-3})$ and converge to 0 in (a', b'), we see (§ 11.42) that $S - \mathfrak{S}''[F\lambda]$ converges uniformly to 0 over (a', b'). This gives the first half of the theorem. To prove the second half we notice that the series conjugate to each of the products on the right of (3) converge uniformly over (a', b') (§ 11.43), and that (2) is the n-th partial sum of the series conjugate to $S - \mathfrak{S}''[F\lambda]$.

Since the series 11.44(1) can be represented as a sum of two trigonometrical series one of which consists of the constant term $\frac{1}{2}a_0$ and the other of the remaining terms, it is sufficient to prove the theorem in the case $S = \frac{1}{2}a_0$. Integrating by parts twice, we see that (1) and (2) are equal to

$$(4)\quad \tfrac{1}{2}a_0 - \frac{1}{\pi}\int_0^{2\pi}\{F(t)\lambda(t)\}''D_n(t-x)\,dt, \quad -\frac{1}{\pi}\int_0^{2\pi}\{F(t)\lambda(t)\}''\bar{D}_n(t-x)\,dt$$

respectively. Since $F(t) = {}^1\!/_4\, a_0 t^2$ and $\{F(t)\lambda(t)\}'' = \frac{1}{2}a_0$ for $a' \leqslant t \leqslant b'$, the simplest criteria for the convergence of Fourier series and conjugate series show that, for $a' \leqslant x \leqslant b'$, the expressions (4) tend uniformly to limits, the limit of the first being 0. This completes the proof of the theorem. We add two remarks.

(a) We supposed that $a' < b'$, but the theorem and the argument are unaffected if $a' = b'$, provided that $\lambda'(x) = \lambda''(x) = 0$ at this point. The last conditions are automatically satisfied in the whole interval (a', b') if $a' < b'$ and the Fourier coefficients of λ are $O(n^{-5})$.

(b) The first of the proofs which we have given in this section and which elucidated the meaning of the Riemann formulae shows in what sense the method of Rajchman is, in certain cases, advantageous over the original method of Riemann. Let S be the series 11.44(1). Following Rajchman, in order to remove the influence of the behaviour of S outside (a, b), we multiply S by $\mathfrak{S}[\lambda]$, where λ is a function which vanishes outside (a, b); the behaviour of $S\,\mathfrak{S}[\lambda]$ is known at every point. Riemann's method consists in integrating S twice, multiplying the resulting function $F(x)$ by $\lambda(x)$, and differentiating the product twice. That the resulting series S_1 is equiconvergent with S in (a', b'), is just the Riemann theorem, and it can easily be shown that S_1 converges to 0 outside (a, b). There remain two intervals, viz. (a, a') and (b', b), and Riemann's theorem tells us nothing about the behaviour of S_1 in them. Using the theorems on formal multiplication, this behaviour can be read from the formula (3), and we see that not only does this involve the series S, but also $\mathfrak{S}'[F]$, which is obtained by formal integration of S.

It must however be emphasized that the Riemann idea of introducing the function F into problems of localization is of fundamental importance. The method of formal multiplication completes it, but can in no way replace it.

11.5. Sets of uniqueness and sets of multiplicity. A point-set $E \subset (0, 2\pi)$ will be called a set of *uniqueness*, or U-set, if every trigonometrical series converging to 0 outside E vanishes identically. In § 11.3 we showed that every enumerable set is a U-set. If E is non-enumerable but does not contain any perfect subset (the existence of such sets E follows from Zermelo's Axiom) E is also a set of type U. This follows from the fact that the set of points where a trigonometrical series does not converge to 0 is a Borel set and so, if it does not contain a perfect subset, it must be at most enumerable[1]); this implies that the series vanishes identically. If E is a set of uniqueness, every set $E_1 \subset E$ is also a U-set.

A set E which is not a U-set will be called a set of *multiplicity*, or M-set. If E is of type M, there is a trigonometrical series which

[1]) See e. g. Hausdorff, *Mengenlehre*, p. 179–180.

converges to 0 outside E but does not vanish identically. Any set E of positive measure is an M-set. For let E_1, $|E_1|>0$, be a perfect subset of E, and $f(t)$ the characteristic function of E_1. The series $\mathfrak{S}[f]$ converges to 0 at every point $x \,\bar{\epsilon}\, E$, and does not vanish identically since its constant term is $|E_1|/2\pi > 0$. It follows that it is only the case of sets of measure 0 which requires investigation, and it is a very curious fact that among perfect sets of measure 0 there exist U-sets as well as M-sets. Whether a given set E, $|E|=0$, is of type U or of type M seems to depend on the arithmetical properties of E, and the problem of necessary and sufficient conditions — expressed in structural terms — is not yet solved.

11.51. *H*-sets are sets of uniqueness. That there exist perfect sets of type U was found independently by Mlle Nina Bary and Rajchman[1]). The latter showed that sets of type H, which we considered in § 11.1 (in particular Cantor's ternary set), are U-sets, and this result will be proved here.

Let Red $x = x-[x]$ = the non-integral part of x. We consider a sequence $\{\alpha_k\}$ of real numbers and an increasing sequence $\{n_k\}$ of positive integers. We fix a number $0<d<1$ and denote by E_k the set of points x where Red $\{n_k(x/2\pi)-\alpha_k\} \leqslant d$. If $E=E_1E_2\ldots E_k\ldots$, the set E will be called an H-set, and the reader will have no difficulty in proving, e. g. geometrically, that this definition is equivalent to that of § 11.1. It will be convenient to place the sets on the circumference of the unit circle. E_k will then consist of n_k equidistant arcs, each of length $2\pi d/n_k$. The complementary set E'_k consists of n_k intervals $I_1^{(k)}, I_2^{(k)}, \ldots, I_{n_k}^{(k)}$ of length $2\pi(1-d)/n_k$.

Let E be the set just defined and let

$$(1)\qquad \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

be any trigonometrical series convergent to 0 outside E. It is convenient to suppose that this series is not necessarily real, i. e. the condition $c_{-n}=\bar{c}_n$ need not be satisfied. Let $F(x)$ be the function obtained by integrating (1) formally twice. $F(x)$ is linear in every interval I contiguous to E, and so, if the points x, $x+2h$, $x-2h$ belong

[1]) N. Bary [1], Rajchman [1]. Another proof, based on a different idea, will be found in Rajchman [3]. See also Verblunsky [$3_{3,4}$], Zygmund [12].

to the same interval I, the expression $\Phi_h(x) = \Delta^2 F(x, 2h)/4h^2$ is equal to 0. Take $h<2\pi(1-d)/4$, and let x_0 be the middle-point of the interval $I_1^{(k)}$. Since the intervals $I_i^{(k)}$ are outside E, the expression $\Phi_{h/\nu}(x_0)$, where $\nu=n_k$, is equal to 0, and the same may be said of $S_{h/\nu}(x_0)$, where

$$S_{h/\nu}(x) = \frac{1}{\nu}\sum_{\mu=0}^{\nu-1}\Phi_{h/\nu}\left(x+\frac{2\pi\mu}{\nu}\right).$$

It is not difficult to see that

$$(2)\qquad S_{h/\nu}(x) = c_0 + \sum_{n=-\infty}^{\infty}{}' c_{n\nu}\, e^{in\nu x}\frac{\sin^2 nh}{n^2h^2},$$

where the dash signifies that the term $n=0$ is omitted in summation. Since the absolute value of the sum on the right does not exceed a constant multiple of Max $|c_m|$ $(m \geqslant \nu)$, we see that $S_{h/\nu}(x) \to c_0$ as $\nu\to\infty$, uniformly in x. Taking for x the point x_0 defined above, and observing that $S_{h/\nu}(x_0)=0$, we obtain $c_0=0$.

To prove that $c_m=0$, we multiply (1) by e^{-imx}. The new series converges to 0 outside E and so its constant term c_m is equal to 0. This completes the proof.

11.52. As a corollary of the previous theorem we shall show that there exist continuous functions of bounded variation with Fourier coefficients $\neq o(1/n)$ (§§ 2.213, 5.7.14). For let E denote the Cantor ternary set constructed on $(0, 2\pi)$, and $\Phi(x)$ any function continuous, of bounded variation, constant in every interval contiguous to E, but not in the whole interval $(0, 2\pi)$. *The Fourier coefficients of Φ are not $o(1/n)$.* For if they were $o(1/n)$, and if 11.44(1) denoted $\mathfrak{S}[\Phi]$ differentiated term by term, we should have $|a_n|+|b_n| = o(1)$. Since the integral of Φ is linear in the intervals contiguous to E, 11.44(1) would be summable R to 0 outside E, and so (§ 11.4) would converge to 0 outside E. Since E is a U-set, we should have $a_0=a_1=b_1=\ldots=0$, $\Phi(x)=$ const., contrary to the assumption[1]).

[1]) See also Carleman [3], Hille and Tamarkin [2].

11.53. Menchoff's example. That there are perfect M-sets of measure 0 was shown by Menchoff[1]), and is a result chronologically prior to those of § 11.51.

Consider the following set. From $E_0 = (0, 2\pi)$ we remove the interior of a concentric interval of length $|E_0|/2$. The rest E_1 consists of two intervals E_1^1 and E_1^2. From each of them we remove the interior of concentric intervals of length $|E_1^i|/3$. The rest E_2 consists of four intervals E_2^i, $i = 1, 2, 3, 4$. Having defined E_{n-1}, consisting of 2^{n-1} intervals E_{n-1}^i, we define E_n by removing the interior of intervals concentric with E_{n-1}^i and of length $|E_{n-1}^i|/(n+1)$. We put $E = E_0 E_1 E_2 \dots$ and, following Menchoff, we shall prove that *E is a perfect M-set of measure* 0.

That E is perfect is plain. Since the measure of E_n is equal to $2\pi(1-1/2)(1-1/3)\dots(1-1/(n+1)) = 2\pi/(n+1)$, we obtain $|E| = 0$. To prove that E is an M-set it is sufficient to construct a function $F(x)$, constant in the intervals contiguous to E, but not equivalent to a constant in $(0, 2\pi)$, which has coefficients $o(1/n)$. For $\mathfrak{S}[F]$ differentiated term by term converges to 0 outside E and does not vanish identically.

The set complementary to E_n consists of $2^n - 1$ intervals, which we shall denote by I_n^k, $k = 1, 2, \dots, 2^n - 1$, counting from the left to the right. We define a sequence of *continuous* functions $F_1(x), F_2(x), \dots, F_n(x), \dots$ $(0 \leqslant x \leqslant 2\pi)$ satisfying the following conditions (i) $F_n(0) = F_n(2\pi) = 0$, $F_n(\pi) = 1$, (ii) $F_n(x)$ is constant in the intervals I_n^k, $k = 1, 2, \dots, 2^n - 1$, and linear in the intervals E_n^i, $i = 1, 2, \dots, 2^n$, (iii) $F_{n+1}(x) = F_n(x)$ in every I_n^k. Moreover, we suppose that (iv) if I_{n+1}^k is contained in an interval E_n^i, the value of $F_{n+1}(x)$ in I_{n+1}^k is equal to the mean value of F_n at the end-points of E_n^i. These conditions determine the functions $F_n(x)$ uniquely (we leave it to the reader to draw the graphs of the curves). It is easy to verify that $|F_n'(x)| \leqslant (n+1)/\pi$, $|F_{n+1}(x) - F_n(x)| \leqslant 1/2^n(n+2)$. It follows that the sequence $\{F_n(x)\}$ converges uniformly to a continuous function $F(x)$, and that $|F(x) - F_n(x)| < 1/n\, 2^{n-1}$.

Let C_n be the complex Fourier coefficients of $F(x)$. To show that $nC_n = o(1)$, we write

[1]) Menchoff [1]; see also N. Bary [1], Rajchman [4], Zygmund [13]. In the last paper it is shown that, if $n_{k+1}/n_k > \lambda > 3$, $\alpha_k \to 0$, $\Sigma \alpha_k^2 = \infty$, the product $\prod_{k=1}^{\infty} (1 + \alpha_k \cos n_k x)$ may be written in the form of a trigonometrical series, which converges to 0 almost everywhere (but not everywhere).

$$
(1)\quad \begin{aligned} n\int_0^{2\pi} F e^{-inx}\,dx &= n\int_0^{2\pi} (F - F_N)\, e^{-inx}\,dx + n\int_0^{2\pi} F_N e^{-inx}\,dx = \\ &= n\int_0^{2\pi} (F - F_N)\, e^{-inx}\,dx - i\int_0^{2\pi} F_N' e^{-inx}\,dx = A + B, \end{aligned}
$$

where n and N are positive. Since $F(x) = F_N(x)$ outside E_N, $|A|$ does not exceed $n|E_N| \cdot \mathrm{Max}\,|F - F_N| < 2\pi n/N^2 2^{N-1} = O(1/\log^2 n)$, if N is defined by the condition $2^{N-1} \leqslant n < 2^N$. Passing to the integral B, we observe that $F_N'(x)$ is equal to $\pm(N+1)/\pi$ in E_N and to 0 elsewhere. To estimate the integral of e^{-inx} over any interval belonging to E_N we have two inequalities: the absolute value of the integral exceeds neither the length of the interval nor $2/n$. The first inequality is more advantageous for intervals not large in comparison with $1/n$, the second for larger intervals. However, neither of these two inequalities alone would enable us to show that $B = o(1)$, and to overcome the difficulty we proceed as follows.

Let $\nu = \nu_N < N$ be a positive integer which we shall define presently; hence $E_N \subset E_\nu$. We write $F_N'(x) = g_N(x) + h_N(x)$, where $g_N(x)$ vanishes outside E_ν and is equal to $\pm(N+1)/\pi$ in E_ν; the sign '+' corresponds to the interval $(0, \pi)$, the sign '—' to $(\pi, 2\pi)$. Then

$$
B = -i\int_0^{2\pi} g_N(x)\, e^{-inx}\,dx - i\int_0^{2\pi} h_N(x)\, e^{-inx}\,dx = B' + B'',
$$

$$
|B'| \leqslant 2^\nu \left(\frac{2}{n}\right) \frac{N+1}{\pi}, \qquad |B''| \leqslant |E_\nu - E_N| \frac{(N+1)}{\pi} = \frac{2(N-\nu)}{\nu+1},
$$

since g_N vanishes outside E_ν, h_N vanishes outside $E_\nu - E_N$, and both $|g_N|$ and $|h_N|$ do not exceed $(N+1)/\pi$. If we put $\nu = N - [\sqrt{N}]$, we obtain $B'' = O(N^{-1/2}) = O(\log^{-1/2} n)$, $B' = O(N 2^{-\sqrt{N}}) = O(\log^{-1/2} n)$ and, collecting the results, $nC_n = O(\log^{-1/2} n) = o(1)$.

11.54. If E_1 and E_2 are sets of uniqueness, their sum $E_1 + E_2$ may be a set of multiplicity. We obtain an example by breaking up the interval $(0, 2\pi)$ into two sets E_1 and E_2, each without a perfect subset. Although E_1 and E_2 are U-sets (§ 11.5), their sum is not. This example may be not entirely convincing and it is natural to ask whether the situation is the same if we restrict ourselves to the domain of Borel sets. The answer to this problem is not known. In the case of closed sets we have the following theorem due to Mlle Bary[1]).

[1]) N. Bary [1].

If $E_1, E_2, \ldots, E_n, \ldots$ are closed U-sets, their sum $E = E_1 + E_2 + \ldots$ is a U-set.

We shall require the following lemma:

Let $\mathcal{E}$ be a closed set of uniqueness and J an open interval. If a trigonometrical series S with coefficients tending to 0 (i) converges to 0 almost everywhere in J, (ii) has partial sums bounded at every point of $J - \mathcal{E}$, the series converges to 0 at every point of J.

We may suppose that $J\mathcal{E} \neq 0$, for otherwise the lemma follows from Theorem 11.46(iii) and the remark of § 11.32. Now let δ be any interval contained in J and without points in common with $\mathcal{E}$. Since S converges to 0 almost everywhere in δ, and has partial sums bounded at every point of δ, S converges to 0 everywhere in δ. Hence S converges to 0 in $J - \mathcal{E}$. Let $\lambda(x)$ be a function vanishing outside J and positive in J. The formal product S_1 of S by $\mathfrak{S}[\lambda]$ converges to 0 outside J and in the set $J - \mathcal{E}$. Since $\mathcal{E}$ is a U-set, S_1 converges to 0 everywhere. Taking into account that $\lambda(x) > 0$ in J, we see that S converges to 0 in J, and the lemma is established.

Suppose now that there is a trigonometrical series S with coefficients tending to 0, converging to 0 outside E, but not everywhere; let R be the set of points at which the partial sums $s_n(x)$ of S are unbounded. R is a product of open sets, for if G_N denotes the set of points where at least one of the functions $|s_n(x)|$ exceeds N, then G_N is an open set and $R = G_1 G_2 \ldots G_N \ldots$ The set R is contained in E; outside E the series converges to 0. Since $|E| = 0$ and S is not identically equal to 0, it follows (§ 11.32) that $R \neq 0$. We may write $R = RE_1 + RE_2 + \ldots$, and since sets which are products of open sets are not of the first category in themselves [1]), there is an n_0 such that RE_{n_0} is not non-dense in R. In other words, there is an open interval J such that $JR \neq 0$ and JRE_{n_0} is dense in JR. From this and from the fact that E_{n_0} is closed, we deduce that $JRE_{n_0} \supset JR$, i. e. $JRE_{n_0} = JR$. We write $E_{n_0} = \mathcal{E}$ and apply the lemma. The series S converges to 0 almost everywhere in J and has partial sums bounded at every point of the set $J - JR = J - JR\mathcal{E} \supset J - \mathcal{E}$. Hence S converges to 0 everywhere in J, contrary to the result $JR \neq 0$ obtained previously. This proves the theorem.

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[1]) See e. g. Hausdorff, *Mengenlehre*, 142 (Satz XI).

11.6. Uniqueness in the case of summable trigonometrical series. In § 11.3 we obtained a number of theorems on the uniqueness of the representation of a function by means of a convergent trigonometrical series. Since however there exist functions whose Fourier series diverge everywhere, it is natural to ask for theorems of uniqueness for summable trigonometrical series. We shall restrict ourselves to Abel's method of summation which has an important function-theoretic significance. Since Abel's method applies to series with coefficients not tending to 0, we begin by investigating what conditions must we impose upon the coefficients of the series considered.

Of the two series

$$\text{(1)} \qquad \text{a)} \ \sum_{n=1}^{\infty} n \sin nx, \qquad \text{b)} \ \tfrac{1}{2} + \sum_{n=1}^{\infty} \cos nx,$$

the first is summable A to 0 for every x; the second for every $x \not\equiv 0 \pmod{2\pi}$. This shows that: ($a$) for series

$$\text{(2)} \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

summable A and having coefficients $\neq o(n)$, the theorem of uniqueness is false, (b) if we drop the condition $a_n \to 0$, $b_n \to 0$, we cannot introduce sets of uniqueness such as the set E of Theorem 11.32.

We write

$$f(r, x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n,$$

$$f_*(x) = \varliminf_{r \to 1} f(r, x), \qquad f^*(x) = \varlimsup_{r \to 1} f(r, x).$$

The functions $f^*(x)$ and $f_*(x)$ may be called the upper and lower Abel sums of the series (2). We shall prove the following two theorems, the second of which is a very special case of the first.

(i) *If the functions $f_*(x)$ and $f^*(x)$ corresponding to the series* (2) *with coefficients $o(n)$ are both finite everywhere, and if $f_*(x) \geqslant \chi(x)$, where χ is integrable,* (2) *is a Fourier series.*

(ii) *If the series* (2) *with coefficients $o(n)$ is, for every x, summable A to 0, then $a_0 = a_1 = b_1 = \ldots = 0$.*

In the case of coefficients tending to 0, propositions (i) and (ii) were established by Rajchman[1]). His method applies, without essential changes, to a slightly more general case, viz. when the periodic part of the series (2) integrated twice is the Fourier series of a continuous function[2]); in particular when we have $|a_n|+|b_n|=O(n^{1-\eta})$, $\eta>0$. The proof of propositions (i) and (ii), as they are stated, requires new devices, and this final step was taken by Verblunsky[3]).

The proof of (i) will be based on a number of lemmas. It will not impair the generality if from the start we assume that $a_0=0$.

11.601. Rajchman's inequalities. These are fundamental for the whole argument and may be stated as follows. *If*

$$C-\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n^2} \tag{1}$$

is the Fourier series of a function $F(x)$, *and if* $f^*(x)$ *and* $f_*(x)$ *are the upper and lower Abel sums of the series* (1) *differentiated twice, then, at every point* x_0 *where* (1) *is summable A, the intervals* $(\underline{D}^2F(x_0),\overline{D}^2F(x_0))$ *and* $(f_*(x_0),f^*(x_0))$ *have points in common, i. e.*

$$\underline{D}^2F(x_0)\leqslant f^*(x_0),\qquad f_*(x_0)\leqslant\overline{D}^2F(x_0)\,^{4}). \tag{2}$$

Let $x_0=0$ and let $F(r,x)$ be the harmonic function corresponding to the series (1). We may assume that $F(0)=0$, i. e. that $F_r=F(r,0)\to 0$ as $r\to 0$. To prove the first inequality (2), it is sufficient to show that, for any m, the inequality $\underline{D}^2F>m$ implies $f^*\geqslant m$. We may also assume that $m=0$, for otherwise we may consider $F(x)-m(1-\cos x)$ instead of $F(x)$. Suppose, contrary to what we want to prove, that $f^*(0)<0$. From the Laplace equation

$$\frac{1}{r^2}\frac{\partial^2F(r,x)}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F(r,x)}{\partial r}\right)=0$$

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[1]) Rajchman [5].

[2]) see e. g. Zygmund [14]; M. Riesz [7] was the first to consider problems of uniqueness in the case of coefficients not tending to 0.

[3]) Verblunsky [3₂].

[4]) Rajchman [5]; Rajchman and Zygmund [1], Verblunsky [3₁]. It can be shown that, if $D^2F(x_0)$ exists and is finite, then $f_*(x_0)=f^*(x_0)=D^2F(x_0)$ (Fatou [1]), but, in the general case, the interval (f_*,f^*) need not be contained in $(\underline{D}^2F,\overline{D}^2F)$; see Rajchman and Zygmund [1].

we obtain that rF_r', where the dash denotes differentiation with respect to r, is an increasing function of r in an interval $r_0\leqslant r<1$. Since $F_r\to 0$ as $r\to 1$, the mean-value theorem gives $F_r/\log r=\rho F_\rho'$, $r_0\leqslant r<\rho<1$, and hence, for a σ contained in $(\rho,1)$,

$$F_r/\log r-F_\rho/\log\rho=\rho F_\rho'-\sigma F_\sigma'<0.$$

To show that this is impossible, it is enough to prove that $\overline{\lim\limits_{r\to1}}\dfrac{d}{dr}\left\{\dfrac{F_r}{\log r}\right\}<0$. Let $\Delta=\Delta(r,t)=1-2r\cos t+r^2$, $P_r(t)=\frac{1}{2}(1-r^2)/\Delta$, $\varphi(t)=\{F(t)+F(-t)-2F(0)\}/\sin^2 t$. From Poisson's formula we obtain

$$\lim_{r\to1}\frac{d}{dr}\left\{\frac{rF_r}{1-r^2}\right\}=\lim_{r\to1}\frac{1}{\pi}\int_0^{\pi}[F(t)+F(-t)]\frac{1-r^2}{\Delta^2}dt= \tag{3}$$
$$=\lim_{r\to1}\frac{1}{\pi}\int_0^{\eta}\varphi(t)\sin^2t\frac{1-r^2}{\Delta^2}dt=\lim_{r\to1}\left\{-\frac{1}{\pi r}\int_0^{\eta}\varphi(t)\sin t\frac{d}{dt}P_r(t)\,dt\right\},$$

where η, $0<\eta\leqslant\pi$, is any fixed number. Taking η so small that $\varphi(t)>h>0$ for $0<t\leqslant\eta$, replacing $\varphi(t)$ by h, and integrating by parts, we find that the right-hand side of (3) exceeds

$$\frac{h}{\pi}\lim_{r\to1}\int_0^{\eta}\cos t\,P_r(t)\,dt=\frac{h}{\pi}\lim_{r\to1}\int_0^{\pi}\cos t\,P_r(t)\,dt=\tfrac{1}{2}h>0.$$

Now, if $c(r)=(1-r^2)/r\log r$, we have

$$\left(\frac{F_r}{\log r}\right)'=c(r)\left(\frac{rF_r}{1-r^2}\right)'+c'(r)\frac{rF_r}{1-r^2}. \tag{4}$$

Since $c(r)\to-2$, $c'(r)=O(1-r)$, the upper limit, for $r\to1$, of $(F_r/\log r)'$ is negative, and the first inequality of (2) follows. Applying this inequality to $-F(x)$, we obtain the remaining inequality.

11.602. If P is a linear set of points, we shall call a *portion* of P, any non-empty product of P by an open interval I.

Let P *be a perfect set and* $\{f_n(x)\}$, $n=1,2,\ldots$, *a sequence of continuous functions defined in* P *and bounded at every point of* P. *Then there is a portion* Π *of* P *in which the sequence* $\{f_n(x)\}$ *is uniformly bounded.*

Let $E_{n,m}(m,n=1,2,\ldots)$ be the set of points where $|f_n(x)|\leqslant m$, and let $H_m=E_{1,m}E_{2,m}E_{3,m}\ldots$ The sets $E_{n,m}$, and so also the sets H_m, are closed. Since P is the sum of all H_m, at least one of the terms, say H_{m_0}, is not non-dense over P, i. e. is dense in a portion Π of P. Being closed, it contains Π. Hence $|f_n(x)| \leqslant m_0$ for $n \geqslant 1$, $x \in \Pi$, and the lemma follows.

11.603. A function $g(x)$ is said to be upper semi continuous if, for every sequence $\{x_n\} \to x$, we have $\overline{\lim\limits_{n\to\infty}}\, g(x_n) \leqslant g(x)$. An important property of an upper semi-continuous function is that it attains its maximum in every finite interval; the proof is immediate.

If $\Phi(x)$ is an upper semi-continuous function satisfying the inequality $\overline{D}^2\Phi \geqslant 0$, the function Φ is convex.

The proof is a mere repetition of the argument of § 11.31(i) (with $E = 0$).

11.604. *Let $\chi_2(x)$ denote the second integral of $\chi(x)$. If, under the hypotheses of Theorem* 11.6(i), *the series* 11.601(1) *is, for $a<x<b$, summable A to a continuous or, more generally, upper semi-continuous function $F(x)$, the difference $F(x) - \chi_2(x)$ is convex for $a < x < b$.*

Taking account of the preceding lemma, the proof is contained in the proof of Theorem 11.31(iv) where we showed that, with the notation of that paragraph, $F(x) - f_2(x)$ was convex; it is sufficient to observe that, in view of Lemma 11.601, we have $\chi(x) \leqslant \overline{D}^2 F(x)$.

The last lemma we shall require is

11.605. *If the series $u_0 + u_1 + u_2 + \ldots$ has Abel's upper and lower sums finite, the series $u_1 + u_2/2 + u_3/3 + \ldots$ is summable A.*

For if $g(r) = u_0 + u_1 r + \ldots$, then $G(r) = \sum\limits_{n=1}^{\infty} \frac{u_n}{n} r^n = \int\limits_0^r \frac{g(\rho) - u_0}{\rho}\, d\rho$. Since the integrand is bounded, we have $|G(r) - G(r')| \to 0$ as $r \to 1$, $r' \to 1$, and the lemma follows.

Suppose that the u_n are functions of a parameter x. *If the function $g(r)$ is uniformly bounded for $0 \leqslant r < 1$ and x belonging to a set E, then the series $u_1 + \frac{1}{2}u_2 + \ldots$ is uniformly summable A for $x \in E$.*

11.606. We now pass on to the proof of Theorem 11.6(i). Applying Lemma 11.605 twice to the series 11.6(2), we see that

$$(1) \qquad -\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

is summable A for every x (using well-known Tauber's theorem that series summable A and having coefficients $o(1/n)$ are convergent[1]), we see that (1) converges for every x; this result will not be required in the proof). The main point of the proof will be to show that the sum $F(x)$ of (1) is continuous, a result which is immediate if e. g. $a_n \to 0$, $b_n \to 0$. Let

$$p_1(r,x) = -\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n} r^n,\quad p_2(r,x) = -\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} r^n.$$

We begin by proving that in every perfect set P there is a portion Π, such that $p_1(r,x)$ is bounded for $0 \leqslant r < 1$, $x \in \Pi$. For, if $r_1 < r_2 < \ldots$ is a sequence tending to 1 sufficiently slowly, then $|p_1(r_n, x) - p_1(r, x)| \leqslant 1$, for $r_{n-1} \leqslant r \leqslant r_n$, $0 \leqslant x \leqslant 2\pi$. In view of Lemma 11.605, $\lim p_1(r,x)$ exists for every x. Since the sequence $p_1(r_n, x)$ is uniformly bounded in a portion Π of P (§ 11.602), the same may be said of the expression $p_1(r,x)$.

From this and the last remark of § 11.605, we see that, in every perfect set P, there is a portion Π in which the function $F(x) = \lim p_2(r,x)$ is continuous. In particular, taking $P = (0, 2\pi)$, we obtain that the set $\varDelta$ of discontinuities of F is nowhere dense in $(0, 2\pi)$.

Suppose, contrary to what we want to prove, that $\varDelta \neq 0$. First of all, $\varDelta$ cannot contain isolated points. For, if x_0 were one, consider the difference $\delta(x) = F(x) - \chi_2(x)$ in the neighbourhood of x_0. Since $\delta(x)$ is convex to the right and to the left of x_0 (§ 11 604), the limits $\delta(x_0 \pm 0)$ exist, and so, in view of Theorem 11.21(ii), $\delta(x_0 + 0) = \delta(x_0 - 0) = \delta(x_0)$. Hence $\delta(x)$ is continuous at x_0, and so is $F(x)$.

$\varDelta$ being dense in itself, the set $\overline{\varDelta}$ of limiting points of $\varDelta$ is perfect. If (α, β) is any interval contiguous to $\overline{\varDelta}$, the function $\delta(x)$ is convex for $\alpha < x < \beta$, and $\delta(\alpha + 0) = \delta(\alpha)$, $\delta(\beta - 0) = \delta(\beta)$. Let $\Pi = J\overline{\varDelta}$ be a portion of $\overline{\varDelta}$ in which $F(x)$, and so also $\delta(x)$, is continuous; J denotes an open interval. Being convex in any interval belonging to $J - \Pi$, the function $\delta(x)$ is upper semi-continuous in J. The same may be said of $F(x)$. Applying Lemma 11.604, we obtain that $\delta(x)$ is convex, and so also continuous, in J. This shows that F is continuous in J. Hence $\varDelta = 0$, i. e. F is everywhere continuous.

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[1]) See e. g. Landau, *Darstellung und Begründung.*

By Lemma 11.604, the difference $\delta(x)=F(x)-\chi_2(x)$ is convex over $(-\infty, \infty)$. To complete the proof of the theorem, we observe that $D^2F(x) = D^2\chi_2(x) + D^2\delta(x) = \chi(x) + D^2\delta(x)$ exists for almost every x and is integrable (§ 11.31(ii), (v)). Let $f(x)=\operatorname{Max}\{f_*(x), \underline{D}^2F(x)\}$. Using Lemma 11.601, we see that $f(x)$, which is contained between $f_*(x)$ and $f^*(x)$, is everywhere finite and satisfies the inequality $\underline{D}^2F(x) \leqslant f(x) \leqslant \overline{D}^2F(x)$. By Lemma 11.31(iv), $F(x)$ is of the form 11.3(1); this, as we know, proves that 11.6(2) is $\mathfrak{S}[f]$, and the theorem is established. Incidentally we obtain that, under the conditions of Theorem 11.6(i), $f_*(x) = f^*(x)$ for almost every x.

11.61 [1]). *If the conditions of Theorem* 11.6(i) *are satisfied, except that $f_*(x)$ and $f^*(x)$ may be infinite at a finite number of points $x_1, x_2, \dots, x_k$, the series* 11.6(2) *differs from a Fourier series by a linear combination of the series $D(x-x_i)$, $i=1,2,\dots,k$, where $D(x)$ denotes the second series* 11.6(1).

We may again assume that $a_0 = 0$. Repeating the proof of Theorem 11.6(i), we obtain that $F(x)$ is everywhere continuous and that, in each of the intervals (x_{i-1}, x_i), $F(x)$ is of the form 11.3(1), with A and B depending on i. The points x_i may be angular points for the function $F(x)$. Let $D_1(x)$ denote the series $\cos x + \cos 2x + \dots$ The sum of the series $D_1(x)$ integrated twice has an angular point for $x=0$ and nowhere else (mod 2π). Therefore, if we substract from 11.6(2) a linear combination of the series $D_1(x-x_i)$, the function F corresponding to the difference has no angular points, i. e. we shall have the formula 11.3(1) with A and B constant throughout the interval $(0, 2\pi)$. It follows that the difference considered is a Fourier series, and the theorem is established. As a corollary we obtain that, if the series 11.6(2), with $|a_n| + |b_n| = o(n)$, is summable A to 0 for $x \neq x_0$, the series is a constant multiple of $D(x-x_0)$.

11.62. *Theorem* 11.6(i) *holds even if the functions $f_*(x)$ or $f^*(x)$, or both, are infinite in a set E, provided that E is at most enumerable and that $F(x)$ is smooth in E.* It is important to observe that *the latter condition is certainly satisfied when $|a_n| + |b_n| \to 0$.* The proof may be left to the reader, since it is wholly similar to that of Theorem 11.6(i), if the lemmas of § 11.31 are used in their complete form.

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[1]) Verblunsky [3_2]; cf. also Zygmund [14].

There are other generalizations of Theorem 11.6(i). The reader interested in the subject will find them in the papers quoted. Here we will only mention one of these generalizations, viz. that *all the theorems of uniqueness established in this chapter hold if integration is understood in the Denjoy-Perron sense* [1]). This is due to the fact that all the lemmas on which our proofs are based hold for the Denjoy-Perron integral. Similarly, the Denjoy-Perron integral may be introduced into theorems on localization. For example, Theorem 11.46(iii) remains true in the new case.

11.7. Miscellaneous theorems and examples.

1. Show that Steinhaus's theorem, i. e. that

$$\overline{\lim_{n\to\infty}} |a_n \cos nx + b_n \sin nx| = \overline{\lim_{n\to\infty}} \sqrt{a_n^2 + b_n^2},$$

except in a set of mesure 0, can be proved by the method of § 11.11.

[Observe that, if m is a positive integer, E an arbitrary set of positive measure, and $n_k \to \infty$, then

$$(1) \qquad \int_E \cos^{2m}(n_k x + \alpha_{n_k})\, dx \to |E| \binom{2m}{m} 2^{-2m},$$

and that, for m large, the right-hand side of (1) is of order $m^{-1/2}$].

2. Theorem 11.21(i) remains true if a_n and b_n are $O(1/n)$. Hardy and Littlewood [20].

[Supposing that 11.1(1) converges to 0, we write

$$\frac{F(x+t)-F(x-t)}{2t} = \sum_{n=1}^{\infty} A_n(x) \frac{\sin nh}{nh} = \sum_{n=1}^{kN} + \sum_{n=kN+1}^{\infty} = P_h + Q_h,$$

where $N=[1/h]$, and $k>0$ is an integer. If k is large, then Q_h is small. Abel's transformation shows that, for fixed k, $P_h \to 0$ with h. Hence 11.1(1) is summable L to 0. Conversely, if that series is summable L, it is summable $(C, 2)$ (§ 3.5) and, as the argument of § 11.21 shows, its partial sums are bounded. Hence it is summable $(C, 1)$ (§ 10.44) and it is sufficient to apply the Hardy theorem of § 3.23].

3. Suppose that $|a_n| + |b_n| = O(1/n)$, so that 11.1(1) is the Fourier series of a function $f(x)$. A necessary and sufficient condition for the convergence, at a point x, of the series conjugate to 11.1(1), is the convergence of the integral

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[1]) Besides the papers quoted, see also P. Nalli [1].

$$-\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt,$$

which represents then the sum of the conjugate series. Hardy and Littlewood [20].

4. Let the series 11.1(1) be summable A, for $a<x<b$, to a non-negative function $f(x)$. A necessary and sufficient condition that the function $f(x)$ should be integrable over (a,b), is that the series

$$(*)\qquad \tfrac{1}{2}a_0x+\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)/n$$

should converge for $x=a$ and $x=b$. Verblunsky [4].

[Let $F(x)$ be the sum of (*). $F(x)$ is monotonic in the interior of (a,b), and $f\in L(a,b)$ if and only if $F(a+0)$ and $F(b-0)$ are finite. Since the coefficients of (*) are $o(1/n)$, it is sufficient to apply Theorem 11.21(ii)].

5. Let S_1 and S_2 be two trigonometrical series with coefficients $o(1/n)$ and $O(1/n)$ respectively. If S_i converges to s_i, $i=1,2$, at a point x, the formal product of S_1 and S_2 converges to s_1s_2 at that point.

As the example $S_1=S_2=\Sigma n^{-1}\sin nx$, $x=0$, shows, the theorem is not true if both factors have coefficients $O(1/n)$.

6. (i) If the sine expansion of a function $f(x)$, $0<x<\pi$, has coefficients $o(1/n)$, the cosine expansion of $f(x)$ converges at the point $x=0$ and has the sum 0. (ii) If the sine. expansion of $f(x)$ has coefficients $o(1/n)$ and converges uniformly in the neighbourhood of $x=0$, the cosine expansion of $f(x)$ also converges uniformly in the neighbourhood of $x=0$. (iii) In the previous theorem the rôle of sine and cosine series may be interchanged, provided that $f(0)=0$.

[To prove (i), consider the product of the sine development of f by the Fourier series of the function $\operatorname{sign}x$, $|x|<\pi$].

7. Given a function $F(x)$, we write

$$\Delta^k F(x,2h)=\sum_{j=0}^{k}\binom{k}{j}F(x+(k-2j)h),$$

Let $F(x)$ be the sum of the series 11.6(2) integrated term by term k times. We shall say that 11.6(2) is summable, at the point x, by the k-th method of Riemann, or summable R_k, to sum s, if the function F exists in the neighbourhood of x, and if

$$(**)\qquad \lim_{h\to 0}\frac{\Delta^kF(x,2h)}{(2h)^k}=\lim_{h\to 0}\left[\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\left(\frac{\sin nh}{nh}\right)^k\right]=s.$$

If $|a_n|+|b_n|=o(n^\alpha)$, $\alpha>-1$, and if the series 11.6(2) is summable (C,α) at the point x, the series is also summable R_k, $k>\alpha+1$, to the same sum. Kogbetliantz [2], Verblunsky [5].

[A consequence of Theorem 10.5.10].

8. If $|a_n|+|b_n|=o(n^r)$, $r=0,1,2,\ldots$, $k=r+1$, and if 11.6(2) is summable (C,r), at the point x, to sum s, we still have the relation (**), where h tends to 0 through a set of points having 0 as a point of density.

See Rajchman and Zygmund [2]. In the same way we can generalize Theorem 10.42.

9. A sequence $\{a_n\}$ is said to be summable R_2' to the limit s, if the expression

$$\frac{2}{\pi}\sum_{n=1}^{\infty}a_n\frac{\sin^2 nh}{n^2h}$$

converges in the neighbourhood of $h=0$ and tends to 0 as $h\to 0$. Show that, if $\{a_n\}$ converges, it is also summable R_2' to the same limit.

[See § 1.8.3; the theorem is practically identical with Theorem 11.2(ii)].

10. The methods R_2 and R_2' are not comparable. See Marcinkiewicz [2].

11. The conditions imposed upon the Fourier coefficients of the function $\lambda(x)$ of Theorem 11.49 are unnecessarily stringent: it is sufficient to suppose that $\lambda''(x)$ is continuous and of bounded variation.

[Consider the formula 11.49(3) and use Theorem 2.531. It is also sufficient to suppose that $\lambda''\in\operatorname{Lip}\alpha$, $\alpha>0$].

12. Let the series 11.44(1) have coefficients $o(n^\alpha)$, $\alpha>-1$, and let k be any integer $>\alpha+\frac{1}{2}$. If $F(x)$ denotes the sum of the series 11.44(1) integrated term by term k times, and if $\lambda(x)$ is a function which is equal to 0 outside (a,b), equal to 1 in (a',b'), $a<a'<b'<b$, and has a sufficient number of derivatives, the differences

$$\sum_{k=0}^{n}A_k(x)-\frac{(-1)^k}{\pi}\int_a^b F(t)\lambda(t)\frac{d^k}{dt^k}D_n(t-x)\,dt,$$

$$-\sum_{k=1}^{n}B_k(x)-\frac{(-1)^k}{\pi}\int_a^b F(t)\lambda(t)\frac{d^k}{dt^k}\bar{D}_n(t-x)\,dt,$$

are uniformly summable (C,α) over (a',b'), the limit of the first being 0.

See Zygmund [11], where the second expression is written in a slightly different form.

13. Let S be any trigonometrical series with coefficients tending to 0, and let $f^*(x)$ and $f_*(x)$ be the upper and lower Abel sum of S. If f^* is integrable, and if f_* and f^* are finite outside a closed set E of measure 0, the difference $S-\mathfrak{S}[f^*]$ converges to 0 outside E. If, in particular, E is a U-set, then $S=\mathfrak{S}[f^*]$.