CHAPTER VIII.

Divergence of Fourier series. Gibbs's phenomenon.

8.1. Continuous functions with divergent Fourier series. In Chapter II we proved several conditions ensuring the convergence of Fourier series. Now we will investigate in what degree those tests represent the best possible results. It will appear that, although some improvements are still possible, the problem of the convergence of Fourier series *at individual points* has reached a stage where we can hardly hope for essentially new positive results, if we only use the classical devices of Chapter II. Such tests as Dini's or Dini-Lipschitz's represent a limit beyond which we encounter actual divergence of Fourier series.

The first negative result in the convergence of Fourier series is due to P. Du Bois Reymond (1876) who proved that

There exist continuous functions with Fourier series diverging at a point [1]).

Since that several other examples have been found, and we intend to reproduce two of them. The first is due do Fejér [2]) and is remarkable for its elegance and simplicity. The second method (§ 8.31), propounded by Lebesgue, lies more at the roots of the matter and can be used in many similar problems.

8.11. Fejér's example. It is based on the use of the trigonometrical polynomial

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[1]) P. Du Bois Reymond [1].

[2]) Fejér [7].

$$(1)\qquad \begin{aligned}&\frac{\cos\mu x}{n}+\frac{\cos(\mu+1)x}{n-1}+\ldots+\frac{\cos(\mu+n-1)x}{1}-\\&-\frac{\cos(\mu+n+1)x}{1}-\ldots-\frac{\cos(\mu+2n)}{n}.\end{aligned}$$

Let us denote it by $Q(x,\mu,n)$, and let $\overline{Q}(x,\mu,n)$ be the conjugate polynomial. Adding up the terms with the same denominator we find that

$$Q(x,\mu,n)=\sin(\mu+n)x\sum_{k=1}^{n}\frac{\sin kx}{k},$$

$$\overline{Q}(x,\mu,n)=-\cos(\mu+n)x\sum_{k=1}^{n}\frac{\sin kx}{k}.$$

Since the partial sums of the series $\sin x+\frac{1}{2}\sin 2x+\ldots$ are less than a constant C in absolute value (§ 3.23(ii), § 5.11), we have $|Q|\leqslant C$, $|Q|\leqslant\overline{C}$, for every x,μ,n. On the other hand, for $x=0$, the sum of the first n terms of $Q(x,\mu,n)$, which is equal to $1/n+1/(n-1)+\ldots+1>\log n$, is large with n.

Let $\{n_k\}$, $\{\mu_k\}$ be sets of integers which we shall define in a moment, and let $\alpha_k>0$, $\alpha_1+\alpha_2+\ldots<\infty$. The series

$$(2)\qquad \text{a)}\ \sum_{k=1}^{\infty}\alpha_k Q(x,\mu_k,n_k),\qquad \text{b)}\ \sum_{k=1}^{\infty}\alpha_k\overline{Q}(x,\mu_k,n_k)$$

converge uniformly to continuous sums which we denote by $f(x)$, $g(x)$ respectively. If $\mu_k+2n_k<\mu_{k+1}$ $(k=1,2,\ldots)$, then $Q(x,\mu_k,n_k)$ and $Q(x,\mu_l,n_l)$ do not overlap for $k\neq l$. Similarly $\overline{Q}(x,\mu_k,n_k)$ and $\overline{Q}(x,\mu_l,n_l)$. Therefore, writing every Q and $\overline{Q}$ in (2) in extenso, we represent (2) in the form of trigonometrical series

$$(3)\quad \text{a)}\ \tfrac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x),\quad \text{b)}\ \sum_{\nu=1}^{\infty}(a_\nu\sin\nu x-b_\nu\cos\nu x).$$

Actually the first of them contains only cosines, the second only sines. Denoting the partial sums of these series by $s_n(x)$, $t_n(x)$, we see that $s_{\mu_k-1}(x)$ and $t_{\mu_k-1}(x)$ converge uniformly, so that (3a) is $\mathfrak{S}[f]$ and (3b) is $\mathfrak{S}[g]$. Since $|s_{\mu_k+n_k}(0)-s_{\mu_k-1}(0)|>\alpha_k\log n_k$, the series (3a) will certainly be divergent at $x=0$ if $\alpha_k\log n_k$ does not tend to 0. Thus

If $\alpha_k=k^{-2}$, $\mu_k=n_k=2^{k^3}$, the continuous function f defined by (2a) *has a divergent Fourier series.*

It is not difficult to see that both series (3) converge uniformly for $\delta\leqslant|x|\leqslant\pi$, whatever $\delta>0$. This follows from the fact that the partial sums of $Q(x,\mu_k,n_k)$ and $\overline{Q}(x,\mu_k,n_k)$ are bounded for $0<\delta\leqslant|x|\leqslant\pi$, uniformly in x,μ_k,n_k (§ 1.22). Since the series (3b), containing only sines, converges for $x=0$, it converges everywhere.

8.12. *If $\alpha_k=k^{-2}$, $\mu_k=n_k=2^{k^3}$, the continuous function g defined by* 8.11(2b) *has a Fourier series which is convergent everywhere, but not uniformly* [1].

In fact, if $x=\pi/4n$ and $\mu=n$, the sum of the first n terms of $\overline{Q}(x,\mu,n)$ exceeds $(1+1/2+\ldots+1/n)\sin(\pi/4)>(\log n)/\sqrt{2}$. Therefore $|t_{\mu_k+n_k}(x)-t_{\mu_k-1}(x)|>\alpha_k(\log n_k)/\sqrt{2}\to\infty$ for some x, and this completes the proof. We add a few remarks.

8.13. (i) If we put $\alpha_k=1/k^2$, $\mu_k=2^{k^2}$ in 8.11(2), the partial sums $s_n(x)$, $t_n(x)$ are uniformly bounded ($|s_n|<A$, $|t_n|<A$) in $(-\pi,\pi)$, but $\{s_n(0)\}$ oscillates finitely and $\{t_n(x)\}$, which converges everywhere, does not converge uniformly in the neighbourhood of $x=0$.

(ii) *There exists a power series $c_0+c_1z+\ldots$ regular for $|z|<1$, continuous for $|z|\leqslant 1$, and divergent at $z=1$.* For $\mathfrak{S}[g]=\overline{\mathfrak{S}}[f]$, and so the power series $c_0+c_1z+\ldots$ which reduces to $\mathfrak{S}[f]+i\overline{\mathfrak{S}}[f]$ for $z=e^{ix}$ is an instance in point [2].

(iii) *There exist continuous functions $F(x)$ and $G(x)$ such that $\mathfrak{S}[F]$ diverges at an everywhere dense set of points, and $\overline{\mathfrak{S}}[G]$ converges everywhere, but in no interval uniformly* [3].

Let $f(x)$, $g(x)$ be the functions considered in (i), and let $r_1,r_2,r_3,\ldots$ be a set E of points everywhere dense in $(0,2\pi)$, $\varepsilon_i>0$, $\varepsilon_1+\varepsilon_2+\ldots<\infty$. We put $F(x)=\varepsilon_1 f(x-r_1)+\varepsilon_2 f(x-r_2)+\ldots$, $G(x)=\varepsilon_1 g(x-r_1)+\varepsilon_2 g(x-r_2)+\ldots$, and denote by $F_k(x)$, $G_k(x)$ the k-th partial sums of these series. Let $F(x)=F_k(x)+R_k(x)$, $G(x)=G_k(x)+R_k^*(x)$. The series defining $F(x)$ converges uniformly and we obtain a partial sum of $\overline{\mathfrak{S}}[F]$ by adding the cor-

[1]) The first example of this singularity is due to Lebesgue.

[2]) Fejér [7].

[3]) For the first part of the theorem see P. Du Bois Reymond [1], Fejér [7], for the second Steinhaus [6].

responding partial sums of $\mathfrak{S}[\varepsilon_i f(x-r_i)]$ for $i=1,2,\ldots$ Suppose that $\eta>0$ is given. The partial sums of $\mathfrak{S}[R_k]$ and $\mathfrak{S}[R_k^*]$ are all less than $A(\varepsilon_{k+1}+\varepsilon_{k+2}+\ldots)<\eta$ in absolute value (see (i)), provided that $k=k(\eta)$ is large enough. Since $\mathfrak{S}[F]=\mathfrak{S}[F_k]+\mathfrak{S}[R_k]$, $\mathfrak{S}[G]=\mathfrak{S}[G_k]+\mathfrak{S}[R_k^*]$, we conclude that (1) $\mathfrak{S}[F]$ diverges at any of the points r_i, $1\leqslant i\leqslant k$, where the oscillation of the partial sums of $\mathfrak{S}[\varepsilon_i f(x-r_i)]$ exceeds η, (2) if $x\,\overline{\in}\,E$, the oscillation of the partial sums of $\mathfrak{S}[F]$ at x is $<\eta$, (3) the oscillation of $\mathfrak{S}[G]$ is less than η at every x. Since η and $1/k$ may be arbitrarily small, we obtain from (1) and (2) that $\mathfrak{S}[F]$ diverges for $x\in E$ and converges for $x\,\overline{\in}\,E$. From (3) we deduce that $\mathfrak{S}[G]$ converges everywhere and it remains only to show that the convergence is non-uniform in the neighbourhood of every r_h. Now, since $\mathfrak{S}[f(x-r_h)]$ converges non-uniformly in the neighbourhood of r_h, so does $\mathfrak{S}[\varepsilon_h g(x-r_h)+R_k^*]=\mathfrak{S}[\varepsilon_h g(x-r_h)]+\mathfrak{S}[R_k^*]$, if $k>h$ is large enough. We have $G=[G_k-\varepsilon_h g(x-r_h)]+[\varepsilon_h g(x-r_h)+R_k^*]$ and, since $\mathfrak{S}[G_k-\varepsilon_h g(x-r_h)]$ converges uniformly in a neighbourhood of r_h, the convergence of $\mathfrak{S}[G]$ cannot be uniform there, and this completes the proof.

8.14. In the preceding section we proved more than we set out to prove since we showed that, for any enumerable set E, there exists a continuous f, such that $\mathfrak{S}[f]$ diverges in E and converges outside E [1]). The problem of existence of a continuous f with $\mathfrak{S}[f]$ divergent everywhere, or almost everywhere, is not solved yet and seems to be exceedingly difficult. However it is a very simple matter to construct a continuous f with $\mathfrak{S}[f]$ divergent in a non-enumerable set of points. Let $r_1, r_2, \ldots$ be now the sequence containing any rational point of the interval $(0,2\pi)$ infinitely many times and let $f(x)=\sum_{k=1}^{\infty} k^{-2}\,Q(x-r_k, 2^{k^3}, 2^{k^3})$. Here f is continuous, and to obtain $\mathfrak{S}[f]$ we simply replace every Q by the expression 8.11(1). At any rational point, $\mathfrak{S}[f]$ will contain infinitely many blocks of terms with sums exceeding $k^{-2}\log 2^{k^3}$ for some, arbitrarily large, values of k. It follows that $\mathfrak{S}[f]$ has the partial sums unbounded at an everywhere dense set of points. We know that the set of points at which a sequence of continuous functions $s_n(x)$ is bounded is a sum $F_1+F_2+\ldots$ of closed

1) Steinhaus [7]. See also Neder [1], Zalcwasser [1].

sets (§ 6.11). In our case no F_i contains an interval, and the sum $F_1+F_2+\ldots$ of non-dense sets is of the first category. It is known that the sets complementary to sets of the first category contain perfect subsets, and therefore are of the power of the continuum.

8.2. A theorem of Faber and Lebesgue. We shall show that the Dini-Lipschitz condition cannot be generalized. *There exist two continuous functions $f(x)$, $g(x)$, both having the modulus of continuity $O(1/\log 1/\delta)$ and such that $\mathfrak{S}[f]$ diverges for $x=0$, $\mathfrak{S}[g]$ converges everywhere but not uniformly* [1]). We define f and g as the sums of the series 8.11(2) respectively, with $\alpha_k=2^{-k}$, $\mu_k=n_k=2^{2^k}$. The argument used in § 8.11 shows that $\mathfrak{S}[f]$ oscillates finitely at $x=0$, and that $\mathfrak{S}[g]$ converges non-uniformly in the neighbourhood of $x=0$. To prove the inequalities for $\omega(\delta; f)$ and $\omega(\delta; g)$, e. g. for the former, let $\nu=\nu(h)$ be the largest integer k such that $2^{2^k}\leqslant 1/h$, where $h>0$. Break up the sum defining f into two parts $f_1(x)$, $f_2(x)$, the latter consisting of terms with indices $>\nu$. We have then $|f_2(x+h)-f_2(x)|\leqslant 2C(2^{-\nu-1}+2^{-\nu-2}+\ldots)=$ $=4C.\,2^{-\nu-1}\leqslant 4C/\log 1/h$. A simple calculation shows that

$$Q'(x,\mu,n)=-(\mu+n)\,\overline{Q}(x,\mu,n)-\sin\mu x-\ldots-\sin(\mu+n-1)\,x+$$
$$+\sin(\mu+n+1)\,x+\ldots+\sin(\mu+2n)\,x,$$

so that $|Q'(x,\mu,n)|\leqslant(\mu+n)\,C+2n<(\mu+n)(C+2)=nC'$ if we suppose that $\mu=n$, $2(C+2)=C'$. By the mean-value theorem we see that $|f_1(x+h)-f_1(x)|$ does not exceed

$$C'h\,[2^{-1}\,2^{2^1}+2^{-2}\,2^{2^2}+\ldots+2^{-\nu}\,2^{2^\nu}]=O(h\,2^{-\nu}\,2^{2^\nu})\ ^{2)}=$$
$$=O(2^{-\nu})=O(1/\log 1/h).$$

Therefore $|f(x+h)-f(x)|\leqslant|f_1(x+h)-f_1(x)|+|f_2(x+h)-f_2(x)|=$ $=O(1/\log 1/h)$ and the theorem is established. Arguing as in § 8.13(iii), we can make $\mathfrak{S}[f]$ diverge in a set everywhere dense, and $\mathfrak{S}[g]$ converge non-uniformly in every interval.

1) Faber [1], Lebesgue [1].

2) We use here the following proposition: if, for a positive sequence $\{m_k\}$, we have $m_{k+1}/m_k>q>1$, then $m_1+m_2+\ldots+m_k=O(m_k)$; for $m_1+m_2+\ldots+m_{k-1}+m_k<m_k(1+q^{-1}+q^{-2}+\ldots+q^{-k})<m_k/(1-q^{-1})$.

8.3. Lebesgue's constants. This name is given to the numbers

$$(1)\qquad L_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt = \frac{2}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{2\sin\frac{1}{2}t}\, dt.$$

Let $s_n(x; f)$ denote the n-th partial sum of $\mathfrak{S}[f]$. It is plain that, if $|f| \leqslant 1$, then $|s_n(x; f)| \leqslant L_n$, and for the function $f(t) = \operatorname{sign} D_n(t)$ we actually have $s_n(0; f) = L_n$. The latter function is discontinuous at a finite number of points, but, smoothing this function slightly at the points of discontinuity, we can obtain a continuous f, $|f| \leqslant 1$, such that $s_n(0; f) > L_n - \varepsilon$, whatever $\varepsilon > 0$. Thus, for a fixed n, L_n is the upper bound of $|s_n(x; f)|$ for all x and continuous f, $|f| \leqslant 1$. For this reason it is interesting to investigate the behaviour of L_n as $n \to \infty$. We will prove that $L_n \simeq (4/\pi^2)\log n$ as $n \to \infty$ [1]).

Since the function $\frac{1}{2}\operatorname{ctg}\frac{1}{2}t - 1/t$ is bounded for $|t| \leqslant \pi$, and $|\sin nt| \leqslant |nt|$, we have

$$L_n = \frac{2}{\pi}\int_0^{\pi} \frac{|\sin nt|}{2\operatorname{tg}\frac{1}{2}t}\, dt + O(1) = \frac{2}{\pi}\int_0^{\pi} \frac{|\sin nt|}{t}\, dt + O(1) =$$

$$= \frac{2}{\pi}\sum_{k=1}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n} \frac{|\sin nt|}{t}\, dt + O(1) = \frac{2}{\pi}\int_0^{\pi/n} \sin nt \left\{\sum_{k=1}^{n-1}\frac{1}{t + k\pi/n}\right\} dt + O(1).$$

The sum in curly brackets, contained between the numbers $n\pi^{-1}(1 + 1/2 + \ldots + 1/(n-1))$ and $n\pi^{-1}(1/2 + 1/3 + \ldots + 1/n)$, is equal to $\pi^{-1} n[\log n + O(1)]$ (§ 1.74). Since the integral of $\sin nt$ over $(0, \pi/n)$ is equal to $2/n$, we have $L_n = (4/\pi^2)\log n + O(1) \simeq (4/\pi^2)\log n$.

8.31. We have proved that, if n is large enough, there exists a continuous $f(x) = f_n(x)$, $|f_n| \leqslant 1$, such that $s_n(0; f)$ is large. This function depends on n. To obtain a fixed f with $s_n(0; f)$ unbounded we appeal to Theorem 4.56(iv). If we replace in it $y_n(t)$ by $D_n(t)$, $x(t)$ by $f(t)$, and use the fact that $L_n \to \infty$, we deduce that there is a continuous function $f(x)$ with $\overline{\lim}\,|s_n(0; f)| = +\infty$, i. e. Theorem 8.1 [2]).

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[1]) F e j é r [8].

[2]) Theorem 4.56(iv) (which is due to L e b e s g u e [2]) lies rather deep, and in the case $y_n(t) = D_n(t)$ it is not difficult to prove it directly. We refer the reader to L e b e s g u e's *Leçons*.

8.32. Let λ_n be any sequence tending to $+\infty$ more slowly than $\log n$. Since the integral of $|D_n(t)|/\lambda_n$ over $(-\pi, \pi)$ tends to $+\infty$, applying Theorem 4.56(iv) again we have:

For any sequence $\lambda_n = o(\log n)$ there exists a continuous f such that $s_n(0; f) > \lambda_n$ for infinitely many n. In § 2.73 we proved that, for any continuous f, $s_n(x; f) = o(\log n)$, uniformly in x. Now we see that this result cannot be improved.

The above theorem can also be established by the method of § 8.11.

8.33. Applying Theorem 4.55 *in its most general form* to the proof of Theorem 4.56(iv), we obtain a result from which we conclude that the set of continuous functions f with $\mathfrak{S}[f]$ convergent at the point 0, or at any fixed point, forms a set of the first category in the space C of all continuous and periodic functions. Thus the set of continuous functions with Fourier series convergent at some rational point or another is again of the first category. In other words, *if we reject from the space C a set of the first category, the Fourier series of the remaining functions have points of divergence everywhere dense.*

8.34. As a last application of Theorem 4.56(iv) we shall show that, in a sense, the Dini condition of § 2.4 cannot be improved: *Given any continuous $\mu(t) \geqslant 0$, such that $\mu(t)/t$ is not integrable in the neighbourhood of $t = 0$, we can find a continuous function f, such that $|f(t) - f(0)| \leqslant \mu(t)$ for small $|t|$, and none the less $\mathfrak{S}[f]$ diverges at $t = 0$.*

Let $s_n^*(x; f)$ be the modified partial sums of $\mathfrak{S}[f]$ (§ 2.3). Put $\chi_n(t) = \mu(t)\sin nt/2\operatorname{tg}\frac{1}{2}t$. If $\mathfrak{M}[\chi_n] \neq O(1)$, we can find a continuous $g(x)$, $|g| \leqslant 1$, such that the integral of $\chi_n(t)\, g(t)$ over $(-\pi, \pi)$ is unbounded as $n \to \infty$. This means that $\mathfrak{S}[f]$, where $f(x) = g(x)\,\mu(x)$, diverges at the point 0. Since we may freely suppose that $\mu(0) = 0$, we have $|f(t) - f(0)| = |f(t)| \leqslant \mu(t)$.

To justify our assumption that $\mathfrak{M}[\chi_n] \neq O(1)$, we prove the following lemma: *If $\alpha(x)$ is bounded, $\beta(x)$ integrable, both periodic, then*

$$(1)\qquad I_n = \int_{-\pi}^{\pi} \alpha(nx)\,\beta(x)\, dx \to \int_{-\pi}^{\pi} \alpha(x)\, dx \int_{-\pi}^{\pi} \beta(x)\, dx$$

as $n \to \infty$ [1]). We begin by the following observation, the proof of which may be left to the reader: If, for every $\varepsilon > 0$, we have $\mathfrak{M}[\beta_1] < \varepsilon$ and the relation (1) holds for β_2 and $\beta = \beta_1 + \beta_2$, where any bounded α, then (1) is true. Now (1) is certainly true if β is the characteristic function of a set E consisting of a finite number of intervals. Therefore it holds true when E is open, or, more generally, measurable. Thence we pass to the case of β assuming only a finite number of values. Since we can approximate uniformly to any bounded β by such functions, we conclude the truth of (1) for β bounded. If β is integrable, we put $\beta = \beta_1 + \beta_2$, where β_2 is bounded and $\mathfrak{M}[\beta_1]$ small.

Let us now put $\alpha(t) = |\sin t|$, $\beta(t) = \mu(t)/|2 \operatorname{tg} \frac{1}{2} t|$ for $0 < \varepsilon \leqslant |t| \leqslant \pi$, $\beta(t) = 0$ elsewhere, and denote the corresponding integrals I_n by $I_n(\varepsilon)$. Since $\mathfrak{M}[\chi_n] \geqslant I_n(\varepsilon)$, we have the inequalities $\underline{\lim}\, \mathfrak{M}[\chi_n] \geqslant \underline{\lim}\, I_n(\varepsilon) = \lim I_n(\varepsilon)$. The function $\mu(t)/2 \operatorname{tg} \frac{1}{2} t$ being non-integrable, we may make $\lim I_n(\varepsilon)$ as large as we please, if only ε is small enough. This shows that $\mathfrak{M}[\chi_n] \to \infty$, and the theorem is established.

The case $\mu(t) = o(\log 1/|t|)^{-1}$ (we may put, for example, $\mu(t) = (\log 1/|t| \log\log 1/|t|)^{-1}$ for small $|t|$) is of special interest in view of the Dini-Lipschitz test (§ 2.71).

Consider a continuous function $f(t)$ with $\mathfrak{S}[f]$ divergent at the point 0, and such that $f(0) = 0$, $f(t) = o(\log 1/|t|)^{-1}$. Let $f_1(t) = f(t)$, $f_2(t) = 0$ for $0 \leqslant t \leqslant \pi$, and $f_1(t) = 0$, $f_2(t) = f(t)$ for $-\pi \leqslant t \leqslant 0$. Since $f = f_1 + f_2$, it follows that at least one of the functions f_1, f_2, say f_1, has a Fourier series divergent at the point 0. Consider the interval $(a, b) = (-\pi/4, 0)$. It is plain that the modulus of continuity of the function f_1 in (a, b) is $o(\log 1/\delta)^{-1}$ and that $f(a-t) - f(a) = o(\log 1/t)^{-1}$, $f(b+t) - f(b) = o(\log 1/t)^{-1}$ as $t \to +0$. In spite of that, $\mathfrak{S}[f_1]$ does not converge uniformly in the interval (a, b). This result justifies the last remark of § 2.72.

8.35. Lebesgue's constants may be defined for any method of summation if we replace $D_n(t)$ in 8.3(1) by the corresponding kernel. In the case of the method $(C, 1)$, or Abel's method, Lebesgue's constants are all equal to 1. As regards the constants

[1]) Fejér [8]. This lemma will be applied only in a special case of α continuous, and β continuous except at a finite number of points. We prove it in the general case since it embraces Theorem 2.211.

$L_n^{(k)}$ corresponding to the metod (C, k), $0 < k < 1$, the following result has been proved. *$L_n^{(k)}$ tends to a finite number $L^{(k)} > 1$, as $n \to \infty$. For any $0 < k < 1$, there exists an $f(x)$, $|f| \leqslant 1$, such that* $\overline{\lim}\, |\sigma_n^k(0; f)| = L^{(k)}$ [1]).

8.4. Kolmogoroff's example. *There exists an integrable function $f(x)$ such that $\mathfrak{S}[f]$ diverges everywhere* [2]).

Let $f_1(x), f_2(x), \dots$ be a sequence of trigonometrical polynomials of orders $\nu_1 < \nu_2 < \dots$, with the following properties (i) $f_n(x) \geqslant 0$, (ii) $\int_0^{2\pi} f_n(x)\, dx = 2\pi$. Suppose, moreover, that to every f_n corresponds an integer λ_n, where $0 < \lambda_n \leqslant \nu_n$, a number $A_n > 0$, and a point set E_n, such that (iii) if $x \in E_n$, there is an integer $k = k_x$, $\lambda_n \leqslant k \leqslant \nu_n$, for which $s_k(x; f_n) > A_n$, (iv) $A_n \to \infty$, (v) $\lambda_n \to \infty$, (vi) $E_1 \subset E_2 \subset \dots$, $E_1 + E_2 + \dots = (0, 2\pi)$. *Under these conditions, if $\{n_k\}$ tends to ∞ sufficiently rapidly, the Fourier series of the function*

$$f(x) = \sum_{k=1}^{\infty} f_{n_k}(x)/\sqrt{A_{n_k}} \tag{1}$$

diverges everywhere.

First of all the series in (1) converges almost everywhere to an integrable sum provided that the series $1/\sqrt{A_{n_1}} + 1/\sqrt{A_{n_2}} + \dots$ converges. This follows from the fact that series with non-negative terms can be integrated term by term. Let us put $n_1 = 1$ and assume that the numbers $n_1, n_2, \dots, n_{i-1}$ have already been defined. The number n_i will be defined as the least integer satisfying the conditions:

$$\text{(a)}\quad \lambda_{n_i} > \nu_{n_{i-1}}, \qquad \text{(b)}\quad A_{n_i} > 4A_{n_{i-1}}, \qquad \text{(c)}\quad \sqrt{A_{n_i}} > \nu_{n_{i-1}}.$$

From (b) we deduce the convergence of $1/\sqrt{A_{n_1}} + 1/\sqrt{A_{n_2}} + \dots$, so that $f(x)$ exists and is integrable. To prove the divergence of $\mathfrak{S}[f]$, let x be an arbitrary point of E_{n_i} and let $f = u + v + w$, where u is the $(i-1)$-st partial sum of the series (1), and $v = f_{n_i}/\sqrt{A_{n_i}}$;

[1]) Cramér [1].

[2]) Kolmogoroff [6]. The construction of the text is slightly different from that of the original paper. The modifications have been suggested to me by Mr. Kolmogoroff.

hence $s_k(x; f) = s_k(x; u) + s_k(x; v) + s_k(x; w)$. In virtue of (iii) there is a $k = k_x$, $\lambda_{n_i} \leqslant k \leqslant \nu_{n_i}$, such that

$$(2) \qquad s_k(x; v) \geqslant \sqrt{A_{n_i}}.$$

From (a) and (i) we see that

$$(3) \qquad s_k(x; u) = u(x) \geqslant 0.$$

Finally, since for any integrable g we have $|s_k(x; g)| \leqslant \leqslant (2k+1)\, \mathfrak{M}[g; 0, 2\pi]/\pi$, we find that $|s_k(x; w)| \leqslant 2(2k+1)\ (1/\sqrt{A_{n_{i+1}}} + 1/\sqrt{A_{n_{i+2}}} + \ldots) < 12k/\sqrt{A_{n_{i+1}}} \leqslant 12\nu_{n_i}/\sqrt{A_{n_{i+1}}} \leqslant 12$. From this and the inequalities (2), (3), we conclude that $s_k(x; f) \geqslant \sqrt{A_{n_i}} - 12$. Since every $x \in (0, 2\pi)$ belongs to E_{n_i} for all i sufficiently large, the result follows.

8.401. It remains to construct the polynomials f_n and to show that they possess the required properties; this is the most fundamental part of the proof. The function $f_n(x)$ will be defined as a sum of two polynomials $\varphi(x)$ and $\psi(x)$.

Let us fix n, put $x_i = 2\pi i/(2n+1)$, $i = 0, 1, \ldots, 2n$, and consider the intervals $I_i = (x_i - \delta, x_i + \delta)$. If δ is small enough, there is a non-negative trigonometrical polynomial $\varphi(x)$ of order $M > n$, with constant term equal to $\frac{1}{2}$, and such that $\varphi(x) \geqslant n$, say, in the intervals I_i. For it is sufficient to put $\varphi(x) = K_m\{(2n+1)x\}$, where K_m denotes Fejér's kernel and m is large enough. Since we may take δ as small as we please, we may suppose that $D_M(x) \geqslant 0$ in the interval $(-\delta, \delta)$, where D_M denotes Dirichlet's kernel.

Next we put

$$\psi(x) = \frac{1}{n+1} \sum_{i=0}^{n} K_{m_i}(x - x_{2i}),$$

where $M \leqslant m_0 < m_1 < \ldots$; the numbers $m_0, m_1, \ldots$ will be defined later. If $m_j \leqslant k < m_{j+1}$, then

$$s_k(x; \psi) = \frac{1}{n+1} \sum_{i=0}^{j} K_{m_i}(x - x_{2i}) +$$

$$+ \frac{1}{n+1} \sum_{i=j+1}^{n} \left\{ \frac{1}{2} + \sum_{l=1}^{k} \frac{m_i - l + 1}{m_i + 1} \cos l(x - x_{2i}) \right\}$$

Since $m_i - l + 1 = (m_i - k) + (k - l + 1)$ and $K_k(x) \geqslant 0$, we obtain

$$(1) \quad s_k(x; \psi) \geqslant \frac{1}{n+1} \sum_{i=j+1}^{n} \frac{m_i - k}{m_i + 1} D_k(x - x_{2i}), \qquad m_j \leqslant k < m_{j+1}.$$

8.402. Let us denote the intervals $(x_i + \delta, x_{i+1} - \delta)$ by I'_i, $i = 0, 1, 2, \ldots, 2n$, and suppose that $x \in I'_{2j}$ or that $x \in I'_{2j+1}$; in particular $x_{2i} < x < x_{2i+2}$. If $2k+1$ is a multiple of $2n+1$, then $\sin(k+\frac{1}{2})(x - x_{2i})$ has the same value for every i, and from 8.401(1) we obtain

$$(1) \quad s_k(x; \psi) \geqslant \frac{\sin(k+\frac{1}{2})(x_{2j+2} - x)}{n+1} \sum_{i=j+1}^{n} \frac{m_i - k}{m_i + 1} \cdot \frac{1}{2\sin\frac{1}{2}(x_{2i} - x)}.$$

It is not difficult to prove that, if the numbers $m_0, m_1, \ldots$ increase sufficiently rapidly, then, to every x belonging either to I'_{2j} or to I'_{2j+1}, corresponds an integer $k = k_x$ satisfying the inequalities $m_j \leqslant k < \frac{1}{2} m_{j+1}$, $\sin(k+\frac{1}{2})(x_{2j+2} - x) \geqslant \frac{1}{2}$, and such that $2k+1$ is a multiple of $2n+1$. Let us take this result for granted; we shall return to it later. Taking such a value for k, we obtain from (1)

$$s_k(x; \psi) \geqslant \frac{1}{2} \cdot \frac{1}{n+1} \sum_{i=j+1}^{n} \frac{1}{2(x_{2i} - x)} > \frac{1}{2n+2} \sum_{i=j+1}^{n} \frac{2n+1}{8\pi(i-j)},$$

i. e. $s_k(x; \psi) \geqslant C_1 \log(n-j)$, $C, C_1, C_2, \ldots$ denoting positive absolute constants. If $j \leqslant n - \sqrt{n}$, then $s_k(x; \psi) \geqslant \frac{1}{2} C_1 \log n = C \log n$.

8.403. Let us put $f_n(x) = \varphi(x) + \psi(x)$. If $x \in I'_{2j}$, or $x \in I'_{2j+1}$, $j \leqslant n - \sqrt{n}$, there is an integer $k \geqslant m_j \geqslant m_0 \geqslant M$ such that $s_k(x; \psi) \geqslant C \log n$. Hence we have $s_k(x; f_n) = s_k(x; \varphi) + s_k(x; \psi) = \varphi(x) + s_k(x; \psi) \geqslant C \log n$.

Now we shall investigate the behaviour of $s_k(x; f_n)$ in the intervals I_l. We shall show that $s_M(x; f_n) \geqslant \frac{1}{2} n$ for $x \in I_l$ and n sufficiently large. The right-hand side of the equation $s_M(x; f_n) = s_M(x; \varphi) + s_M(x; \psi)$ consists of two terms, the first of which exceeds n for $x \in I_l$, and we will show that, if $x \in I_l$, the second term is dominated by the first (this is just the reason why we define f_n as $\varphi + \psi$). More precisely, we shall prove the inequality $s_M(x; \psi) > -C_2 \log n$ for $x \in I_l$ and $n > 1$, so that $s_M(x; f_n) \geqslant n - C_2 \log n > \frac{1}{2} n$ for $x \in I_l$, $n > n_0$.

We first suppose that l is even, $l = 2h$. If $k = M = m_0$, we have the formula 8.401(1) with $j = -1$. If $x \in I_{2h}$, the term $i = h$ in the sum on the right is positive in virtue of the condition imposed on the intervals I_i. If this term is omitted, the inequality 8.401(1) holds à fortiori. Since $|D_k(u)| \leqslant \pi/|u|$, for $|u| < \pi$, and since $|x - x_{2i}| > 2\pi |h - i|/(2n+1)$, we obtain that

$$(1)\quad s_M(x;\psi) \geqslant -\frac{1}{n+1}\sum_{i=0}^{n}{}' \frac{2n+1}{2|h-i|} > -C_3 \log n, \quad n > 1, \quad x \in I_{2h},$$

where $'$ denotes that $i \neq h$.

If l is odd, $l = 2h+1$, we again have the inequality 8.401(1) with $j = -1$, $k = M$. It is not difficult to see that $|x - x_{2i}|$ exceeds a constant multiple of $|h-i|/(2n+1)$, and, arguing as in the previous case, we obtain that $s_M(x;\psi) > -C_4 \log n$, for $x \in I_{2h+1}$, $n > 1$. This, together with (1), gives $s_M(x;\psi) > -C_2 \log n$, where $x \in I_l$, $n > 1$, $C_2 = \mathrm{Max}\,(C_3, C_4)$. Hence, as we have already observed, $s_M(x; f_n) > \frac{1}{2} n$ if $x \in I_l$, $n > n_0$.

Collecting the results and observing that $C \log n < \frac{1}{2} n$ for n sufficiently large, we obtain that *to every x in the interval* (E_n) $0 \leqslant x \leqslant 4\pi\,(n - \sqrt{n})/(2n+1)$ *corresponds an integer $k > n$, such that* $s_k(x; f_n) > C \log n$, $n > n_1$. The reader will have no difficulty in verifying that the functions f_n satisfy the conditions of the lemma established in § 8.4, at least for n sufficiently large.

8.404. There is one point in the preceding argument which requires explanation. We must show that, if the numbers $m_0, m_1, \ldots$ increase sufficiently rapidly, then, to every x belonging to $I'_{2j} + I'_{2j+1}$, corresponds an integer k satisfying the inequalities $m_j \leqslant k < \frac{1}{2} m_{j+1}$, $\sin (k + \frac{1}{2})\,(x_{2j+2} - x) \geqslant \frac{1}{2}$, and such that $2k+1$ is divisible by $2n+1$. Let us put $2k+1 = \rho\,(2n+1)$, so that ρ is odd, and $x_{2j+2} - x = 4\pi\theta/(2n+1)$. Then $\sin\,(k+\frac{1}{2})\,(x_{2j+2} - x) = \sin 2\pi\rho\theta$, and x belongs to $I'_{2j} + I'_{2j+1}$ if and only if θ belongs to the sum of intervals $\eta \leqslant \theta \leqslant \frac{1}{2} - \eta$, $\frac{1}{2} + \eta \leqslant \theta \leqslant 1 - \eta$, where η is positive and depends on δ and n.

Let $m_0 = M$, and suppose that $m_0, m_1, \ldots, m_j$ have already been defined. It is sufficient to show that, if ρ_0 is a fixed odd integer, then there is a number ν such that, if θ belongs to $(\eta, \frac{1}{2} - \eta) + (\frac{1}{2} + \eta, 1 - \eta)$, we have $\sin 2\pi\rho\theta \geqslant \frac{1}{2}$ for an odd integer ρ satisfying the inequality $\rho_0 \leqslant \rho \leqslant \rho + \nu$. For, if m'_j denotes a number such that for every $x \in I'_{2j} + I'_{2j+1}$ there is an integer k, $m_j \leqslant k \leqslant m'_j$ such that $2k+1$ is a multiple of $(2n+1)$, and that $\sin\,(k+\frac{1}{2})\,(x_{2j+2} - x) \geqslant \frac{1}{2}$, then we may take for m_{j+1} any integer greater than $2m'_j$.

Now consider the points $\rho\theta$ where ρ runs through the sequence $\rho_0, \rho_0 + 2, \rho_0 + 4, \ldots$ If ρ is increased by 2, $\theta\rho$ increases by 2θ, i. e. by a number the fractional part of which belongs to the interval $(2\eta, 1-2\eta)$. Consider the following three cases (i) $2\theta \in (2\eta, {}^1/_3)$, (ii) $2\theta \in ({}^1/_3, {}^2/_3)$, (iii) $2\theta \in ({}^2/_3, 1 - 2\eta)$. In case (i) the situation is fairly simple; for the length of the interval $({}^1/_{12}, {}^5/_{12})$ is equal to ${}^1/_3$, and so after a bounded number of steps the point $\rho\theta$ will certainly fall into this interval i. e. we shall have $\sin 2\pi\rho\theta \geqslant \frac{1}{2}$. In case (iii) the argument is similar.

In case (ii) the situation is slightly less simple for, if $\theta\rho_0$ and 2θ are both very near (mod 1) to the number ${}^1/_2$, the sequence $\theta\rho$, $\rho \geqslant \rho_0$, may stay outside the interval $({}^1/_{12}, {}^5/_{12})$ for a long time. Consider the cases (ii') $2\theta \in ({}^1/_3, {}^5/_{12})$, (ii'') $2\theta \in ({}^5/_{12}, {}^7/_{12})$, (ii''') $2\theta \in ({}^7/_{12}, {}^2/_3)$. In cases (ii') and (ii'''), 4θ belongs to the intervals $({}^2/_3, {}^5/_6)$, $({}^1/_6, {}^1/_3)$ respectively, and so, arguing as before, we see that, after an e v e n number of steps, $\theta\rho$ will fall into the interval $({}^1/_{12}, {}^5/_{12})$.

Now suppose that $2\theta \in ({}^5/_{12}, {}^7/_{12}) \subset ({}^1/_3, {}^2/_3)$, i. e. θ belongs either to $({}^2/_{12}, {}^4/_{12})$ or to $({}^8/_{12}, {}^{10}/_{12})$, e. g. to the former interval. It is easy to see that, if m is even and positive, and if $m\theta$ belongs either to $({}^1/_{12}, {}^5/_{12})$ or to $({}^7/_{12}, {}^{11}/_{12})$, then, after a bounded number of steps, the point $\rho\theta$, $\rho \geqslant \rho_0$, will reach the interval $({}^1/_{12}, {}^5/_{12})$. Now we observe that the numbers $\rho_0 - 1$, $\rho_0 + 1$, $2\rho_0$ are even and that (a) if $\rho_0\theta \in ({}^1/_{12}, {}^5/_{12})$, we may put $\rho = \rho_0$, (b) if $\rho_0\theta \in ({}^5/_{12}, {}^7/_{12})$, then $(\rho_0 - 1)\,\theta \in ({}^1/_{12}, {}^5/_{12})$, (c) if $\rho_0\theta \in ({}^7/_{12}, {}^8/_{12})$, then $2\rho_0\theta \in ({}^2/_{12}, {}^4/_{12})$, (d) if $\rho_0\theta \in ({}^8/_{12}, {}^{10}/_{12})$, then $(\rho_0 + 2)\,\theta \in ({}^1/_{12}, {}^5/_{12})$, (e) if $\rho_0\theta \in ({}^{10}/_{12}, {}^{11}/_{12})$, then $2\rho_0\theta \in ({}^8/_{12}, {}^{10}/_{12})$, (f) if $\rho_0\theta \in ({}^{11}/_{12}, 1) + (0, {}^1/_{12})$, then $(\rho_0 + 1)\,\theta \in ({}^1/_{12}, {}^5/_{12})$.

The case $2\theta \in ({}^5/_{12}, {}^7/_{12})$, $\theta \in ({}^8/_{12}, {}^{10}/_{12})$ may be dealt with in the same way and Theorem 8.4 is established completely.

8.5. Gibbs's phenomenon. We shall now investigate the behaviour of the partial sums $d_n(x)$ of the series

$$(1)\qquad \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu} = d\,(x) = \tfrac{1}{2}\,(\pi - x) \qquad (0 < x < 2\pi)$$

in the neighbourhood of $x = 0$. Suppose, as we may, that $x > 0$. Since $\frac{1}{2} \operatorname{ctg} \frac{1}{2} t - 1/t$ is of bounded variation over $(0, \pi)$, we have

$$\tfrac{1}{2} x + d_n(x) = \int_0^x D_n(t)\,dt = \int_0^x \frac{\sin nt}{2 \operatorname{tg} \frac{1}{2} t}\,dt + o\,(1) =$$

$$= \int_0^x \frac{\sin nt}{t}\,dt + o\,(1) = \int_0^{nx} \frac{\sin t}{t}\,dt + o\,(1),$$

where the last term tends to 0 uniformly in x (§ 2.213). From this we deduce the approximate formula

$$d_n(x) = \int_0^{nx} \frac{\sin t}{t}\, dt + o(1), \tag{2}$$

where the error is $<\varepsilon$, provided that $x<\varepsilon$, $n>n_0(\varepsilon)$. Let us put $\varphi(u) = \int_0^u \frac{\sin t}{t}\, dt$. The integrals of $(\sin t)/t$ over the intervals $(k\pi, (k+1)\pi)$ decrease in absolute value and are of alternating sign when k runs through the values $0, 1, 2, \ldots$ This shows that the curve $y=\varphi(x)$ has a wave-like shape with maxima $M_1 > M_3 > M_5 > \ldots$ attained at the points $\pi, 3\pi, 5\pi, \ldots$ and minima $m_2 < m_4 < m_6 < \ldots$ at $2\pi, 4\pi, \ldots$ From the relation $d_n(x) \to \frac{1}{2}(\pi - x)$, and the equation $d_n(x) = -\frac{1}{2}x + \varphi(nx) + o(1)$, we see that $\varphi(u) \to \frac{1}{2}\pi$ as $u\to\infty$, i. e.

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}\cdot$$

Substituting $x=\pi/n$ in the formula (2), we obtain that $d_n(\pi/n) \to \varphi(\pi) > > \varphi(\infty) = \frac{1}{2}\pi$. Thus, although $d_n(x)$ tends to $d(x) \leqslant \frac{1}{2}\pi$ for every fixed x, $0<x<\pi$, the curves $y=d_n(x)$, which pass through the point $(0, 0)$, condense to the interval $0 \leqslant y \leqslant \varphi(\pi)$ on the y-axis, transcending the interval $0 \leqslant y \leqslant d(+0)$ in the ratio

$$\frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\, dt = 1.089490\ldots$$

Since the $d_n(x)$ are odd functions of x, a similar situation occurs in the left-hand neighbourhood of $x=0$, where the curves $y=d_n(x)$ condense to the interval $-\varphi(\pi) \leqslant y \leqslant 0$ [1]. This phenomenon is called Gibbs's phenomenon and may be described, quite generally, as follows. Let a sequence $\{f_n(x)\}$ converge to a function $f(x)$ for $x_0 < x \leqslant x_0 + h$, say. If, for n and $1/(x-x_0)$ tending to $+\infty$ independently of each other, $\overline{\lim}\, f_n(x) > f(x_0+0)$, or if $\underline{\lim}\, f_n(x) < f(x_0+0)$, we say that $\{f_n(x)\}$ presents Gibbs's phenomenon in the right-hand neighbourhood of the point x_0. A similar definition holds for the left-hand neighbourhood [2].

[1]) For interesting graphs and a more detailed discussion we refer the reader to C a r l s l a w's, *Introduction to the Theory of Fourier Series and Integrals.*

[2]) See Z a l c w a s s e r [1], where a discussion of some problems connected with Gibbs's phenomenon is given.

8.51. Let $f(x)$ be an arbitrary function having a simple discontinuity at a point ξ: $f(\xi+0) - f(\xi-0) = l \neq 0$. The function $\Delta(x) = f(x) - l\cdot d(x-\xi)/\pi$ is continuous at ξ. Suppose that $\mathfrak{S}[\Delta]$ converges uniformly at the point ξ (§ 2.601). The behaviour of $s_n(x; f)$ in the neighbourhood of ξ will then, in a sense, be dominated by the behaviour of $s_n(x; l\cdot d(x-\xi)/\pi)$, and so Gibbs's phenomenon will occur. Thus, in particular, *if f is of bounded variation, $\mathfrak{S}[f]$ will present Gibbs's phenomenon at every point of simple discontinuity of f* [1]).

8.52. The formula 8.5(2) has interesting applications [2]). Suppose that $f(x)$ is of bounded variation and ξ a point of discontinuity of f. Let $\{h_m\}$ be a sequence of numbers such that $mh_m \to H$. Making the decomposition $f(x) = \Delta(x) + l\cdot d(x-\xi)/\pi$, we find the formula

$$s_n(\xi + h_n) \to \frac{f(\xi+0)+f(\xi-0)}{2} + \frac{f(\xi+0)-f(\xi-0)}{2}\cdot\frac{2}{\pi}\int_0^H \frac{\sin t}{t}\, dt,$$

where $s_n(x) = s_n(x; f)$. Taking for H one of the infinitely many roots of the equation $\varphi(u) = \pi/2$ (in particular $H=\infty$), we obtain the formulae: $s_n(\xi+h_n) \to f(\xi+0)$, $s_n(\xi-h_n) \to f(\xi-0)$, where $h_n = H/n$ if H is finite and, for example, $h_n = 1/\sqrt{n}$ if $H=\infty$. From these formulae we obtain, in particular, the value of the jump $f(\xi+0) - f(\xi-0)$.

8.6. Theorems of Rogosinski [3]).

In the preceding paragraph we obtained certain results concerning the behaviour of $s_n(\xi+h_n; f)$, provided that f was of bounded variation. It will appear that similar results hold in the general case if we consider the symmetric expressions $\frac{1}{2}[s_n(\xi+h_n) + s_n(\xi-h_n)]$ instead of $s_n(\xi+h_n)$.

8.61. (i) *If $\alpha_n = O(1/n)$ and if the series* 8.11(3a) [4]) *converges at a point ξ, to s, then $\frac{1}{2}[s_n(\xi+\alpha_n) + s_n(\xi-\alpha_n)] \to s$* (ii) *If this series is summable $(C, 1)$ at the point ξ to the value s, and if $\alpha_n = O(1/n)$, then*

$$\tfrac{1}{2}[s_n(\xi+\alpha_n) + s_n(\xi-\alpha_n)] - (s_n(\xi) - s)\cos n\alpha_n \to s. \tag{1}$$

[1]) F e j é r [3], R o g o s i n s k i [2].

[2]) D u B o i s-R e y m o n d [2], F e j é r [3].

[3]) R o g o s i n s k i [3], [4].

[4]) not necessarily a Fourier series.

Abel's transformation shows that

$$(2)\quad \tfrac{1}{2}[s_n(\xi+\alpha_n)+s_n(\xi-\alpha_n)]=\sum_{k=0}^{n-1}s_k(\xi)\,\Delta\cos k\alpha_n+s_n\cos n\alpha_n.$$

Here we have a linear transformation of $\{s_n(\xi)\}$, and the reader will verify that Toeplitz's conditions (§ 3.1) are satisfied. In particular, the condition (iii) of Toeplitz follows from the inequality $|\Delta\cos k\alpha_n|\leqslant\alpha_n=O(1/n)$.

This completes the proof of (i). Making Abel's transformation once more, we obtain, for the left-hand side of (2), the expression

$$(3)\quad \sum_{k=0}^{n-2}(k+1)\sigma_k\Delta^2\cos k\alpha_n+\sigma_{n-1}\,n\Delta\cos(n-1)\,\alpha_n+s_n\cos n\alpha_n,$$

where $\sigma_k=\sigma_k(\xi)$ are the first arithmetic means of the series considered. This expression without its last term is a linear transformation of $\{\sigma_n\}$. Toeplitz's conditions (i) and (iii) are again satisfied. Supposing, in particular, that $s_0=s_1=s_2=\ldots=1$, we find that the sum of the coefficients of σ_k in (3) is equal to $(1-\cos n\alpha_n)$. It follows that the expression (3) deprived of its last term and divided by $1-\cos n\alpha_n$ tends to s, and this is just (1). As a corollary we obtain

If 8.11(3a) is a $\mathfrak{S}[f]$, ξ a point of continuity of f, and p any fixed odd number, then $\frac{1}{2}[s_n(\xi+p\pi/2n)+s_n(\xi-p\pi/2n)]\to f(\xi)$. This relation holds uniformly in any interval of continuity of f.

8.62. We know that, if ξ is a point of continuity of f, then $|\sigma_k(\xi+h)-f(\xi)|<\varepsilon$ for $k>\nu$, $|h|<\delta$ [1]). Hence, for any sequence $\{h_n\}\to 0$, we have $|\sigma_k(\xi+h_n)-f(\xi)|\leqslant\varepsilon_k$, $k\leqslant n<\infty$, where $\varepsilon_k\to 0$. It follows that, if $\sigma_k=\sigma_k(\xi+h_n)$, $1\leqslant k<n$, $\alpha_n=\pi/2n$, the expression 8.61(3) is $f(\xi)+o(1)$, and so

If $s_n(x)=s_n(x;f)$ and ξ is a point of continuity of f, we have

$$\tfrac{1}{2}[s_n(\xi+h_n+\pi/2n)+s_n(\xi+h_n-\pi/2n)]\to f(\xi)$$

for every $\{h_n\}\to 0$.

In § 8.11 we learnt that $s_n(x;f)$ may be unbounded in the neighbourhood of a point of continuity of f. The last theorem detects a certain regularity in the behaviour of the curves $y=s_n(x)$: for $|x-\xi|<\varepsilon$, the arithmetic mean of the values of $s_n(x)$ at the

1) See footnote 1) on p. 52.

ends of intervals of length π/n differs very little from $f(\xi)$, and the less the smaller ε and $1/n$ are.

8.7. Cramér's theorem. We shall now study Gibbs's phenomenon for the method (C,r). From the inequality 3.22(1) we deduce that Fejér's sums cannot present Gibbs's phenomenon. Moreover it is easy to see that, if this phenomenon does not exist for a value r_1 of r, it cannot exist for any larger value of r. For, if $\sigma_n^r(x)$ denote the Cesàro means for $\mathfrak{S}[f]$ and if we have $m-\varepsilon\leqslant\sigma_n^{r_1}(x)\leqslant M+\varepsilon$ for $|x-\xi|\leqslant\eta$, $n>n_0$, and if $r>r_1$, then $m-2\varepsilon\leqslant\sigma_n^r(x)\leqslant M+2\varepsilon$ for $|x-\xi|\leqslant\eta$, $n\geqslant n_1$ (§ 3.13). It is therefore sufficient to consider the case $0<r<1$.

There exists a number $0<r_0<1$ with the following property: If f is simply discontinuous at a point ξ, the (C,r) means $\sigma_n^r(x)$ of $\mathfrak{S}[f]$ present Gibbs's phenomenon at ξ for $r<r_0$, but not for $r\geqslant r_0$ [1]).

8.701. It is sufficient to prove the theorem for the series 8.5(1), for which we have the formulae

$$(1)\quad \sigma_n^r(x)=-\tfrac{1}{2}x+\int_0^x K_n^r(t)\,dt,\qquad \sigma_n^r(x)=\tfrac{1}{2}(\pi-x)-\int_x^\pi K_n^r(t)\,dt,$$

where K_n^r denotes the (C,r) kernel. Let us consider first the case $r=1$. Replacing the denominator $4\sin^2\frac{1}{2}t$ by t^2, we find, as in § 8.5, that

$$(2)\qquad \sigma_n(x)=-\tfrac{1}{2}x+\int_0^{(n+1)\frac{x}{2}}\frac{\sin^2 t}{t^2}\,dt+R_n(x),$$

where $\sigma_n(x)=\sigma_n^1(x)$, $R_n(x)=O(n^{-1})=o(1)$ uniformly in x. Since $\sigma_n(x)\to(\pi-x)/2$ for $0<x<2\pi$, we obtain from (2) that

$$(3)\qquad \int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt=\tfrac{1}{2}\pi.$$

From (2) and (3) we deduce the following proposition which will be used presently. *Given any number $l>0$, there exists an $\varepsilon=\varepsilon(l)>0$ and an integer $n_0=n_0(l)$, such that $\sigma_n(x)<\pi/2-\varepsilon$ for $0\leqslant x\leqslant l/n$, $n>n_0$.*

1) Cramér [1]. Gronwall [2] showed that $r_0=0.4395516\ldots$

8.702. Next we require a formula for $K_n^r(t)$. Such a formula was found in § 3.3(3). Applying Abel's transformation to the last term of it, we find that $K_n^r(t)$ is equal to

$$\Im\left\{\frac{e^{i(n+\frac{1}{2})t}}{2A_n^r \sin\frac{1}{2}t}\left[\frac{1}{(1-e^{-it})^r} - A_{n+1}^{r-1}\frac{e^{-i(n+1)t}}{1-e^{-it}} - \sum_{\nu=n+1}^{\infty} A_{\nu+1}^{r-2}\frac{e^{-i(\nu+1)t}}{1-e^{-it}}\right]\right\} =$$

$$= \frac{1}{A_n^r}\cdot\frac{\sin[(n+\frac{1}{2}+\frac{1}{2}r)t - \frac{1}{2}\pi r]}{(2\sin\frac{1}{2}t)^{r+1}} + \frac{r}{n+1}\cdot\frac{1}{(2\sin\frac{1}{2}t)^2} + \frac{\theta}{n^2}\cdot\frac{8r(1-r)}{(2\sin\frac{1}{2}t)^3}\text{ [1]},$$

where $|\theta| \leqslant 1$ (see § 1.22). Integrating this expression over (x, π) and applying the second mean-value theorem to the first integral, we obtain from the second equation 8.701(1) that $\sigma_n^r(x)$ is equal to

$$(1)\qquad \tfrac{1}{2}(\pi - x) - \frac{r}{n+1}\,\tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2}x + \frac{2\theta_1}{nA_n^r(2\sin\frac{1}{2}x)^{r+1}} + \frac{B}{n^2x^2},$$

where $|\theta_1| \leqslant 1$, and $|B|$ is less than an absolute constant.

It was implicitly proved in § 3.12 that there exists an absolute constant C such that $A_n^r \geqslant Cn^r$ if $n \geqslant 1$, $0 \leqslant r \leqslant 1$. This shows that, if nx is large, of the last three terms in (1) the first is the largest in absolute value. Therefore there exists a number l such that $|\sigma_n^r(x)| \leqslant \pi/2$ for $l/n \leqslant x \leqslant \pi$, $1/2 \leqslant r \leqslant 1$, $n \geqslant n_1$.

Now we will show that, if $1-r$ is small enough, we have $|\sigma_n^r(x)| \leqslant \pi/2$ for $0 \leqslant x \leqslant l/n$. Taking into account the inequality $A_k^r/A_n^r \geqslant A_k^s/A_n^s$, which is true for $0 \leqslant k \leqslant n$, $-1 < r < s$, we find that $|\sigma_n^r(x) - \sigma_n^s(x)|$ is less than

$$(2)\qquad \sum_{\nu=1}^{n}\left(\frac{A_{n-\nu}^r}{A_n^r} - \frac{A_{n-\nu}^s}{A_n^s}\right)\left|\frac{\sin\nu x}{\nu}\right| \leqslant x\left[\frac{A_n^{r+1}}{A_n^r} - \frac{A_n^{s+1}}{A_n^s}\right] = \frac{nx(s-r)}{(r+1)(s+1)}.$$

If $s=1$, the last expression is less than $\frac{1}{2}nx(1-r)$, and so it is sufficient to take r such that $\frac{1}{2}(1-r)l < \varepsilon(l)$ (§ 8.701).

8.703. We have proved that, if r is sufficiently near to 1, σ_n^r cannot present Gibbs's phenomenon. To show that, if $r>0$ is small enough, Gibbs's phenomenon does occur, we consider the expression $|\sigma_n^r(x) - s_n(x)|$ which, in view of the inequality 8.702(2), is less than $x\,n\,r/(r+1)$. Since $s_n(\pi/n) \to \varphi(\pi) > \pi/2$ (§ 8.5), we conclude that Gibbs's phenomenon certainly occurs if we have $\pi r/(r+1) < \varphi(\pi) - \pi/2$.

[1] For a different proof, based on complex integration, of this formula, see Kogbetliantz [1].

8.704. In the previous sections we established the existence of a number r_0, $0 < r_0 < 1$, such that for any $r > r_0$ we have Gibbs's phenomenon, whereas for $r < r_0$ we have not. It remains only to show that for $r = r_0$ the phenomenon does not occur.

Let r_1 be any positive number less than r_0. From the formula 8.702(1) for σ_n^r we see that there is a number l_1 such that $|\sigma_n^r(x)| \leqslant \frac{1}{2}\pi$ for $r_1 \leqslant r \leqslant 1$, $l_1/n \leqslant x \leqslant \pi$. From the inequality 8.702(2) for $|\sigma_n^r - \sigma_n^s|$ we see that $\sigma_n^r(x)$ is a uniformly continuous function of r in the range $r \geqslant 0$, $0 \leqslant x \leqslant l_1/n$, $n = 1, 2, \ldots$ If the Gibbs phenomenon occurs for a value $r > r_1$, that is if there is a sequence $\{x_n\} \to +0$ such that $|\sigma_n^r(x_n)| > \frac{1}{2}\pi + \varepsilon$, then $0 \leqslant x_n \leqslant l_1/n$ and so, if $|s-r|$ is small enough, $|\sigma_n^s(x)| > \frac{1}{2}\pi + \frac{1}{2}\varepsilon$. This shows that the set of r for which the Gibbs phenomenon occurs is an open set, and the theorem is established.

8.8. Miscellaneous theorems and examples.

1. The Lebesgue constant L_n is equal to

$$\frac{16}{\pi^2}\sum_{\nu=1}^{\infty}\{1 + 1/3 + 1/5 + \ldots + 1/[2\nu(2n+1)-1]\}/(4\nu^2-1).$$

From this formula we see that $\{L_n\}$ is an increasing sequence. Szegö [2].

[Consider $\mathfrak{S}[|\sin x|]$ (§ 1.8.2) and the formula

$$(\sin kx)^2/\sin x = \sin x + \sin 3x + \ldots + \sin(2k-1)x].$$

2. Theorems 3.5(i) and 3.5(ii) are false for $r=1$.

[To prove the first part of this assertion, show that $\int_0^\pi \sin t\,|K_n'(t)|\,dt \neq O(1)$. where K_n denotes Fejér's kernel, and apply an argument similar to that of § 8.31. For the second part we refer the reader to Hahn [2]].

3. A series $u_0 + u_1 + \ldots$ is said to be summable by Borel's method, or summable B, to sum s, if $e^{-x}\sum_{n=0}^{\infty} s_n x^n/n! \to s$ as $x \to \infty$, where $s_n = u_0 + \ldots + u_n$. Show that

(i) If a series is convergent, it is summable B to the same sum.

(ii) A power series may be summable B outside its circle of convergence, so that the method B is rather strong. Nevertheless,

(iii) There exist continuous functions with Fourier series non-summable B at some points. Moore [1].

(iv) If $[f(x_0+h) - f(x_0)]\log 1/|h| \to 0$ with h, $\mathfrak{S}[f]$ is summable B at the point x_0, to the value $f(x_0)$. Hardy and Littlewood [2].

[Ad (i): apply Toeplitz's theorem (§ 3.1). Ad (ii): the series $1+z+z^2+\dots$ is summable B for $\Re z<1$. To prove (iii) it is sufficient to observe that the Lebesgue constants corresponding to the method B form an unbounded *function*. These constants, which are equal to

$$\frac{2}{\pi}\int_0^{\pi} e^{-x(1-\cos t)}\frac{|\sin(x\sin t+\tfrac{1}{2}t)|}{2\sin\tfrac{1}{2}t}\,dt,$$

are of order $\log x$. Propositions (ii) and (iii) show that the methods B and (C,k), $k>0$, are not comparable].

4. Consider a sequence $p_0, p_1, \dots$ of positive numbers, with the properties that $P_n=p_0+p_1+\dots+p_n\to\infty$, $p_n/P_n\to 0$. A series $u_0+u_1+\dots$ is said to be summable by Nörlund's method corresponding to $\{p_\nu\}$, or summable $N\{p_\nu\}$, to sum s, if

$$\sigma_n=(s_0p_n+s_1p_{n-1}+\dots+s_np_0)/P_n=(u_0P_n+\dots+u_nP_0)/P_n\to s$$

as $n\to\infty$. If $P_n=A_n^{\alpha}$, $\alpha>0$, we obtain, as a special case, Cesàro's method of summation (§ 3.11). Show that

(i) If Σu_n converges, it is summable $N\{p_\nu\}$ to the same sum.

(ii) If $0<p_0\leqslant p_1\leqslant\dots$ and if Σu_n is summable $(C,1)$, it is also summable $N\{p_\nu\}$ to the same sum. Tamarkin, *Fourier series*, p. 156.

5. Let $p_\nu>p_{\nu+1}\to 0$, $P_\nu\to\infty$. A necessary and sufficient condition that the method $N\{p_\nu\}$ should sum $\mathfrak{S}[f]$, to the value $f(x)$, at every point of continuity of f, is that the sequence

$$\lambda_n=P_n^{-1}\sum_{\nu=1}^{n}\frac{P_\nu}{\nu}$$

should be bounded Hille and Tamarkin [1₂], Tamarkin, *Fourier Series*, 190.

[In the first place we show that, if $\lambda_n=O(1)$, then the $N\{p_\nu\}$ kernel is quasi-positive (§ 3.201). Conditions (ii) and (iii) follow immediately. To prove (i') we argue as in § 3.3 and obtain, for the kernel, the expression

$$\frac{\sin(n+1)t}{2\sin\frac{1}{2}t}P_n^{-1}\sum_{\nu=0}^{n}p_\nu\cos(\nu+\tfrac{1}{2})t-\frac{\cos(n+1)t}{2\sin\frac{1}{2}t}P_n^{-1}\sum_{\nu=0}^{n}p_\nu\sin(\nu+\tfrac{1}{2})t=U_n-V_n.$$

Applying Abel's transformation to V_n, and denoting by K_n Fejér's kernel, we find that $\mathfrak{M}[V_n;\,0,\pi]$ does not exceed

$$P_n^{-1}\int_0^{\pi}\{(n+1)p_nK_n+\sum_{\nu=0}^{n-1}(\nu+1)\Delta p_\nu K_\nu\}\,dt=\tfrac{1}{2}\pi P_n^{-1}\{(n+1)p_n+\sum_{\nu=0}^{n-1}(\nu+1)\Delta p_\nu\}.$$

The expression in curly brackets is equal to P_n, so that $\mathfrak{M}[V_n;\,0,\pi]\leqslant\frac{1}{2}\pi$, and everything depends on the behaviour of $\mathfrak{M}[U_n;\,0,\pi]=\mathfrak{M}[U_n;\,0,1/n]+\mathfrak{M}[U_n;\,1/n,\pi]$. It is easy to see that $U_n=O(n)$ in the interval $0\leqslant t\leqslant 1/n$, so that $\mathfrak{M}[U_n;\,0,1/n]=O(1)$. Now Abel's transformation gives, for U_n, the value

$$(1)\qquad \frac{\sin(n+1)t}{P_n}\left\{p_n\frac{\sin(n+1)t}{4\sin^2\frac{1}{2}t}+\sum_{\nu=0}^{n-1}\Delta p_\nu\frac{\sin(\nu+1)t}{4\sin^2\frac{1}{2}t}\right\}.$$

Observing that $\displaystyle\int_{1/n}^{\pi}\frac{|\sin(\nu+1)t|}{4\sin^2\frac{1}{2}t}dt=\int_{1/n}^{1/\nu}+\int_{1/\nu}^{\pi}\leqslant(\nu+1)\int_{1/n}^{1/\nu}\frac{t\,dt}{4\sin^2\frac{1}{2}t}+$ $\displaystyle+\int_{1/\nu}^{\pi}\frac{dt}{4\sin^2\frac{1}{2}t}<A(\nu+1)\log(n/\nu)+B(\nu+1)$, $\nu\geqslant 1$, where A and B are constants, we see that the absolute value of (1) integrated over $(1/n,\pi)$ gives less than

$$\frac{B}{P_n}\{(n+1)p_n+\sum_{\nu=0}^{n-1}(\nu+1)\Delta p_\nu\}+\frac{A}{P_n}\left\{\sum_{\nu=1}^{n-1}\Delta p_\nu\cdot(\nu+1)\log(n/\nu)\right\}+O(P_n^{-1}\log n).$$

Here the first term is equal to B. Making Abel's transformation, we see that the second term is equal to

$$\frac{2Ap_1}{P_n}\log n+\frac{A}{P_n}\left\{\sum_{\nu=2}^{n}P_\nu\log(n/\nu)\right\}+\frac{A}{P_n}\left\{\sum_{\nu=2}^{n}\nu P_\nu\log\left(1-\frac{1}{\nu}\right)\right\}=A_n+B_n+C_n.$$

It is not difficult to verify that the condition $\lambda_n=O(1)$ implies $\log n=O(P_n)$, i. e. $A_n=O(1)$. Since $\log(1-1/\nu)\simeq-1/\nu$, we obtain $C_n=O(1)$. Applying Abel's transformation, we see that $B_n=O(\lambda_n)+O(P_n^{-1}\log n)=O(1)$. Hence $\mathfrak{M}[U_n;\,1/n,\pi]=O(1)$, $\mathfrak{M}[U_n;\,0,\pi]=O(1)$, and the first half of the theorem is established.

To prove the second half, it is sufficient to show that, if $\mathfrak{M}[U_n]=O(1)$, then $\lambda_n=O(1)$. Applying Abel's transformation to U_n and observing that $|\sin(n+1)t|\geqslant\sin^2(n+1)t$, we see that the relation $\mathfrak{M}[U_n;0,\pi]=O(1)$ implies

$$(2)\qquad P_n^{-1}\int_0^{\pi}\sin^2(n+1)t\left\{P_n\frac{\cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}+\sum_{\nu=0}^{n-1}P_\nu\sin(\nu+1)t\right\}dt=O(1).$$

It is not difficult to see that the integral, extended over $(0,\pi)$, of the function $\sin^2(n+1)t\cdot\cos(n+\frac{1}{2})t/2\sin\frac{1}{2}t$ is bounded. Hence, using the equation $2\sin^2(n+1)t=1-\cos 2(n+1)t$, and the fact that the integral over $(0,\pi)$ of $\sin(\nu+1)t\cos 2(n+1)t$ is $O(1/n)$ for $0\leqslant\nu<n$, we see that (2) may be written

$$\tfrac{1}{2}P_n^{-1}\int_0^{\pi}\left\{\sum_{\nu=0}^{n-1}P_\nu\sin(\nu+1)t\right\}dt+R=O(1),$$

where $R = O(P_0 + P_1 + \dots + P_{n-1})/n\,P_n = O(1)$. From this we obtain $\lambda_n = O(1)$].

6. The partial sums $d_n(x)$ of the series $\sin x + \frac{1}{2}\sin 2x + \dots$ are positive for $0 < x < \pi$. Jackson [1]; Landau [1].

[Suppose that the theorem has been established for $n-1$ and that $d_n(x)$, $0 \leqslant x \leqslant \pi$, attains its minimum at a point x_0, $0 < x_0 < \pi$. Since

$$d_n'(x_0) = [\sin(n+\tfrac{1}{2})\,x_0 - \sin\tfrac{1}{2}x_0]/2\sin\tfrac{1}{2}x_0 = 0,$$

we obtain that $\sin(n+\frac{1}{2})\,x_0 = \sin\frac{1}{2}x_0$ and so also $|\cos(n+\frac{1}{2})\,x_0| = \cos\frac{1}{2}x_0$. This shows that $\sin nx_0 = \sin(n+\frac{1}{2})\,x_0\cos\frac{1}{2}x_0 - \cos(n+\frac{1}{2})\,x_0\sin\frac{1}{2}x_0 \geqslant 0$, $d_n(x_0) \geqslant d_{n-1}(x_0)$, which is impossible since the theorem is true for $n-1$].