CHAPTER X.

Further theorems on the summability and convergence of Fourier series.

10.1. An extension of Fejér's theorem. Let $f(x)$ be an integrable and periodic function, and let $s_n(x)$ be the n-th partial sum of $\mathfrak{S}[f]$. Fejér's theorem asserts that, if f is continuous at the point x, then

$$(1) \qquad \frac{1}{n+1}\sum_{\nu=0}^{n}\{s_\nu(x)-f(x)\}\to 0$$

as $n\to\infty$. We shall prove a result from which it will follow in particular that, at every point of continuity of f,

$$(2) \qquad \frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu(x)-f(x)|\to 0.$$

The relation (2) tells us that the mean value of $s_\nu(x)-f(x)$ tends to 0 not because of the interference of positive and negative terms, but because the indices ν for which $|s_\nu(x)-f(x)|$ is not small are comparatively sparse.

We shall require the following lemma.

If $f\in L^r$, $r\geqslant 1$, then, for almost every x, and h tending to 0,

$$\int_0^h |f(x\pm t)-f(x)|^r dx = o(h).$$

The case $r=1$ was considered in § 2.703, and the proof of the general result is not essentially different. For let α be any rational number, and let E_α be the set of x such that $\frac{1}{h}\int_0^h |f(x\pm t)-\alpha|^r dt$ does not tend to $|f(x)-\alpha|^r$ as $h\to 0$. Every set E_α, and so their sum E, is of measure 0. If $x \,\bar{\epsilon}\, E$ and if β is a rational number such that $|f(x)-\beta|<\frac{1}{2}\varepsilon$, then, by Minkowski's inequality,

$$\left\{\frac{1}{h}\int_0^h |f(x\pm t)-f(x)|^r dt\right\}^{1/r} \leqslant \left\{\frac{1}{h}\int_0^h |f(x\pm t)-\beta|^r dt\right\}^{1/r} + $$
$$+\left\{\frac{1}{h}\int_0^h |\beta-f(x)|^r dt\right\}^{1/r}.$$

Since the first term on the right tends to $|f(x)-\beta|$ as $h\to 0$, and the following term is equal to $|f(x)-\beta|$, the left-hand side of this inequality is less than ε for h sufficiently small. Since $\varepsilon>0$ is arbitrary and $|E|=0$, the lemma follows.

Let $\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$; in view of the relation $|\varphi_x(t)|\leqslant|f(x+t)-f(x)|+|f(x-t)-f(x)|$, and applying Minkowski's inequality, we obtain that $\Phi_{x,r}(h)=\int_0^h|\varphi_x(t)|^r dt$ is $o(h)$ for almost every x. The chief object of this paragraph is the following theorem [1]).

(i) *If $f\in L^r$, $r>1$, and if k is any positive number, then, at every point x where $\Phi_{x,r}(h)=o(h)$, we have*

$$(3)\qquad \frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu(x)-f(x)|^k\to 0 \quad \textit{as } n\to\infty.$$

(ii) *If $f\in L$, and if f is continuous at every point of an interval $a\leqslant x\leqslant b$, the relation* (3) *holds uniformly in the interval* (a,b).

In the first place we observe that, if (3) is established for a certain value of k, it holds à fortiori for any smaller k; this follows from the fact that, if $c_1, c_2, \dots c_m$ are arbitrary numbers, the expression $\{(|c_1|^k+|c_2|^k+\dots+|c_m|^k)/m\}^{1/k}$ is a non-decreasing function of k (this expression is equal to $\mathfrak{M}_k[g;\,0,m]$, where $g(x)=c_j$ for $j-1<x\leqslant j$, $j=1,2,\dots,m$; § 4.15). Secondly, it is sufficient to prove (3) for $k=r'=r/(r-1)$; for $\{\Phi_{x,r}(h)/h\}^{1/r}$ is a non-decreasing function of r and so, if $\Phi_{x,r}(h)=o(h)$ for a certain value of r, this relation remains true for any smaller r; taking r sufficiently near to 1 we obtain k as large as we please. Finally, it is

[1]) See Hardy and Littlewood [16] (for the case $r=k=2$), Carleman [2], Sutton [1].

sufficient to prove (3) for the modified partial sums s_ν^* (§ 2.3); for $|s_\nu-f|^k\leqslant(|s_\nu^*-f|+|s_\nu-s_\nu^*|)^k$; hence, applying Jensen's inequality (§ 4.14), we obtain that $|s_\nu-f|^k\leqslant 2^{k-1}(|s_\nu^*-f|^k+|s_\nu-s_\nu^*|^k)$ and it is enough to observe that $|s_\nu-s_\nu^*|^k$ tends uniformly to 0.

Now, if $0<\nu\leqslant n$,

$$s_\nu^*(x)-f(x)=\frac{1}{\pi}\int_0^\pi \varphi_x(t)\frac{\sin\nu t}{2\operatorname{tg}\frac{1}{2}t}\,dt=\frac{1}{\pi}\left(\int_0^{1/n}+\int_{1/n}^{\pi}\right)=\alpha_\nu^{(n)}+\beta_\nu^{(n)},$$

$$\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu^*-f|^k\right\}^{1/k}\leqslant\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|\alpha_\nu^{(n)}|^k\right\}^{1/k}+\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|\beta_\nu^{(n)}|^k\right\}^{1/k},$$

and (i) will be established when we have shown that each of the terms on the right in the last inequality tends to 0 as $n\to\infty$. Since $|\sin\nu t/2\operatorname{tg}\frac{1}{2}t|<\nu$ for $0<t\leqslant\pi$, we obtain that $|\alpha_\nu^{(n)}|$ does not exceed $\pi^{-1}\nu\,\Phi_{x,1}(1/n)\leqslant\nu\,\Phi_{x,1}(1/\nu)=\eta_\nu$. The relation $\Phi_{x,r}(h)=o(h)$ implies $\Phi_{x,1}(h)=o(h)$. Hence $\eta_\nu\to 0$ and

$$(4)\qquad \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|\alpha_\nu^{(n)}|^k\right\}^{1/k}\leqslant\left\{\frac{1}{n+1}\sum_{\nu=1}^{n}\eta_\nu^k\right\}^{1/k}\to 0.$$

Now observe that the β's are Fourier coefficients of the function equal to $\varphi_x(t)\,\frac{1}{2}\operatorname{ctg}\frac{1}{2}t$ for $1/n\leqslant t\leqslant\pi$, and to 0 for $-\pi\leqslant t<1/n$. Applying the Hausdorff-Young inequality (§ 9.9.2) and supposing, as we may, that $r\leqslant 2$, we have

$$(5)\qquad \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|\beta_\nu^{(n)}|^k\right\}^{1/k}\leqslant\frac{1}{(n+1)^{1/k}}\left(\frac{1}{\pi}\int_{1/n}^{\pi}\left|\frac{\varphi_x(t)}{2\operatorname{tg}\frac{1}{2}t}\right|^r dt\right)^{1/r},$$

where $k=r'$. Replacing $2\operatorname{tg}\frac{1}{2}t$ by t, and integrating by parts, we see that the right-hand side of (5) does not exceed

$$\frac{1}{(n+1)^{1/k}}\left\{\left[\frac{\Phi_{x,r}(t)}{t^r}\right]_{1/n}^{\pi}+r\int_{1/n}^{\pi}\frac{\Phi_{x,r}(t)}{t^{r+1}}\,dt\right\}^{1/r}=$$
$$=\frac{1}{(n+1)^{1/k}}\left\{o(n^{r-1})+\int_{1/n}^{\pi}o(t^{-r})\,dt\right\}^{1/r}=$$
$$=(n+1)^{-1/k}[o(n^{r-1})+o(n^{r-1})]^{1/r}=o(1).$$

Hence the left-hand side of (5) tends to 0 and this, together with (4), proves (i).

The reader has no doubt noticed a curious feature of the above argument, namely, the less we suppose about the function, i. e. the smaller the number $r>1$ is, the larger value for k we obtain. The argument however breaks down for $r=1$ and the problem whether (3) is true for integrable functions remains unsolved, even when $k=1$.

It is also of some interest to observe that it is sufficient to consider the values of r of the form $2l/(2l-1)$, $l=1,2,3,\ldots$, in which case the proof of the Hausdorff-Young theorem is simple (§ 9.12).

If $f\in L^r$, $r>1$, the proof of (ii) is essentially the same as that of (i). We need only observe that, if $a\leqslant x\leqslant b$, then $\Phi_{x,r}(h)=o(h)$, $\Phi_{x,1}(h)=o(h)$ uniformly in x, and that the estimates we obtain are also uniform in x. If $f\in L$, we can find an interval (a_1,b_1), $a_1<a\leqslant b<b_1$ such that f is bounded in (a_1,b_1). Let $f(x)=f'(x)+f''(x)$, where $f'(x)=f(x)$ in (a_1,b_1) and $f'(x)=0$ elsewhere. If s'_ν and s''_ν denote the partial sums of $\mathfrak{S}[f']$ and $\mathfrak{S}[f'']$, then $s_\nu=s'_\nu+s''_\nu$ and

$$\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu-f|^k\right\}^{1/k}\leqslant\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|s'_\nu-f'|^k\right\}^{1/k}+\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|s''_\nu-f''|^k\right\}^{1/k}.$$

The first term on the right tends to 0 uniformly in x, $a\leqslant x\leqslant b$, since f' is bounded and so belongs to every L^r. Since $f''(x)=0$ for $a_1<x<b_1$, the expression $|s''_\nu-f''|^k$ tends uniformly to 0 for $a\leqslant x\leqslant b$. Hence the second term on the right in the last inequality tends uniformly to 0 for $a\leqslant x\leqslant b$, and the proof of (ii) is complete.

We add that (3) is true if f is integrable and is continuous at the point x. This is a special case of (ii) when the interval (a,b) reduces to one point. The result holds if f has a simple discontinuity at x and if $2f(x)=f(x+0)+f(x-0)$.

10.11. When $r=k=2$, Theorem 10.1(i) may be proved by a different argument which also works for general orthogonal systems of functions.

Let $\varphi_0(x)$, $\varphi_1(x)$, ... be a system of functions orthogonal and normal in an interval (a,b). If Σc_k^2 converges, and if the series $c_0\varphi_0(x)+c_1\varphi_1(x)+\ldots$, with partial sums $s_n(x)$, is summable $(C,1)$ in a set E, $|E|>0$, to a function $s(x)$, then

$$\frac{1}{n+1}\sum_{k=0}^{n}[s_k(x)-s(x)]^2\to 0 \qquad as\ n\to\infty,$$

for almost every $x\in E$[1]).

Let $\sigma_n(x)$ be the first arithmetic means of $\{s_n(x)\}$. We shall prove the following lemma: *If $\Sigma c_n^2<\infty$, the series $\Sigma[s_n(x)-\sigma_n(x)]^2/n$ converges for almost every $x\in(a,b)$.* In view of Theorem 4.2(ii), it is sufficient to show that the latter series, integrated term by term over (a,b), is convergent. But

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_a^b(s_n-\sigma_n)^2\,dx=\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2}\sum_{k=1}^{n}k^2c_k^2=$$

$$=\sum_{k=1}^{\infty}k^2c_k^2\sum_{n=k}^{\infty}\frac{1}{n(n+1)^2}\leqslant\sum_{k=1}^{\infty}k^2c_k^2\cdot\frac{1}{k^2}=\sum_{k=1}^{\infty}c_k^2,$$

and the lemma follows. Observing that, for every convergent series Σu_n, we have $u_1+2u_2+\ldots+nu_n=o(n)$ (§ 3.13(1)), we obtain that $(s_1-\sigma_1)^2+(s_2-\sigma_2)^2+\ldots+(s_n-\sigma_n)^2=o(n)$ for almost every x. Now

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}(s_k-s)^2\right\}^{1/2}\leqslant\left\{\frac{1}{n+1}\sum_{k=0}^{n}(s_k-\sigma_k)^2\right\}^{1/2}+\left\{\frac{1}{n+1}\sum_{k=0}^{n}(\sigma_k-s)^2\right\}^{1/2},$$

and since of the two terms on the right the first is $o(1)$ for almost every x, and the second for every $x\in E$, the theorem is established.

10.2. In this paragraph we shall prove a number of theorems on the Abel and Cesàro means of Fourier series. The results will mostly bear on the behaviour of Fourier series in the whole interval $(0,2\pi)$ and not at individual points.

10.21. An inequality for integrals. Let $f(x)$ be a non-negative function defined in an interval $(0,a)$, where for simplicity we suppose that $a<\infty$, and let $f^*(x)$, $0<x\leqslant a$, be the function equimeasurable with f and non-increasing (§ 9.42). We put

$$(1)\qquad \theta(x;f)=\operatorname*{Sup}_{\xi}\frac{1}{x-\xi}\int_\xi^x f(t)\,dt,\quad 0\leqslant\xi<x,\ 0<x\leqslant a,$$

and similarly define $\theta(x;f^*)$. It is easy to see that for non-in-

[1]) Borgen [1], Zygmund [10].

creasing f, and in particular for f^*, the upper bound in (1) is attained when $\xi = 0$. The following theorem has important applications.

For any non-decreasing and non-negative function $s(t)$, $t \geqslant 0$,

$$\int_0^a s\{\theta(x;f)\}\,dx \leqslant \int_0^a s\{\theta(x;f^*)\}\,dx \tag{2}$$ [1].

Given a non-negative function $g(x) \in L(0,a)$, let $e(y) = |E(y)|$, where $E(y)$ is the set of points x for which $g(x) > y$; then

$$\int_0^a g(x)\,dx = -\int_0^\infty y\,de(y) = \int_0^\infty e(y)\,dy, \tag{3}$$

the second integral being a Riemann-Stieltjes integral. When g is bounded, the first equation follows at once if we observe that the approximate Lebesgue sums for the first integral are approximate Riemann-Stieltjes sums for the second integral. To obtain the result in the general case we apply the formula to the function $g_n(x) = \text{Max}\{g(x), n\}$ and then make $n \to \infty$. The equality of the second and third integral follows by an integration by parts if we notice that $ye(y) \to 0$ as $y \to \infty$. This last relation is, in turn, a consequence of the fact that $ye(y)$ does not exceed the integral of $g(x)$ extended over the set of x for which $g(x) > y$.

Let $E(y_0)$ and $E^*(y_0)$ denote the sets of points where $\theta(x;f) > y_0$ and $\theta(x;f^*) > y_0$ respectively. Comparing the extreme terms of (3) we see that (2) will be established if we show that $|E(y_0)| \leqslant |E^*(y_0)|$ for every y_0. We break up the proof of this inequality into three stages.

(a) Given a continuous function $F(x)$, $0 \leqslant x \leqslant a$, let H denote the set of points x for each of which there is a point ξ, $0 \leqslant \xi < x$, such that $F(\xi) < F(x)$. Then H is an open set and is a sum of an at most enumerable system of open and non-overlapping intervals (α_k, β_k) such that $F(\alpha_k) \leqslant F(\beta_k)$ (it can easily be shown that actually we have $F(\alpha_k) = F(\beta_k)$, but this will not be required).

That H is open follows from the fact that the inequality $F(\xi) < F(x)$ is not impaired by slight changes of x. Let (α_k, β_k) be any of the open and non-overlapping intervals whose sum is E. Suppose that $F(\alpha_k) > F(\beta_k)$, and let x_0 be the least number belonging to (α_k, β_k) and such that $F(x_0) = \frac{1}{2}[F(\alpha_k) + F(\beta_k)]$. No point ξ corresponding to x_0 can belong to (α_k, x_0), for the points x of this interval satisfy the inequality $F(x) \geqslant F(x_0)$. Hence $\xi < \alpha_k$, and the inequalities $F(\xi) < F(x_0)$, $F(x_0) < F(\alpha_k)$ give $F(\xi) < F(\alpha_k)$. Here we have a contradiction since the last inequality and the inequality $\xi < \alpha_k$ imply that $\alpha_k \in H$, which is false.

[1]) Hardy and Littlewood [17]; F. Riesz [7].

(b) If E is an arbitrary set contained in $(0,a)$, $|E| > 0$, then $\int_E f\,dx \leqslant \int_0^{|E|} f^*\,dx.$

This is a special case of a more general result established in § 9.42. An independent proof runs as follows. Let $f_1(x)$ be the function which is equal to $f(x)$ in E and to 0 elsewhere. Since $f_1(x) \leqslant f(x)$, we have $f_1^*(x) \leqslant f^*(x)$ and

$$\int_E f\,dx = \int_0^a f_1\,dx = \int_0^a f_1^*\,dx = \int_0^{|E|} f_1^*\,dx \leqslant \int_0^{|E|} f^*\,dx.$$

(c) Let $E_1^*(y_0)$ denote the set of points where $\theta(x;f^*) \geqslant y_0$. Having fixed y_0 we shall write E, E^*, E_1^* instead of $E(y_0)$, $E^*(y_0)$, $E_1^*(y_0)$. If we put $F(x) = \int_0^x f\,dt - y_0 x$, the set E becomes the set H of (a). If $\{(\alpha_k, \beta_k)\}$ is the sequence of open and non-overlapping intervals of which E consists, then, using the results obtained in (a) and (b),

$$\int_{\alpha_k}^{\beta_k} f\,dx \geqslant y_0(\beta_k - \alpha_k), \qquad \int_E f\,dx \geqslant y_0|E|, \qquad \int_0^{|E|} f^*\,dx \geqslant y_0|E|.$$

Now $\theta(x;f^*) = \dfrac{1}{x}\displaystyle\int_0^x f^*\,dt$; since the right-hand side of this equation is a non-increasing function of x, $|E_1^*|$ may be defined as the largest number x satisfying the inequality $\dfrac{1}{x}\displaystyle\int_0^x f^*\,dt \geqslant y_0$. From this and the preceding inequality we infer that $|E| \leqslant |E_1^*|$. Therefore, if $\varepsilon > 0$, we have $|E(y_0+\varepsilon)| \leqslant |E_1^*(y_0+\varepsilon)|$ and, making $\varepsilon \to 0$, we obtain $|E(y_0)| \leqslant |E^*(y_0)|$. This completes the proof of (2).

10.211. We shall change the notation slightly. The function which we denoted by $\theta(x;f)$ will now be written $\theta_1(x;f)$. By $\theta_2(x;f)$ we shall denote $\operatorname{Sup}_{\xi} \dfrac{1}{\xi - x}\displaystyle\int_x^{\xi} f\,dt$ for $x<\xi\leqslant a$. If f_* denotes the function equimeasurable with f and non-decreasing, then

$$\int_0^a s\{\theta_1(x;f)\}\,dx \leqslant \int_0^a s\{\theta_1(x;f^*)\}\,dx, \qquad \int_0^a s\{\theta_2(x;f)\} \leqslant \int_0^a s\{\theta_2(x;f_*)\}.$$

The second inequality follows from the first by a simple transformation of the variable x. Let $\theta=\operatorname{Max}(\theta_1,\theta_2)$. It is not difficult to see that the inequality 10.21(2) holds for the new function θ if we introduce the factor 2 into the right-hand side. For $s(\theta)=\operatorname{Max}\{s(\theta_1),\, s(\theta_2)\}\leqslant s(\theta_1)+s(\theta_2)$ and so

$$\int_0^a s\{\theta(x;f)\}\,dx \leqslant \int_0^a s\{\theta_1(x;f^*)\}\,dx + \int_0^a s\{\theta_2(x;f_*)\}\,dx = 2\int_0^a s\{\theta(x;f^*)\}\,dx.$$

Thence, by a change of variable, we obtain

If (a,b) is a finite interval and

$$\theta(x;f)=\theta(x;f,a,b)=\operatorname{Sup}_{\xi}\frac{1}{x-\xi}\int_{\xi}^{x} f(t)\,dt, \qquad a\leqslant\xi\leqslant b,$$

then

$$\int_a^b s\{\theta(x;f)\}\,dx \leqslant 2\int_a^b s\left\{\frac{1}{x-a}\int_a^x f^*(t)\,dt\right\}dx,$$

where $f^*(x)$ is the function equimeasurable with $f(x)$ and non-decreasing.

10.22. Theorems of Hardy and Littlewood [1].

(i) *If $f\in L^r(a,b)$, $r>1$, then $\theta(x;|f|)\in L^r(a,b)$ and*

(1) $$\int_a^b \theta^r(x;|f|)\,dx \leqslant 2\left(\frac{r}{r-1}\right)^r\int_a^b |f|^r\,dx.$$

This follows from the remarks made in the previous section and from Theorem 4.17.

[1] Hardy and Littlewood [17]; see also Paley [6].

The example of the function $f(x)=1/x\log^2 x$ considered in the interval $(0,a)$, $0<a<1$, shows that, if $f\in L$, the function $\theta(x;|f|)$ need not be integrable.

(ii) *If $f\in L(a,b)$, then $\theta(x;|f|)\in L^{\alpha}(a,b)$ for every $0<\alpha<1$, and*

(2) $$\left\{\int_a^b \theta^{\alpha}(x;|f|)\,dx\right\}^{1/\alpha}\leqslant A_{\alpha}\int_a^b |f|\,dx,$$

where A_α depends on α and $b-a$ only.

(iii) *If $|f|\log^+|f|\in L(a,b)$, then $\theta(x;|f|)\in L(a,b)$ and*

(3) $$\int_a^b \theta(x;|f|)\,dx\leqslant B\int_a^b |f|\log^+|f|\,dx + C,$$

where B and C depend on $b-a$ only.

It is sufficient to prove (ii) and (iii) in the case of functions which are non-negative and non-increasing. We may also suppose that the interval (a,b) is of the form $(0,a)$. Then, applying Hölder's inequality,

$$\int_0^a \theta^{\alpha}(x;f)\,dx=\int_0^a\frac{dx}{x^{\alpha(1-\alpha)}}\left\{\frac{1}{x^{\alpha}}\int_0^x f\,dt\right\}^{\alpha}\leqslant\left\{\int_0^a\frac{dx}{x^{\alpha}}\right\}^{1-\alpha}\left\{\int_0^a\frac{dx}{x^{\alpha}}\int_0^x f\,dt\right\}^{\alpha}=$$

$$=\frac{a^{(1-\alpha)^2}}{(1-\alpha)^{1-\alpha}}\left\{\int_0^a f\,dt\int_t^a\frac{dx}{x^{\alpha}}\right\}^{\alpha}\leqslant\frac{a^{1-\alpha}}{1-\alpha}\left\{\int_0^a f\,dt\right\}^{\alpha},$$

so that in the general case we have (2) with $A_\alpha^\alpha=2a^{(1-\alpha)}/(1-\alpha)$.

To prove (3), let $I=\int_0^a f\,dx$, $J=\int_0^a f\log^+ f\,dx$; we shall denote by $B_1, B_2, \ldots$ constants which depend on a only. If f is non-negative and non-increasing, the left-hand side of (3) is equal to

(4) $$\int_0^a\frac{1}{x}\int_0^x f\,dt=\int_0^a f\log\frac{a}{x}dx\leqslant I\log^+ a+\int_0^a f\log^+\frac{1}{x}\,dx.$$

Observing that $f\leqslant\operatorname{Max}(e, f\log^+ f)\leqslant e+f\log^+ f$, we find that $I\leqslant J+ae=J+B_1$. On the other hand, since the monotonic functions $\Phi(x)=(x+1)\log(x+1)-x\leqslant(x+1)\log(x+1)$ and $\Psi(y)=e^y-y-1<e^y$ are complementary functions in the sense of Young (§ 4.11), an application of Young's inequality gives

$$\int_0^a 2f\cdot\tfrac{1}{2}\log^+\frac{1}{x}\,dx<\int_0^a(2f+1)\log(2f+1)\,dx+\int_0^a e^{\frac{1}{2}\log^+ 1/x}\,dx.$$

Since $2f+1 \leqslant \text{Max}(3, 3f)$, the first integral on the right is less than $B_2 J + B_3$. Collecting the results, we see that the left-hand side of (4) does not exceed $BJ + C$, and (3) is established.

Suppose now that $f(x)$ is of period 2π and integrable over $(0, 2\pi)$. Let

$$M(x;f) = \sup_{0<|t|\leqslant\pi} \frac{1}{t}\int_0^t |f(x+u)|\,du = \sup_{0<|t|\leqslant\pi} \frac{1}{t}\int_x^{x+t} |f(u)|\,du,$$

for $-\pi \leqslant x \leqslant \pi$. If we replace the condition $0 < |t| \leqslant \pi$ by $-2\pi - x \leqslant t \leqslant 2\pi - x$, we increase the upper bound and we obtain, instead of $M(x;f)$, the function $\theta(x;|f|)$ formed for the interval $(-2\pi, 2\pi)$, and so

$$\int_{-\pi}^{\pi} s\{M(x;f)\}\,dx \leqslant \int_{-2\pi}^{2\pi} s\{\theta(x;|f|, -2\pi, 2\pi)\}\,dx.$$

Thence we easily obtain that

(iv) *The inequality* (1) *remains true if we replace the interval of integration* (a, b) *by* $(-\pi, \pi)$, *the function* $\theta(x;|f|)$ *by* $M(x;f)$, *and the factor* 2 *on the right by* 4.

(v) *The inequalities* (2) *and* (3) *hold if* (a, b) *is replaced by* $(-\pi, \pi)$, *and* $\theta(x;|f|)$ *by* $M(x;f)$. *The constant* A_α *will now depend only on* α, *and* B *and* C *will be absolute constants.*

Applications of the previous results to the theory of Fourier series are based on the following lemma.

(vi) *Let* $\chi(t,p)$, $-\pi \leqslant t \leqslant \pi$, *be a non-negative function depending on a parameter* p *and satysfying the conditions*

$$\text{(5a)}\quad \int_{-\pi}^{\pi} \chi(t,p)\,dt \leqslant K, \qquad \text{(5b)}\quad \int_{-\pi}^{\pi} \left| t\frac{\partial}{\partial t}\chi(t,p)\right| dt \leqslant K_1,$$

where K *and* K_1 *are independent of* p. *If*

$$h(x,p) = \int_{-\pi}^{\pi} f(x+t)\,\chi(t,p)\,dt,$$

then $\sup_p |h(x,p)| \leqslant AM(x;f)$, *where the constant* A *is independent of* f.

For let $F_x(t)$ be the integral of $|f(x+u)|$ over the interval $0 \leqslant u \leqslant t$ or $t \leqslant u \leqslant 0$. Integrating the formula defining $h(x,p)$ by parts and observing that $|F_x(t)| \leqslant |t|\,M(x;f)$, we find

$$|h(x,p)| \leqslant M(x;f)\left\{\int_{-\pi}^{\pi}\left| t\frac{\partial}{\partial t}\chi(t,p)\right| dt + \pi[\chi(\pi,p) + \chi(-\pi,p)]\right\}.$$

Integrating the integral of (5a) by parts and taking into account (5b), we see that $\pi[\chi(\pi,p)+\chi(-\pi,p)] \leqslant K + K_1$. Hence $|h(x,p)| \leqslant (2K_1 + K)\,M(x;f)$ and the lemma is established.

It is useful to observe that, if $t\,\partial\chi/\partial t$ is of constant sign, and if $\chi(\pm\pi, p)$ are bounded functions of p, then the inequality (5b) is a consequence of (5a). This follows at once if we drop the sign of absolute value in (5b) and integrate by parts.

If for $\chi(t,p)$ we take the Poisson kernel $P_r(t)$, the inequality (5a) is satisfied; also (5b) is true, for $t\,dP_r(t)/dt \leqslant 0$ and $P_r(\pm\pi) = O(1)$. Therefore,

(vii) *If* $N(x;f)$ *is the upper bound of* $|f(r,x)|$ *for* $0 \leqslant r < 1$, *where* $f(r,x)$ *denotes the Poisson integral of an integrable function* $f(x)$, *then* $N(x;f) \leqslant AM(x;f)$, *where* A *is an absolute constant.*

From this and from (iv) and (v) we obtain:

(viii) *The function* $N(x;f)$ *satisfies the inequalities*

$$\text{(6)}\quad \begin{aligned} &\int_{-\pi}^{\pi} N^r(x;f)\,dx \leqslant A_r \int_{-\pi}^{\pi} |f|^r\,dx, \quad r > 1,\\ &\int_{-\pi}^{\pi} N^\alpha(x;f)\,dx \leqslant A_\alpha \int_{-\pi}^{\pi} |f|\,dx, \quad 0 < \alpha < 1,\\ &\int_{-\pi}^{\pi} N(x;f)\,dx \leqslant B\int_{-\pi}^{\pi} |f|\log^+|f|\,dx + C, \end{aligned}$$

where A_r *depends only on* r, A_α *only on* α, *and* B *and* C *are absolute constants.*

The Fejér kernel $K_n(t)$ satisfies (5a) but, as can easily be shown, not (5b). The same may be said of the kernel $K_n^\delta(t)$, $0 < \delta < 1$, which, besides, is not of constant sign. The kernel $K_n^\delta(t)$, $0 < \delta \leqslant 1$, can however be majorised by a function which satisfies the inequalities (5). For

(7) $$|K_n^\delta(t)| \leqslant L_n^\delta(t) = \frac{c(\delta)\, n}{1 + (n\,|t|)^{\delta+1}}, \quad 0 < \delta \leqslant 1, \quad n \geqslant 1, \quad |t| \leqslant \pi,$$

where $c(\delta)$ depends on δ only. To prove this inequality, which is due substantially to Fejér[1]), it is sufficient to observe that $L_n^\delta(t) \geqslant \frac{1}{2} c(\delta)\, n$ for $n|t| \leqslant 1$, $L_n^\delta(t) \geqslant \frac{1}{2} c(\delta)/n^\delta |t|^{\delta+1}$ for $n|t| \geqslant 1$, and to take into account the inequalities 3.3(2). The reader will verify that the function $\chi(t, p) = L_n^\delta(t)$ satisfies (5a); that the inequality (5b) is also satisfied follows from the fact that $t \cdot dL_n^\delta(t)/dt \leqslant 0$ and that $L_n^\delta(\pm\pi) = O(1)$.

Let $\sigma_n^\delta(x; f)$ be the Cesàro means of order δ for $\mathfrak{S}[f]$. Observing that $|\sigma_n^\delta(x; f)| \leqslant \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+t)|\, L_n^\delta(t)\, dt$, we obtain:

(ix) *If* $N_\delta(x; f)$, $0 < \delta \leqslant 1$, *is the upper bound of* $|\sigma_n^\delta(x; f)|$ *for* $1 \leqslant n < \infty$, *then* $N_\delta(x; f) \leqslant AM(x; f)$ *with A depending only on* δ; *the function* $N_\delta(x; f)$ *satisfies inequalities similar to* (6), *where the constants* A_r, A_α, B, C *will now depend also on* δ [2]).

The theorem remains true for $\delta > 1$. This follows from the fact, which is easy to verify (§ 3.13), that $N_\delta(x; f)$ is a non-increasing function of δ.

We return to the case of harmonic functions $f(r, x)$. If $0 \leqslant \varphi < \frac{1}{2}\pi$, we denote by $S_\varphi(x)$ the part of the unit circle limited by two chords through e^{ix} at angles α to the radius, and the perpendiculars upon them from the origin. Let $N(x; f, \varphi)$ be the upper bound of $|f(r, \theta)|$ for $z = re^{i\theta}$, $r < 1$, belonging to $S_\varphi(x)$.

(x) *There is a number A depending only on* φ *such that* $N(x; f, \varphi) \leqslant AM(x; f)$. *The function* $N(x; f, \varphi)$ *satisfies inequalities similar to* (6), *except that the constants* A_r, A_α, B, C *will now depend also on* φ.

It is only the first part of this theorem which needs a proof. If $z = re^{i\theta}$, $r < 1$, is any point belonging to $S_\varphi(x)$, and $\zeta = re^{i(\theta - x)}$, then

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[1]) Fejér [10]. If we replace n by $n+1$ in the numerator of the last ratio, the inequality will hold for $n \geqslant 0$.

[2]) The theorem remains true if in the definition of $N_\delta(x; f)$ we suppose that n runs from 0 to ∞. It suffices to modify the definition of $L_n^\delta(t)$ slightly (see the preceding footnote).

$$f(r, \theta) = \int_{-\pi}^{\pi} f(x+t)\, \chi(t, \zeta)\, dt, \quad \text{where } \chi(t, \zeta) = \frac{1}{\pi} P_r(t + x - \theta).$$

The expression $\chi(t, \zeta)$ here depends on the variable t and the parameter ζ belonging to the region $S_\varphi(0)$. That the inequality (5a) is satisfied, is apparent. The left-hand side of (5b) takes the form $\frac{1}{\pi} \int_{-\pi}^{\pi} \left| t \frac{d}{dt} P_r(t + \xi) \right| dt$, where $\xi = x - \theta$. Supposing, to fix ideas, that $\xi > 0$, we break up the interval of integration into three parts $(-\pi, -\xi)$, $(-\xi, 0)$, $(0, \pi)$, in each of which the expression under the sign of absolute value is of constant sign. Integrating by parts, and observing that $P_r(0) = O(1/(1-r))$, $\xi = O(1-r)$, we obtain the desired inequality.

Proposition (x), suitably modified, can be extended to general classes H^p, $p > 0$ (§ 7.51):

(xi) *If* $F(z)$ *is a function regular for* $|z| < 1$, *and if* $\mu_p(r; F) \leqslant \lambda^p$, $0 \leqslant r < 1$, $p > 0$, *then* $\mathfrak{M}_p^p[N(x; F, \varphi)] \leqslant A_\varphi \lambda^p$, *where* A_φ *depends on* φ *only*.

This theorem is a consequence of (x) if $p = 2$. In the general case we have $F(z) = G(z)\, B(z)$, where $|B(z)| \leqslant 1$, $G(z)$ is regular and non-vanishing, and $\mu_p(r; G) \leqslant \lambda^p$ (§ 7.53(v)). The function $G^{p/2}(z)$ is regular and belongs to H^2. Since $\mu_2(r; G^{p/2}) = \mu_p(r; G) \leqslant \lambda^p$, we obtain $\mathfrak{M}_2^2[N(x; G^{p/2}, \varphi)] \leqslant A_\varphi \lambda^p$, and it is sufficient to observe that the left-hand side of the last inequality is equal to the expression $\mathfrak{M}_p^p[N(x; G, \varphi)] \geqslant \mathfrak{M}_p^p[N(x; F, \varphi)]$.

The most important special case of (xi) is when $\varphi = 0$ and A_φ reduces to a radius of the circle.

The theorems established in this section elucidate certain results of Chapter IV. To prove, for example, that, if $f \in L^r$, $r > 1$, then $\mathfrak{M}_r[f - \sigma_n] \to 0$ (§ 4.35), it is sufficient to observe that $|f(x) - \sigma_n(x)|^r$ tends almost everywhere to 0 and is dominated by an integrable function. Similarly Theorem 7.56(iii) is an easy consequence of (xi).

10.23. We conclude this paragraph by a few remarks on the function $\tilde{f}(x) = \operatorname{Sup}_h |f_h(x)|$, where

icm

$$\bar{f}_h(x) = -\frac{1}{\pi}\int_h^\pi \frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt, \qquad 0<h\leqslant\pi,$$

f denoting an integrable and periodic function. In § 7.11 we showed that $\tilde{f}(x)$ is finite almost everywhere. Completing the results of §§ 7.21, 7.24, we shall show that

$$\mathfrak{M}_r[\tilde{f}] \leqslant A_r\,\mathfrak{M}_r[f],\ \ r>1;\quad \mathfrak{M}_\alpha[\tilde{f}] \leqslant A_\alpha\,\mathfrak{M}\,[f],\quad 0<\alpha<1;$$

$$\mathfrak{M}\,[\tilde{f}] \leqslant B\,\mathfrak{M}\,[f\log^+|f|]+C,$$

where A_r depends only on r, A_α only on α, and B and C are absolute constants. It is sufficient to prove the first of these inequalities only, the proof of the remaining being similar. Let us put $\psi_x(t)=f(x+t)-f(x-t)$; then

$$\bar{f}_{1-r}(x)-\bar{f}(r,x) = \frac{1}{\pi}\int_0^{1-r}\psi_x(t)\,Q_r(t)\,dt - \frac{1}{\pi}\int_{1-r}^{\pi}\psi_x(t)\,R_r(t)\,dt = G_r(x)+H_r(x),$$

where $Q_r(t) = r\sin t/(1-2r\cos t+r^2)$, $R_r(t)$ denotes the ratio $(1-r)^2/2\operatorname{tg}\frac{1}{2}t\cdot(1-2r\cos t+r^2)$, and $\bar{f}(r,x)$ is the harmonic function conjugate to $f(r,x)$. Since $|Q_r(t)|<1/(1-r)$, we have $|G_r(x)|\leqslant M(x;f)$. Integrating by parts and observing that $t\,R_r(t)=O(1)$ for $t=1-r$, and that $\int_{1-r}^{\pi}\left|t\frac{d}{dt}R_r(t)\right|dt=O(1)$, we find that $|H_r(x)|$, and so also $|\bar{f}_{1-r}(x)-\bar{f}(r,x)|$, does not exceed a multiple of $M(x;f)$.

Suppose now that $f\in L^r$, $r>1$; then the function $\bar{f}(x)=\bar{f}(x;+0)$ belongs to L^r, and $\bar{f}(r,x)$ is the Poisson integral of $\bar{f}(x)$. Hence $|\bar{f}_{1-r}(x)| \leqslant |\bar{f}_{1-r}(x)-\bar{f}(r,x)|+|\bar{f}(r,x)| \leqslant D\{M(x;f)+M(x;\bar{f})\}$, where D is an absolute constant. This inequality gives $|\tilde{f}(x)|\leqslant D\{M(x;f)+M(x;\bar{f})\}$, $\mathfrak{M}_r[\tilde{f}]\leqslant D\{\mathfrak{M}_r[M(x;f)]+\mathfrak{M}_r[M(x;\bar{f})]\}$. In view of Theorem 10.22(iv), the right-hand side of the last inequality does not exceed a multiple of $\mathfrak{M}_r[f]+\mathfrak{M}_r[\bar{f}]$ and it suffices to apply Theorem 7.21.

10.3. Partial sums of $\mathfrak{S}[f]$ for $f\in L^2$. The theory of summability of Fourier series by Abel's method, or Cesàro's methods of positive order, is in a state which may be described as satisfactory. The situation is adequately represented when we

say that what we need there most are problems, that is interesting problems. Achievements of the modern theory of real functions have left means at our disposal which seem to be sufficient to cope with problems of summability, although the latter may in some cases be fairly difficult.

The situation is different when we consider the behaviour of partial sums. Several results have been obtained for the convergence at individual points, but as regards convergence or divergence almost everywhere, our knowledge is still very scanty. Problems which suggest themselves to the beginner (for example the problem whether $\mathfrak{S}[f]$ must converge at one point at least when f is continuous) seem to be far from being solved. It is true that in the last few years a number of important results have been obtained, connected with the names of Kolmogoroff and Seliverstoff, Plessner, and Littlewood and Paley, but much more still remains to be done.

10.31. Theorems of Kolmogoroff [1]**.** Let $f(x)$ be a function of the class L^2 and let $s_n(x)$ be the partial sums of the Fourier series

(1) $$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

of $f(x)$. Since $\mathfrak{M}_2[f-s_n]\to 0$, there is a subsequence $\{s_{n_k}(x)\}$ of $\{s_n(x)\}$ which converges almost everywhere to $f(x)$ (§ 4.2). We shall now prove that for $\{n_k\}$ we may take a sequence independent of f.

(i) *If $n_{k+1}/n_k>\lambda>1$, $k=1,2,\ldots$, the partial sums $s_{n_k}(x)$ of $\mathfrak{S}[f]$, $f\in L^2$, converge almost everywhere to $f(x)$.*

A series Σc_i is said to possess a gap (u,v) if $c_i=0$ for $u<i\leqslant v$. We shall require the following lemma. *If a series Σc_i, with partial sums s_n, possesses infinitely many gaps (m_k, m'_k) such that $m'_k/m_k>\lambda>1$, and is summable $(C,1)$ to sum s, then s_{m_k}, and so also $s_{m'_k}$, converges to s.*

Let $s=0$, $s_0+s_1+\ldots+s_n=(n+1)\,\sigma_n$. Then

(2) $$\begin{aligned}(m'_k-m_k)\,s_{m_k} &= s_{m_k+1}+s_{m_k+2}+\ldots+s_{m'_k}=\\ &=(m'_k+1)\,\sigma_{m'_k}-(m_k+1)\,\sigma_{m_k}=o\,(m'_k)+o\,(m_k)=o\,(m'_k),\end{aligned}$$

whence $s_{m_k}=o(1)$ and the lemma is established. In particular

[1]) Kolmogoroff [8]; Marcinkiewicz [1].

(ii) *If the Fourier series of an integrable function $f(x)$ possesses infinitely many gaps (m_k, m'_k) such that $m'_k/m_k > \mu > 1$, the partial sums $s_{m_k}(x)$ converge almost everywhere to $f(x)$.*

Now, in order to prove (i), we split (1) into consecutive blocks of terms $n_k < n \leqslant n_{k+1}$, $n_0 = 0$, including $\frac{1}{2} a_0$ in the first block; we then break up the whole series into two, one consisting of blocks with even, the other with odd, indices. By the Riesz-Fischer theorem, these series are Fourier series of functions f' and f'' respectively. For each series the terms with indices n_k are either at the end of or immediately preceding a gap, and so, by (ii), the partial sums of the two series, viz. $s'_{n_k}(x)$ and $s''_{n_k}(x)$, converge almost everywhere. The same is true for $s_{n_k}(x) = s'_{n_k}(x) + s''_{n_k}(x)$.

(iii) *Let* $s(x) = \operatorname{Sup}_k |s_{n_k}(x)|$. *Under the hypothesis of* (i), $s(x)$ *belongs to* L^2 *and* $\mathfrak{M}_2[s] \leqslant A_\lambda \mathfrak{M}_2[f]$, *where* A_λ *depends on* λ *only.*

Denoting by $B_1, B_2, \ldots$ constants depending exclusively on λ, we obtain from (2) that $\operatorname{Sup} |s_{m_k}| \leqslant B_1 \operatorname{Sup} |\sigma_{m_k}|$. Hence, under the hypothesis of (ii), $\operatorname{Sup} |s_{m_k}(x)| \leqslant B_1 \operatorname{Sup} |\sigma_{m_k}(x)| \leqslant B_2 M(x; f)$ (§ 10.2(ix)). Therefore, if $f', f'', s'_{n_k}, s''_{n_k}$ have the same meaning as before,

$$s(x) \leqslant \operatorname{Sup} |s'_{n_k}(x)| + \operatorname{Sup} |s''_{n_k}(x)| \leqslant B_2\{M(x; f') + M(x; f'')\},$$

$$\mathfrak{M}_2[s] \leqslant B_2\{\mathfrak{M}_2[M(x; f')] + \mathfrak{M}_2[M(x; f'')]\} \leqslant B_3\{\mathfrak{M}_2[f'] + \mathfrak{M}_2[f'']\},$$

and it is sufficient to observe that, in view of Parseval's relation, the last expression in curly brackets does not exceed the sum $\mathfrak{M}_2[f] + \mathfrak{M}_2[f] = 2\mathfrak{M}_2[f]$.

10.32. Convergence of a class of trigonometrical series [1]). An immediate consequence of Theorem 3.71 is that, if $\Sigma (a_n^2 + b_n^2) \log^2 n < \infty$, the series 10.31(1) converges almost everywhere. For from the last inequality and the Riesz-Fischer theorem we see that the trigonometrical series with coefficients $a_n \log n$, $b_n \log n$, is a Fourier series and, applying the first part

[1]) Kolmogoroff and Seliverstoff [1], [2], Plessner [4]. The method of the proof seems to have been used first by Jerosch and Weyl [1], to obtain much weaker results.

of Theorem 3.71 to it, we obtain the desired result. Now we shall prove a more general theorem.

(i) *If the series* $\sum_{n=2}^{\infty} (a_n^2 + b_n^2) \log n$ *converges, the series* 10.31(1) *converges almost everywhere.*

The argument which we shall use to prove this theorem is not less interesting than the result itself, and may be used in many problems.

Without loss of generality we may suppose that $a_0 = a_1 = b_1 = 0$. Let $E_n(x)$ and $H_n(x)$, $n = 0, 1, \ldots$, denote the partial sums of the series

$$\sum_{n=2}^{\infty} \frac{\cos nx}{\sqrt{\log n}}, \qquad \sum_{n=2}^{\infty} \frac{\cos nx}{\log n}$$

respectively. Let $n(x)$, $0 \leqslant x \leqslant 2\pi$, be any measurable function taking non-negative integral values and bounded above by some integer N. If $s_\nu(x)$ are the partial sums of 10.31(1), and if the series $\sum_{2}^{\infty} (a_n \cos nx + b_n \sin nx) \sqrt{\log n}$ is $\mathfrak{S}[g]$, then

$$s_\nu(x) = \frac{1}{\pi} \int_0^{2\pi} g(t)\, E_\nu(t - x)\, dt.$$

Putting $\nu = n(x)$, integrating over the interval $(0, 2\pi)$, and using Schwarz's inequality, we obtain

$$\left| \int_0^{2\pi} s_{n(x)}(x)\, dx \right| = \left| \frac{1}{\pi} \int_0^{2\pi} dx \int_0^{2\pi} g(t)\, E_{n(x)}(t - x)\, dt \right| =$$

$$= \left| \frac{1}{\pi} \int_0^{2\pi} g(t)\, dt \int_0^{2\pi} E_{n(x)}(t - x)\, dx \right| \leqslant \mathfrak{M}_2[g]\, \mathfrak{M}_2\left[\frac{1}{\pi} \int_0^{2\pi} E_{n(x)}(t - x)\, dx\right].$$

The square of the last factor is equal to

$$\frac{1}{\pi} \int_0^{2\pi} dt \left[\int_0^{2\pi} E_{n(x)}(t - x)\, dx\right] \left[\int_0^{2\pi} E_{n(x')}(t - x')\, dx'\right] =$$

$$(1) \qquad = \int_0^{2\pi}\!\!\int_0^{2\pi} dx\, dx' \left\{\frac{1}{\pi} \int_0^{2\pi} E_{n(x)}(x - t)\, E_{n(x')}(x' - t)\, dt\right\}.$$

The expression in curly brackets is equal to $H_m(x - x')$, where $m = m(x, x') = \operatorname{Min}\{n(x), n(x')\}$, and so the right-hand side of (1) does not exceed

$$(2)\qquad \begin{aligned} &\int_0^{2\pi}\int_0^{2\pi} \{|H_{n(x)}(x-x')| + |H_{n(x')}(x-x')|\}\, dx\, dx' = \\ &\qquad = 2\int_0^{2\pi}\int_0^{2\pi} |H_{n(x)}(x-x')|\, dx\, dx'. \end{aligned}$$

In § 5.12 we saw that $\mathfrak{M}[H_\nu] = O(1)$. Hence, integrating first with respect to x' and then with respect to x, we see that the right-hand side of (2) is less than an absolute constant A, and

$$(3)\qquad \left|\int_0^{2\pi} s_{n(x)}(x)\, dx\right| \leqslant A\,\mathfrak{M}_2[g] = A\left\{\pi \sum_{\nu=2}^{\infty}(a_\nu^2 + b_\nu^2)\log \nu\right\}^{1/2}.$$

This is a fundamental inequality from which the theorem follows comparatively easily. For let $\varphi_N(x) = \operatorname{Sup} s_n(x)$, $0 \leqslant n \leqslant N$, $\psi_N(x) = \operatorname{Sup}\{-s_n(x)\}$, $0 \leqslant n \leqslant N$. Since $s_0(x) = 0$, the functions φ_N and ψ_N are non-negative. By choosing suitable functions $n(x)$, the inequality (3) gives $\mathfrak{M}[\varphi_N] \leqslant A\,\mathfrak{M}_2[g]$, $\mathfrak{M}[\psi_N] \leqslant A\,\mathfrak{M}_2[g]$. The sequences $\{\varphi_N(x)\}$ and $\{\psi_N(x)\}$ are non-decreasing and so, putting $\Phi(x) = \lim \varphi_N(x)$, $\Psi(x) = \lim \psi_N(x)$, we have $\mathfrak{M}[\Phi] \leqslant A\,\mathfrak{M}_2[g]$, $\mathfrak{M}[\Psi] \leqslant A\,\mathfrak{M}_2[g]$. The functions Φ and Ψ, being integrable, are finite almost everywhere and, since $\Phi(x) = \operatorname{Sup} s_n(x)$, $\Psi(x) = \operatorname{Sup}\{-s_n(x)\}$, the sequence $\{s_n(x)\}$ is bounded for almost every x.

If $\Omega(x)$ denotes the upper bound of $|s_m(x) - s_n(x)|$ for all values of m and n, then $\Omega(x) \leqslant \Phi(x) + \Psi(x)$, and so we have $\mathfrak{M}(\Omega) \leqslant 2A\,\mathfrak{M}_2[g]$.

To prove that $\{s_n(x)\}$ converges almost everywhere, let $\Omega_M(x) = \operatorname{Sup}|s_n(x) - s_m(x)|$ for all possible values of $m \geqslant M$ and $n \geqslant M$, and let $g_M(x) \sim \sum_{M+1}^{\infty}(a_n \cos nx + b_n \sin nx)\sqrt{\log n}$. The function Ω_M is the Ω corresponding to g_M, so that $\mathfrak{M}[\Omega_M] \leqslant 2A\,\mathfrak{M}_2[g_M]$. In view of Parseval's formula, $\mathfrak{M}_2[g_M] \to 0$ as $M \to \infty$, and so we also have $\mathfrak{M}[\Omega_M] \to 0$. Since $\{\Omega_M\}$ is a non-increasing sequence, we conclude that $\mathfrak{M}[\lim \Omega_M] = 0$, i. e. $\lim \Omega_M(x) = 0$ for almost every x. In other words, the sequence $\{s_n(x)\}$ converges for almost every x, and (i) is established.

(ii) *If the series* 10.31(1) *belongs to* L^2, *the partial sums* $s_n(x)$ *of the series are* $o(\sqrt{\log n})$ *for almost every* x.

For if $a_n/\sqrt{\log n} = a'_n$, $b_n/\sqrt{\log n} = b'_n$, $n = 2, 3, \ldots$, then $\Sigma(a_n'^2 + b_n'^2)\log n < \infty$. Hence the series $\Sigma(a'_n \cos nx + b'_n \sin nx)$ converges for almost every x, and it is sufficient to prove the following lemma. *If* $0 < l_2 \leqslant l_3 \leqslant \ldots \to \infty$, *and if the series* $u_2/l_2 + u_3/l_3 + \ldots$ *converges, then* $u_2 + u_3 + \ldots + u_n = o(l_n)$.

Let $s_n = u_2 + \ldots + u_n$, $r_n = u_n/l_n + u_{n+1}/l_{n+1} + \ldots$ Taking m such that $|r_k| < \varepsilon$ for $k > m$, and applying Abel's transformation, we have

$$s_n - s_m = \sum_{m+1}^{n} \frac{u_k}{l_k} l_k = r_{m+1} l_{m+1} + \sum_{m+2}^{n} r_k(l_k - l_{k-1}) - r_{n+1} l_n$$

for $n > m$. The last expression does not exceed $2\varepsilon\, l_n$ in absolute value. Hence $|s_n| \leqslant |s_n - s_m| + |s_m| \leqslant 2\varepsilon\, l_n + |s_m| < 3\varepsilon\, l_n$ if n is large enough. Since ε is arbitrary, the lemma is established.

(iii) *If the series* 10.31(1) *belongs to* L^2, *the series*

$$\sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{\sqrt{\log n}}$$

converges almost everywhere.

Since the sequence $\mu_n = 1/\sqrt{\log n}$, $n = 2, 3, \ldots$ is convex, (iii) follows from (ii) and the lemma established in § 3.71.

10.33. The theorems of the previous sections have been extended by Littlewood and Paley to the case of functions belonging to L^r, $r > 1$. In this case the arguments are more difficult and require new devices. We shall state here, without proof, the most important of the Littlewood-Paley results [1]). Let $s_n(x)$ denote the partial sums of the series 10.31(1), which is the Fourier series of a function $f(x)$; then

(i) *If* $f \in L^r$, $r > 1$, *and if the sequence* $\{n_k\}$ *satisfies an inequality* $n_{k+1}/n_k > \lambda > 1$, $k = 1, 2, \ldots$, *the sequence* $\{s_{n_k}(x)\}$ *converges to* $f(x)$ *for almost every* x; *the function* $\operatorname{Sup}_k |s_{n_k}(x)|$ *belongs to* L^r.

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[1]) See Littlewood and Paley [1]. Detailed proofs have not yet been published, but some indications as to the methods of proofs will be found in Paley [1], where similar results are obtained for the orthogonal system defined in § 1.8.5.

(ii) *If* $f \in L^r$, $1 < r \leqslant 2$, *then, almost always,* $s_n(x) = o\,(\log n)^{1/r}$ *and the series*

$$\sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{(\log n)^{1/r}}$$

converges.

(iii) *If* $\{\varepsilon_k\}$ *is any sequence of numbers of which each has one of the three values* $0, 1, -1$, *and if* $f \in L^r$, $r > 1$, $n_{k+1}/n_k > \lambda > 1$, *the series*

$$\sum_{k=1}^{\infty} \varepsilon_k \sum_{n=n_k+1}^{n_{k+1}} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function $g \in L^r$.

We add a few remarks.

Proposition (i) is false for $r = 1$; more precisely: *for any sequence* $\{\lambda_k\}$ *of positive numbers there is an integrable function* $f(x)$, *and a sequence* $\{n_k\}$ *such that* $n_{k+1}/n_k > \lambda_k$ *and that* $s_{n_k}(x)$ *diverges almost everywhere.* For the proof we refer the reader to Kolmogoroff [7]. Although the result is not stated there explicitely, it is an easy consequence of the argument used.

Theorem (ii) is established for $r \leqslant 2$ only, so that for functions $f \in L^s$, $s > 2$, and in particular for continuous functions, it does not give more than Theorems 10.32(ii), (iii). It is not excluded that proposition (ii) is false for $r > 2$.

The meaning of (iii) will be understood better, if the reader compares this result with the theorems established in § 5.6.

10.4. Summability C of Fourier series. In Chapter II we studied various tests ensuring the convergence of the Fourier series of a function $f(x)$ at a given point. All those tests represent sufficient conditions only, and the problem of finding a *necessary and sufficient* condition (which would not be a more or less disguised tautology) remains unsolved. The situation is the same when, instead of ordinary convergence, we consider summability by an assigned Cesàro mean, e. g. summability $(C, 1)$: Fejér's fundamental theorem (§ 3.21) gives a sufficient condition only. We therefore change the problem and ask not when $\mathfrak{S}[f]$ is summable by *some particular* mean, but when it is summable by *some mean or another*, i. e. when it is *summable C*. In this form the problem was first stated by Hardy and Littlewood, who also gave a complete solution. This solution has been precised

at certain points by a number of writers, in particular by Bosanquet. A new approach to the problem was found by Plessner.

We begin by proving a number of auxiliary theorems which are interesting and important in themselves.

10.41. Suppose that $f(x)$ is defined in the neighbourhood of a point x_0 and that, for small values of $|t|$,

$$f(x_0 + t) = \alpha_0 + \frac{1}{1!}\alpha_1 t + \frac{1}{2!}\alpha_2 t^2 + \ldots + \frac{1}{(r-1)!}\alpha_{r-1} t^{r-1} + \frac{1}{r!}(\alpha_r + \varepsilon_t) t^r,$$

where the α's are constants and $\varepsilon_t = \varepsilon_{r,t} \to 0$ with t. The number α_s, $1 \leqslant s \leqslant r$, will then be called the s-th *generalized derivative* of f at the point x_0. It is plain that, if $f^{(s)}(x_0)$, $s = 1, 2, \ldots$, exists and is finite, then the s-th generalized derivative α_s exists and is equal to $f^{(s)}(x_0)$. For applications to the theory of trigonometrical series it is convenient to modify this definition and to consider the cases of even and odd suffixes separately. Let $\varphi_{x_0}(t) = \frac{1}{2}[f(x_0+t) + f(x_0-t)]$, $\psi_{x_0}(t) = \frac{1}{2}[f(x_0+t) - f(x_0-t)]$. If either

$$\varphi_{x_0}(t) = \beta_0 + \frac{\beta_2}{2!}t^2 + \ldots + \frac{\beta_{2k-2}}{(2k-2)!}t^{2k-2} + (\beta_{2k} + \varepsilon_t)\frac{t^{2k}}{(2k)!}, \quad \text{or}$$

$$\psi_{x_0}(t) = \beta_1 t + \frac{\beta_3}{3!}t^3 + \ldots + \frac{\beta_{2k-1}}{(2k-1)!}t^{2k-1} + (\beta_{2k+1} + \varepsilon_t)\frac{t^{2k+1}}{(2k+1)!},$$

where $\varepsilon_t \to 0$ as $t \to 0$, and the β's are constants, then β_j will be called the j-th *generalized symmetric derivative* of $f(x)$ at the point x_0, and will be denoted by $f_{(j)}(x_0)$ [1]. The existence of $f_{(j)}(x_0)$ involves that of $f_{(j-2)}(x_0)$. The following theorem is a generalization of Theorem 3.5.

If $f_{(r)}(x_0)$ *exists, the Fourier series of* $f(x)$, *differentiated term by term* r *times, is, at the point* x_0, *summable* (C, α), $\alpha > r$, *to the value* $f_{(r)}(x_0)$ [2].

We observe that, given $2s+1$ numbers $\xi_0, \xi_1, \ldots, \xi_{2s}$, there is a trigonometrical polynomial $T(x)$ of order $\leqslant s$, such that $T^{(j)}(x_0) = \xi_j$, $0 \leqslant j \leqslant 2s$. This is easily seen when we represent $T(x)$ in the complex form and write equations for the coefficients. Since the

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[1]) The generalized derivatives were first introduced by de la Vallée-Poussin [4].

[2]) de la Vallée-Poussin [4], Gronwall [3], Young [4], Zygmund [15].

theorem is obvious in the case of trigonometrical polynomials, we may, by substracting a polynomial $T(x)$ from $f(x)$, suppose that $f_{(r)}(x_0)=f_{(r-2)}(x_0)=\ldots=0$. If $K_n^\alpha(t)$ denotes the (C,α) kernel, and $\sigma_n^\alpha(x)$ are the (C,α) means of $\mathfrak{S}[f]$, the (C,α) means of $\mathfrak{S}^{(r)}[f]$ are equal to $\{\sigma_n^\alpha(x)\}^{(r)}$, i. e. to

$$\frac{(-1)^r}{\pi}\int_{-\pi}^{\pi} f(t)\frac{d^r}{dt^r}K_n^\alpha(x-t)\,dt=\frac{2(-1)^r}{\pi}\int_0^\pi \tfrac12[f(x+t)+(-1)^r f(x-t)]\,\frac{d^r}{dt^r}K_n^\alpha(t)\,dt.$$

In what follows, $C, C_1, C_2, \ldots$ will denote positive constants independent of the variables t and n. The proof of the theorem is an easy consequence of the following lemma:

If $0\leqslant r<\alpha$, *then* (i) $\int_0^\pi t^r\left|\frac{d^r}{dt^r}K_n^\alpha(t)\right|dt\leqslant C$, *and* (ii) *the expression* $\frac{d^r}{dt^r}K_n^\alpha(t)$ *tends uniformly to* 0 *in any interval* $0<\eta\leqslant t\leqslant\pi$.

Let us take this lemma for granted for the moment, and let $\delta>0$ be an arbitrary number. If $f_{(r)}(x_0)=f_{(r-2)}(x_0)=\ldots=0$, then, since $2/\pi r!<1$, the expression $|\{\sigma_n^\alpha(t)\}^{(r)}|$ does not exceed

$$\int_0^\pi|\varepsilon_t|\,t^r\left|\frac{d^r}{dt^r}K_n^\alpha(t)\right|dt=\int_0^\eta+\int_\eta^\pi=A+B,$$

where η is so chosen that $|\varepsilon_t|\leqslant\delta/2C$ for $0<t\leqslant\eta$. Then $|A|\leqslant C\cdot\delta/2C=\frac12\delta$, and since, in view of (ii), $B\to0$ as $n\to\infty$, we obtain that $|\{\sigma_n^\alpha(x_0)\}^{(r)}|<\delta$ for $n\geqslant n_0$. Hence $\{\sigma_n^\alpha(x_0)\}^{(r)}\to0$ and the theorem is established.

Let $u(\beta,n,t)=\sum_{\nu=0}^{n}A_\nu^\beta e^{i\nu t}$. Abel's transformation shows that $u(\beta,n,t)=[-A_n^\beta e^{i(n+1)t}+u(\beta-1,n,t)]/(1-e^{it})$, and so

(1) $$u(\beta,n,t)=-e^{i(n+1)t}\sum_{j=1}^{s}\frac{A_n^{\beta-j+1}}{(1-e^{it})^j}+\frac{u(\beta-s,n,t)}{(1-e^{it})^s}.$$

To prove the lemma we use, besides (1), the relations 3.3(3) and the first formula in 3.11(1). Then

$$K_n^\alpha(t)=\frac{1}{2A_n^\alpha\sin\frac12 t}\,\Im\,[e^{i(n+\frac12)t}u(\alpha-1,n,-t)]=$$

$$=\frac{1}{2A_n^\alpha\sin\frac12 t}\,\Im\left[-e^{-\frac12 t}\sum_{j=1}^{s}\frac{A_n^{\alpha-j}}{(1-e^{-it})^j}+\frac{u(\alpha-s-1,n,-t)}{(1-e^{-it})^s}\,e^{i(n+\frac12)t}\right]=$$

$$=\frac{1}{A_n^\alpha}\,\Im\left[-\frac{e^{-\frac12 t}}{2\sin\frac12 t}\sum_{j=1}^{s}\frac{A_n^{\alpha-j}}{(1-e^{-it})^j}+\frac{e^{i(n+\frac12)t}}{2\sin\frac12 t\cdot(1-e^{-it})^\alpha}-\frac{\sum_{\nu=n+1}^{\infty}A_\nu^{\alpha-s-1}\,e^{-i(\nu-n-\frac12)t}}{2\sin\frac12 t\cdot(1-e^{-it})^s}\right],$$

provided that the last series converges. So far the value of s has not been defined. Now we take s so large that the last series differentiated r times is still absolutely convergent. It is sufficient to suppose that $s>\alpha+r$. Since $A_n^\gamma=O(n^\gamma)$, and since each of the expressions

$$\left|\frac{d^h}{dt^h}\left(\frac{1}{2\sin\frac12 t\cdot(1-e^{-it})^\gamma}\right)\right|,\qquad\left|\frac{d^h}{dt^h}\left(\frac{e^{-\frac12 t}}{2\sin\frac12 t\cdot(1-e^{-it})^\gamma}\right)\right|$$

$(\gamma\geqslant0)$ is less than $C_1/t^{\gamma+h+1}$, we obtain that $A_n^\alpha|\{K_n^\alpha(t)\}^{(r)}|$ is less than the sum of three expressions

$$C_2\sum_{j=1}^{s}\frac{n^{\alpha-j}}{t^{j+1+r}},\qquad C_3\sum_{\mu=0}^{r}\frac{n^\mu}{t^{\alpha+1+r-\mu}},\qquad C_4\sum_{\mu=0}^{r}\frac{n^{\alpha-s+\mu}}{t^{s+r-\mu+1}},$$

and the second part of the lemma follows at once. If $t\geqslant1/n$, the second sum is $<C_5n^r/t^{\alpha+1}$, and the third is $<C_6n^{\alpha-s+r}/t^{s+1}$. Hence

$$\int_{1/n}^{\pi}t^r\left|\frac{d^r}{dt^r}K_n^\alpha(t)\right|dt\leqslant C_7\int_{1/n}^{\pi}\left[\frac{n^{r-\alpha}}{t^{\alpha-r+1}}+\frac{n^{-s+r}}{t^{s-r+1}}+\sum_{j=1}^{s}\frac{n^{-j}}{t^{j+1}}\right]dt<C_8.$$

On the other hand, from the formula

$$K_n^\alpha(t)=\{\tfrac12A_n^\alpha+A_{n-1}^\alpha\cos t+\ldots\}/A_n^\alpha$$

we easily deduce that $|\{K_n^\alpha(t)\}^{(r)}|$ does not exceed the expression $n^rA_n^{\alpha+1}/A_n^\alpha<C_9n^{r+1}$. It follows that $\int_0^{1/n}t^r|\{K_n^\alpha(t)\}^{(r)}|\,dt<C_9$, and we obtain the first part of the lemma with $C=C_8+C_9$.

10.42. *Let the series*

$$(1)\qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

be summable (C, α), $\alpha = 0, 1, 2, \dots$, *for* $x = x_0$, *to sum* s. *Let* r *be an integer* $> \alpha + 1$, *and suppose that the series* (1) *integrated term by term* r *times converges, in the neighbourhood of* x_0, *to a function* $F(x)$. *Then* $F_{(r)}(x_0)$ *exists and is equal to* s [1]).

To fix ideas we suppose that r is even; for r odd the proof would be similar. Increasing α, if necessary, we may suppose that either $r = \alpha + 2$, or $r = \alpha + 3$. We have [2])

$$(2)\qquad F(x) = \frac{a_0 x^r}{2r!} + (-1)^{r/2}\sum_{n=1}^{\infty}\frac{a_n \cos nx + b_n \sin nx}{n^r}.$$

Without loss of generality we may assume that $x_0 = 0$, $s = 0$, $a_0 = 0$. We may also assume that (1) is a purely cosine series; for the sine component of (2) is an odd function of x, and so its r-th symmetric derivative at the point 0 is equal to 0. Let us put $\gamma(u) = (\cos u)/u^r$, $s_n = s_n^0 = a_1 + a_2 + \dots + a_n, \dots$, $s_n^k = s_1^{k-1} + \dots + s_n^{k-1}, \dots$ Since $s_n^\alpha = o(n^\alpha)$, and so also $s_n^{\alpha-1} = o(n^\alpha)$, $s_n^{\alpha-2} = o(n^\alpha), \dots$, Abel's transformation applied $(\alpha + 1)$ times gives

$$F(t) = (-1)^{r/2} t^r \sum_{n=1}^{\infty} a_n \gamma(nt) = (-1)^{r/2} t^r \sum_{n=1}^{\infty} s_n^\alpha \Delta^{\alpha+1}\gamma(nt).$$

where the $(\alpha+1)$-st difference $\Delta^{\alpha+1}$ is defined by the following conditions: for any sequence $\{u_m\}$ we write $\Delta u_n = \Delta^1 u_n = u_n - u_{n+1}$, $\Delta^j u_n = \Delta(\Delta^{j-1} u_n)$. It is well known that, if $u(x)$ is a function differentiable j times, x_0 and $h > 0$ are fixed numbers, and $u_n = u(x_0 + nh)$, $n = 1, 2, \dots$, then

$$(3)\qquad \Delta^j u_n = (-1)^j h^j u^{(j)}(x_0 + nh + \theta jh), \qquad 0 < \theta < 1\ ^{3)}.$$

Let
$$P(x) = \sum_{\nu=0}^{\frac{1}{2}r-1}(-1)^\nu \frac{x^{2\nu}}{(2\nu)!}, \qquad \lambda(x) = \frac{\cos x - P(x)}{x^r}.$$

[1]) See Plessner, *Trigonometrische Reihen*, p. 1381. This result is a generalization of a theorem of Riemann (§ 11.21). The series (2) is certainly convergent if, for example, $|a_n| + |b_n| = o(n^\alpha)$.

[2]) Into the right-hand side of (2) we might introduce an arbitrary polynomial of order $(r-1)$; this would not affect the result.

[3]) The proof of (3) will be found in many treatises of Analysis. See e. g. de la Vallée-Poussin, *Cours d'Analyse*, 1, p. 72.

Then $\gamma(nt) = \lambda(nt) + P(nt)/(nt)^r$, and so

$$F(t) = \sum_{\nu=0}^{\frac{1}{2}r-1}\frac{A_\nu}{(2\nu)!}t^{2\nu} + t^r R(t),$$

where $A_\nu = (-1)^{\frac{1}{2}r+\nu}\sum s_n^\alpha \Delta^{\alpha+1} n^{2\nu-r}$, $R(t) = (-1)^{r/2}\sum s_n^\alpha \Delta^{\alpha+1}\lambda(nt)$. Since, in view of (3), $\Delta^{\alpha+1} n^{2\nu-r} = O(n^{2\nu-r-\alpha-1}) = O(n^{-\alpha-3})$, the series defining the numbers A_ν converge absolutely; it follows that the series defining $R(t)$ is also absolutely convergent. The theorem will have been established when we have shown that $R(t) = o(1)$ as $t \to 0$. Let $N = [1/t]$, $0 < t \leqslant 1$. Then

$$|R(t)| = \sum_{n=1}^{\infty}|s_n^\alpha \Delta^{\alpha+1}\lambda(nt)| = \sum_{n=1}^{N} + \sum_{n=N+1}^{\infty} = U + V.$$

The function $\lambda(u)$ is regular in the whole plane, and so, on account of (3), $|\Delta^{\alpha+1}\lambda(nt)| \leqslant Ct^{\alpha+1}$ for $n \leqslant N$, where $C, C_1, \dots$ denote constants independent of n and t. It follows that U does not exceed $Ct^{\alpha+1}\sum_{n=1}^{N}|s_n^\alpha| = Ct^{\alpha+1}\cdot o(N^{\alpha+1}) = o(1)$ as $t \to 0$. On the other hand, an easy calculation shows that $|\gamma^{(\alpha+1)}(u)| \leqslant C_1 u^{-r}$, and so $|\lambda^{(\alpha+1)}(u)| \leqslant C_2 u^{-r}$, for $u \geqslant 1$. Using (3) again, we therefore obtain

$$V \leqslant C_2 t^{\alpha-r+1}\sum_{n=N+1}^{\infty}|s_n^\alpha| n^{-r} = C_2 t^{\alpha-r+1}\sum_{n=N+1}^{\infty} o(n^{\alpha-r}) = C_2 t^{\alpha-r+1}\cdot o(N^{r-\alpha-1}).$$

Hence $V = o(1)$, $U + V = o(1)$, and the theorem follows.

10.43. An immediate corollary of Theorems 10.41 and 10.42 is:

(i) *Suppose that the series* 10.42(1) *has coefficients* $O(n^k)$ *for some* k. *A necessary and sufficient condition that the series should be summable* C *for* $x = x_0$, *to sum* s, *is that there should exist an integer* $r > 0$ *such that, if* $F(x)$ *is the function obtained by integrating* 11.42(1) *term by term* r *times, then* $F_{(r)}(x_0)$ *exists and is equal to* s [1]).

When 10.42(1) is a Fourier series, the above result may be stated in a different form.

Given a function $\varphi(t)$, defined to the right of $t = 0$, we shall say that the number s is the (C, r) limit of $\varphi(t)$ as $t \to 0$, if

$$(1)\qquad \frac{r}{t^r}\int_0^t \varphi(u)(t-u)^{r-1}\,du \to s \quad \text{as} \quad t \to 0 \quad (r > 0).$$

[1]) Plessner, loc. cit.

A more detailed discussion of this notion will be found in § 12.3. The relation (1) will be written $(C, r)\ \varphi(t) \to s$. If $(C, \alpha)\ \varphi(t) \to s$ for some α, we shall write $(C)\ \varphi(t) \to s$.

(ii) *A necessary and sufficient condition that the Fourier series of a function $f(x)$ should be summable C, for $x = x_0$, to the sum $f(x_0)$, is $(C)\ \varphi_{x_0}(t) \to f(x_0)$, where $\varphi_{x_0}(t) = \frac{1}{2}[f(x_0+t)+f(x_0-t)]$*[1].

Let 10.42(1) be $\mathfrak{S}[f]$. Since $\mathfrak{S}[f]$ at the point $x = x_0$ is the same thing as $\mathfrak{S}[\varphi_{x_0}(t)]$ at the point $t = 0$, we may assume that $x_0 = 0$ and that $f(t)$ is an even function of t; we also assume that $f(0) = 0$. Fourier series may be integrated term by term, and so, if $F(x)$ is the result of integrating $\mathfrak{S}[f]$ r times, we have an equation

$$(2) \qquad F(t) + P(t) = \frac{1}{(r-1)!}\int_0^t \varphi(u)\,(t-u)^{r-1}\,du,$$

where $P(t)$ is a polynomial of order $\leqslant r-1$, and $\varphi(u) = \varphi_{x_0}(u) = f(u)$. From this we see that, if $(C, r)\ \varphi(t) \to 0$ as $t \to 0$, then $F_{(r)}(0)$ exists and is equal to 0. Conversely, if $F_{(r)}(0)$ exists and is equal to 0, then $F(t) = o(t^r)$ + a polynomial of order $r-2$; since the right-hand side of (2) is, in any case, $o(t^{r-1})$, it must be $o(t^r)$, i. e. $(C, r)\ \varphi_0(t) \to 0$. To complete the proof of (ii), we apply (i).

Proposition (ii) may be precised as follows.

(iii) *If $(C, \alpha)\ \varphi_{x_0}(t) \to f(x_0)$ as $t \to 0$, then $\mathfrak{S}[f]$ is summable (C, β), for $x = x_0$, to the value $f(x_0)$, where $\beta > \alpha \geqslant 0$.*

(iv) *If $\mathfrak{S}[f]$ is summable (C, β) to the sum $f(x_0)$, for $x = x_0$, then $(C, \alpha)\ \varphi_{x_0}(t) \to f(x_0)$ as $t \to 0$, where $\beta > -1$, $\alpha > \beta + 1$.*

For the proofs we refer the reader to Bosanquet [1], where also a further bibliography will be found. Here we intend to apply proposition (ii) to obtain an important result due to Hardy and Littlewood. For the proof we require the following theorem:

10.44. *If Σu_n is finite (C, α) and summable (C, β), $\beta > \alpha > -1$, then it is summable $(C, \alpha+\delta)$ for any $\delta > 0$*[2].

We may suppose that $\beta = \alpha + 1$, $0 < \delta < 1$, for the general result can be obtained by repeated application of this special case. Assuming, as we may, that the sum of Σu_n is 0, we have to prove that, with the notation of § 3.11, $s_n^{\alpha+\delta}/A_n^{\alpha+\delta} \to 0$. Now

[1]) Hardy and Littlewood [7].

[2]) Andersen [1].

$$s_n^{\alpha+\delta} = \sum_{k=0}^{n} A_{n-k}^{\delta-1} s_k^{\alpha} = \sum_{k=0}^{[n\vartheta]} + \sum_{k=[n\vartheta]+1}^{n} = P_n + Q_n \qquad (1/2 < \vartheta < 1).$$

Observing that $|s_k^\alpha| < C_1 k^\alpha$, where $C_1, C_2, \ldots$ denote constants, we have

$$|Q_n| \leqslant C_1 n^\alpha \sum_{k=[n\vartheta]+1}^{n} A_{n-k}^{\delta-1} = C_1 n^\alpha A_{n-[n\vartheta]-1}^{\delta} \simeq C_2 n^{\alpha+\delta}(1-\vartheta)^{\delta}.$$

(Since $\vartheta > \frac{1}{2}$, the first inequality is true for $\alpha < 0$ also). Hence, if ϑ is sufficiently near to 1, we have $|Q_n|/A_n^{\alpha+\delta} < \frac{1}{2}\varepsilon$, where ε is arbitrarily given and $n > n_0$. Having fixed ϑ, we shall prove that $P_n = o(n^{\alpha+\delta})$; for, making Abel's transformation,

$$|P_n| \leqslant \Big|\sum_{k=0}^{[n\vartheta]} A_{n-k}^{\delta-2} s_k^{\alpha+1}\Big| + o(n^{\alpha+\delta}) < C_3[n(1-\vartheta)]^{\delta-2} \sum_{k=0}^{[n\vartheta]} o(k^{\alpha+1}) + o(n^{\alpha+\delta}) =$$
$$= C_3[n(1-\vartheta)]^{\delta-2} o(n^{\alpha+2}) = o(n^{\alpha+\delta}) < \tfrac{1}{2}\varepsilon A_n^{\alpha+\delta}$$

for $n > n_1$. Hence $|s_n^{\alpha+\delta}/A_n^{\alpha+\delta}| < \varepsilon$ for $n > \mathrm{Max}(n_0, n_1)$, and the theorem is established.

10.45 [1]. (i) *If f is non-negative and if $\mathfrak{S}[f]$ is summable C at a point x, then $\mathfrak{S}[f]$ is summable (C, ε) at that point, for every positive ε.*

(ii) *If $f \geqslant 0$, a necessary and sufficient condition that $\mathfrak{S}[f]$ should be summable C at a point x, to $f(x)$, is that $(C, 1)\ \varphi_x(t) \to f(x)$.*

Under the hypothesis of (i), we have 10.43(1), with $\varphi(u) = \varphi_x(u)$, for some $r > 0$. Since $\varphi_x(u) \geqslant 0$, the left-hand side of 10.43(1) is not less than

$$\frac{r}{t^r}\int_0^{t/2} \varphi_x(u)\,(t-u)^{r-1}\,du \geqslant \frac{r\,2^{1-r}}{t}\int_0^{t/2} \varphi_x(u)\,du, \text{ i. e. } \frac{1}{t}\int_0^t \varphi_x(u)\,du = O(1).$$

Let $\xi_x(t) = \frac{1}{t}\int_0^t |f(x+t)+f(x-t)-2f(x)|\,dt$. In § 3.3 we proved that, at any point x where $\xi_x(t) = o(1)$, $\mathfrak{S}[f]$ is summable (C, α), $\alpha > 0$, to the sum $f(x)$. Exactly the same argument shows that, if $\xi_x(t) = O(1)$, then $\mathfrak{S}[f]$ is finite (C, α) at the point x (it must be remembered that $\varphi_x(t)$ has a slightly different meaning in § 3.3, viz. $f(x+t)+f(x-t)-2f(x)$). Since the conditions $\varphi_x(t) \geqslant 0$ and $\frac{1}{t}\int_0^t \varphi_x(u)\,du = O(1)$ imply $\xi_x(t) = O(1)$, $\mathfrak{S}[f]$ is, in our case, finite

[1]) Hardy and Littlewood [5].

(C, α) and so, in view of Theorem 10.44, summable $(C, \alpha+\delta)$ for every $\alpha > 0$ and $\delta > 0$. Putting $\alpha+\delta=\varepsilon$, we obtain (i).

We write $\varphi_x(t)=\varphi(t)=\Phi_0(t)$, and denote by $\Phi_k(t)$, $k=1,2,\dots$, the integral of $\Phi_{k-1}(u)$ over $0 \leqslant u \leqslant t$. The relation 10.43(1), with $s=f(x)$, may be written $\Phi_r(t) \simeq f(x)\, t^r/r!$, and to prove (ii) we have to show that $\Phi_1(t) \simeq f(x)\, t$. Since $\Phi_k(t)$, $k=1,2,\dots$, is a non-decreasing function of t, proposition (ii) follows by repeated application of the following lemma:

Let $s(t)$, $t \geqslant 0$, be an everywhere differentiable function of t. If $s'(t)$ is non-decreasing and if $s(t) \simeq st^\alpha$ as $t \to 0$, then $s'(t) \simeq s\alpha t^{\alpha-1}$.

Let $0 < \theta < 1$ be a fixed number; by the mean-value theorem,

$$(1) \qquad (1-\theta)\, t\, s'(\theta t) < s(t) - s(\theta t) < (1-\theta)\, t\, s'(t).$$

Since $s(t)-s(\theta t) \simeq s \cdot (1-\theta^\alpha)\, t^\alpha$, we obtain from (1)

$$\varliminf_{t\to 0} \frac{s'(t)}{t^{\alpha-1}} \geqslant s\,\frac{1-\theta^\alpha}{1-\theta},$$

$$\varlimsup_{t\to 0} \frac{s'(\theta t)}{(\theta t)^{\alpha-1}} \leqslant s\,\frac{1-\theta^\alpha}{(1-\theta)\,\theta^\alpha}, \text{ i. e. } \varlimsup_{t\to 0} \frac{s'(t)}{t^{\alpha-1}} \leqslant s\,\frac{1-\theta^\alpha}{(1-\theta)\,\theta^{\alpha-1}}.$$

Since θ may be taken as near to 1 as we please, we obtain $\varliminf s'(t)/t^{\alpha-1} \geqslant s\alpha$, $\varlimsup s'(t)/t^{\alpha-1} \leqslant s\alpha$, i. e. $s'(t) \simeq s\alpha t^{\alpha-1}$.

It is plain that (i) and (ii) hold when f is bounded below, and so, in particular, when f is bounded.

10.46. Miscellaneous theorems and examples.

1. If $f \in L^r$, $r > 1$, and if $s_n(x)$ are the partial sums of $\mathfrak{S}[f]$, then

$$\frac{1}{n+1} \sum_{\nu=0}^{n} e^{\lambda |f(x)-s_\nu(x)|} \to 1$$

for every $\lambda > 0$ and almost every x. In particular, $s_n(x) = o(\log n)$ for almost every x. Carleman [2].

[Use the equation $e^{\lambda u} = 1 + \lambda u + \dots$ and argue as in § 10.1].

2. If $f \in L^2$ and $s_n(x)$ are the partial sums of $\mathfrak{S}[f]$, then, for almost every x, the sequence $1, 2, 3, \dots$ can be broken up into two complementary sequences $\{m_k\}$ and $\{n_k\}$ (depending in general on x) such that $s_{m_k}(x) \to f(x)$, $\Sigma\, 1/n_k < \infty$.

[Use the lemma of § 10.11].

3. A series Σu_n is said to be *absolutely summable A*, if the function $g(r) = \Sigma u_n r^n$ is of bounded variation over $0 \leqslant r < 1$. Show that, if $\Sigma |u_n| < \infty$ then Σu_n is absolutely summable A.

4. $\mathfrak{S}[f]$ is absolutely summable A for $x = x_0$ provided that either (i) f satisfies Dini's test (§ 2.4) at the point x_0, or (ii) $f(x)$ is of bounded variation in the neighbourhood of x_0. See Whittaker [1], Prasad [2].

5. Let $s_n(x)$ and $\bar{s}_n(x)$ denote the partial sums of $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ respectively, and suppose that there is a function $g(x) \geqslant 0$, $g \in L$, such that $s_n(x) \geqslant -g(x)$, $n=0,1,2,\dots$ Then (i) there is a function $h(x)$ belonging to $L^{1-\varepsilon}$ for every $\varepsilon > 0$ and such that $s_n(x) \leqslant h(x)$, $|\bar{s}_n(x)| \leqslant h(x)$. Moreover, (ii) if $f \in L^r$, $g \in L^r$, $r > 1$, then $h \in L^r$, (iii) if $f \log^+ |f| \in L$, $g \log^+ |g| \in L$, then $h \in L$.

For this and the following theorem see Paley and Zygmund [2].

6. (i) If $|f| \leqslant 1$, and $s_n(x) > -A$, $0 \leqslant x \leqslant 2\pi$, $n=0,1,\dots$, where A is a constant, then there is a constant $B = B(A)$ such that $s_n(x) \leqslant B$. (ii) If $f(x)$ is continuous and, for any $\varepsilon > 0$, we have $s_n(x) > f(x) - \varepsilon$, $n \geqslant n(\varepsilon)$, $0 \leqslant x \leqslant 2\pi$, then $\mathfrak{S}[f]$ converges uniformly to $f(x)$.

7. Let $\{a_n\}$ be a positive decreasing sequence such that $\{na_n\}$ is monotonic and $\Sigma a_n/n < \infty$; let $s_n(x)$ and $t_n(x)$ denote the partial sums of the series $\Sigma a_n \cos nx$ and $\Sigma a_n \sin nx$ respectively; then the functions $s(x) = \operatorname*{Sup}_n |s_n(x)|$ and $t(x) = \operatorname*{Sup}_n |t_n(x)|$ are both integrable.

8. If a_n and b_n, $n=1,2,\dots$, are the Fourier coefficients of an integrable function, the partial sums of the series

$$\sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{(\log n)^{1+\delta}}, \qquad \sum_{n=2}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{(\log n)^{1+\delta}} \qquad (\delta > 0)$$

can be majorized by integrable functions. For $\delta = 0$ this is no longer true.

9. If $a_k \geqslant 0$, $k = 1, 2, \dots$, and if the series $\Sigma a_k \sin kx$ is the Fourier series of a bounded function $f(x)$, the partial sums of the series are uniformly bounded; if f is continuous, the series converges uniformly. Paley [7].

[Let $\sigma_n(x)$ be the first arithmetic means of the series considered. To prove the first part of the theorem, observe that, if $|f(x)| \leqslant M$, then $|\sigma_{2n}(x)| \leqslant M$, $|\sigma'_{2n}(x)| \leqslant 4nM$ (§ 7.31), and so, taking $x = 0$,

$$\sum_{k=1}^{2n} \left(1 - \frac{k}{2n+1}\right) k a_k \leqslant 4Mn.$$

Taking the first n terms on the left, we obtain $a_1 + 2a_2 + \dots + na_n \leqslant 8Mn$, and it is sufficient to apply 3.13(1)].

10. Theorem 10.42 holds for α fractional and > -1.

[For $-1 < \alpha < 0$, $r = 1$, the theorem was established by Hardy and Littlewood [1]. The general result can be obtained by combining the Hardy-Littlewood argument with that of § 10.42].

11. The results concerning summability C holds, *mutatis mutandis*, for Fourier-Stieltjes series; in particular, if $F(x)$ is non-decreasing, summability C involves summability (C, ε) for any $\varepsilon > 0$; a necessary and sufficient condition that $\mathfrak{S}[dF]$ should be summable C for $x = x_0$, is that $F_{(1)}(x_0)$ should exist.

12. Power series on the circle of convergence may be considered as trigonometrical series, so that Theorem 10.43(i) remains true for power series

$$(1) \qquad \sum_{n=0}^{\infty} \alpha_n e^{inx}.$$

It may however then be stated in a slightly different form, viz. it holds if by $F_{(r)}(x_0)$ we mean the r-th *unsymmetric* generalized derivative defined at the beginning of § 10.41. Plessner. *Trigonometrische Reihen*, p. 1382; see also Hardy and Littlewood [7].

[Theorem 10.42 holds if $\alpha > -1$, $r > \alpha + 1$, and $F_{(r)}(x)$ is the r-th unsymmetric generalized derivative, provided that 10.42(1) is of the form (1)].

13. If 10.42(1) is the Fourier series of a bounded function $f(x)$, the conjugate series is summable C if and only if it is summable (C, ε) for every $\varepsilon > 0$. A necessary and sufficient condition that $\bar{\mathfrak{S}}[f]$ should be summable C for $x = x_0$, is the existence of the integral

$$-\frac{1}{\pi}\int_0^{\pi} \frac{f(x_0+t)-f(x_0-t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt,$$

which represents, then, the sum of $\bar{\mathfrak{S}}[f]$ for $x = x_0$. Prasad [3], Hardy and Littlewood [19].

[To prove the first part of the theorem, we show that the difference 3.32(1) is bounded for every $r > 0$ (that it is bounded for $0 < r \leqslant 1$, was implicitely proved in 3.32). For then $\bar{\sigma}_n^r(x_0) - \bar{\sigma}_n^s(x_0) = O(1)$ for every $r > 0$ and $s > 0$, and it is sufficient to apply Theorem 10.44. For the second part of the theorem we refer the reader to the papers quoted [1])].

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[1]) A theory of summability C of the series conjugate to general trigonometrical series will be found in Plessner, loc. cit.