CHAPTER IX.

Further theorems on Fourier coefficients. Integration of fractional order.

9.1. Remarks on the theorems of Hausdorff-Young and F. Riesz. It has been proved in Chapter IV that, for any complex function $f(t)$ with Fourier coefficients c_n, we have

$$(1) \qquad \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt = \sum_{n=-\infty}^{+\infty} |c_n|^2.$$

This formula contains two propositions: (i) If $f \in L^2$, the series on the right converges to the sum equal to the integral on the left (*Parseval*), (ii) If c_n is an arbitrary sequence such that $\Sigma |c_n|^2$ converges, there is an $f \in L^2$ with complex Fourier coefficients c_n satisfying (1) *(Riesz-Fischer)*. It is natural to inquire how far these results can be extended to exponents other than 2. It appears that such extensions are possible, but only partly. Here we shall only state the results and make a few remarks about them. Complete proofs will be given in § 9.3.

Given any function $f(t)$, $0 \leqslant t \leqslant 2\pi$, and any sequence $\{c_n\}$, $-\infty < n < +\infty$, we write

$$\mathfrak{M}_r[f] = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^r\,dt\right\}^{1/r}, \qquad \mathfrak{N}_r[c] = \left\{\sum_{n=-\infty}^{+\infty} |c_n|^r\right\}^{1/r}.$$

We assume that f and c_n may take complex values. Using the letters p and q, we shall suppose, unless a statement to the contrary is made, that $1 < p \leqslant 2 \leqslant q < \infty$. For any $r \geqslant 1$ we define r' by the condition $1/r + 1/r' = 1$, so that p' is a q, q' is a p.

The following theorem is due to Hausdorff and Young[1]).

(a) *If* $f \in L^p$ *and* c_n *are the Fourier coefficients of* f, *then* $\mathfrak{N}_{p'}[c]$ *is finite and* $\mathfrak{N}_{p'}[c] \leqslant \mathfrak{A}_p[f]$.

(b) *Given any sequence of numbers* c_n, $-\infty < n < +\infty$, *such that* $\mathfrak{N}_p[c] < \infty$, *there is a function* $f \in L^{p'}$ *with Fourier coefficients* c_n *and such that* $\mathfrak{A}_{p'}[f] \leqslant \mathfrak{N}_p[c]$.

Theorem (a) is an extension of Parseval's theorem, the sign $=$ being replaced by $\leqslant$; Theorem (b) is an extension of the Riesz-Fischer theorem. In both (a) and (b) the argument goes from p to p', i. e. from the smaller number to the larger. The theorem would be false if we replaced p by q. For (i) there is a continuous function f such that $\mathfrak{N}_p[c] = \infty$ for every $p < 2$ (§§ 5.33, 5.61), (ii) there exist trigonometrical series which are not Fourier series and have coefficients c_n such that $\mathfrak{N}_q[c] < \infty$ for every $q > 2$; the series $\Sigma\, n^{-1/2} \cos 2^n x$ is an instance in point (§ 5.4). Roughly speaking, the theorems of Parseval and of Riesz-Fischer are the best: we can neither strengthen the thesis of the former, nor weaken the hypothesis of the latter.

The reader will observe that between the two parts of the Hausdorff-Young theorem there is a certain analogy. The second part may be obtained from the first if the function f, depending on the variable t, is replaced by the function c depending on the variable n, integration is replaced by summation and vice versa. This fact is explained by the theory of Fourier integrals, where both parts of the theorem corresponding to that of Hausdorff-Young coincide. The analogy just stated can be detected in various theorems of the theory of Fourier series and is an important guide in the search of new results. We shall not investigate this subject systematically.

9.11. The Hausdorff-Young theorem can be extended to general systems of complex functions $\varphi_1, \varphi_2, \ldots$ which are orthogonal, normal, and uniformly bounded ($|\varphi_n| \leqslant M$, $n = 1, 2, \ldots$) in an interval (a, b). Let us consider an arbitrary function $f(t)$, $a \leqslant t \leqslant b$, and an arbitrary sequence of numbers $c_1, c_2, \ldots$, and put

$$\mathfrak{M}_r[f] = \mathfrak{M}_r[f;\, a, b] = \left(\int_a^b |f|^r\, dt\right)^{1/r}, \quad \mathfrak{N}_r[c] = \left(\sum_{n=1}^{\infty} |c_n|^r\right)^{1/r}.$$

[1]) Young [12], [13] proved the theorem in the case $p' = 2k$, $k = 1, 2, ..$ The general result is due to Hausdorff [2].

F. Riesz's extension of the Hausdorff-Young theorem may be stated as follows[1]).

(a) *If* $f \in L^p(a, b)$ *and if* c_n *are the Fourier coefficients of* f *with respect to* $\{\varphi_n\}$, *then* $\mathfrak{N}_{p'}[c]$ *is finite and* $\mathfrak{N}_{p'}[c] \leqslant M^{(2-p)/p}\mathfrak{M}_p[f]$.

(b) *If, for a given* $\{c_n\}$, *we have* $\mathfrak{N}_p[c] < \infty$, *there is an* $f \in L^{p'}(a, b)$ *whose Fourier coefficient with respect to* φ_n *is* c_n, $n = 1, 2, \ldots$, *and such that* $\mathfrak{M}_{p'}[f] \leqslant M^{(2-p)/p}\mathfrak{N}_p[c]$.

Applying this theorem to the system of functions e^{ikx}, $k = 0, \pm 1, \ldots$, $0 \leqslant x \leqslant 2\pi$, we obtain the Hausdorff-Young theorem.

9.12. The Hausdorff-Young theorem will be established, as a corollary of F. Riesz's theorem, in § 9.3. Here we give an independent proof of the former theorem in the case $p' = 2k$, i. e. $p = 2k/(2k-1)$, $k = 1, 2, \ldots$ This case is fairly easy and, what is more important, in certain interesting applications of the Hausdorff-Young theorem it suffices entirely.

Given an $f \in L$, we put $f(x) = f_1(x)$ and

$$f_i(x) = \frac{1}{2\pi}\int_0^{2\pi} f_{i-1}(x+t)\, f(-t)\, dt, \qquad i = 2, 3, \ldots$$

From Theorem 2.11 we see that, if c_n are the Fourier coefficients of f, those of f_i are c_n^i. From § 4.16(ii) we obtain, by induction, that, if $\alpha_i > 0$, $i = 1, 2, \ldots, j$, $\alpha_1 + \alpha_2 + \ldots + \alpha_j < 1$, then

$$\mathfrak{A}_{1/(1-\alpha_1-\ldots-\alpha_j)}[f_j] \leqslant \prod_{i=1}^{j} \mathfrak{A}_{1/(1-\alpha_i)}[f].$$

Putting $j = k$, $\alpha_1 = \alpha_2 = \ldots = 1/2k$, and supposing that $f \in L^{2k/(2k-1)}$, we obtain $\mathfrak{A}_2[f_k] \leqslant \mathfrak{A}^k_{2k/(2k-1)}[f]$. Hence, observing that the Fourier coefficients of f_k are c_n^k and applying Parseval's theorem, we have

$$\mathfrak{A}_{2k/(2k-1)}[f] \geqslant \mathfrak{A}_2^{1/k}[f_k] = \mathfrak{N}_2^{1/k}[c^k] = \mathfrak{N}_{2k}[c],$$

i. e. $\mathfrak{A}_{2k/(2k-1)}[f] \geqslant \mathfrak{N}_{2k}[c]$; this is just Theorem 9.1(a) for $p' = 2k$.

9.121. Theorem 9.1(b) may be obtained by a similar argument, using, instead of the results of § 4.16, analogous results for sequences. We prefer to follow a different way and to deduce Theorem 9.1(b) from Theorem 9.1(a), or, more generally, Theorem 9.11(b) from Theorem 9.11(a).

Suppose that $\mathfrak{N}_p[c] < \infty$, and let $f_n = c_1\varphi_1 + \ldots + c_n\varphi_n$, $n = 1, 2, \ldots$ For every function g with Fourier coefficients $d_1, d_2, \ldots$ we have

[1]) F. Riesz [6].

$$\left|\int_a^b \overline{f}_n g\,dx\right| = \left|\sum_1^n \overline{c}_\nu d_\nu\right| \leqslant \left(\sum_1^n |c_\nu|^p\right)^{1/p} \left(\sum_1^n |d_\nu|^{p'}\right)^{1/p'} \leqslant \mathfrak{N}_p[c]\, M^{(2-p)/p}\, \mathfrak{M}_p[g],$$

the last inequality being an application of Theorem 9.11(a). The upper bound of the left-hand side, for all g with $\mathfrak{M}_p[g] \leqslant 1$, is equal to $\mathfrak{M}_{p'}[\overline{f}_n] = \mathfrak{M}_{p'}[f_n]$ (§ 4.7.2), so that

$$(1) \qquad \mathfrak{M}_{p'}[f_n] \leqslant M^{(2-p)/p}\, \mathfrak{N}_p[c].$$

Since the inequality $\mathfrak{N}_p[c] < \infty$ implies $\mathfrak{N}_2[c] < \infty$, the series $c_1\varphi_1 + c_2\varphi_2 + \dots$ is the Fourier series of a function f (§ 4.21). If n tends to ∞ through a particular sequence of values, then $f_n(x) \to f(x)$ almost everywhere (§ 4.2) and, applying Fatou's lemma, we deduce from (1) that $\mathfrak{M}_{p'}[f] \leqslant M^{(2-p)/2}\, \mathfrak{N}_p[c]$, i. e. Theorem 9.11(b).

In a similar way we could deduce Theorem 9.11(a) from Theorem 9.11(b), so that both theorems are in reality equivalent.

9.2. M. Riesz's convexity theorems [1]). Consider a system of numbers a_{jk}, $1 \leqslant j \leqslant m$, $1 \leqslant k \leqslant n$, and the linear forms $X_j = a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n$, $j = 1, 2, \dots, m$, of the variables $x_1, x_2, \dots, x_n$. Let $M_{\alpha\beta}$ denote the upper bound of the expression $(\sigma_1|X_1|^{1/\beta} + \dots + \sigma_m|X_m|^{1/\beta})^\beta$ for the values of $x_1, x_2, \dots x_n$, satisfying the inequality $(\rho_1|x_1|^{1/\alpha} + \dots + \rho_n|x_n|^{1/\alpha})^\alpha \leqslant 1$, that is

$$(1) \qquad M_{\alpha\beta} = \operatorname*{Max}_{x_1,\dots,x_n} \left(\sum_{j=1}^m \sigma_j |X_j|^{1/\beta}\right)^\beta \Big/ \left(\sum_{k=1}^n \rho_k |x_k|^{1/\alpha}\right)^\alpha, \qquad (\alpha, \beta > 0),$$

where σ_j and ρ_k are arbitrary but fixed positive numbers. It is easy to see that the maximum is attained for every $\alpha, \beta \geqslant 0$.

$M_{\alpha\beta}$ is a multiplicatively convex function of the variables α, β in the triangle (Δ) $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \alpha$.

We mean by this that on an arbitrary segment l which lies entirely in Δ, $M_{\alpha\beta}$, considered as a function of a point, is multiplicatively convex (§ 4.14). To show this it is sufficient to prove that, for every point $P(\alpha, \beta)$ lying inside l, there exist on l, arbitrarily near P, two points $P_1(\alpha_1, \beta_1)$ and $P_2(\alpha_2, \beta_2)$, such that $P = t_1P_1 + t_2P_2$, $t_1 > 0$, $t_2 > 0$, $t_1 + t_2 = 1$, and that $M_{\alpha\beta} \leqslant M^{t_1}_{\alpha_1\beta_1} M^{t_2}_{\alpha_2\beta_2}$ [2]).

1) M. Riesz [3]; Paley [2].

2) If a function $y = \varphi(x)$ is not convex, there is an arc $y = \varphi(x)$, $x_1 < x < x_2$, lying totally above its chord $y = l(x)$, $x_1 < x < x_2$. Let x_0 be the largest value of the argument x, $x_1 < x < x_2$, for which $\varphi(x) - l(x)$ attains its maximum. Then, for any numbers x_1' and x_2' such that $x_1 < x_1' < x_0 < x_2' < x_2$, the point $(x_0, \varphi(x_0))$ lies above the chord joining $(x_1', \varphi(x_1'))$ and $(x_2', \varphi(x_2'))$.

Since $M_{\alpha\beta}$ is a continuous function of α, β [1]), we may restrict ourselves to the case of l lying entirely inside Δ. We may also suppose that l is not parallel to the β-axis.

Let us fix α, β, and put $a = 1/\alpha$, $b = 1/\beta$. Let $x_1, x_2, \dots, x_n$ be a system of values for which the maximum in (1) is attained; $y_1, y_2, \dots, y_n$ denotes a system of numbers which will be defined presently, and $Y_1, Y_2, \dots, Y_m$ are the corresponding values of the linear forms. The expression

$$(2) \qquad (\Sigma\sigma_j|X_j + \varepsilon Y_j|^b)^\beta / (\Sigma\rho_k|x_k + \varepsilon y_k|^a)^\alpha,$$

considered as a function of ε, attains its maximum for $\varepsilon = 0$. Let $x = x' + ix''$, $y = y' + iy''$. It is easy to see that, if $a > 1$, the expression $|x + \varepsilon y|^a = [(x' + \varepsilon y')^2 + (x'' + \varepsilon y'')^2]^{a/2}$ is a differentiable function of ε, and its derivative at the point $\varepsilon = 0$ is $\Re\, a|x|^{a-1}\overline{(\operatorname{sign} x)}\, y$. Hence the ratio (2) is also differentiable and, equating its derivative at the point 0 to 0, we obtain the formula

$$(3) \quad \Sigma\sigma_j|X_j|^b / \Sigma\rho_k|x_k|^a = \Re\,\Sigma\sigma_j|X_j|^{b-1}\overline{(\operatorname{sign} X_j)}\,Y_j \big/ \Re\,\Sigma\rho_k|x_k|^{a-1}\overline{(\operatorname{sign} x_k)}\,y_k.$$

Let us put $y_k = |x_k|^\lambda \operatorname{sign} x_k$; thence $|x_k| = |y_k|^{1/\lambda}$ and the denominator on the right may be written in the form of a product $(\Sigma\rho_k|x_k|^{a-1+\lambda})^{\theta_1}(\Sigma\rho_k|y_k|^{(a-1+\lambda)/\lambda})^{\theta_2}$, where the numbers $\lambda, \theta_1 > 0$, $\theta_2 > 0$, $\theta_1 + \theta_2 = 1$, will be fixed presently. Let us represent the coefficient of σ_j on the right in (3) in the form $|X_j|^{b-a} \cdot |X_j|^{a-1} \cdot \overline{(\operatorname{sign} X_j)}\, Y_j$. Applying Hölder's inequality with exponents k, k_1, k_2, where $1/k + 1/k_1 + 1/k_2 = 1$, $k = b/(b-a)$, we obtain from (3)

$$\frac{\Sigma\sigma_j|X_j|^b}{\Sigma\rho_k|x_k|^a} \leqslant \frac{(\Sigma\sigma_j|X_j|^b)^{(b-a)/b}\,(\Sigma\sigma_j|X_j|^{(a-1)k_1})^{1/k_1}\,(\Sigma\sigma_j|Y_j|^{k_2})^{1/k_2}}{(\Sigma\rho_k|x_k|^{1/\alpha_1})^{\theta_1}\,(\Sigma\rho_k|y_k|^{1/\alpha_2})^{\theta_2}}.$$

Here $\alpha_1 = 1/(a-1+\lambda)$, $\alpha_2 = \lambda/(a-1+\lambda)$, whence $(a-1)\alpha_1 + \alpha_2 = 1$, that is $(1-\alpha)\alpha_1 + \alpha\alpha_2 = \alpha$. Let us put

1) Considering separately the cases $\alpha > 0$ and $\alpha = 0$, we prove that the denominator in (1) is a continuous function of α, $x_1, \dots, x_n$ in the range $\alpha \geqslant 0$, $x_1, \dots x_n$ arbitrary. Hence, denoting the ratio in (1) by $f(\alpha, \beta, x_1, \dots, x_n)$, we see that f is continuous in the range $\alpha \geqslant 0$, $\beta \geqslant 0$, $|x_1|^2 + \dots + |x_n|^2 \neq 0$. Since we may plainly define $M_{\alpha\beta}$ as the maximum of f on the 'sphere' (S) $|x_1|^2 + \dots + |x_n|^2 = 1$, and since f is uniformly continuous on S, $M_{\alpha\beta}$ is a continuous function of α, β. It must be remembered that, if $\alpha = 0$, the denominator of (1) is equal to $\operatorname{Max}(|x_1|, |x_2|, \dots |x_n|)$ (§ 4.12).

$$\theta_1 = \alpha_1 (a-1), \quad \theta_2 = \alpha_2, \quad (a-1) k_1 = 1/\beta_1, \quad k_2 = 1/\beta_2,$$

so that $\theta_1+\theta_2=1$. The relation $1/k_1+1/k_2=a/b$ gives $(a-1)\beta_1+\beta_2=a/b$, that is $(1-\alpha)\beta_1+\alpha\beta_2=\beta$. From the last inequality we obtain easily

(4) $$M_{\alpha\beta} \leqslant M_{\alpha_1\beta_1}^{1-\alpha} M_{\alpha_2\beta_2}^{\alpha}.$$

The formulae $(1-\alpha)\alpha_1+\alpha\alpha_2=\alpha$, $(1-\alpha)\beta_1+\alpha\beta_2=\beta$ show that (α,β) lies on the segment l' joining (α_1,β_1) and (α_2,β_2). If for λ we take a value sufficiently near to 1, it follows from the definitions that α_1 and α_2 will be near α. When k_2 runs from the smallest possible value, viz. $b/a=\alpha/\beta$ corresponding to $k_1=\infty$, to infinity, then β_2 varies from β/α to 0. Since $\beta/\alpha > \beta$ and since α_2 is as near α as we please, we find, taking for k_2 a suitable value, that the point (α_2,β_2) lies on l. Then the directions of l and l' coincide, and the formula (4) shows $M_{\alpha\beta}$ to be a multiplicatively convex function on l. This proves the theorem.

9.21. So far, whenever we spoke of the Stieltjes integral $\int_a^b f(x)\,d\varphi(x)$, we understood this integral in the Stieltjes-Riemann sense. Now we shall introduce the Stieltjes-Lebesgue integral, restricting ourselves to the case when $\varphi(x)$ is a non-decreasing function.

Let $y=\varphi(x)$ be a function non-decreasing in an interval $a \leqslant x \leqslant b$, and let $\psi(y)$, $c \leqslant y \leqslant d$, be the inverse function, where $c=\varphi(a)$, $d=\varphi(b)$. If $\varphi(x)$ takes a constant value y_0 for $\alpha < x < \beta$, we assign to $\psi(y_0)$ any value from the interval (α,β). If $\varphi(x_0-0) < \varphi(x_0+0)$, we put $\psi(y)=x_0$ for y belonging to the interval $(\varphi(x_0-0), \varphi(x_0+0))$. Let $f(x)=f(\psi(y))=g(y)$. If $g(y)$ is integrable over (c,d), we say that f is integrable with respect to φ over (a,b) and define the integral by the formula

(1) $$\int_a^b f(x)\,d\varphi(x) = \int_{\varphi(a)}^{\varphi(b)} g(y)\,dy\ ^{1)}.$$

Since the number of stretches of invariability of $\varphi(x)$ is at most enumerable, the values of $\psi(y)$ corresponding to these stretches have no influence upon the value of the integral.

A set E of points x is said to be of measure 0 with respect to φ, if the variation of φ over E is equal to 0, that is if we can cover E by a finite or enumerable system of intervals (a_i,b_i) such that $\Sigma\{\varphi(b_i)-\varphi(a_i)\}$ is arbitrarily small. This is the same thing as to say that the set E on the x-axis is transformed by the function $y=\varphi(x)$ into a set of ordinary measure 0 on the y-axis[1]. It is plain that, if E is of measure 0 with respect to φ, the left-hand side of (1) is not affected if we change the values of $f(x)$ in E. The function f may even be undefined in E. If $f(x)=f_1(x)$ outside E, we shall not distinguish f from f_1.

[1]) For a detailed discussion we refer the reader to Lebesgue's, *Leçons sur l'intégration.*

A function $\varphi(x)$, $a \leqslant x \leqslant b$, is called a *step*-function if (a,b) can be broken up into a finite number of intervals in the interior of which $\varphi(x)$ is constant. If $x_1, x_2, \dots, x_k$ are the points of discontinuity of a step-function φ, then $\int_a^b f(x)\,d\varphi(x) = \Sigma\,\sigma_i f(x_i)$, where $\sigma_i=\varphi(x_i+0)-\varphi(x_i-0)$. For such functions a set is of measure 0 with respect to φ if it does not contain any of the points x_i. It can be proved that, if $\varphi(x)$ is absolutely continuous and non-decreasing, the left-hand side of (1) is equal to $\int_a^b f(x)\varphi'(x)\,dx$, but we shall not require this result, except in very special cases such as $\varphi(x)=-1/x$.

As regards the applications we have in view, the Stieltjes-Lebesgue integration is not really necessary and we could work with Lebesgue's definition of an integral. The use of the Lebesgue-Stieltjes integral has however certain advantages, the chief of them being that it enables us to treat series ($\varphi(x)=$ a step function) and integrals ($\varphi(x)=x$) in the same way, so that the arguments and results can be stated in a concise form.

We shall denote by $L^{r,\varphi}=L^{r,\varphi}(a,b)$ the class of functions $f(x)$ such that $|f(x)|^r$ is integrable with respect to $\varphi(x)$ over (a,b), and write

$$\mathfrak{M}_r[f]=\mathfrak{M}_{r,\varphi}[f]=\mathfrak{M}_{r,\varphi}[f;a,b]=\left\{\int_a^b |f(x)|^r\,d\varphi(x)\right\}^{1/r}.$$

From (1) and §§ 4.12, 4.13, we deduce the generalized Hölder and Minkowski inequalities

$$\mathfrak{M}[fg] \leqslant \mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[f], \qquad \mathfrak{M}_r[f_1+f_2] \leqslant \mathfrak{M}_r[f_1]+\mathfrak{M}_r[f_2], \qquad r \geqslant 1,$$

where $\mathfrak{M}_r=\mathfrak{M}_{r,\varphi}$. If f is a step-function, then $\mathfrak{M}_{\infty,\varphi}[f]$ is equal to the upper bound of $|f|$.

Let S denote the class of step-functions $s(x)$, $a \leqslant x \leqslant b$, which vanish in the intervals where $\varphi(x)$ is unbounded. It is plain that such intervals, if they exist, must be extreme intervals.

(i) *The set S is everywhere dense in every class $L^{r,\varphi}$ $1 \leqslant r < \infty$.*

Suppose first that the intervals (a,b) and $(\varphi(a),\varphi(b))$ are both finite, and let $a=a_0<a_1<a_2<\dots<a_n=b$ be a subdivision of the interval (a,b) such that

[1]) We define the image of a point x as the interval $\varphi(x-0) \leqslant y \leqslant \varphi(x+0)$ of the y-axis.

the points $a_1, a_2, \ldots, a_{n-1}$ are points of continuity of φ. Let $l_i = \varphi(a_i) - \varphi(a_{i-1})$; we define a step-function $s(x)$ by the following conditions: if $l_i \neq 0$, we put

$$(2)\qquad s(x) = \frac{1}{l_i}\int_{a_{i-1}}^{a_i} f(x)\,d\varphi(x), \qquad a_{i-1} \leqslant x < a_i, \qquad i = 1, 2, \ldots, n;$$

if $l_i = 0$, we put $s(x) = 0$ for $a_{i-1} \leqslant x < a_i$; in any case $s(b) = s(a_{n-1})$. Applying Hölder's inequality, we obtain that $\mathfrak{M}_{r,\varphi}[s; a_{i-1}, a_i] \leqslant \mathfrak{M}_{r,\varphi}[f; a_{i-1}, a_i]$, and so

$$(3)\qquad \mathfrak{M}_{r,\varphi}[s; a, b] \leqslant \mathfrak{M}_{r,\varphi}[f; a, b],$$

an inequality which will be used in a moment.

Now let us consider a sequence of subdivisions of the interval (a, b) such that $\operatorname{Max}(a_i - a_{i-1})$ tends to 0, and the sequence $s_1, s_2, \ldots$ of the corresponding functions s. If x_0 is a point of discontinuity of φ, then $s_m(x_0) \to f(x_0)$, i. e. $g_m(y) \to g(y)$ for $\varphi(x_0 - 0) \leqslant y \leqslant \varphi(x_0 + 0)$, where $g_m(y) = s_m[\psi(y)]$. Let E be the set of the points y which correspond to the intervals of constancy of φ; E is at most enumerable. If y corresponds to a point of continuity of φ and does not belong to E, then $g_m(y) \to g(y)$ provided that $g(y)$ is the derivative, at the point y, of the integral of g. It follows that $g_m(y) \to g(y)$ for almost every y. Hence, if f is bounded,

$$\int_{\varphi(a)}^{\varphi(b)} |g(y) - g_m(y)|^r\,dy \to 0, \quad \text{i. e.} \quad \int_a^b |f(x) - s_m(x)|^r\,d\varphi(x) \to 0.$$

If $f \in L^{r,\varphi}$, we write $f = f' + f''$, where f' is bounded and $\mathfrak{M}_{r,\varphi}[f''] < {}^1/_3\,\varepsilon$. Correspondingly $s_m(x) = s'_m(x) + s''_m(x)$ and

$$\mathfrak{M}_{r,\varphi}[f - s_m] \leqslant \mathfrak{M}_{r,\varphi}[f' - s'_m] + \mathfrak{M}_{r,\varphi}[f''] + \mathfrak{M}_{r,\varphi}[s''_m] < \mathfrak{M}_r[f' - s'_m] + {}^2/_3\,\varepsilon < \varepsilon$$

for m sufficiently large. This shows that $\mathfrak{M}_{r,\varphi}[f - s_m] \to 0$, and (i) is established in the case considered.

To prove (i) in the general case, we again write $f = f' + f''$, where $f' = 0$ outside an interval (a', b') completely interior to (a, b), $f'(x) = f(x)$ in (a', b'), and $\mathfrak{M}_{r,\varphi}[f''] < \frac{1}{2}\varepsilon$. Let $h(x)$ be a step-function vanishing outside (a', b') and such that $\mathfrak{M}_{r,\varphi}[f' - h; a', b'] < \frac{1}{2}\varepsilon$. Then

$$\mathfrak{M}_{r,\varphi}[f - h; a, b] \leqslant \mathfrak{M}_{r,\varphi}[f' - h; a, b] + \mathfrak{M}_{r,\varphi}[f''; a, b] < \varepsilon$$

and this proves the theorem in the general case.

We shall now prove the following result, which will be required in the next section.

(ii) *Given a finite number of functions $f_1, f_2, \ldots, f_n$ belonging to $L^{r,\varphi}$, $1 \leqslant r < \infty$, and a number $\varepsilon > 0$, we can find step-functions $h_1, h_2, \ldots, h_n$ such that $\mathfrak{M}_{r,\varphi}[f_i - h_i] < \varepsilon$ and that, for every sequence of constants $c_1, c_2, \ldots, c_n$, we have $\mathfrak{M}_{k,\varphi}[h] \leqslant \mathfrak{M}_{k,\varphi}[f]$, where $f = c_1 f_1 + \ldots + c_n f_n$, $h = c_1 h_1 + \ldots + c_n h_n$, $1 \leqslant k \leqslant \infty$.*

If the intervals (a, b) and $(\varphi(a), \varphi(b))$ are both finite, this is immediate. For if h_i is a function of type s (see (2)) corresponding to f_i, and if the sub-

division $a_0, a_1, a_2, \ldots$ is sufficiently dense, then $\mathfrak{M}_{r,\varphi}[f_i - h_i] < \varepsilon$. If the subdivision is the same for all f_i, then h is a step-function of type s corresponding to f and so, in view of (3), $\mathfrak{M}_{k,\varphi}[h] \leqslant \mathfrak{M}_{k,\varphi}[f]$ for every $1 \leqslant k \leqslant \infty$ (if $\mathfrak{M}_{k,\varphi}[f] = \infty$, there is nothing to prove).

To prove (ii) in the general case, we proceed as in the last stage of the proof of (i). We write $f_i = f'_i + f''_i$, where f'_i is equal to f_i in an interval (a', b') and vanishes outside it, $f' = c_1 f_1' + c_2 f_2' + \ldots$, $f'' = c_1 f_1'' + c_2 f_2'' + \ldots$ Let h'_i be a function of type s corresponding to the function f_i in the interval (a', b'); outside (a', b') we put $h'_i = 0$. If $h = c_1 h_1' + c_2 h_2' + \ldots$, then we may suppose that h corresponds to the function f' in (a', b'), and so

$$\mathfrak{M}_{k,\varphi}[h; a, b] = \mathfrak{M}_{k,\varphi}[h; a', b'] \leqslant \mathfrak{M}_{k,\varphi}[f'; a', b'] \leqslant \mathfrak{M}_{k,\varphi}[f; a, b].$$

9.22. Let us fix two intervals $u \leqslant t \leqslant u_1$, $v \leqslant t \leqslant v_1$, and two non-decreasing functions $\varphi(t)$, $u \leqslant t \leqslant u_1$ and $\psi(t)$, $v \leqslant t \leqslant v_1$. We suppose that we have an operation T associating with every function $f(t)$, $u \leqslant t \leqslant u_1$, belonging to a class $\mathfrak{F}$, another function $g(t) = T[f]$, $v \leqslant t \leqslant v_1$. The functions f and g may even be undefined in sets of measure 0, the former with respect to φ, the latter with respect to ψ. As regards the class $\mathfrak{F}$, we suppose that, if $f_1 \in \mathfrak{F}$, $f_2 \in \mathfrak{F}$, and if c_1 and c_2 are arbitrary constants, then $c_1 f_1 + c_2 f_2 \in \mathfrak{F}$. The operation T is to be an *additive* operation, that is $T[c_1 f_1 + c_2 f_2] = c_1\,T[f_1] + c_2\,T[f_2]$ for any constants c_1, c_2.

T will be said to be of type (a, b) if $T[f]$ is defined for every $f \in L^{a,\varphi}(u, u_1)$, and if

$$(1)\qquad \mathfrak{M}_{b,\psi}[T[f];\ v, v_1] \leqslant M\,\mathfrak{M}_{a,\varphi}[f;\ u, u_1],$$

where M is independent of f; in particular $T[f] \in L^{b,\psi}(v, v_1)$. The least value of M satisfying (1) will be called the *modulus* of the operation and denoted by $M_{\alpha\beta}$, where $\alpha = 1/a$, $\beta = 1/b$. The operation T is a linear operation in the sense of § 4.52.

It may happen that an operation T is defined not for all $f \in L^{a,\varphi}$ but only for a set S of f everywhere dense in $L^{a,\varphi}$ (the distance of two functions f_1 and f_2 being defined as $\mathfrak{M}_{a,\varphi}[f_1 - f_2]$), and that (1) is satisfied for all $f \in S$. Moreover suppose that S contains linear combinations of its elements. Then, without changing the values of $T[f]$ for $f \in S$, the operation T may be defined by continuity in the whole space $L^{a,\varphi}$ in such a way that it becomes of type (a, b) and that, moreover,

$$M_{\alpha\beta} = \operatorname{Sup} \mathfrak{M}_{b,\psi}[T[f]]/\mathfrak{M}_{a,\varphi}[f] \quad \text{for} \quad f \in S.$$

For if $f \in L^{a,\varphi}$, $f_n \in S$, $n = 1, 2, \ldots$, $\mathfrak{M}_{a,\varphi}[f - f_n] \to 0$, then $\mathfrak{M}_{a,\varphi}[f_m - f_n] \to 0$ as $m, n \to \infty$, and hence, by (1), $\mathfrak{M}_{b,\psi}[T[f_m] - T[f_n]] \to 0$. From

Theorem 4.2, using the definition 9.21(1), we deduce that there is a function $g(t)$, which we may denote by $T[f]$, such that $\mathfrak{M}_{b,\psi}[T[f]-T[f_n]]\to 0$. The function $T[f]$ is defined outside a set of measure 0 with respect to ψ and is independent of the choice of $\{f_n\}$. If (1) is satisfied with f replaced by f_n, it holds for f also.

A particularly important case is the one in which S is the set S of § 9.21.

9.23. *Let T be an operation which is simultaneously of type (a_1, b_1) and of type (a_2, b_2), where $a_i = 1/\alpha_i$, $b_i = 1/\beta_i$, and the points $P_i = (\alpha_i, \beta_i)$ belong to the triangle (Δ) $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \alpha$. Then T may be extended in such a way as to become of type (a, b) for every (α, β) on the segment l joining (α_1, β_1) and (α_2, β_2). Moreover the function $M_{\alpha\beta}$ is multiplicatively convex on l.*

Suppose that $\beta > 0$, i. e. that l does not lie on the α-axis.

Let $P = P(\alpha, \beta)$, $P = t_1 P_1 + t_2 P_2$, $t_i > 0$, $t_1 + t_1 = 1$. From what we have said it follows that it is enough to consider functions belonging to the set S of § 9.21. This set S is everywhere dense in every class $L^{a,\varphi}$, $1 \leqslant a < \infty$ [1]). If f belongs to S, then $f = x_1 f_1 + x_2 f_2 + \dots + x_n f_n$, where f_i is the characteristic function of an interval over which the variation of φ is equal to ρ_i. If $g = T[f]$, $g_i = T[f_i]$, then $g = x_1 g_1 + \dots + x_n g_n$. Since $f_i \in L^{a_1,\varphi}$, $f_i \in L^{a_2,\varphi}$, hence $g_i \in L^{b_1,\psi}$, $g_i \in L^{b_2,\psi}$; since b is contained between b_1 and b_2, we obtain, by Hölder's inequality, that $g_i \in L^{b,\psi}$. We can therefore find a step-function g_i^* such that we shall have $\mathfrak{M}_{b,\psi}[g_i - g_i^*] < \varepsilon$. Let $g^* = x_1 g_1^* + \dots + x_n g_n^*$; we may also suppose that $\mathfrak{M}_{k,\psi}[g^*] \leqslant \mathfrak{M}_{k,\psi}[g]$, $1 \leqslant k \leqslant \infty$, for all values of $x_1, x_2, \dots, x_n$. (§ 9.21(ii)).

Let ω be the maximum, with respect to the variables $x_1, x_2, \dots, x_n$, of the ratio $(|x_1| + \dots + |x_n|)/(\rho_1 |x_1|^a + \dots + \rho_n |x_n|^a)^\alpha$ at the point P. Let $\eta = \omega\varepsilon$; since, by Minkowski's inequality, $|\mathfrak{M}_{b,\psi}[g] - \mathfrak{M}_{b,\psi}[g^*]|$ does not exceed $\varepsilon(|x_1| + \dots + |x_n|)$ we see that

$$(1) \qquad \mathfrak{M}_{b,\psi}[g]/\mathfrak{M}_{a,\varphi}[f] \leqslant \eta + \mathfrak{M}_{b,\psi}[g^*]/(\Sigma \rho_k |x_k|^a)^\alpha.$$

Denoting by $X_1, X_2, \dots, X_m$ certain linear forms of the variables $x_1, x_2, \dots, x_n$, and by $\sigma_1, \sigma_2, \dots, \sigma_m$ certain positive constants, we may represent the numerator of the last fraction in the form

[1]) This is not true if $a = \infty$.

$(\sigma_1 |X_1|^b + \dots + \sigma_m |X_m|^b)^\beta$. Using Theorem 9.2, we see that this fraction does not exceed

$$\operatorname{Sup}\left\{\frac{\mathfrak{M}_{b_1,\psi}[g^*]}{\mathfrak{M}_{a_1,\varphi}[f]}\right\}^{t_1} \operatorname{Sup}\left\{\frac{\mathfrak{M}_{b_2,\psi}[g^*]}{\mathfrak{M}_{a_2,\varphi}[f]}\right\}^{t_2} \leqslant \operatorname{Sup}\left\{\frac{\mathfrak{M}_{b_1,\psi}[g]}{\mathfrak{M}_{a_1,\varphi}[f]}\right\}^{t_1} \operatorname{Sup}\left\{\frac{\mathfrak{M}_{b_2,\psi}[g]}{\mathfrak{M}_{a_2,\varphi}[f]}\right\}^{t_2}$$

where the upper bounds are taken with respect to $x_1, \dots, x_n$. Thus the left-hand side of (1) does not exceed $\eta + M_{\alpha_1\beta_1}^{t_1} M_{\alpha_2\beta_2}^{t_2}$ and, η being arbitrarily small with ε, we obtain

$$(2) \qquad M_{\alpha\beta} \leqslant M_{\alpha_1\beta_1}^{t_1} M_{\alpha_2\beta_2}^{t_2}.$$

From this we deduce the first part of the theorem. Since (α_1, β_1) and (α_2, β_2) may be any pair of points on the segment l, the inequality (2) proves the second part of the theorem also.

It remains to prove the theorem in the case of l lying on the α-axis. This case has no interesting application and we consider it for the sake of completness only. Suppose first that the number $l = \psi(v_1) - \psi(v)$ is finite. If the operation T is of type (a_1, ∞) and of type (a_2, ∞), where $0 \leqslant 1/a_1 = \alpha_1 < \alpha_2 = 1/a_2$, then T is also of type $(a_2, 1/\eta)$ for every $\eta > 0$. Since the expression $\left(\frac{1}{l}\int_v^{v_1} |g|^{1/\eta} d\psi\right)^\eta$ increases as η decreases to 0 (§ 4.15) and tends to the essential upper bound of g (with respect to the function ψ), we deduce that $M_{\alpha_2\eta} \leqslant l^\eta M_{\alpha_2 0}$. Hence, if $f \in S$, $g = T[f]$, and if (α, η_1), $\alpha = t_1\alpha_1 + t_2\alpha_2$, $t_1 + t_2 = 1$, is a point on the segment joining $(\alpha_1, 0)$ and (α_2, η), then

$$\left(\int_v^{v_1} |g|^{1/\eta_1} d\psi\right)^{\eta_1} \leqslant M_{\alpha_1 0}(l^\eta M_{\alpha_2 0})^{t_2} \left(\int_u^{u_1} |f|^{1/\alpha} d\varphi\right)^\alpha$$

and, making $\eta \to 0$, we obtain $M_{\alpha 0} \leqslant M_{\alpha_1 0}^{t_1} M_{\alpha_2 0}^{t_2}$.

To remove the condition $l < \infty$, let (v', v_1') be an interval interior to (v, v_1). Considering the function g in (v', v_1') only, we have a linear operation with norm $M'_{\alpha 0} \leqslant M_{\alpha 0}$. We have $M'_{\alpha 0} \leqslant M'^{t_1}_{\alpha_1 0} M'^{t_2}_{\alpha_2 0} \leqslant M^{t_1}_{\alpha_1 0} M^{t_2}_{\alpha_2 0}$ and, making $v' \to v$, $v_1' \to v_1$, we obtain $M_{\alpha 0} \leqslant M^{t_1}_{\alpha_1 0} M^{t_2}_{\alpha_2 0}$.

9.24. It is natural to inquire how far the condition imposed upon the point (α, β) to remain within the triangle Δ is essential for the truth of the theorem. The results are mostly negative. For details we refer the reader to M. Riesz [3].

Having in view definite applications we supposed in Theorem 9.2 that the coefficients of the linear forms X_j, as well as the variables x_k, were complex numbers. Similarly in Theorem 9.23 the functions f and $T[f]$ were complex functions of a real variable. In some cases however it is important to have those theorems for real variables. Theorem 9.2 holds, and its proof is unaffected, if we assume that the numbers a_{jk}, x_k are real. Similarly Theorem 9.23, which follows from Theorem 9.2 by passages to limits, remains true in the domain of real variables.

9.25. As an application of Theorem 9.23 we shall prove the following theorem, stated without proof in § 4.63. *If $r<s<r'$, the class (L^r, L^r) is contained in (L^s, L^s).* Consider the series

$$(1)\quad \frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),\qquad (1a)\quad \frac{1}{2}a_0\lambda_0+\sum_{n=1}^{\infty}\lambda_n(a_n\cos nx+b_n\sin nx),$$

and suppose that, whenever (1) is the Fourier series of a function $f\in L^r$, (1a) is the Fourier series of a function $g=T[f]\in L^r$. We shall prove first that $g=T[f]$ is an operation of type (r,r) in the sense of § 9.22. It is plain that $T[f]$ is an additive operation and it remains to prove the existence of a constant M such that $\mathfrak{M}_r[g]\leqslant M\,\mathfrak{M}_r[f]$. Let $\sigma_n^*(x)$ and $l_n(x)$ denote the $(C,1)$ means of the series (1a) and of the series $\frac{1}{2}\lambda_0+\lambda_1\cos x+\dots$ respectively. From the formula

$$(2)\qquad \sigma_n^*(x)=\frac{1}{\pi}\int_0^{2\pi}f(x+t)\,l_n(t)\,dt$$

(§ 4.64), we obtain $|\sigma_n^*(x)|\leqslant\pi^{-1}\mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[l_n]$, $\mathfrak{M}_r[\sigma_n^*]\leqslant 2\mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[l_n]$, so that, for fixed n, (2) is an operation associating with every $f\in L^r$ a function $\sigma_n^*\in L^r$. Let M_n be the modulus of the operation (2). Since, by hypothesis, $\mathfrak{M}_r[\sigma_n^*]$ is bounded for every $f\in L^r$, the sequence $\{M_n\}$ is bounded (§ 4.55). If $M=\operatorname{Sup}M_n$, we have $\mathfrak{M}_r[\sigma_n^*]\leqslant M\,\mathfrak{M}_r[f]$ and, making $n\to\infty$, $\mathfrak{M}_r[g]\leqslant M\,\mathfrak{M}_r[f]$. This shows that $T[f]$ is of type (r,r).

Now it is easy to complete the proof. In view of Theorem 4.63(ii), T is also of type (r',r'), and from Theorem 9.23 we see that T is of type (s,s), where s is any number such that $1/s$ is contained between $1/r$ and $1/r'$.

9.3. Proof of F. Riesz's theorem.

To prove Theorem 9.11(a) let

$$(1)\qquad c_n=\int_a^b f(t)\,\overline{\varphi_n(t)}\,dt,\qquad n=1,2,\dots$$

be the n-th Fourier coefficient of f (§ 1.31). Then

$$(2)\qquad \sum_{n=1}^{\infty}|c_n|^2\leqslant\int_a^b|f(t)|^2\,dt,\qquad |c_n|\leqslant M\int_a^b|f(t)|\,dt,$$

where the first inequality is Bessel's inequality and the second follows from (1) and the inequality $|\varphi_n|\leqslant M$. Let us put $\varphi(x)=x$, and let $\psi(x)=[x]$ for $x>0$, $\psi(x)=0$ elsewhere. If $c(x)$ is equal to c_n for $x=n$ and is arbitrary elsewhere, the inequalities (2) may be written

$$\mathfrak{M}_{2,\psi}[c]\leqslant\mathfrak{M}_{2,\varphi}[f],\quad \mathfrak{M}_{\infty,\psi}[c]\leqslant M\,\mathfrak{M}_{1,\varphi}[f],$$

so that the operation $c(x)=T[f]$ is of types $(2,2)$ and $(1,\infty)$. In view of Theorem 9.23, T is also of type (p,p'), where $p=1/\alpha$, $p'=1/(1-\alpha)$, $\frac{1}{2}\leqslant\alpha\leqslant 1$. Since $M_{1/2,1/2}\leqslant 1$, $M_{1,0}\leqslant M$, we find, using Theorem 9.23 again, that

$$M_{\alpha,1-\alpha}\leqslant M_{1,0}^{(\alpha-1/2)/(1-1/2)}\,M_{1/2,1/2}^{(1-\alpha)/(1-1/2)}\leqslant M^{2\alpha-1}=M^{(2-p)/p}.$$

Hence $\mathfrak{M}_{p',\psi}[c]\leqslant M^{(2-p)/p}\,\mathfrak{M}_{p,\varphi}[c]$, and this is just Theorem 9.11(a).

To prove Theorem 9.11(b) we argue similarly, starting from the inequalities

$$\int_a^b|f(t)|^2\,dt\leqslant\sum_{n=1}^{\infty}|c_n|^2,\quad |f(t)|\leqslant M\sum_{n=1}^{\infty}|c_n|,$$

where f is the function the existence of which is assured by the Riesz-Fischer theorem (§ 4.21(1)). The details may be left to the reader.

9.31. We complete the above proof by a few general remarks. In the first place we observe that the apparatus of the Stieltjes-Lebesgue integral was not really necessary in the proof of Theorem 9.11(a). For, if we put $c(x)=c_n$ for $n-1\leqslant x<n$, $n=1,2,\dots$, the inequalities 9.3(2) may be writen $\mathfrak{M}_2[c]\leqslant\mathfrak{M}_2[f]$, $\mathfrak{M}_\infty[c]\leqslant M\,\mathfrak{M}[f]$, where the integrals are ordinary Lebesgue integrals, and we may apply Theorem 9.23 in the case $\varphi(x)=x$, $\psi(x)=x$. This course is slightly less simple in the case of Theorem 9.11(b); but, as we know, both parts of Theorem 9.11 can easily be deduced from each other (see also § 9.9.1).

The proof of F. Riesz's Theorem can be made more elementary by basing it on Theorem 9.2 instead of Theorem 9.23. But the application of the latter theorem has two advantages. The first of them is that it clearly shows the proper place of Theorem 9.11, which turns out to be not a generalization but a consequence of the Riesz-Fischer theorem. Besides, Theorem 9.23 is of fundamental character and may be applied, so to speak, automatically in many cases where an application of Theorem 9.2 would require certain calculations, which would amount substantially to a proof of Theorem 9.23.

We also observe that in § 9.3 we applied the Bessel inequality and the Riesz-Fischer theorem for a complex system $\{\varphi_n\}$, whereas the proofs given in §§ 1.6, 4.21 bear on the case of real φ_n. The reader will have no difficulty in adapting those proofs to the case of complex φ_n.

9.4. Theorems of Paley. The Hausdorff-Young theorems are not the only results which connect the type of integrability of a function with the exponent of convergence of its coefficients. Further results in this direction have been obtained by Hardy and Littlewood. The simplest way to them seems to lead through theorems of Paley which partly generalize the Hardy-Littlewood theorems and bear on general orthogonal and normal systems of uniformly bounded functions.

Given any sequence of complex numbers $c_1, c_2, \ldots$ tending to 0, we denote by $c_1^*, c_2^*, \ldots$ the sequence $|c_1|, |c_2|, \ldots$ rearranged in descending order of magnitude. If several $|c_n|$ are equal, then there are corresponding repetitions in the c_n^*. We put

$$\left\{\sum_{n=1}^{\infty} |c_n|^r n^{r-2}\right\}^{1/r} = \mathfrak{B}_r[c], \qquad \left\{\sum_{n=1}^{\infty} c_n^{*r} n^{r-2}\right\}^{1/r} = \mathfrak{B}_r[c^*].$$

Let $\varphi_1(x), \varphi_2(x), \ldots$ be a system of functions which are orthogonal, normal, and uniformly bounded ($|\varphi_n| \leqslant M$, $n = 1, 2, \ldots$) in an interval (a, b). Writing $\mathfrak{M}_r[f] = \mathfrak{M}_r[f; a, b]$, Paley's theorems may be stated as follows [1]).

(i) *If, for a sequence of numbers $c_1, c_2, \ldots$, the expression $\mathfrak{B}_q[c^*]$ is finite, there is a function $f \in L^q$ such that c_n is the Fourier coefficient of f with respect to φ_n, $n = 1, 2, \ldots$, and*

$$\mathfrak{M}_q[f] \leqslant A_q\, \mathfrak{B}_q[c^*], \tag{1}$$

where A_q depends only on q and M.

---

1) Paley [4].

(ii) *If $f \in L^p$, and if $c_1, c_2, \ldots$ are the Fourier coefficients of f with respect to $\{\varphi_n\}$, then $\mathfrak{B}_p^*[c] < \infty$ and*

$$\mathfrak{B}_p^*[c] \leqslant A_p' \,\mathfrak{M}_p[f], \tag{2}$$

where A_p' depends only on p and M.

The reason why we introduced the starred sequence $\{c_n^*\}$ becomes clear from the following considerations. Let $a_1, a_2, \ldots, b_1, b_2, \ldots$ be two sequences of non-negative numbers, and let S be the sum $a_1 b_1 + a_2 b_2 + \ldots$; S may also be infinite. We suppose that $\{a_n\}$ is either non-increasing or non-decreasing. *Rearranging $\{b_n\}$ in all possible manners, we obtain for S the largest value when $\{a_n\}$ and $\{b_n\}$ vary in the same sense,* i. e. if they are either both non-increasing or both non-decreasing; *S is a minimum when $\{a_n\}$ and $\{b_n\}$ vary in opposite senses.* To fix ideas we assume that $a_1 \geqslant a_2 \geqslant \ldots$ To prove the first part of the proposition we observe that, if e. g. $a_1 > a_2$ and $b_1 < b_2$, then, replacing $a_1 b_1 + a_2 b_2$ by $a_1 b_2 + a_2 b_1$, we increase S by $(a_1 - a_2)(b_2 - b_1) > 0$. Similarly we prove the second part.

Hence, considering all possible rearrangements of $\{|c_n|\}$, we see that $\mathfrak{B}_q[c]$ is a minimum when $\{|c_n|\}$ is arranged in descending order of magnitude. With this arrangement the expression $\mathfrak{B}_p[c]$ attains its maximum. It follows that, if (1) and (2) are true, the inequalities which we obtain by replacing $\mathfrak{B}_q^*[c], \mathfrak{B}_p^*[c]$ by $\mathfrak{B}_q[c], \mathfrak{B}_q[c]$, hold à fortiori. On the other hand, since the order of the functions φ_n within the sequence $\{\varphi_n\}$ is irrelevant, we may change this order, if necessary, and suppose from the very beginning that $c_n^* = |c_n|$. It is therefore sufficient to prove (1) and (2) with c_n^* replaced by $|c_n|$, and in the subsequent proof we shall write $|c_n|$ instead of c_n^*.

9.401. Since, by Hölder's inequality,

$$\Sigma |c_n|^2 = \Sigma |c_n|^2 n^{2(q-2)/q} n^{-2(q-2)/q} \leqslant (\Sigma |c_n|^q n^{q-2})^{2/q} (\Sigma n^{-2})^{(q-2)/q},$$

we see that, under the hypothesis of Theorem 9.4(i), the numbers c_n are the Fourier coefficients, with respect to $\{\varphi_n\}$, of a function $f(x) \in L^2$. Let $s_n(x)$ be the n-th partial sum of the series $c_1 \varphi_1(x) + c_2 \varphi_2(x) + \ldots$ It is sufficient to prove that $\mathfrak{M}_q[s_{2^N-1}] \leqslant A_q \mathfrak{B}_q[c]$, $N = 1, 2, \ldots$, for, since $\mathfrak{M}_2[f - s_{2^{N_i}-1}] \to 0$, there is a sequence of integers $\{N_i\}$ such that $s_{2^{N_i}-1}(x)$ converges almost everywhere to

$f(x)$ (§ 4.2), and an application of Fatou's lemma to the last inequality gives 9.4(1).

Let $C_\mu = \sum_{m=2^{\mu-1}}^{2^\mu-1} |c_m|^q m^{q-2}$, $\Phi_\mu = \sum_{m=2^{\mu-1}}^{2^\mu-1} c_m \varphi_m$, $\mu = 1, 2, \dots$, and let $\nu \geqslant \mu$. We begin by proving that

(1) $$\int_a^b |\Phi_\mu \Phi_\nu|^{q/2} dx \leqslant B_q C_\mu^{1/2} C_\nu^{1/2} 2^{-\frac{1}{2}(\nu-\mu)}, \quad q \geqslant 4,$$

where B_q is independent of $\{c_n\}$. For, since $|\varphi_m| \leqslant M$, the left-hand side of (1) does not exceed

$$\operatorname{Max}\{|\Phi_\mu|^{\frac{1}{2}q} |\Phi_\nu|^{\frac{1}{2}q-2}\} \int_a^b |\Phi_\nu|^2 dx \leqslant$$

$$\leqslant M^{q-2} \Big(\sum_{m=2^{\mu-1}}^{2^\mu-1} |c_m|\Big)^{\frac{1}{2}q} \Big(\sum_{n=2^{\nu-1}}^{2^\nu-1} |c_n|\Big)^{\frac{1}{2}q-2} \Big(\sum_{p=2^{\nu-1}}^{2^\nu-1} |c_p|^2\Big).$$

Writing $|c_m| = |c_m| m^{(q-2)/q} \cdot m^{-(q-2)/q}$, $|c_n| = |c_n| n^{(q-2)/q} \cdot n^{-(q-2)/q}$, $|c_p|^2 = |c_p|^2 p^{2(q-2)/q} \cdot p^{-2(q-2)/q}$, applying Hölder's inequalities so as to introduce the sums C_μ, C_ν, and observing that

$$\sum_{m=2^{\mu-1}}^{2^\mu-1} m^{-\alpha} < \int_0^{2^\mu} x^{-\alpha} dx, \ \alpha > 0; \quad \sum_{p=2^{\nu-1}}^{2^\nu-1} p^{-2} < \frac{1}{2^{2(\nu-1)}} + \int_{2^{\nu-1}}^\infty \frac{dx}{x^2} < 4.2^{-\nu},$$

we easily obtain the inequality (1) with B_q not exceeding

$$M^{q-2} 4^{(q-2)/q} (q-1)^{\frac{1}{2}(q-1)} (q-1)^{(q-1)(q-4)/2q} < M^{q-2} q^q.$$

Now, supposing that $q \geqslant 4$ is an integer, we have

$$\mathfrak{M}_q^q[s_{2^N-1}] = \int_a^b \Big|\sum_{\nu=1}^N \Phi_\nu\Big|^q dx \leqslant \sum_{\nu_1=1}^N \sum_{\nu_2=1}^N \dots \sum_{\nu_q=1}^N \int_a^b |\Phi_{\nu_1} \dots \Phi_{\nu_q}| dx.$$

Writing $|\Phi_{\nu_1}\Phi_{\nu_2}\dots\Phi_{\nu_q}| = |(\Phi_{\nu_1}\Phi_{\nu_2})(\Phi_{\nu_1}\Phi_{\nu_3})\dots(\Phi_{\nu_1}\Phi_{\nu_q})\dots(\Phi_{\nu_{q-1}}\Phi_{\nu_q})|^{1/(q-1)}$, where the number of bracketed factors is $Q = \frac{1}{2}q(q-1)$, and applying Hölder's inequality with the exponents Q (§ 4.141), we obtain

$$\int_a^b |\Phi_{\nu_1} \Phi_{\nu_2} \dots \Phi_{\nu_q}| dx \leqslant \prod_{\substack{i,j=1\\ i<j}}^q \Big\{\int_a^b |\Phi_{\nu_i}\Phi_{\nu_j}|^{\frac{1}{2}q} dx\Big\}^{1/Q} \leqslant$$

$$\leqslant B_q \prod_{\substack{i,j=1\\ i<j}}^q C_{\nu_i}^{1/2Q} C_{\nu_j}^{1/2Q} 2^{-|\nu_i-\nu_j|/2Q} = B_q \prod_{i=1}^q C_{\nu_i}^{1/q} \Big\{\prod_{j=1}^{q\,(i)} 2^{-|\nu_i-\nu_j|/4Q}\Big\}.$$

Here the upper suffix (i) indicates that the factor $j = i$ (which, by the way, is equal to 1) is omitted. Substituting this in the right-hand side of the inequality for $\mathfrak{M}_q^q[s_{2^N-1}]$, and applying Hölder's inequality with the exponents q, we obtain

$$\mathfrak{M}_q^q[s_{2^N-1}] \leqslant B_q \prod_{i=1}^q \Big\{\sum_{\nu_1=1}^N \dots \sum_{\nu_q=1}^N C_{\nu_i} \prod_{j=1}^{q\,(i)} 2^{-|\nu_i-\nu_j|/2(q-1)}\Big\}^{1/q}.$$

Consider the multiple sum in curly brackets. Summing first with respect to $\nu_1, \nu_2, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_q$, and then with respect to ν_i, we obtain that the sum considered does not exceed

$$\Big(\sum_{\nu=1}^N C_\nu\Big) \Big\{\sum_{\nu=-\infty}^{+\infty} 2^{-|\nu|/2(q-1)}\Big\}^{q-1}, \text{ and so}$$

$$\mathfrak{M}_q^q[s_{2^N-1}] \leqslant B_q \Big\{\sum_{\nu=1}^\infty C_\nu\Big\} \Big\{\sum_{\nu=-\infty}^\infty 2^{-|\nu|/2(q-1)}\Big\}^{q-1} = A_q^q \mathfrak{V}_q^q[c],$$

where $A_q^q = B_q\{\Sigma 2^{-|\nu|/2(q-1)}\}^{q-1}$. Thus the theorem is proved for $q = 4, 5, \dots$; and it is plainly true also for $q = 2$.

To prove the theorem in the general case we observe that the inequality 9.4(1) may be written

$$\mathfrak{M}_q^q[f] \leqslant A_q^q \sum_{n=1}^\infty (|c_n| n)^q n^{-2},$$

and that $f(t) = \Sigma n c_n \cdot \varphi_n(t)/n$ is obtained by a linear transformation from the numbers nc_n. Thus, arguing as in § 9.3, we may interpolate by means of Theorem 9.23, and Theorem 9.4(i) is established completely.

9.402. Theorem 9.4(ii) may be obtained by an argument similar to that of § 9.121. We put $p' = q$, fix an integer $N > 0$, and denote by $g(x)$ a sum $d_1\varphi_1(x) + d_2\varphi_2(x) + \dots + d_N\varphi_N(x)$, where the numbers $d_1, d_2, \dots, d_N$ will be defined in a moment. Then

(1) $$\int_a^b f\bar{g}\, dx = \sum_{n=1}^N c_n \bar{d}_n = \sum_{n=1}^N c_n n^{(p-2)/p} \cdot \bar{d}_n n^{(q-2)/q}.$$

Let us apply Hölder's inequality, with the exponents p and q, to the last sum. If $\operatorname{sign} d_n = \operatorname{sign} c_n$, $|c_n|^p n^{p-2} = |d_n|^q n^{q-2}$, the inequality degenerates into equality (§ 4.12); hence, applying Hölder's inequality to the integral in (1), we obtain

$$\Big(\sum_{n=1}^N |c_n|^p n^{p-2}\Big)^{1/p} \Big(\sum_{n=1}^N |d_n|^q n^{q-2}\Big)^{1/q} \leqslant \mathfrak{M}_p[f]\, \mathfrak{M}_q[g].$$

In virtue of 9.4(1), the second factor on the right does not exceed $A_q \mathfrak{B}_q[d]$, so that

$$\sum_{n=1}^{N} |c_n|^p n^{p-2} \leqslant A_q^p \int_a^b |f|^p \, dx.$$

Making $N \to \infty$ we obtain the inequality 9.4(2) with $A'_p = A_q$.

The reader will easily convince himself that $A_q \leqslant M^{(q-2)/q} \alpha_q$, and so $A'_p \leqslant M^{(2-p)/p} \alpha'_p$, where α_q depends only on q, and α'_p only on p.

9.41. It is an interesting fact that Paley's theorems contain the theorems of F. Riesz as special cases, although in a slightly less precise form: into the right-hand sides of the inequalities $\mathfrak{M}_{p'}[f] \leqslant M^{(2-p)/p} \mathfrak{N}_p[c]$, $\mathfrak{N}_{p'}[c] \leqslant M^{(2-p)/p} \mathfrak{M}_p[f]$ we shall have to introduce a numerical factor β_p depending on p. In view of the last remark of 9.402 it is sufficient to show that

(1) $$\mathfrak{B}_q[c^*] \leqslant \gamma_q \mathfrak{N}_{q'}[c], \quad \mathfrak{B}_p[c^*] \geqslant \gamma'_p \mathfrak{N}_{p'}[c],$$

where γ_q depends only on q, and γ'_p only on p. We shall prove the first of these inequalities only; the proof of the second is similar.

First of all we observe that, if $x, y, \dots$ are non-negative numbers, then $(x+y+\dots)^r \leqslant x^r + y^r + \dots$ for $0 < r \leqslant 1$, and $(x+y+\dots)^r \geqslant x^r + y^r + \dots$ for $r \geqslant 1$. The first of these inequalities has already been established in the case of two terms (§ 4.13), and in the general case the proof follows by induction; the second inequality may be obtained in the same way. Now

$$\sum_{n=1}^{\infty} c_n^{*q} n^{q-2} = \sum_{\nu=0}^{\infty} \sum_{n=2^\nu}^{2^{\nu+1}-1} c_n^{*q} n^{q-2} \leqslant$$

$$\leqslant 2^{q-2} \sum_{\nu=0}^{\infty} c_{2^\nu}^{*q} 2^{\nu(q-1)} = 2^{q-2} \sum_{\nu=0}^{\infty} (c_{2^\nu}^{*q'} 2^\nu)^{q-1} \leqslant$$

$$\leqslant 2^{q-2} \left(\sum_{\nu=0}^{\infty} c_{2^\nu}^{*q'} 2^\nu \right)^{q-1} \leqslant 2^{2q-3} \left(c_1^* + \sum_{\nu=1}^{\infty} c_{2^\nu}^{*q'} 2^{\nu-1} \right)^{q-1} \leqslant$$

$$\leqslant 2^{2q-3} \left(c_1^* + \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}}^{2^\nu - 1} c_n^{*q'} \right)^{q-1} \leqslant 2^{2q-3} \left(2 \sum_{n=1}^{\infty} c_n^{*q'} \right)^{q-1} = 2^{3q-4} \mathfrak{N}_{q'}^q[c],$$

and the first inequality (1) is established.

This result might suggest that, perhaps, the theorems of Paley and those of F. Riesz are, roughly speaking, equivalent. But this is not so. For if we put e. g. $\varphi_n(x) = \cos nx$, $(a, b) = (0, \pi)$, $c_n = n^{-3/4} \log^{-3/4}(n+1)$, $n = 1, 2, \dots$, then $\mathfrak{B}_4[c] < \infty$, and so, by Theorem 9.4(i), the function $f(x) = c_1 \cos x + c_2 \cos 2x + \dots$ belongs to L^4. Since $\mathfrak{N}_{4/3}[c] = \infty$, this result cannot be obtained from Theorem 9.11(b).

9.42. Given a real function $f(x)$, $a \leqslant x \leqslant b$, we shall denote by $\mathrm{E}(f > y)$ the set of points where $f(x) > y$. The functions f and φ will be called *equimeasurable* functions if $|\mathrm{E}(f > y)| = |\mathrm{E}(\varphi > y)|$ for every y[1]. Each of these functions may be thought of as obtained from the other by a sort of 'rearrangement' of the argument x, although we should find some difficulty if we tried to define this rearrangement precisely. It is plain that if one of two functions equimeasurable in the interval (a, b) is integrable, so is the other and their integrals over (a, b) are equal.

For every measurable function $f(x)$ defined in a finite interval $a \leqslant x \leqslant b$, there is a function $f^*(x)$, $a \leqslant x \leqslant b$, equimeasurable with $f(x)$ and non-increasing. For let $m(y) = |\mathrm{E}(f > y)|$ and suppose for simplicity that $a = 0$; then $f^*(x)$ may be defined as the function inverse to $m(y)$. The function f^* is defined uniquely except at its points of discontinuity. To fix ideas we may suppose that $f^*(x+0) = f(x)$. Similarly there is a function $f_*(x)$ equimeasurable with $f(x)$ and non-decreasing.

We shall require the following lemma.

If $f(x)$ is non-negative, then, for any function $g(x)$ which is non-negative and non-increasing, we have

(1) $$\int_a^b g f_* \, dx \leqslant \int_a^b g f \, dx \leqslant \int_a^b g f^* \, dx.$$

First of all we observe that, if $f_n(x)$ tends almost everywhere to $f(x)$, then $f_n^*(x) \to f^*(x)$, $f_{*n}(x) \to f_*(x)$, except at a set of points which is at most enumerable. This follows from the fact that, for every y, $|\mathrm{E}(f_n > y)| \to |\mathrm{E}(f > y)|$. Secondly, if $\{f_n\}$ is monotonic and tends to a limit $f(x)$, and if (1) is true for f_n, $n = 1, 2, \dots$, it is true also for f. This follows from the preceding remark and from Lebesgue's theorem on the integration of monotonic

[1]) This notion has been introduced by F. Riesz [8].

sequences. Now (1) is certainly true if (a, b) can be broken up into a number of intervals of equal length in each of which f, and so also f^* and f_*, is constant, for then the integrals (1) reduce to sums (§ 9.4). Since, starting with such functions, we may, by monotonic passages to limits, obtain any measurable function $f(x)$ [1] (more precisely, a function equivalent to $f(x)$) the inequalities (1) are true in the general case.

9.43. Now we shall show that, if we invert the rôles of $f(x)$ and $\{c_n\}$ in Theorems 9.4, we obtain theorems which are equally true. It will simplify the proofs slightly if we suppose that the interval (a, b) is finite, but the proofs in the general case undergo but little change. We suppose for simplicity that (a, b) is of the form $(0, h)$. By f^* we shall denote the function which is non-increasing and equimeasurable with $|f|$, and write

$$\mathfrak{U}_r[f] = \left\{\int_0^h |f|^r x^{r-2}\, dx\right\}^{1/r}, \qquad \mathfrak{U}_r[f^*] = \left\{\int_0^h f^{*r} x^{r-2}\, dx\right\}^{1/r} {}^{2)}.$$

If the functions φ_n satisfy the same conditions as before, then

(i) *If $\mathfrak{U}_q[f^*]$ is finite and if c_n is the Fourier coefficient of f with respect to φ_n, then $\mathfrak{N}_q[c]$ is finite and*

$$\mathfrak{N}_q[c] \leqslant A_q\, \mathfrak{U}_q[f^*], \tag{1}$$

where A_q depends only on q and M.

(ii) *If, for a sequence $\{c_n\}$, we have $\mathfrak{N}_p[c] < \infty$, the numbers c_n are the Fourier coefficients of a function f such that*

$$\mathfrak{U}_p[f^*] \leqslant A'_p\, \mathfrak{N}_p[c], \tag{2}$$

where $A'_p = A_{p'}$.

Since the proofs follow the same lines as those of Theorems 9.4, we shall condense some parts. We begin by proving (1) in a weaker form, with f^* replaced by $|f|$ on the right.

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[1] See e. g. Hobson, *Theory of functions*, 2, 376.

[2] In the case $(a, b) = (-\infty, +\infty)$ it is convenient to define f^* as a function which is equimeasurable with $|f|$, even, and non-increasing in $(0, \infty)$, and to put $\mathfrak{U}^*_r[f] = \left\{\int_{-\infty}^{+\infty} f^{*r} |x|^{r-2}\, dx\right\}^{1/r}$.

The inequality is true for $q = 2$, and so, if we prove it for $q = 4, 5, \ldots$, an application of Theorem 9.23 yields the result for general q. Let $f_\nu(x)$ be the function equal to $f(x)$ in the interval $(h2^{-\nu}, h2^{-\nu+1})$ and to 0 elsewhere, $\nu = 1, 2, \ldots$, and let c_n^ν be the Fourier coefficient of f_ν with respect to φ_n, so that $c_n = c_n^1 + c_n^2 + \ldots$ We fix an integer $N > 0$ and observe that

$$\sum_{n=1}^N |c_n|^q = \sum_{n=1}^N |c_n^1 + c_n^2 + \ldots|^q \leqslant \sum_{\nu_1=1}^\infty \ldots \sum_{\nu_q=1}^\infty \left\{\sum_{n=1}^N |c_n^{\nu_1} \ldots c_n^{\nu_q}|\right\},$$

and that

$$\sum_{n=1}^N |c_n^{\nu_1} \ldots c_n^{\nu_q}| \leqslant \prod_{\substack{i,j=1\\ i<j}}^q \left\{\sum_{n=1}^N |c_n^{\nu_i} c_n^{\nu_j}|^{q/2}\right\}^{1/Q},$$

where $Q = \frac{1}{2} q(q-1)$. Now we prove that

$$\sum_{n=1}^N |c_n^\mu c_n^\nu|^{q/2} \leqslant B_q\, \eta_\mu^{1/2} \eta_\nu^{1/2}\, 2^{-\frac{1}{2}|\mu-\nu|}, \tag{3}$$

where $B_q \leqslant M^{q-2} \beta_q$ with β_q depending only on q, and η_ν equal to $\mathfrak{M}[|f|^q x^{q-2};\ h2^{-\nu},\ h2^{-\nu+1}]$. For the left-hand side of (3) is equal to

$$\sum_{n=1}^N \left|\int_{h2^{-\mu}}^{h2^{-\mu+1}} f\bar{\varphi}_n\, dx\right|^{\frac{1}{2}q} \left|\int_{h2^{-\nu}}^{h2^{-\nu+1}} f\bar{\varphi}_n\, dx\right|^{\frac{1}{2}q} \leqslant$$

$$\leqslant M^{q-2} \left(\int_{h2^{-\mu}}^{h2^{-\mu+1}} |f|\, dx\right)^{\frac{1}{2}q} \left(\int_{h2^{-\nu}}^{h2^{-\nu+1}} |f|\, dx\right)^{\frac{1}{2}q-2} \sum_{n=1}^N \left|\int_{h2^{-\nu}}^{h2^{-\nu+1}} f\bar{\varphi}_n\, dx\right|^2,$$

and, by Bessel's inequality, the last factor on the right does not exceed $\mathfrak{M}_2^2[f;\ h2^{-\nu},\ h2^{-\nu+1}]$. Writing $|f| = |f|\, x^{(q-2)/q} x^{-(q-2)/q}$, $|f|^2 = |f|^2 x^{2(q-2)/q} x^{-2(q-2)/q}$, and applying Hölder's inequalities, we obtain (3). Hence, arguing as in § 9.401, we obtain the inequality $\left(\sum_{n=1}^N |c_n|^q\right)^{1/q} \leqslant A_q\, \mathfrak{U}_q[f]$, and (1) follows on making N tend to ∞.

So far we have proved (1) with f^* replaced by $|f|$. To obtain the exact inequality (1) let us assume first that f is a step-function. Rearranging the order of the intervals where f is constant, which amounts to an one-to-one transformation of the interval $(0, h)$ into itself, we transform $|f|$ into f^*. At the same time $f(x)$ is transformed into a function $h(x)$, and the functions φ_n are transformed into functions ψ_n, which again form an orthogonal and normal system. Since the Fourier coefficient of f with respect to φ_n is equal to that of h with respect to ψ_n, (1) follows, in our case, from the weaker inequality previously established.

To prove (1) in the general case, let $\{f_k\}$ be a sequence of functions for each of which (1) is true, so that

$$\sum_{n=1}^{N} |c_n^k|^q \leqslant A_q^q \int_0^h f_k^{*q}\, x^{q-2}\, dx, \tag{4}$$

where $N>0$ is fixed, f_k^* is non-increasing and equimeasurable with $|f_k|$, and $c_1^k, c_2^k, \dots$ are the Fourier coefficients of f_k. Since any bounded f is the limit of a uniformly bounded and almost everywhere convergent sequence $\{f_k\}$ of step-functions, and since $c_n^k \to c_n$, $f_k^*(x) \to f^*(x)$ as $k \to \infty$, we may replace c_n^k, f_k by c_n, f in (4). If f is arbitrary, we put $f_k(x) = f(x)$ if $|f(x)| \leqslant k$ and $f_k(x) = 0$ if $|f(x)| > k$. Hence again $c_n^k \to c_n$, $f_k^*(x) \leqslant f_{k+1}^*(x) \to f^*(x)$, and, since the f_k are bounded, (4) is true for f. The inequality (1) follows on making $N \to \infty$.

To prove (2) let us fix $N>0$ and put $f_N = c_1\varphi_1 + \dots + c_N\varphi_N$. We verify that

$$\mathfrak{U}_p[f_N^*] = \operatorname{Sup} \int_0^h f_N^* g\, dx \qquad \text{for all } g \geqslant 0 \text{ with } \mathfrak{U}_{p'}[g] \leqslant 1.$$

It is even sufficient to restrict g to the domain of step-functions. A moment's consideration shows that, then, $\int_0^h f_N^* g\, dx = \int_0^N \bar{f}_N \gamma\, dx$, where the absolute value of the function $\gamma(x) = \gamma(x; g, N)$ is equimeasurable with g. Denoting the Fourier coefficients of γ by d_n, we have

$$\mathfrak{U}_p[f_N^*] = \operatorname*{Sup}_g \int_0^h \bar{f}_N \gamma\, dx = \operatorname*{Sup}_g \left| \sum_{n=1}^{N} \bar{c}_n d_n \right| \leqslant \tag{5}$$

$$\leqslant \operatorname*{Sup}_g \left(\sum_{n=1}^{N} |c_n|^p \right)^{1/p} \left(\sum_{n=1}^{N} |d_n|^{p'} \right)^{1/p'} \leqslant \operatorname*{Sup}_g \mathfrak{N}_p[c]\, A_{p'}\, \mathfrak{U}_{p'}[\gamma^*] =$$

$$= \operatorname*{Sup}_g A_{p'}\, \mathfrak{N}_p[c]\, \mathfrak{U}_{p'}[g^*] \leqslant \operatorname*{Sup}_g A_{p'}\, \mathfrak{N}_p[c]\, \mathfrak{U}_{p'}[g] \leqslant A_{p'}\, \mathfrak{N}_p[c].$$

Since $\mathfrak{N}_p[c] < \infty$ involves $\mathfrak{N}_2[c] < \infty$, there is a sequence $\{f_{N_k}(x)\}$ which converges almost everywhere to $f(x)$, and so $f_{N_k}^*(x) \to f^*(x)$ for almost every x. Comparing the extreme terms of (5) and putting $N = N_k$, we obtain (2) by an application of Fatou's lemma.

The reader will easily convince himself that $A_q \leqslant M^{(q-2)/q}\alpha_q$ and $A'_p \leqslant M^{(2-p)/p}\alpha'_p$, where α_q depends only on q, and α'_p only on p.

9.5. Theorems of Hardy and Littlewood [1]. The theorems established in the previous paragraph are extensions to general orthogonal systems of results which had been obtained previously for the system $1, e^{ix}, e^{-ix}, e^{2ix}, \dots$ by Hardy and Littlewood. This special case, however, is of independent interest, for the results may be stated in a different form and give the solution of an interesting problem. It will be convenient to change the notation of the previous paragraph slightly.

Given a sequence $c_0, c_1, c_{-1}, c_2, c_{-2}, c_3, \dots$ let $c_0^* \geqslant c_1^* \geqslant c_{-1}^* \geqslant c_2^* \geqslant \dots$ be the sequence $|c_0|, |c_1|, |c_{-1}|, \dots$ arranged in the descending order of magnitude. Similarly, given a function $f(x)$, $-\pi \leqslant x \leqslant \pi$, we shall denote by $f^*(x)$, $-\pi \leqslant x \leqslant \pi$, the function which is equimeasurable with $|f(x)|$ and even; for $0 \leqslant x \leqslant \pi$, $f^*(x)$ may be defined as the function inverse to $\frac{1}{2}|\mathrm{E}(|f| > y)|$. We put

$$\mathfrak{B}_r[c] = \left\{ \sum_{n=-\infty}^{+\infty} |c_n|^r (|n|+1)^{r-2} \right\}^{1/r}, \quad \mathfrak{U}_r[f] = \left\{ \int_{-\pi}^{\pi} |f|^r |x|^{r-2}\, dx \right\}^{1/r}. \tag{1}$$

If, for a moment, we denote the sequence $c_0^*, c_1^*, c_{-1}^*, c_2^*, \dots$ by $d_1^*, d_2^*, d_3^*, \dots$, then the ratio $\sum_{-\infty}^{+\infty} c_n^{*r}(|n|+1)^{r-2} \Big/ \sum_1^{\infty} d_n^{*r} n^{r-2}$ is contained between two positive numbers depending exclusively on r. Thence we see that Theorems 9.4 remain true for the system $1, e^{ix}, e^{-ix}, \dots$ if $\mathfrak{B}_r$ is given by the first formula (1). Similarly Theorems 9.43 are true for this system if the interval $(0, h)$ is replaced by $(-\pi, \pi)$ and $\mathfrak{U}_r$ is defined by the second formula (1).

We know that a necessary and sufficient condition that a sequence $c_0, c_1, c_{-1}, \dots$ should be that of Fourier coefficients of an $f \in L^2$ is that $\Sigma |c_n|^2 < \infty$. This condition bears on the moduli of the c_n, so that a necessary and sufficient condition that the numbers $c_0, c_1, c_{-1}, \dots$ should be, *for every variation* of their arguments, the Fourier coefficients of an $f \in L^2$, is again $\Sigma |c_n|^2 < \infty$. We ask whether anything similar is true for other classes L^r. The answer is negative: there can be no such condition for $r \neq 2$. For let us consider the series

[1]) Hardy and Littlewood [10], [15]; see also Gabriel [1], Mulholland [1].

$$(2) \qquad \sum_{n=1}^{\infty} n^{-\alpha} e^{inx}, \qquad \sum_{n=1}^{\infty} \pm n^{-\alpha} e^{inx} \qquad (0 < \alpha < 1).$$

If $\alpha = {}^3/_4$, the first series belongs to L^q if $q < 4$ only (§ 5.7.3), while the second belongs, for a special sequence of signs, to every L^q (§§ 5.6, 5.61) so that two functions, one of which belongs to L^q while the other does not, may have the same $|c_n|$. If $\alpha = {}^1/_4$, the first series in (2) belongs to L^p for $p < {}^4/_3$, while the second need not be a Fourier series.

These facts suggest a change in the problem. Now we shall vary not only the arguments of the c_n but also their order, and we ask when the new sequences will be those of Fourier coefficients, with respect to the system $1, e^{ix}, e^{-ix}, \dots,$ of functions belonging to L^r.

(i) *A necessary and sufficient condition that the c_n should be, for every variation of their arguments and arrangement, the Fourier coefficients of a function $f \in L^q$, is that $\mathfrak{B}_q[c^*] < \infty$; and then*

$$(3) \qquad \mathfrak{M}_q[f] \leqslant A_q \mathfrak{B}_q[c^*]$$

for every such f, where A_q depends on q only.

(ii) *A necessary and sufficient condition that the c_n should be, for some variation of their arguments and arrangement, the Fourier coefficients of an $f \in L^p$, is that $\mathfrak{B}_p[c^*] < \infty$; and then*

$$(4) \qquad \mathfrak{B}_p[c^*] \leqslant A'_p \mathfrak{M}_p[f]$$

for every such f, where A'_p depends on p only.

For the proof we shall require the following lemmas:

9.501. (i) *If $a_1 \geqslant a_2 \geqslant \dots \to 0$, a necessary and sufficient condition that the function $g(x) = \Sigma a_n \cos nx$ should belong to L^r, $r > 1$, is that the expression $S_r = \Sigma a_n^r n^{r-2}$ should be finite*

(ii) *The result remains true for sine series.*

Let $G(x)$ denote the integral of g, and $H(x)$ the integral of $|g|$, over $(0, x)$; let $A_n = a_1 + a_2 + \dots + a_n$. By $B_1, B_2, \dots$ we shall denote positive numbers which are either absolute constants or depend on r only. If $g \in L^r$, the series defining g is $\mathfrak{S}[g]$ (this is a corollary of the following proposition which will be established in Chapter XI: if a trigonometrical series converges,

except at a finite number of points, to an integrable function f, the series is $\mathfrak{S}[f]$), and so

$$G(x) = \int_0^x g(t)\,dt = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx, \quad G\left(\frac{\pi}{n}\right) =$$

$$= \sum_{m=1}^{n-1} \left(\frac{a_m}{m} - \frac{a_{m+n}}{m+n} + \frac{a_{m+2n}}{m+2n} - \dots\right) \sin\frac{m\pi}{n} \geqslant \sum_{m=1}^{n-1} \left(\frac{a_m}{m} - \frac{a_{m+n}}{m+n}\right) \sin\frac{m\pi}{n} \geqslant$$

$$\geqslant B_1 \sum_{[n/3]+1}^{[2n/3]} \left(\frac{a_m}{m} - \frac{a_{m+n}}{m+n}\right) \geqslant B_2 \sum_{[n/3]+1}^{[2n/3]} \frac{a_m}{m} \geqslant B_3\, a_n,$$

$$\sum_{n=2}^{\infty} a_n^r n^{r-2} \leqslant B_4 \sum_{n=2}^{\infty} n^{r-2} G^r\left(\frac{\pi}{n}\right) \leqslant B_4 \sum_{n=2}^{\infty} n^{r-2} H^r\left(\frac{\pi}{n}\right) \leqslant$$

$$\leqslant B_5 \sum_{n=2}^{\infty} \int_{\pi/n}^{\pi/(n-1)} \left\{\frac{H(x)}{x}\right\}^r dx = B_5 \int_0^{\pi} \left\{\frac{H(x)}{x}\right\}^r dx \leqslant B_6 \int_0^{\pi} |f|^r dx$$

(§ 4.17, $s = 0$) and the necessity of the condition in (i) is established. To show that the condition is sufficient we observe that

$$|g(x)| \leqslant \left|\sum_{\nu=1}^{n} a_\nu\right| + \left|\sum_{\nu=n+1}^{\infty} a_\nu \cos \nu x\right| \leqslant A_n + \frac{\pi a_n}{x}$$

(§ 1.22), and so $|g(x)| \leqslant B_7 A_n$ if $\pi/(n+1) \leqslant x \leqslant \pi/n$. Hence

$$(1) \qquad \int_0^{\pi} |g|^r dx = \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} |g|^r dx \leqslant B_8 \sum_{n=1}^{\infty} A_n^r n^{-2},$$

and it remains to show that the last series converges whenever $S_r < \infty$. Let $f(x)$ denote the function which is equal to a_n for $n-1 \leqslant x < n$, $n = 1, 2, \dots$, and let $F(x)$ be the integral of f over $(0, x)$. $S_r < \infty$ implies that $f^r(x)\, x^{r-2} \in L(0, \infty)$, and so, by Theorem 4.17 with $s = r - 2$, $\{F(x)/x\}^r x^{r-2} = F^r(x)\, x^{-2} \in L(0, \infty)$. Since the last relation is equivalent to the convergence of the series $\Sigma A_n^r n^{-2}$, lemma (i) follows. Lemma (ii) may be obtained by a similar argument, or, still simpler, may be deduced from (i) using Theorem 7.21.

9.502. Now we are in a position to prove Theorems 9.5. That the condition of Theorem 9.5(i) is sufficient follows from Theorem 9.4(i), whence we also deduce the inequality 9.5(3). To prove that the condition is necessary, consider the series $\Sigma c_n^* e^{inx}$ and $\Sigma c_{-n}^* e^{inx}$. If both of them belong to L^q, so does their sum

$$\sum_{n=-\infty}^{+\infty} (c_n^* + c_{-n}^*)\, e^{inx} = 2\left[c_0^* + \sum_{n=1}^{\infty} \tfrac{1}{2}(c_n^* + c_{-n}^*)\cos nx\right],$$

and from § 9.501(i) we obtain $\mathfrak{V}_q[c^*] < \infty$.

Theorem 9.4(ii) shows that the condition of Theorem 9.5(ii) is necessary. That it is also sufficient follows from the fact that the series $\sum_{n=-\infty}^{+\infty} c_n^* e^{inx}$ belongs to L^p if $\mathfrak{V}_p[c^*] < \infty$ (§ 9.501).

9.51 [1]). The following two theorems, in which we consider 'rearrangements' not of the Fourier coefficients but of the values the function, are, in a sense, reciprocals of Theorems 9.5.

(i) *A necessary and sufficient condition that* $\mathfrak{N}_q[c]$ *should be finite for* a l l $f(x)$ *having the same* $f^*(x)$, *is that* $\mathfrak{U}_q[f^*]$ *should be finite, and then*

(1) $$\mathfrak{N}_q[c] \leqslant A_q\, \mathfrak{U}_q[f^*].$$

(ii) *A necessary and sufficient condition that* $\mathfrak{N}_p[c]$ *should be finite for* s o m e $f(x)$ *with a given* $f^*(x)$, *is that* $\mathfrak{U}_p[f^*]$ *should be finite, and then*

(2) $$\mathfrak{U}_p[f^*] \leqslant A'_p\, \mathfrak{N}_p[c].$$

The proofs of (i) and (ii) are similar to those of Theorems 9.5 and are even a little easier since $f^*(x)$, unlike c_n^*, is a symmetrical function of its argument. The only thing we need is the following lemma: *if a function* $g(x)$, $|x| \leqslant \pi$, *is non-negative, even, and decreases in* $(0, \pi)$, *and if* a_n *are the cosine coefficients of* g, *then a necessary and sufficient condition that* $\mathfrak{N}_r[a] < \infty$, $r > 1$, *is that the function* $g^r(x)\, x^{r-2}$ *should be integrable.* We shall only sketch the proof which runs on the same line as in § 9.501. Denoting by $G(x)$ the integral of g over $(0, x)$, we shall show that

(3) $$|a_n| \leqslant 2\, G\left(\frac{\pi}{n}\right), \quad A_n \geqslant B_{10}\, g\left(\frac{\pi}{n}\right),$$

where $A_n = |a_0| + |a_1| + \ldots + |a_n|$. The first inequality follows from the formula

$$\frac{\pi}{2} a_n = \int_0^{\pi/n} g(x) \cos nx\, dx + \int_{\pi/n}^{\pi} g(x) \cos nx\, dx,$$

[1]) H a r d y and L i t t l e w o o d [10], [15].

where the last term on the right is, by the second mean-value theorem, less than $g(\pi/n)\cdot(2/n) \leqslant G(\pi/n)$ in absolute value. To prove the second inequality we notice that $\frac{1}{2}a_0 + a_1 + \ldots + a_{n-1} + \frac{1}{2}a_n$ is equal to

$$\frac{2}{\pi}\int_0^{\pi} g(t)\, \frac{\sin nt}{2\operatorname{tg}\frac{1}{2}t}\, dt \geqslant \frac{2}{\pi}\int_0^{\pi/n} \left[\frac{g(t)}{2\operatorname{tg}\frac{1}{2}t} - \frac{g(t+\pi/n)}{2\operatorname{tg}\frac{1}{2}\cdot(t+\pi/n)}\right] \sin nt\, dt \geqslant$$

$$\geqslant B_9 \int_0^{\pi/2n} \frac{g(t)}{t} \sin nt\, dt \geqslant B_{10}\, g\left(\frac{\pi}{2n}\right) \geqslant B_{10}\, g\left(\frac{\pi}{n}\right).$$

Now it is sufficient to observe that, if $g^r x^{r-2}$ is integrable, so is $G^r(x)\, x^{-2}$, hence $\Sigma\, G^r(\pi/n) < \infty$, and, in view of the first inequality in (3), $\mathfrak{N}_r[a] < \infty$. Conversely, if $\mathfrak{N}_r[a] < \infty$, then $\Sigma\, \{A_n/n\}^r < \infty$, (this is an easy consequence of Theorem 4.17 with $s = 0$) and the second inequality in (3) gives $\Sigma\, n^{-r} g^r(\pi/n) < \infty$. Since $g(x)$ is non-increasing we obtain that $g^r(x)\, x^{r-2}$ is integrable.

9.6. Banach's theorems on lacunary coefficients [1]). We know that a necessary condition for a sequence $\{a_n, b_n\}$ to be that of the Fourier coefficients of an integrable function f, is $|a_n| + |b_n| \to 0$. If a_n, b_n are to be the Fourier coefficients of a continuous f, the series $a_1^2 + b_1^2 + a_2^2 + b_2^2 + \ldots$ must converge. The converse propositions are obviously false, but we will prove that, at least for some values of n, the Fourier coefficients of integrable, or continuous, functions may be prescribed, roughly speaking, arbitrarily.

(i) *Let* $\{n_i\}$ *be any sequence of positive integers such that* $n_{i+1}/n_i > \lambda > 1$, $i = 1, 2, \ldots$, *and let* $\{x_i, y_i\}$ *be an arbitrary sequence such that* $(x_1^2 + y_1^2) + (x_2^2 + y_2^2) + \ldots < \infty$. *Then there exists a continuous* f *with Fourier coefficients* a_n, b_n *satisfying the equations* $a_{n_i} = x_i$, $b_{n_i} = y_i$, $i = 1, 2, \ldots$

(ii) *If* $\{n_i\}$ *satisfies the same conditions as above and if* $x_i \to 0$, $y_i \to 0$, *there exists an integrable* f *such that* $a_{n_i} = x_i$, $b_{n_i} = y_i$, $i = 1, 2, \ldots$

We begin the proof of (i) by two remarks.

[1]) B a n a c h [1], S z i d o n [3], [4], V e r b l u n s k y [2].

(a) It is sufficient to prove the existence of a *bounded* f with the prescribed coefficients. For let $\{\varepsilon_k\}$ be a convex sequence tending to 0 and such that the series with terms $(x_i^2+y_i^2)/\varepsilon_{n_i}^2$ converges[1]. If we can find a bounded function $g\sim\frac{1}{2}a_0+(a_1\cos x+b_1\sin x)+\dots$, such that $a_{n_i}=x_i/\varepsilon_{n_i}$, $b_{n_i}=y_i/\varepsilon_{n_i}$, then $\frac{1}{2}a_0\,\varepsilon_0+(a_1\cos x+b_1\sin x)\,\varepsilon_1+\dots$ is the Fourier series of a continuous function (§ 4.65), and the terms with indices n_i in this series are $(x_i\cos n_i x+y_i\sin n_i x)$.

(b) It is sufficient to prove that, for every integer $k>0$, there exists a function $f_k(x)\sim\frac{1}{2}a_0^k+(a_1^k\cos x+b_1^k\sin x)+\dots$, such that $a_{n_i}^k=x_i$, $b_{n_i}^k=y_i$, $1\leqslant i\leqslant k$, and that $|f_k(x)|\leqslant C$, where C is a constant independent of k. In fact, let us assume, as we may, that $a_0^j=0$, $j=1,2,\dots$, and let $F_k(x)$ be the integral of f_k over $(0,x)$. Since the f_k are uniformly bounded, the functions F_k are uniformly continuous and we may find a subsequence $\{F_{m_k}\}$ converging uniformly to an $F(x)\in\text{Lip}\,1$. The Fourier coefficients of F are limits of the corresponding Fourier coefficients of F_{m_k} as $k\to\infty$, and so the bounded function $f(x)=F'(x)$ has the prescribed coefficients for all the indices n_i.

Now we shall prove a number of lemmas.

9.601. *If* $n_{k+1}/n_k>\lambda>1$, *and if the series* $\Sigma\,(a_k^2+b_k^2)$ *converges, then*

$$(1)\qquad \sum_{k=1}^{\infty}(a_k\cos n_k x+b_k\sin n_k x)$$

is the Fourier series of a function $f(x)$ *belonging to every class* L^r, *and*

$$(2)\qquad \left\{\frac{1}{\pi}\int_0^{2\pi}|f(x)|^r dx\right\}^{1/r}\leqslant A_{r,\lambda}\left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2},$$

where $A_{r,\lambda}$ *depends only on* r *and* λ.

This lemma will be required only in the case $r=4$, but the proof does not become simpler by considering any special value of r. Since the left-hand side of (2), multiplied by $2^{-1/r}$, is an increasing function of r (§ 4.15), it is sufficient to consider the

[1] We may find first a sequence $\{\varepsilon'_k\}$, $\varepsilon'_k>0$, tending to 0 and then majorise it by a convex $\{\varepsilon_k\}$.

values $r=2h$, $h=1,2,\dots$ Suppose first that the series (1) converges absolutely, and let $F(z)=\Sigma\,c_k z^{n_k}$ be the power series the real part of which, for $z=e^{ix}$, is (1). Then

$$F^h(z)=\sum_{\nu=0}^{\infty}d_\nu z^\nu,$$

where $d_\nu=0$ if ν is not of the form

$$(3)\quad \alpha_1 n_{k_1}+\alpha_2 n_{k_2}+\dots,\quad \text{with } n_{k_1}>n_{k_2}>\dots,\quad \alpha_i>0,\quad \alpha_1+\alpha_2+\dots=h.$$

Now we observe that, if λ is sufficiently large, $\lambda\geqslant\lambda_0$, then every positive integer can be represented at most once in the form (3). For otherwise we should have an equation $\beta_1 n_{k_1}+\beta_2 n_{k_2}+\dots=0$, where $n_{k_1}>n_{k_2}>\dots$, $0\leqslant|\beta_i|\leqslant h$, $\beta_1\neq 0$, and so also $n_{k_1}\leqslant h(n_{k_2}+n_{k_3}+\dots)$, $1<h(\lambda^{-1}+\lambda^{-2}+\dots)$, which is impossible if $\lambda\geqslant\lambda_0=h+1$.

By Parseval's theorem, $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}|F^h(e^{ix})|^2dx=\sum_{\nu=0}^{\infty}|d_\nu|^2$, where, if ν is of the form (3),

$$d_\nu=\frac{h!}{\alpha_1!\,\alpha_2!\dots}c_{k_1}^{\alpha_1}c_{k_2}^{\alpha_2}\dots,\qquad |d_\nu|^2\leqslant h!\,\frac{h!}{\alpha_1!\,\alpha_2!\dots}|c_{k_1}|^{2\alpha_1}|c_{k_2}|^{2\alpha_2}\dots$$

Hence, if $\lambda\geqslant\lambda_0$, $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}|F(e^{ix})|^{2h}\leqslant h!\left(\sum_{k=1}^{\infty}|c_k|^2\right)^h$, and since we have $|f(x)|\leqslant|F(e^{ix})|$, $c_k=a_k-ib_k$, the inequality (2) follows with $A_{2h,\lambda}^{2h}=2h!$

To remove the condition concerning the absolute convergence of (1), we apply (2) to the function $f(r,x)=\Sigma\,(a_k\cos n_k x+b_k\sin n_k x)\,r^{n_k}$ and then make $r\to 1$.

To prove (2) for general $\lambda>1$, we break up (1) into a finite number, say s, of series, for each of which the number λ is $\geqslant h+1$. Correspondingly $f=f_1+f_2+\dots+f_s$. Since

$$\mathfrak{M}_{2h}[f_i]\leqslant(2h!)^{1/2h}\left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2},\qquad \mathfrak{M}_{2h}[f]\leqslant\sum_{i=1}^{s}\mathfrak{M}_{2h}[f_i],$$

we obtain (2) with $A_{2h,\lambda}=s\,(2h!)^{1/2h}$.

9.602. *Under the conditions of the preceding lemma,*

$$(1)\qquad \frac{1}{\pi}\int_0^{2\pi}|f(x)|\,dx \geqslant B_\lambda\left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2},$$

where B_λ depends only on λ.

If J_r denotes the left-hand side of 9.601(2) then, by Hölder's inequality, $J_2 \leqslant J_1^{1/3} J_4^{2/3}$ and so $J_1 \geqslant J_2^3/J_4^2$.

To prove (1) we apply the preceding lemma and observe that $A_{2,\lambda}=1$.

9.603. Let $n_1, n_2, \dots, n_k, \dots$ be the sequence of Theorem 9.6(i). Let us fix an integer $k>0$ and let B denote the set of all periodic functions f, $|f|\leqslant 1$. We put

$$x_i=\frac{1}{\pi}\int_0^{2\pi} f\cos n_i x\,dx,\qquad y_i=\frac{1}{\pi}\int_0^{2\pi} f\sin n_i x\,dx,\qquad 1\leqslant i\leqslant k,$$

and denote by E the set, situated in the $2k$-dimensional space, of points $P(x_1, y_1, \dots, x_k, y_k)$ obtained in this way. This set is convex, that is, if two points P_1, P_2 belong to it, so does every point $tP_1+(1-t)P_2$, $0\leqslant t\leqslant 1$, of the segment P_1P_2. An argument similar to that used in § 9.6(b) shows that E is closed. We will now prove the following lemma.

E contains a whole 'sphere' $x_1^2+y_1^2+\dots+x_k^2+y_k^2\leqslant R^2$, where $R=R_\lambda$ is a constant depending on λ but not on k.

Let $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k$ be an arbitrary set of numbers such that $\alpha_1^2+\dots+\beta_k^2=1$ and let

$$T(x)=(\alpha_1\cos n_1x+\beta_1\sin n_1x)+\dots+(\alpha_k\cos n_kx+\beta_k\sin n_kx).$$

If $P(x_1, \dots, y_k)$ corresponds to an $f\in B$, we have the Parseval equation

$$(\alpha_1x_1+\beta_1y_1)+\dots+(\alpha_kx_k+\beta_ky_k)=\frac{1}{\pi}\int_0^{2\pi} fT\,dx.$$

For $f=\operatorname{sign} T\in B$ the last integral becomes $\pi^{-1}\mathfrak{M}[T]\geqslant R^2$, where $R=B_\lambda^{1/2}$ (§ 9.602). If we put $f=\theta\operatorname{sign} T$, where θ has a suitable value between 0 and 1, we obtain that $\alpha_1x_1+\dots+\beta_ky_k=R^2$.

This fact may be interpreted geometrically[1]) as follows: on every 'plane' $\alpha_1x_1+\dots+\beta_ky_k=R^2$, 'tangent' to the 'sphere' $(S_1)\,x_1^2+\dots+y_k^2\leqslant R^2$, there exists a point $P\in E$.

Let us assume, contrary to what we intend to prove, that not all points of S_1 belong to E, and let P_0 be a point on the boundary of E nearest to the origin O. Let S_0 be the sphere with centre at the origin, having P_0 on its 'surface', P_1 the point where the radius OP_0 meets the surface of S_1, P a point belonging to E and situated on the plane Π_1 tangent to S_1 at P_1. It is obvious that $S_0\subset E$, and that no point $Q\neq P_0$ on the segment P_0P_1 belongs to E (for, otherwise, it would follow from the convexity of E that P_0 is a point interior to E[2])). The line PP_1 lies on Π_1, and so PP_0 cannot lie on the plane Π_0 tangent to S_0 at P_0 since Π_0 and Π_1 have no point in common. Thus the line PP_0 meets S_0 in more than one point. Thence we deduce, by continuity, that if $Q\neq P_0$ is a point on P_0P_1 sufficiently near P_0, the line QP must have a point P_0' in common with S_0. It is easy to see that Q lies *between* P and P_0' (for P_0' and Q lie on different sides of Π_0), and since $P_0'\in E$, $P\in E$, so does Q. Here we have a contradiction since no point $Q\neq P_0$ on the segment P_0P_1 belongs to E. This establishes the lemma.

9.604. Now we are in a position to prove Theorem 9.6(i). We put $(x_1^2+y_1^2)+\dots+(x_k^2+y_k^2)=h_k^2$. From the last lemma follows the existence of a function $f_k(x)$, $|f_k(x)|\leqslant h_k/R$, such that the Fourier coefficients of f_k on the places n_i, $1\leqslant i\leqslant k$, are equal to x_i, y_i. In virtue of remark (b) of § 9.6, this completes the proof of the theorem.

Corollary. Let $\varphi(u)$ be an arbitrary function tending to $+\infty$ with u. Then there exists a continuous function f having the Fourier coefficients a_n, b_n such that the series $\Sigma r_n^2\varphi(1/r_n)$, where $r_n^2=a_n^2+b_n^2$, diverges[3]).

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[1]) We use the geometrical language to make more intuitive the argument, which might be given a purely analytic form.

[2]) If P' is an arbitrary point situated sufficiently near to P_0, the line QP' meets S_0, and so $P'\in E$.

[3]) Gronwall [1], Szidon [4], Paley [3]. Putting $\varphi(u)=\log u$, we obtain an f such that $r_1^{2-\varepsilon}+r_2^{2-\varepsilon}+\dots=\infty$ for every $\varepsilon>0$ (§ 5.33).

For let $\{\alpha_k, \beta_k\}$ be an arbitrary sequence of numbers such that $\rho_1^2+\rho_2^2+\ldots<\infty$, $\rho_1^2\varphi(1/\rho_1)+\rho_2^2\varphi(1/\rho_2)+\ldots=\infty$, where $\rho_k^2=\alpha_k^2+\beta_k^2$. There exists a continuous f such that $a_{2^k}=\alpha_k$, $b_{2^k}=\beta_k$, say. Since $\rho_1^2\varphi(1/\rho_1)+\rho_2^2\varphi(1/\rho_2)+\ldots$ diverges, so does $r_1^2\varphi(1/r_1)+r_2^2\varphi(1/r_2)+\ldots$

9.61. The proof of Theorem 9.6(ii) is easier than that of Theorem 9.6(i) since we are able to give the required series explicitely [1]). First we prove the following lemma: *For any bounded sequence $\{x_i, y_i\}$ there exists a Fourier-Stieltjes series having x_i, y_i as the coefficients with the indices n_i.* It will be convenient to write x_{n_i}, y_{n_i} instead of x_i, y_i. We may suppose that $\rho_{n_i}^2=x_{n_i}^2+y_{n_i}^2\leqslant 1$. Let us assume first that $\lambda>3$. We put $x_{n_i}\cos n_i x+y_{n_i}\sin n_i x=\rho_{n_i}\cos(n_i x+\varphi_{n_i})$ and consider the partial products p_k of the product

$$(1)\qquad P=\prod_{i=1}^{\infty}\{1+\rho_{n_i}\cos(n_i x+\varphi_{n_i})\}.$$

Multiplying out these products, we see that no reduction of terms takes place (§ 6.4) and that the polynomial p_k is a partial sum of p_{k+1}. Making $k\to\infty$ we obtain, quite formally, a trigonometrical series. Since some partial sums, viz. p_k, are non-negative, this series is a Fourier-Stieltjes series (§ 4.39). Moreover the coefficients with suffixes n_i are x_{n_i}, y_{n_i}. It is important to observe that, if λ is large enough, $\lambda>\lambda_0(\varepsilon)$, the indices of terms different from 0 belong all to the intervals $(n_i(1-\varepsilon),\ n_i(1+\varepsilon))$, for every $0<\varepsilon<1$ (§ 6.4).

In the general case $\lambda>1$, we break up $\{n_i\}$ into r sequences $n_1^1, n_2^1, \ldots;\ n_1^2, n_2^2, \ldots;\ \ldots;\ n_1^r, n_2^r, \ldots$ in such a way that $n_{i+1}^s/n_i^s>\mu$, $i=1,2,\ldots$, $1\leqslant s\leqslant r$, $\mu>3$ being a large number which we shall define in a moment. Let P_s denote the product analogous to (1), consisting of factors $1+\rho_m\cos(mx+\varphi_m)$, where m runs through the sequence $n_1^s, n_2^s, \ldots$ We shall prove that $P_1+P_2+\ldots+P_r$ gives the required Fourier-Stieltjes series. In fact, if μ is large enough, the indices occuring in the series obtained from P_s all belong to the intervals $(n_i^s/\sqrt{\lambda},\ n_i^s\sqrt{\lambda})$, $i=1,2,\ldots$, so that the series $P_1, P_2, \ldots, P_r$ do not overlap. Since in the series P_s the terms with indices n_i^s have the coefficients $x_{n_i^s}, y_{n_i^s}$, the lemma follows.

1) Szidon [3].

To prove the theorem, let $\{\varepsilon_k\}$ be a convex sequence tending to 0 and such that $\{x_{n_i}/\varepsilon_{n_i}\}$ and $\{y_{n_i}/\varepsilon_{n_i}\}$ are bounded. If for a Fourier-Stieltjes series $\frac{1}{2}a_0+(a_1\cos x+b_1\sin x)+\ldots$ we have $a_{n_i}=x_{n_i}/\varepsilon_{n_i}$, $b_{n_i}=y_{n_i}/\varepsilon_{n_i}$, the series $\frac{1}{2}a_0\varepsilon_0+(a_1\cos x+b_1\sin x)\varepsilon_1+\ldots$ is the required Fourier series (§§ 4.64, 5.12).

9.7. Wiener's theorem on functions of bounded variation. Let f be a function of bounded variation, a_n, b_n its Fourier coefficients, and $\rho_n^2=a_n^2+b_n^2$, $\rho_n\geqslant 0$. We know that, if f is discontinuous, then $n\rho_n\neq o(1)$ (§ 2.632), but since this inequality may occur also for f continuous (§ 5.7.14), it is not a necessary and sufficient condition for the discontinuity of f. It is interesting that such a condition may be obtained if the expressions $n\rho_n$ are replaced by their arithmetic means:

A necessary and sufficient condition that a function f of bounded variation be continuous is that $A_n=(\rho_1+2\rho_2+\ldots+n\rho_n)/n\to 0$ [1]).

We first prove the theorem in the following form: A necessary and sufficient condition for a function f of bounded variation to be continuous, is

$$(1)\qquad n\sum_{k=1}^{\infty}\rho_k^2\sin^2\frac{k\pi}{2n}\to 0\quad\text{as } n\to\infty.$$

Let $\varphi_n(u)=[f(u+\pi/n)-f(u)]^2+[f(u+2\pi/n)-f(u+\pi/n)]^2+\ldots+[f(u+2\pi)-f(u+\pi(2n-1)/n)]^2$. Using Parseval's formula, we obtain

$$(2)\qquad \int_0^{2\pi}\varphi_n(u)\,du=8\pi n\sum_{k=1}^{\infty}\rho_k^2\sin^2\frac{k\pi}{2n}$$

(§ 6.31). If f is continuous, $\omega(\delta)$ the modulus of continuity, and V the total variation of f, then, for every n, we have $\varphi_n(u)\leqslant\omega(\pi/n)\,V\to 0$ as $n\to\infty$, so that the right-hand side of (2) tends to 0, i. e. we have (1). Conversely, if f is discontinuous at a point ξ, $f(\xi+0)-f(\xi-0)=d\neq 0$, $2f(\xi)=f(\xi+0)+f(\xi-0)$, then, if n is large enough and (α,β) is any interval of length π/n containing ξ, we have $|f(\beta)-f(\alpha)|>d/3$. It follows that, if n is large, $\varphi_n(u)\geqslant d^2/9$ for every u and so the right-hand side of (2) does not tend to 0 as $n\to\infty$.

1) Wiener [2].

We shall now show that, if C_n is the left-hand side of (1), the relations $A_n \to 0$, $C_n \to 0$ are equivalent. Let B_n denote the ratio $(\rho_1^2 + 2^2\rho_2^2 + \dots n^2\rho_n^2)/n$. We shall show first that the relations $A_n \to 0$, $B_n \to 0$ are equivalent. Since the expressions $k\rho_k$ are bounded, the formula $A_n \to 0$ implies $B_n \to 0$. Applying Schwarz's inequality to the sum $1\cdot\rho_1 + 1\cdot 2\rho_2 + \dots + 1\cdot n\rho_n$, we obtain that $A_n \leqslant B_n^{1/2}$, so that $B_n \to 0$ implies $A_n \to 0$.

It remains to prove that the relations $B_n \to 0$, $C_n \to 0$ are equivalent. Let us take only the first n terms in the series (1). Since $\sin u \geqslant 2u/\pi$ for $0 \leqslant u \leqslant \pi/2$, we see that $B_n \leqslant C_n$, and so, if $C_n \to 0$, then $B_n \to 0$. Observing that $\rho_k \leqslant \sqrt{2}\, V/k$ (§ 2.213) and breaking up the sum C_n into two, the first consisting of terms with indices $\leqslant nr$, where $r > 0$ is an integer, we see that

$$C_n \leqslant n \sum_{k=1}^{nr} \rho_k^2 \left(\frac{k\pi}{2n}\right)^2 + 2V^2 n \sum_{k=nr+1}^{\infty} \frac{1}{k^2}.$$

The first term on the right is equal to $B_{nr}\cdot \pi^2 r/4 \to 0$, if $B_n \to 0$. The second term is $< 2V^2/r$ and so is small for r large but fixed. This shows that $C_n \to 0$ if $B_n \to 0$, and the proof is complete.

9.8. Integrals of fractional order. Let $f(x)$ be integrable in an interval (a, b). Let $F_1(x)$ denote the integral of $f(t)$ over (a, x), $F_\alpha(x)$ the integral of $F_{\alpha-1}(t)$ over (a, x), $\alpha = 2, 3, \dots$ It can be verified by induction that

$$(1) \qquad F_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \qquad a \leqslant x \leqslant b,$$

where $\Gamma(\alpha) = (\alpha-1)!$. If $\Gamma(\alpha)$ denotes the Euler Gamma function, the formula (1) may be taken as a definition of $F_\alpha(x)$ for every $\alpha > 0$. From the results of § 2.11 we deduce that $F_\alpha(x)$ exists for almost every x and is itself integrable [1]); for $\alpha \geqslant 1$ it is even continuous.

This definition of a fractional integral is due to Riemann and Liouville [2]). In the theory of periodic functions it is not entirely satisfatory since $F_\alpha(x)$ is not, in general, a periodic func-

[1]) For $\Gamma(\alpha) F_\alpha(x) = \int_a^b g(x-t) f(t)\, dt$, where $g(u) = u^{\alpha-1}$ for $u > 0$ and $g(u) = 0$ elsewhere.

[2]) Riemann [2], Liouville [1].

tion if f is one. Moreover it makes $F_\alpha(x)$ depend on a particular value of a. For this reason we shall consider another definition, propounded by Weyl, and more convenient in the theory of trigonometrical series [1]).

Let $f(x)$ be an integrable function having the period 1. (It simplifies the notation slightly if we consider functions of period 1 and not 2π, but this point is plainly without importance). *We assume that the mean value of f over* $(0, 1)$ *is equal to* 0, so that the constant term of $\mathfrak{S}[f]$ vanishes. It follows that the integral f_1 of f is also periodic, whatever the constant of integration. If we choose this constant of integration in such a way that the integral of f_1 over $(0, 1)$ vanishes, then the integral f_2 of f_1 will also be periodic, and so on. Generally, having defined the periodic functions $f_1, f_2, \dots f_{\alpha-1}$, we define $f_\alpha(x)$ as that of the primitives of $f_{\alpha-1}$, whose integral over $(0, 1)$ vanishes. Hence, the Fourier expansion of $f_\alpha(x)$ does not contain the constant term. In other words, if $f \sim \sum_{n=-\infty}^{+\infty} c_n e^{2\pi i n x}$, $c_0 = 0$, then

$$(2) \qquad f_\alpha(x) = \sum_{n=-\infty}^{+\infty} c_n \frac{e^{2\pi i n x}}{(2\pi i n)^\alpha} = \int_0^1 f(t)\, \Psi_\alpha(x-t)\, dt,$$

$\Psi_\alpha(x)$ being the function which has the complex Fourier coefficients $\gamma_n^{(\alpha)} = (2\pi i n)^{-\alpha}$, $\gamma_0 = 0$ (§ 2.15 [2]), where the actual function Ψ_α, corresponding to the interval $0 \leqslant x \leqslant 2\pi$, is denoted by f_k). The formula (2) may be considered as a definition of $f_\alpha(x)$ for every $\alpha > 0$, if we put $\gamma_n = (2\pi n)^{-\alpha} \exp(-\frac{1}{2}\alpha\pi i)$, $\gamma_{-n} = \bar{\gamma}_n$, $n > 0$, $\gamma_0 = 0$. From Theorem 5.12 we see that there really exists an integrable function $\Psi_\alpha(x)$ with Fourier coefficients γ_n. The integral in (2) exists for almost every x (§ 2.11), and the series converges almost everywhere. This last fact follows easily from the results of § 3.7, if we apply them not to the factors $1/\log n$ as in § 3.71 but to the factors $n^{-\alpha}$.

Let us denote $f_\alpha(x)$ by $I_\alpha[f]$. From (2) we see that $I_\beta[I_\alpha[f(x)]] = I_{\alpha+\beta}[f]$, $\alpha > 0$, $\beta > 0$. Since, for $\alpha = 1, 2, \dots$, $I_\alpha[f]$ coincides with the ordinary integral, the most interesting is the

[1]) Weyl [1].

[2]) See Errata.

case $0<\alpha<1$. To find the actual form of $\Psi_\alpha(x)$ we consider the formula

$$(3)\qquad \int_0^\infty t^{\alpha-1} e^{-it}\,dt = e^{-\frac{\pi i\alpha}{2}}\,\Gamma(\alpha),\qquad 0<\alpha<1,$$

which is easily obtained from the equation $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx$ by integrating round the contour

$$0<\varepsilon\leqslant z\leqslant R;\quad z=Re^{i\vartheta},\quad 0\leqslant\vartheta\leqslant\tfrac12\pi;\quad z=ir,\quad R\geqslant r\geqslant\varepsilon;$$
$$z=\varepsilon e^{i\vartheta},\quad \tfrac12\pi\geqslant\vartheta\geqslant 0,$$

and then making ε and $1/R$ tend to 0. Making the substitution $t=2\pi mu$ in (3), and taking into account the last remark of § 2.85, we see that, for $0<x<1$,

$$(4)\qquad \Gamma(\alpha)\,\Psi_\alpha(x)=$$
$$=\lim_{n\to\infty}\{x^{\alpha-1}+(x+1)^{\alpha-1}+\ldots+(x+n)^{\alpha-1}-n^\alpha/\alpha\},\quad 0<\alpha<1.$$

It is easy to see that, if we omit the term $x^{\alpha-1}$ in the expression on the right, the limit, which we shall denote then by $\Gamma(\alpha)\,r_\alpha(x)$, exists uniformly in $0\leqslant x\leqslant 1$. Taking this into account and observing that in the integral in (2) we may substitue $f(t)\,\Psi_\alpha(x-t)$ for $f(x-t)\,\Psi_\alpha(t)$, we obtain from (2) and (4) that

$$(5)\qquad f_\alpha(x)=\frac{1}{\Gamma(\alpha)}\int_0^\infty f(x-t)\,t^{\alpha-1}\,dt=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^x f(t)\,(x-t)^{\alpha-1}\,dt.$$

It appears that the new definition differs from (1) in that the lower limit of integration is equal to $-\infty$. It must be remembered that the integrals (5) only converge owing to the fact that the mean value of f over $(0,1)$ vanishes.

Let $\Psi^*_\alpha(x)$, $-1<x<1$, be the function equal to 0 in $(-1,0)$ and to $x^{\alpha-1}/\Gamma(\alpha)$ in $(0,1)$. Since $\Psi_\alpha(x+1)=\Psi_\alpha(x)$, considering the cases $-1<x\leqslant 0$ and $0<x<1$ separately, we see that $\Psi_\alpha(x)-\Psi^*_\alpha(x)$ is regular and equal to the function $r_\alpha(x)$ for $-1<x<1$. If we replace Ψ_α by Ψ^*_α in the integral (2), the function f_α is changed into F_α from (1) (with $a=0$). Thence we conclude that the function $f_\alpha(x)-F_\alpha(x)$ is regular for $0<x<1$, and so the two definitions of a fractional integral are, after all, not so essentially different.

It is easy to define *derivatives* $f^\alpha(x)$ of fractional order. For the sake of simplicity, we confine ourselves to the case $0<\alpha<1$, which is the most interesting in applications; and we put $f^\alpha(x)=\dfrac{d}{dx}f_{1-\alpha}(x)$. It is easy to prove that, if $f_{1-\alpha}(x)$ is absolutely continuous (in particular, if f^α is continuous), then $f(x)$ is the α-th integral of f^α. In fact, from the definition of f^α we see that $\mathfrak{S}[f^\alpha]$ is obtained by term-by-term differentation of $\mathfrak{S}[f_{1-\alpha}]$. In other words, $\mathfrak{S}[f^\alpha]$ may be obtained from $\mathfrak{S}[f]$ by introducing into the latter series the factors $\gamma_n^{(-\alpha)}=(2\pi in)^\alpha$, and this shows that f is the α-th integral of $f^\alpha(x)$.

9.81. Integration of functions satisfying Lipschitz conditions [1]. (i) *Let* $0\leqslant\alpha<1$, $\beta>0$, $\alpha+\beta<1$. *If* $f\in\mathrm{Lip}\,\alpha$, *then* $f_\beta\in\mathrm{Lip}\,(\alpha+\beta)$. (ii) *Let* $0<\gamma<\alpha\leqslant 1$. *If* $f\in\mathrm{Lip}\,\alpha$, *then* f^γ *exists and belongs to* $\mathrm{Lip}\,(\alpha-\gamma)$.

Let $F(t)$ denote the integral of f over $(0,t)$, so that $F(x)-F(x-t)$ is a primitive function of $f(x-t)$ with respect to t. Integrating by parts the first integral in 9.8(5) and observing that $F(x)-F(x-t)$ vanishes for $t=0$, we obtain

$$(1)\qquad \Gamma(\beta)\,f_\beta(x)=(1-\beta)\int_0^\infty[F(x)-F(x-t)]\,t^{\beta-2}\,dt.$$

Let us write a similar equation for $f_\beta(x+h)$, $h>0$, and substract (1) from it. We have $\Gamma(\beta)\,[f_\beta(x+h)-f_\beta(x)]=A_h+B_h$, where A_h, B_h denote the integrals over $(0,h)$, (h,∞) respectively. The integrand of A_h may be represented in the form

$$(2)\qquad \begin{aligned}(1-\beta)\,t^{\beta-2}\{[F(x+h)-F(x)]-[F(x+h-t)-F(x-t)]\}&=\\ =(1-\beta)\,t^{\beta-1}[f(x+h-\theta t)-f(x-\theta t)]\ [2],&\end{aligned}$$

where $\theta,\theta_1,\ldots$ are numbers contained between 0 and 1. Since $|f(u_1)-f(u_2)|\leqslant M\,|u_1-u_2|^\alpha$, M denoting a constant, we find

[1]) Hardy and Littlewood [6]. A special case of (ii) will be found in Weyl [1].

[2]) Here we employ the mean-value theorem.

that (2) does not exceed $M(1-\beta)\,t^{\beta-1}h^{\alpha}$ in absolute value, and so $|A_h|\leqslant M(1-\beta)\,h^{\alpha+\beta}/\beta$.

The left-hand side of (2) may also be written in the form

$$(1-\beta)\,t^{\beta-2}\{[F(x+h)-F(x+h-t)]-[F(x)-F(x-t)]\}=$$
$$=(1-\beta)\,t^{\beta-2}h\,[f(x+\theta_1 h)-f(x+\theta_1 h-t)].$$

This expression does not exceed $M(1-\beta)\,t^{\alpha+\beta-2}h$ in absolute value and so $|B_h|\leqslant Mh^{\alpha+\beta}(1-\beta)/(1-\alpha-\beta)$. Collecting the results we see that $|f_\beta(x+h)-f_\beta(x)|\leqslant M_1 h^{\alpha+\beta}$, where M_1 is independent of x and h. This completes the proof of (i).

If $\alpha+\beta=1$, it is not difficult to obtain that the modulus of continuity $\omega(\delta; f_\beta)$ of f_β is $O(\delta\log 1/\delta)$.

Passing to the proof of (ii), we observe that, since $f^{\gamma}(x)=\dfrac{d}{dx}f_{1-\gamma}$, we have to prove that $f_{1-\gamma}$ possesses a derivative belonging to Lip $(\alpha-\gamma)$. Let us put $\beta=1-\gamma$ in the formula (1); differentiating the integral on the right with respect to x, we obtain

$$(3)\qquad \gamma\int_0^\infty [f(x)-f(x-t)]\,t^{-\gamma-1}\,dt.$$

Since $|f(x)-f(x-t)|\leqslant Mt^{\alpha}$ and f is bounded, the integral (3) converges uniformly in the neighbourhoods of $t=0$ and $t=\infty$, and so represents a continuous function $\varphi(x)$. It remains to show that $\varphi\in\text{Lip}(\alpha-\gamma)$. Let us replace x by $x+h$, $h>0$, in (3) and substract (3) from the new integral. Breaking up the interval of integration $(0,\infty)$ into two, $(0,h)$ and (h,∞), we have, as in the proof of (i), $\varphi(x+h)-\varphi(x)=A_h+B_h$. The integrand in A_h does not exceed $\gamma t^{-\gamma-1}[|f(x+h)-f(x+h-t)|+|f(x)-f(x-t)|]\leqslant 2M\gamma\, t^{\alpha-\gamma-1}$ in absolute value, and, consequently, $|A_h|\leqslant 2Mh^{\alpha-\gamma}\gamma/(\alpha-\gamma)$. The integrand of B_h does not exceed $2M\gamma h^{\alpha}t^{-\gamma-1}$, and $|B_h|\leqslant 2Mh^{\alpha-\gamma}$. Hence $f^{\gamma}\in\text{Lip}(\alpha-\gamma)$.

It has been proved by Hardy [4] that the Weierstrass series considered in § 2.9.3 is nowhere differentiable if $ab=1$. If $a=1/b$, that series may be considered as the $(1-\alpha)$-th integral of a trigonometrical series which is a linear combination of the series

$$(4)\qquad \sum_{n=1}^{\infty} b^{-\alpha n}\cos b^n x,\qquad \sum_{n=1}^{\infty} b^{-\alpha n}\sin b^n x.$$

Each of the series (4) belongs to Lip α (for the first of them this

was actually proved in § 2.9.3; the proof for the second remains essentially the same). This shows that the proposition (ii) is false for $\gamma=\alpha$: for a function $f\in\text{Lip}\,\alpha$, $0\leqslant\alpha<1$, there may be no point at which the derivative $f^{\alpha}(x)$ exists. The same example shows that proposition (i) fails for $\alpha+\beta=1$.

9.82. Integration of functions belonging to a class L^p [1]**.** In the rest of this chapter we abandon our convention concerning the use of the letters p, q, which may now denote any numbers greater than 1.

(i) *If $f\in L^p$, $p>1$, and $0<\alpha<1/p$, then $f_\alpha\in L^q$, where q is given by the formula $1/p-1/q=\alpha$. Moreover $\mathfrak{M}_q[f_\alpha; 0,1]\leqslant K\,\mathfrak{M}_p[f;0,1]$, where $K=K(p,q)$ depends only on p and q.*

(ii) *If $p>1$, $1/p<\alpha<1/p+1$, then $f_\alpha\in\text{Lip}\,(\alpha-1/p)$.*

We begin by proving (ii), which is comparatively easy. In virtue of Theorem 9.81(i), it is sufficient to consider the case $1/p<\alpha<1$. Applying Hölder's inequality, we see that the left-hand side of the equation

$$f_\alpha(x+h)-f_\alpha(x)=\int_0^1 f(x-t)\,[\Psi_\alpha(t+h)-\Psi_\alpha(t)]\,dt$$

does not exceed $\mathfrak{M}_p[f]\,\mathfrak{M}_{p'}[\Psi_\alpha(t+h)-\Psi_\alpha(t)]$ in absolute value, and we have only to show that the second factor is $O(h^{\alpha-1/p})$. Supposing that $0<h<1/2$, we may write

$$(1)\qquad \int_0^1 |\Psi_\alpha(t+h)-\Psi_\alpha(t)|^{p'}\,dt=\int_0^h+\int_h^{1-h}+\int_{1-h}^1=P+Q+R.$$

Denoting by $C, C_1, \ldots$ constants which depend only on α and p, we may write the following inequalities, true for $0<t\leqslant 1$ and $0<\alpha<1$:

$$(2)\qquad |\Psi_\alpha(t)|\leqslant Ct^{\alpha-1},\qquad |\Psi'_\alpha(t)|\leqslant C_1 t^{\alpha-2}.$$

The second of them is an immediate corollary of the formula $\Gamma(\alpha)\,\Psi'_\alpha(t)=\lim\,[t^{\alpha-2}+(t+1)^{\alpha-2}+\ldots+(t+n)^{\alpha-2}]$ $(n\to\infty)$ which, in turn, follows from 9.8(4). Returning to the equation (1) we see that, if $0<t\leqslant h$, then $|\Psi_\alpha(t+h)-\Psi_\alpha(t)|\leqslant 2Ct^{\alpha-1}$, and so $P\leqslant C_2 h^{(\alpha-1)p'+1}$

[1]) Hardy and Littlewood [6]; see also Hardy, Littlewood, and Pólya, *Inequalities*, Chapter X.

(the reader will observe that $(\alpha-1)p' > -1$ since $\alpha > 1/p$). Similarly, since $R = \int_0^h |\Psi_\alpha(t) - \Psi_\alpha(t-h)|^{p'} dt$, and since for $0 < t \leqslant h$ we have $|\Psi_\alpha(t) - \Psi_\alpha(t-h)| = |\Psi_\alpha(t) - \Psi_\alpha(1+t-h)| \leqslant 2Ct^{\alpha-1}$, the expression R satisfies the same inequality as P. Finally, if $h \leqslant t \leqslant 1-h$, we obtain, by the mean-value theorem, that $|\Psi_\alpha(t+h) - \Psi_\alpha(t)| \leqslant C_1 h t^{\alpha-2}$ and so $Q \leqslant C_3 h^{p'} h^{(\alpha-2)p'+1}$. Collecting the results we see that $P+Q+R \leqslant C_4 h^{(\alpha-1)p'+1}$. Thus $\mathfrak{M}_p[\Psi_\alpha(t+h) - \Psi_\alpha(t)] = O(h^{\alpha-1/p})$. This completes the proof of the second part of the theorem.

Remarks. (a) Putting $f = f_1 + f_2$, where f_1 is a trigonometrical polynomial and $\mathfrak{M}_p[f_2]$ is very small, it is easy to see that $\omega(\delta; f_\alpha) = o(\delta^{\alpha-1/p})$.

(b) The theorem which we have proved holds also for $p=1$, $1 \leqslant \alpha < 2$. This follows from Theorem 9.81(i) and the fact that the integral of f is continuous

9.83. Theorem 9.82(i) is rather deep; its proof is long and will be based on a series on lemmas. Before we pass on to these lemmas we observe that a theorem less general than Theorem 9.82(i), viz. that $f_\alpha \in L^{q-\varepsilon}$ for every $\varepsilon > 0$, is trivially true. For $\Psi_\alpha(t) = O(t^{\alpha-1})$ in the neighbourhood of $t=0$, so that $\Psi_\alpha(t) \in L^{1/(1-\alpha)-\varepsilon}$, and we need only apply Theorem 4.16.

9.831. The first of the lemmas is as follows: *Let* $f(x) \geqslant 0$, $g(x) \geqslant 0$ *belong respectively to* $L^p(0,\infty)$, $L^q(0,\infty)$, *where* $p>1$, $q>1$. *If* $\lambda = 1/p + 1/q - 1 = 1 - 1/p' - 1/q' > 0$, $\mathfrak{M}_p[f; 0, \infty] = A$, $\mathfrak{M}_q[g; 0, \infty] = B$, *and if* $F(x)$ *denotes the integral of* f *over* $(0,x)$, *then*

(1) $$\int_0^\infty \frac{F(t)}{t} g(t) t^\lambda dt \leqslant K_1 AB, \qquad (K_1 = p'^{p/q'}).$$

Applying Hölder's inequality we see that the left-hand side does not exceed B multiplied by

(2) $$\left(\int_0^\infty F^{q'} t^{-\frac{q'}{p'}-1} dt\right)^{1/q'}.$$

From the inequality $1/p + 1/q > 1 = 1/q + 1/q'$ we see that $q' > p$. Hölder's inequality applied to the integral defining F gives $F(t) \leqslant At^{1/p'}$. Hence, writing $F^{q'} = F^{q'-p} F^p$, we see that (2) does not exceed

$$A^{\frac{q'-p}{q'}} \left(\int_0^\infty F^p t^{\frac{q'-p}{p'}} t^{-\frac{q'}{p'}-1} dt\right)^{1/q'} = A^{\frac{q'-p}{q'}} \left(\int_0^\infty \left(\frac{F}{t}\right)^p dt\right)^{1/q'},$$

and, by Theorem 4.17, the right-hand side does not exceed $Ap'^{p/q'}$.

9.832. The second lemma is: *Let* f, g *satisfy the conditions of the preceding lemma and be, in addition, non-increasing. Putting* $\mu = 2 - 1/p - 1/q$, *we have*

$$I = \int_0^\infty \int_0^\infty \frac{f(x)\, g(t)}{|x-t|^\mu} dx\, dt \leqslant K_2 AB,$$

where K_2 *depends only on* p *and* q. Since

$$I = \int_0^\infty g(t)\, dt \left[\int_0^t f(x)(t-x)^{-\mu} dx\right] + \int_0^\infty f(x)\, dx \left[\int_0^x g(t)(x-t)^{-\mu} dt\right] = I_1 + I_2,$$

it suffices, in virtue of the symmetrical rôle of f and g, to consider e. g. I_1. Let $\lambda = 1-\mu$; decomposing the inner integral in I_1 into two, taken over $(0, t/2)$ and $(t/2, t)$, and remembering that f is monotonic, we find that this integral does not exceed

$$(\tfrac{1}{2}t)^{-\mu} F(\tfrac{1}{2}t) + f(\tfrac{1}{2}t)\ (\tfrac{1}{2}t)^\lambda/\lambda \leqslant 2\lambda^{-1} (\tfrac{1}{2}t)^\lambda F(\tfrac{1}{2}t)/(\tfrac{1}{2}t) \leqslant 4\lambda^{-1} t^\lambda F(t)/t,$$

since $f(u) \leqslant F(u)/u$. It remains to apply the preceding lemma.

9.833. The third lemma, which is the most fundamental, may be enunciated as follows:

Let $f(x)$, $g(x)$, $h(x)$ *be three non-negative functions defined in* $(-\infty, +\infty)$. *Let* $f^*(x)$, $g^*(x)$, $h^*(x)$ *denote three functions, even, non-increasing in* $(0,\infty)$, *and equimeasurable* [1] *with* f, g, h *respectively. If*

(1) $$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x)\, g(t)\, h(x+t)\, dx\, dt$$

and I^* *is the corresponding integral formed with* f^*, g^*, h^*, *then* $I \leqslant I^*$.

This lemma asserts that, among all functions equimeasurable with f, g, h, the maximum of I is attained when the functions are even and non-increasing in the interval $(0, \infty)$.

(i) We start with the case in which f, g, h are characteristic functions of sets F, G, H consisting of a finite number of inter-

[1] § 9.42. Let $m(y) = |E(f>y)|$. We may define $f^*(x)$, $0 < x < \infty$, as the function inverse to $\frac{1}{2}m(y)$. We assume that $f^*(x) = f^*(x+0)$ for $x \geqslant 0$.

vals, each of the form $(n, n+1)$, $n = 0, \pm 1, \pm 2, \ldots$ We shall suppose that the numbers of these intervals in which f, g, h non vanish are $2\alpha, 2\beta, 2\gamma$, respectively, α, β, γ being even. Let

$$(2)\quad \varphi(x) = \int_{-\infty}^{+\infty} g(t)\, h(x+t)\, dt, \qquad \psi(x) = \int_{-\infty}^{+\infty} g^*(t)\, h^*(x+t)\, dt.$$

The continuous curves $y = \varphi(x)$, $y = \psi(x)$ are linear in the intervals $(n, n+1)$, and $y = 0$ for $|x|$ large. The function $\psi(x)$ is even, vanishes for $x \geqslant \gamma + \beta$, is equal to 2β for $0 \leqslant x \leqslant \gamma - \beta$ (assuming, as we may, that $\gamma \geqslant \beta$), and is linear in $(\gamma - \beta, \gamma + \beta)$; $\varphi(x)$ never exceeds 2β. Integrating (2) we find that the areas of the two curves are the same, viz. $4\beta\gamma$. Multiplying $\varphi(x)$ by $f(x)$, $\psi(x)$ by $f^*(x)$, and integrating over $(-\infty, +\infty)$, we deduce the lemma from geometrical considerations if $\alpha \leqslant \gamma - \beta$ or $\alpha \geqslant \gamma + \beta$.

Suppose then that $\gamma - \beta < \alpha < \gamma + \beta$. We can find two integers $\beta_0 < \beta$, $\gamma_0 < \gamma$ such that $\gamma_0 - \beta_0 = \gamma - \beta$, $\gamma_0 + \beta_0 = \alpha$. The lemma is true for $\alpha, \beta_0, \gamma_0$. Thence we will deduce it for $\alpha, \beta_0 + 1, \gamma_0 + 1$. For the values of $\psi(x)$ in the interval $(-\alpha, \alpha)$ will increase exactly by 2, and the result will be established when we have shown that the values of $\varphi(x)$ in $(-\infty, +\infty)$ will increase at most by 2. Since φ is linear in the intervals $(n, n+1)$, it suffices to consider integral values of x.

If H_x denotes the set H translated by x, then $\varphi(x) = |GH_x|$ represents the number of intervals of length 1 common to G and H_x. Now we may plainly suppose that one of the two intervals which we add to G (and similarly to H_x) is extreme on the left, and the other extreme on the right, with respect to G. Then the reader will easily convince himself that GH_x will increase by at most two intervals, each of length 1. For let J', J'' be the intervals which are added on the left to G and H_x respectively; then $(G + J')(H_x + J'') - GH_x = J'(H_x + J'') + GJ''$. If J' does not belong $H_x + J''$, then $|GH_x|$ remains unchanged when $|J''G| = 0$, and increases by 1 otherwise. If J' belongs to $H_x + J''$, then J'' lies to the left of G; hence $|J''G| = 0$ and $|GH_x|$ increases by 1.

The same argument gives the result for $\alpha, \beta_0 + 2, \gamma_0 + 2$, and so on, and finally for α, β, γ.

(ii) Changing variables we establish the truth of the lemma when the intervals have rational end-points. The restriction that the number of intervals in each set is divisible by 4 can now be

removed, since, if this is not so, each of the intervals may be divided into four equal parts.

(iii) To prove the lemma in the case when F, G, H are arbitrary measurable sets, we observe that F (and similarly G, H) is a difference between an open set and a set of arbitrarily small measure; hence, for every $\varepsilon > 0$, we have $F = \mathcal{F} + F_1 - F_2$ where $\mathcal{F}$ consists of a finite number of intervals with, say, rational end-points, and $|F_1| < \varepsilon$, $|F_2| < \varepsilon$. The reader will have no difficulty in reducing the present case to the case (ii), observing that, roughly speaking, if one of the numbers $|F|, |G|, |H|$ is small, the integral I in (1) is small.

In the above argument we tacitly assumed that each of the numbers $|F|, |G|, |H|$ is finite. That the result holds without this assumption will follow from proposition (v) below.

(iv) If $f \geqslant 0$ is any function which only takes a finite number of values $\alpha_1, \alpha_2, \ldots \alpha_m$, then $f = u_1 f_1 + u_2 f_2 + \ldots + u_m f_m$, where $u_1, \ldots, u_m$ are positive constants and $f_1, f_2, \ldots, f_m$ are the characteristic functions of sets $F_1 \subset F_2 \subset \ldots \subset F_m$. Then $f^* = u_1 f_1^* + u_2 f_2^* + \ldots + u_m f_m^*$. If, in the same way, $g = v_1 g_1 + \ldots + v_n g_n$, $h = w_1 h_1 + \ldots + w_p h_p$, then

$$I = \Sigma\, u_i v_j w_k I_{ijk} \leqslant \Sigma\, u_i v_j w_k I_{ijk}^* = I^*,$$

where I_{ijk} are formed with f_i, g_j, h_k. This proves the lemma when f, g, h assume only a finite number of values.

(v) Let $\{f_n\}, \{g_n\}, \{h_n\}$ be three increasing sequences of non-negative functions and let $f_n \to f$, $g_n \to g$, $h_n \to h$. If the lemma is true for f_n, g_n, h_n, it is also true for f, g, h. In fact, $f_n(x)\, g_n(t)\, h_n(x+t)$ tends, increasing, to $f(x)\, g(t)\, h(x+t)$, and so, using an obvious notation, we have, by Lebesgue's theorem, $I_n \to I$. On the other hand, $f^* \geqslant f_n^*$, $g^* \geqslant g_n^*$, $h^* \geqslant h_n^*$; hence $I^* \geqslant I_n^* \geqslant I_n$ and, consequently, $I^* \geqslant I$.

(vi) Every non-negative function f is the limit of an increasing sequence of functions assuming only a finite number of values; e. g. we may put $f_n(x) = 2^{-n} k$, $0 \leqslant k < n 2^n$, where $k 2^{-n} \leqslant f_n(x) < (k+1) 2^{-n}$, and $f_n(x) = n 2^n$ elsewhere. From this and (iv), (v), we conclude the truth of the lemma in the general case.

Changing t into $-t$ in (1) we obtain a similar result for integrals (1) with $h(x-t)$ instead of $h(x+t)$.

9.84. Completion of the proof of Theorem 9.82(i). Let us replace, as we may, the interval of integration $(0, 1)$ in the formula 9.8(2) by $(-\frac{1}{2}, \frac{1}{2})$, and let $g(x) \in L^{q'}$ be an arbitrary periodic function such that $\mathfrak{M}_{q'}[g] = 1$. Then (§ 4.7.2)

$$\mathfrak{M}_q[f_\alpha] = \operatorname{Max}_g \int_{-1/2}^{1/2} f_\alpha(x)\, g(x)\, dx \leqslant \operatorname{Max}_g \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |f(t)\, g(x)\, \Psi_\alpha(x-t)|\, dx\, dt.$$

Let $f^*(x)$ be even, non-increasing in the interval $0 < x < \infty$, and equimeasurable with the function equal to $|f(x)|$ for $|x| < \frac{1}{2}$ and to 0 elsewhere; similarly $g^*(x)$. Since $|\Psi_\alpha(u)| \leqslant C |u|^{\alpha-1}$ for $|u| < 1$, where C depends only on α, we deduce from Lemma 9.833 (with $h(u) = |u|^{\alpha-1}$ and $h(x+t)$ replaced by $h(x-t)$) and Lemma 9.832 that $\mathfrak{M}_q[f_\alpha]$ does not exceed

$$\operatorname{Max}_g 4C \int_0^\infty\int_0^\infty \frac{f^*(t)\, g^*(x)}{|x-t|^{1-\alpha}}\, dx\, dt \leqslant 4C K_2\, \mathfrak{M}_p[f^*; 0, \infty]\, \mathfrak{M}_{q'}[g^*; 0, \infty],$$

if $1 - \alpha = 2 - 1/p - 1/q'$, i. e. if $\alpha = 1/p - 1/q$. Putting $4CK_2 = K$ we obtain that $\mathfrak{M}_q[f_\alpha] \leqslant K\mathfrak{M}_p[f]$.

9.85. Theorem 9.82(i) is false for $p = 1$, that is if $f \in L$, $q = 1/(1-\alpha)$, then f_α need not necessarily belong to L^q. In fact, if $f(t) = -C + t^{-1}(\log 1/t)^{-1-1/q}$ for $0 < t < 1/2$, $f(t) = 0$ for $1/2 \leqslant t < 1$, where C is a constant such that the mean value of f over $(0, 1)$ vanishes, we have

$$f_\alpha(x) = \int_0^{1/2} f(t)\, \Psi_\alpha(x-t)\, dt = \int_0^{1/2} f(t)\, \Psi^*_\alpha(x-t)\, dt + R(x),$$

where R is a function regular in a neighbourhood of $x = 0$, $\Psi^*_\alpha(u) = u^{\alpha-1}/\Gamma(\alpha)$ for $u > 0$, $\Psi^*_\alpha(u) = 0$ otherwise (§ 9.8). If $0 < x < 1/2$, the last integral exceeds

$$-\frac{Cx^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha-1}}{\Gamma(\alpha)} \int_0^x t^{-1}(\log 1/t)^{-1-1/q}\, dt.$$

Hence, for x small, $f_\alpha(x) \geqslant C_1 x^{\alpha-1}(\log 1/x)^{-1/q}$, and so $f_\alpha \,\bar{\in}\, L^q$. To show that Theorem 9.81(i) is false for $\alpha = 1/p$, i. e. that if $f \in L^p$, then $f_{1/p}$ need not be bounded, we may argue as follows. Multiplying the integral in 9.8(2) by $g(x) \in L$, integrating over $(0, 1)$, and inverting the order of integration, we see that if, for every $f \in L^p$, $f_{1/p}$ were bounded, then, for every $g(x) \in L$, we should have $g_{1/p} \in L^{p'}$, which we know to be false.

9.86. It is of some interest to investigate whether Theorem 9.82(i) is a corollary of the theorems on Fourier coefficients established in the first part of this chapter. We shall show that this is really the case when $p \leqslant 2 \leqslant q$,

p, q having the meaning of Theorem 9.82(i), and only then. Assuming f real, consider the inequalities

$$(1)\quad \sum_{n=1}^\infty c_n^{*p}\, n^{p-2} < \infty, \qquad (2)\quad \sum_{n=1}^\infty \frac{|c_n|^{q'}}{n^{\alpha q'}} < \infty,$$

where $c_1^*, c_2^*, \dots$ is the sequence $|c_1|, |c_2|, \dots$ rearranged in descending order of magnitude. The inequality (1) is implied by the relation $f \in L^p$, and (2) implies that $f_\alpha \in L^q$. Now (2) is certainly true if the series with terms $c_n^{*q'} n^{-\alpha q'}$ converges. We have $c_n^{*q'} n^{-\alpha q'} = c_n^{*p} n^{p-2} c_n^{*q'-p} n^{-\alpha q'+2-p}$ and, since

$$-\alpha q' + 2 - p = -(1/q' - 1/p')\, q' + 2 - p = q'/p' - p/p',$$

we obtain that $c_n^{*q'} n^{-\alpha q'} = c_n^{*p} n^{p-2} (c_n^* n^{1/p'})^{q'-p}$. Since the terms on the left in (1) decrease monotonically, the expression $c_n^{*p} n^{p-2} \cdot n$ is bounded, i. e. $c_n^* n^{1/p'} = O(1)$, and this, together with the last formula and the inequality (1), ensures the inequality (2), provided that $p \leqslant 2 \leqslant q$, $p \leqslant q'$. To get rid of the last condition assume that $p \leqslant 2 \leqslant q$ and $q' < p$. We have then $q' \leqslant 2 \leqslant p'$, $q' < p$. Since $\alpha = 1/p - 1/q = 1/q' - 1/p'$, we see, by the result already obtained, that integration of order α transforms $L^{q'}$ into $L^{p'}$, and this is equivalent to the fact that the said integration transforms L^p into L^q (§ 4.63(ii))[1].

We have only proved that $\mathfrak{M}_q[f_\alpha] < \infty$, but in the same way we can obtain the complete result $\mathfrak{M}_q[f_\alpha] \leqslant K\mathfrak{M}_p[f]$.

It is easy to see why the above argument fails in the cases $p < q < 2$ or $2 < p < q$ (which are equivalent) e. g. in the latter. Integration of order α consists in introducing the factors $\gamma_n = |n|^{-\alpha}\varepsilon_n$ into $\mathfrak{S}[f]$, where $\{\varepsilon_n\}$ is a *special* sequence of unit numbers. The proof given above shows that, if $p \leqslant 2 \leqslant q$, the theorem holds when ε_n is an *arbitrary* bounded sequence. To show that such an extension is impossible for $2 < p < q$, let us suppose that the Fourier expansion of f is the cosine series with coefficients $\varepsilon_n/\sqrt{n}\log n$, $n = 2, 3, \dots$, where $\varepsilon_n = \pm 1$. Choosing for $\{\varepsilon_n\}$ a special sequence, we may have $f \in L^p$, $p > 2$ (§ 5.6). Introducing into $\mathfrak{S}[f]$ the factors ε_n/n^α, $0 < \alpha < \frac{1}{2}$, we obtain the series $\Sigma(\cos nx)/n^{1/2+\alpha}\log n$. In the neighbourhood of $x = 0$ the sum of this series behaves like $x^{-1/2+\alpha}/\log x$ and so it does not belong to L^γ if $\frac{1}{2} - 1/\gamma > \alpha$. If $\alpha = 1/p - 1/q < \frac{1}{2} - 1/q$, the series does not belong to L^q.

9.9. Miscellaneous theorems and examples.

1. Let $\omega_1(t), \omega_2(t), \dots, \omega_n(t)$ be a system of functions measurable and bounded in a finite interval $a \leqslant t \leqslant b$, and let

$$M_{\alpha\beta} = \operatorname{Sup}_{x_1, \dots, x_n} \left\{\int_a^b \left|\sum_{i=1}^n x_i\, \omega_i(t)\right|^{1/\beta} dt\right\}^\beta \Bigg/ \left\{\sum_{i=1}^n |x_i|^{1/\alpha}\right\}^\alpha.$$

---

[1]) Theorem 4.63(ii) holds in the case of complex factors.

Show that (i) $M_{\alpha\beta}$ is a multiplicatively convex function in the triangle $(\Delta)\ 0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \beta \leqslant \alpha$, (ii) Theorem 9.11(b) is a consequence of (i). M. Riesz [3].

[Once the continuity of $M_{\alpha\beta}$ in the triangle Δ has been established, (i) may be proved by an argument similar to that of § 9.2, independently of the more difficult Theorem 9.23. To prove Theorem 9.11(b), we put $\omega_i(t) = \varphi_i(t)$, compute $M_{1,0}$ and $M_{1/_2,\,1/_2}$, and obtain

$$(1)\qquad \mathfrak{M}_{p'}[s_n; a, b] \leqslant M^{(2-p)/p} \left\{ \sum_{i=1}^{n} |c_i|^p \right\}^{1/p},$$

where s_n is the n-th partial sum of the series $(S)\ c_1\varphi_1 + c_2\varphi_2 + \dots$ Since $\mathfrak{N}_2[c] < \infty$, S is the Fourier series of a function $f(t)$ and a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ tends almost everywhere to $f(t)$. An application of Fatou's lemma to (1) completes the proof. If the interval where the functions φ_i are orthogonal is infinite, observe that the inequality (1) is true for any interval (a_1, b_1) completely interior to (a, b), and so it holds for (a, b) also].

2. Let $f(x)$ be a real function belonging to L^p, $1 < p \leqslant 2$, with Fourier coefficients a_n, b_n; the inequality of Theorem 9.1(a) then gives

$$\left\{ \left| \frac{a_0}{\sqrt{2}} \right|^{p'} + \sum_{n=1}^{\infty} (|a_n|^{p'} + |b_n|^{p'}) \right\}^{1/p'} \leqslant \left\{ \frac{1}{\pi} \int_0^{2\pi} |f(t)|^p\, dt \right\}^{1/p}.$$

Inverting this inequality and interchanging the numbers p and p', we obtain the inequality corresponding to Theorem 9.1(b).

3. Let $1 < p \leqslant 2 \leqslant q$, $p \leqslant r \leqslant p'$, $q' \leqslant s \leqslant q$, $\lambda = 1/p + 1/r - 1$, $\mu = 1/q + 1/s - 1$. Then (i) Under the hypothesis of Theorem 9.4(ii),

$$(1)\qquad \left\{ \sum_{n=1}^{\infty} (c_n^* n^{-\lambda})^r \right\}^{1/r} \leqslant A'_p\, \mathfrak{M}_p[f],$$

where A'_p depends on p and M only.

(ii) If $\Sigma (c_n^* n^{-\mu})^s < \infty$, the series $\Sigma c_n \varphi_n$ is the Fourier series of a function $f \in L^q$, and

$$(2)\qquad \mathfrak{M}_q[f] \leqslant A_q \left\{ \sum_{n=1}^{\infty} (c_n^* n^{-\mu})^s \right\}^{1/s},$$

where A_q depends on q and M only.

The results are due, in substance, to Hardy and Littlewood [10], who considered the case of trigonometrical series.

[Proposition (i) is, so to speak, an intermediate result between Theorem 9.11(a) and Theorem 9.4(ii), and is a consequence of those theorems. To prove it, we observe that $r = t_1 p + t_2 p'$, $t_i \geqslant 0$, $t_1 + t_2 = 1$, apply Hölder's inequality to the left-hand side of (1) and use the theorems just quoted. To prove (2), we show that $(\Sigma c_n^{*q} n^{q-2})^{1/q}$ does not exceed $\{\Sigma (c_n^* n^{-\mu})^s\}^{1/s}$, and apply Theorem 9.4(i)].

4. Let $\{\varphi_n\}$ be a set of functions orthogonal, normal, and uniformly bounded $(|\varphi_n| \leqslant M)$ in an interval (a, b). If $|c_n| \leqslant 1/n$, $n = 1, 2, \dots$, the c's are the Fourier coefficients, with respect to $\{\varphi_n\}$, of a function f such that $\exp \lambda f$ is integrable for every $\lambda < 1/eM$.

[Assuming for simplicity that $c_1 = 0$, observe that

$$(*)\qquad \frac{\lambda^k}{k!} \int_a^b |f|^k\, dx \leqslant \frac{\lambda^k}{k!} M^{k-2} \left(\sum_{n=2}^{\infty} n^{-k/(k-1)} \right)^{k-1} < \frac{\lambda^k}{k!} M^{k-2} \frac{(k-1)^{k-1}}{k!}$$

for $k \geqslant 2$, and that $\exp \lambda u = 1 + \lambda u + \tfrac{1}{2}\lambda^2 u^2 + \dots$].

5. If the functions φ_n satisfy the conditions of the previous theorem, the interval (a, b) is finite, $|f| \log^+ |f|$ is integrable over (a, b), and γ_n are the Fourier coefficients of f, then the series $\Sigma |\gamma_n|/n$ converges.

[This follows from the previous theorem by an application of Young's inequality. Observe that the inequality (*) holds if we replace f by any partial sum of the series $c_1\varphi_1 + c_2\varphi_2 + \dots$].

6. Under the conditions of the previous theorem we have

$$\sum_{n=1}^{\infty} \frac{\gamma_n^*}{n} < \infty, \qquad \sum_{n=1}^{\infty} e^{-k/|\gamma_n|} < \infty,$$

where $\{\gamma_n^*\}$ is the sequence $\{|\gamma_n|\}$ arranged in descending order of magnitude, and k is any positive number. For a similar result see Hardy and Littlewood [15].

[The second inequality follows from the first].

7. When $1 < p < 2$, equality in Theorem 9.1(a) occurs if and only if f is a trigonometrical monomial, i. e. if $f(x) = Ce^{inx}$, where C is a constant and $n = 0, \pm 1, \dots$ Similarly equality in Theorem 9.1(b) can occur only if all the c_n, except perhaps one, are equal to 0.

[For the proof (which is not quite simple) see Hardy and Littlewood [10]. The special case $p = 2k/(2k-1)$ is comparatively easy and may be proved by the argument of § 9.12, investigating cases of equality in Young's inequality 4.16(2). See also Hardy, Littlewood, and Pólya, *Inequalities*, Chapter VIII].

8. Let $P_1 = (\alpha_1, \beta_1)$ and $P_2 = (\alpha_2, \beta_2)$ be two points in the triangle $(\Delta)\ 0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \beta \leqslant \alpha$. If a sequence $\{\lambda_n\}$ belongs to $(L^{1/\alpha_1}, L^{1/\beta_1})$ and to $(L^{1/\alpha_2}, L^{1/\beta_2})$ (§ 4.6), then it belongs also to $(L^{1/\alpha}, L^{1/\beta})$ for every point (α, β) on the segment $P_1 P_2$. M. Riesz [3].

[The proof follows the same line as in § 9.25].

9. (i) Let a_n, b_n be the Fourier coefficients of a function $f(x) \in L^p$, $p > 1$; then, if $n_{i+1}/n_i > \lambda > 1$, the series $\Sigma (a_{n_i}^2 + b_{n_i}^2)$ converges. More generally

(ii) If the power series $\Sigma \alpha_n z^n$ belongs to H (§ 7.51), the series $\Sigma |\alpha_{n_i}|$ converges. Paley [5]; Zygmund [5].

Proposition (i) shows that, if $\Sigma (x_i^2 + y_i^2) = \infty$, the function $f(x)$ of Theorem 9.6(ii) does not belong to any class L^p, $p > 1$.

[By Theorem 7.53(vi), $F(z) = F_1(z) F_2(z)$, where $F_1(z) = \Sigma \beta_n z^n$ and $F_2(z) = \Sigma \gamma_n z^n$ belong to H^2. Let $\Sigma |\beta_n|^2 = B^2$, $\Sigma |\gamma_n|^2 = C^2$. Then

$$|\alpha_{n_i}| \leqslant \left(\sum_{k=0}^{n_{i-1}} + \sum_{k=n_{i-1}+1}^{n_i}\right) |\beta_k \gamma_{n_i-k}| \leqslant B\left(\sum_{n_i-n_{i-1}}^{n_i} |\gamma_k|^2\right)^{1/2} + C\left(\sum_{n_{i-1}+1}^{n_i} |\beta_k|^2\right)^{1/2},$$

whence we easily deduce (ii)].

10. (i) If $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta > 1$, and if $f \in \text{Lip}\,\alpha$, then $f_\beta(x)$ has a derivative $f'(x) \in \text{Lip}\,(\alpha + \beta - 1)$. (ii) If $0 < \alpha < \gamma < 1$, and if $f(x)$ has a derivative $f'(x) \in \text{Lip}\,\alpha$, then $f^\gamma(x) \in \text{Lip}\,(1 + \alpha - \gamma)$.

[Corollaries of Theorems 9.81].

11. Theorem 9.82(i) holds for $p = 1$ provided that $\bar{\mathfrak{S}}[f]$ is a Fourier series.

For the proof, which is rather difficult, see Hardy and Littlewood [6_2].

12. Let $r > 1$, $r' = r/(r-1)$. If $f \in L^r$, $g \in L^{r'}$, we have the formula 7.3(3), the series on the right being convergent. If $r = 2$ the series converges absolutely. Show that this last result is false for any other value of r. M. Riesz [4].

[Suppose that $1 < r < 2$, and let $r < 1/(1-\alpha)$, $0 < \alpha < 1/2$. There is a function $h(x) \in \text{Lip}\,\alpha$ such that $\mathfrak{S}[h]$ does not converge absolutely. We may assume that $\mathfrak{S}[h] = \Sigma\, a_n \cos nx$ is a purely cosine series, for otherwise, if x_0 is a point where $\mathfrak{S}[h]$ does not converge absolutely, we may consider $\frac{1}{2}[h(x_0 + x) + h(x_0 - x)]$ instead of $h(x)$. Let

$$f(x) = \sum_{n=1}^{\infty} n^{-\alpha} \cos nx, \qquad g(x) = h^\alpha(x) = \sum_{n=1}^{\infty} n^\alpha a_n \cos(nx + \tfrac{1}{2}\alpha\pi).$$

Since $\Sigma |a_n| = \infty$, the Parseval series for f and g does not converge absolutely, although $f \in L^r$ (§§ 5.221, 5.7.2), and g is continuous and so belongs to $L^{r'}$].

13. Let $f \sim \Sigma\, c_n e^{inx}$, $g \sim \Sigma\, d_n e^{inx}$. If $f \in L^p$, $g \in L^r$, where $1 < p \leqslant 2$, $p \leqslant r \leqslant p'$, the series

$$(1) \qquad \sum_{n=-\infty}^{\infty} |n|^{-\lambda} e^{-\frac{1}{2}\lambda\pi i\,(\text{sign}\, n)} c_n d_n, \qquad \lambda = 1/p + 1/r - 1,$$

converges. If in addition $r \leqslant 2$, the series (1) converges absolutely. Hardy and Littlewood [14].

[If $r = p'$, the theorem follows from M. Riesz's equation 7.3(3). Applying this special case to the functions $f_\lambda(x)$ and $g(x)$, and taking account of Theorem 9.82(i), we obtain the convergence of (1). To obtain the second part of the theorem, apply Theorems 9.9.3(i) and 9.1(a)].