CHAPTER VII.

Conjugate series and complex methods in the theory of Fourier series.

7.1. Summability of conjugate series [1]. In Chapter III we proved some results on the summability (C, r) of Fourier series. As regards the conjugate series our results were less complete. The obstacle was that we knew nothing about the existence of the integral

$$(1)\qquad -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt = \lim_{h\to 0}\left(-\frac{1}{\pi}\int_h^\pi \frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt\right).$$

We proved that, almost everywhere, the existence of (1) was equivalent to the summability A of $\bar{\mathfrak{S}}[f]$. We now intend to prove the latter fact using complex methods, independently of the behaviour of (1). This will just enable us to prove the existence of (1) for almost every x. The proof will be based on the following lemma.

Let $G(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots$, $z = re^{ix}$, *be a function which is regular, bounded, and non-vanishing in the circle* $|z| < 1$. *The function* $l(x) = \lim_{r\to 1} G(re^{ix})$ *may vanish only in a set of measure* 0.

Suppose that $|G(z)| < 1$. That $l(x)$ exists for almost every x follows from the fact that the real and imaginary parts of $\alpha_0 + \alpha_1 e^{ix} + \dots$ are Fourier series of bounded functions (§ 4.36). Let us take any branch of the function $\log G(z) = \log|G(re^{ix})| + i \arg G(re^{ix})$. Since $G(z) \neq 0$ for $|z| < 1$, $\log G(z)$ is regular

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[1]) Privaloff [2], Plessner [2], Hardy and Littlewood [4], Zygmund [2].

and $\log|G(re^{ix})|\leqslant 0$. It follows that the harmonic function $\log|G(re^{ix})|$ is a Poisson-Stieltjes integral (§ 4.36), and so tends, for almost every x, to a finite limit as $r\to 1$. This shows that $l(x)\neq 0$ for almost every x, and the lemma is established.

For any integrable $f(x)$, $\tilde{\mathfrak{S}}[f]$ is summable A almost everywhere. It is sufficient to suppose that $f\geqslant 0$. Let $f(r,x)$ be the Poisson integral for $f(x)$, and $\bar{f}(r,x)$ the conjugate harmonic function. The values of the function $F(z)=f(r,x)+i\bar{f}(r,x)$, $z=re^{ix}$, belong to the right half-plane, so that the function $G(z)=1/(F(z)+1)$ is regular and less than 1 in absolute value for $|z|<1$. Hence, by the lemma, $\lim\limits_{r\to1} G(re^{ix})$ exists and is different from 0 for almost every x. Thence we deduce that, for almost every x, $\lim F(re^{ix})$, and therefore $\lim\bar{f}(r,x)$, exists and is finite. As corollaries we obtain the following propositions.

(i) *For any integrable f the integral* (1) *exists almost everywhere.*

(ii) *$\tilde{\mathfrak{S}}[f]$ is summable (C,r), $r>0$, at almost every point, to the value* (1) (§ 3.32).

The integral (1) will be denoted troughout by $\bar{f}(x)$. The function $\bar{f}(x)$ is called the *conjugate* function of $f(x)$. Considering the points where $\mathfrak{S}[f]$ and $\tilde{\mathfrak{S}}[f]$ are both summable $(C,1)$, we obtain the following proposition (§ 3.14):

(iii) *Given any integrable $f(x)$, the conjugate harmonic function $\bar{f}(r,x)$, tends, for almost every x_0, to the value $\bar{f}(x_0)$ as the point (r,x) approaches $(1,x_0)$ along any path not touching the circle.*

7.11. *If $F(x)$, $0\leqslant x\leqslant 2\pi$, is a function of bounded variation, $\tilde{\mathfrak{S}}[dF]$ is, at almost every point, summable (C,r), $r>0$, to the value*

$$(1)\quad -\frac{1}{\pi}\int_0^\pi\frac{F(x+t)+F(x-t)-2F(x)}{4\sin^2\frac12 t}dt=\lim_{h\to0}\left\{-\frac{1}{\pi}\int_h^\pi\frac{F(x+t)+F(x-t)-2F(x)}{4\sin^2\frac12 t}\right\}dt.$$

The proof runs on the same lines as in the case when F is absolutely continuous. Supposing, as we may, that $F(x)$ is non-decreasing, let $f(r,x)>0$ be the Poisson-Stieltjes integral for dF, and $\bar{f}(r,x)$ the conjugate harmonic function. Since $\bar{f}(r,x)>0$, we prove, as before, that $\lim\limits_{r\to1}\bar{f}(r,x)$ exists and is finite for almost every x. Combining the arguments of §§ 3.45, 3.8, it can easily be shown (the details of the proof we leave to the reader) that, at any point where $F'(x)$ exists and is finite, $\tilde{\mathfrak{S}}[dF]$ is summable A if and only if the integral (1) exists. An appeal to the second part of Theorem 3.8 completes the proof.

If F' is absolutely continuous and $F'=f$, an integration by parts shows the integrals (1) and 7.1(1) to be equal.

7.2. Conjugate series and Fourier series. We shall now be concerned with the very important problem of conditions under which the conjugate series is itself a Fourier series. A special result was established in § 4.22, but the method used there cannot be extended to more general cases. The following important result is due to M. Riesz.

7.21. *If $f\in L^p$, $p>1$, then $\bar{f}\in L^p$ and there exists a constant A_p depending only on p and such that $\mathfrak{M}_p[\bar{f};\,0,2\pi]\leqslant A_p\,\mathfrak{M}_p[f;\,0,2\pi]$. Moreover, $\tilde{\mathfrak{S}}[f]=\mathfrak{S}[\bar{f}]$*[1].

In virtue of Theorem 4.36 (iii), and of Fatou's lemma, the theorem which we have to prove is a corollary of, and is in reality equivalent to, the following proposition.

Let $F(z)=u(z)+iv(z)$, $v(0)=0$, be an arbitrary function regular inside the unit circle. Then

$$(1)\quad \mathfrak{M}_p[v(re^{ix})]\leqslant A_p\,\mathfrak{M}_p[u(re^{ix})],\quad 0\leqslant r<1,\quad p>1.$$

It is sufficient to prove the truth of (1) in the case when $\Re f(z)=u(z)>0$ for $|z|<1$. In fact, having fixed r, let $\varphi_1(x)=\mathrm{Max}\{u(r,x),0\}$, $\varphi_2(x)=\mathrm{Min}\{u(r,x),0\}$, so that $u(re^{ix})=\varphi_1(x)+\varphi_2(x)=\varphi(x)$, say. The functions φ_1,φ_2 are continuous and possess first derivatives which are continuous, except at a finite number of points where they have simple discontinuities. It follows that the conjugate functions $\bar\varphi(x)$, $\bar\varphi_1(x)$, $\bar\varphi_2(x)$ are also continuous. Let $\varphi(\rho,x)$, $\varphi_j(\rho,x)$, $\bar\varphi_j(\rho,x)$, $j=1,2$, denote the corresponding harmonic functions. Since $\varphi_j(\rho,x)>0$, we have, assuming the truth of (1) for $u>0$, that $\mathfrak{M}_p[\bar\varphi_j(\rho,x)]\leqslant A_p\,\mathfrak{M}_p[\varphi_j(\rho,x)]$, and, making $\rho\to1$, $\mathfrak{M}_p[\bar\varphi_j(x)]\leqslant A_p\,\mathfrak{M}_p[\varphi_j(x)]\leqslant A_p\,\mathfrak{M}_p[\varphi(x)]$. By Minkowski's inequality we obtain: $\mathfrak{M}_p[\bar\varphi(x)]\leqslant\mathfrak{M}_p[\bar\varphi_1(x)]+\mathfrak{M}_p[\bar\varphi_2(x)]\leqslant 2A_p\,\mathfrak{M}_p[\varphi(x)]$. This is just (1) with the constant twice as large, which is, of course, immaterial.

Passing to the proof of the theorem, let us consider the branch of the function $F^p(z)$ which is positive at the origin. Writing u,v instead of $u(re^{ix})$, $v(re^{ix})$, we have, by Cauchy's theorem,

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[1] M. Riesz [4].

$$(2)\qquad \frac{1}{2\pi i}\int\limits_{|z|=r}\frac{F^p(z)}{z}\,dz=\frac{1}{2\pi}\int\limits_0^{2\pi}(u+iv)^p\,dx=u^p(0)=\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}u\,dx\right\}^p.$$

The difference $(u+iv)^p-(iv)^p$ is equal to the integral of the function pz^{p-1}, which is regular in the right half-plane, taken along the straight line between iv and $u+iv$, and so its modulus does not exceed the length u of the path of integration, multiplied by the maximal modulus of the function integrated, viz. $p\,(u^2+v^2)^{\frac{1}{2}(p-1)}\leqslant p2^{\frac{1}{2}(p-1)}\,(u^{p-1}+v^{p-1})$. Using this and the fact that the last term in (2) is equal to $\mathfrak{A}^p[u]\leqslant\mathfrak{A}^p_p[u]$ (§ 4.15), we obtain from (2) the inequality

$$(3)\qquad \left|\frac{1}{2\pi}\int\limits_0^{2\pi}(iv)^p\,dx\right|\leqslant\frac{p2^{\frac{1}{2}(p-1)}}{2\pi}\int\limits_0^{2\pi}(u^p+u\,|v|^{p-1})\,dx+\frac{1}{2\pi}\int\limits_0^{2\pi}u^p\,dx.$$

Now $(iv)^p=|v|^p\exp(\pm\frac{1}{2}\pi\,ip)$, where the sign in the exponent is that of v; it follows that $\Re\,(iv)^p=|v|^p\cos\frac{1}{2}p\pi$. Let I denote the integral on the left of (3). Then the inequality will hold à fortiori if we replace $|I|$ by $|\Re I|$; and so, applying Hölder's inequality to the product $u\,|v|^{p-1}$, we obtain the inequality $|\cos\frac{1}{2}p\pi|\,\mathfrak{M}_p[v]\leqslant p\,2^{\frac{1}{2}(p-1)}\{\mathfrak{M}_p^p[u]+\mathfrak{M}_p[u]\,\mathfrak{M}_p^{p-1}[v]\}+\mathfrak{M}_p^p[u]$. Denoting the ratio $\mathfrak{M}_p[v]/\mathfrak{M}_p[u]$ by X, we see that

$$(4)\qquad |\cos\tfrac{1}{2}p\pi|\,X^p\leqslant p2^{\frac{1}{2}(p-1)}\,(X^{p-1}+1)+1.$$

It follows that, if only $\cos\frac{1}{2}p\pi\neq 0$, X cannot exceed a constant A_p and thus the theorem is established for $p\neq 3, 5, 7, \ldots$

It would not be difficult to supply a special proof for these exceptional values, but it is more convenient to use another, more illuminating, argument, which will give us, besides, information about the constants A_p.

7.22. *If the inequality* 7.21(1) *is true for a certain* $p>1$, *it is also true for the complementary exponent* p'; *moreover* $A_p=A_{p'}$.

Let $g(x)$ be any trigonometrical polynomial with $\mathfrak{M}_p[g]\leqslant 1$, and $\bar{g}(x)$ the conjugate polynomial. From Parseval's relation we have

$$\int\limits_0^{2\pi}vg(x)\,dx=-\int\limits_0^{2\pi}u\bar{g}(x)\,dx.$$

It is not difficult to see that $\mathfrak{M}_{p'}[v]$ is the upper bound of the expression on the left for all possible g (§ 4.7.2). The expression on the right does not exceed, in absolute value, $\mathfrak{M}_{p'}[u]\,\mathfrak{M}_p[\bar{g}]\leqslant\mathfrak{M}_{p'}[u]\,A_p\,\mathfrak{M}_p[g]\leqslant A_p\,\mathfrak{M}_{p'}[u]$, so that $\mathfrak{M}_{p'}[v]\leqslant A_p\,\mathfrak{M}_{p'}[u]$ and the theorem follows. At the same time, since Theorem 7.21 was established for $1<p\leqslant 2$, it holds for $p\geqslant 2$, and in particular for $p=3, 5, \ldots$

7.23. Stein's proof. The preceding proof of Theorem 7.21 is due to M. Riesz. An alternative proof, based on a different idea, has been obtained by Stein[1]). We shall reproduce it here since it is very simple and yields a good estimate for the constants A_p. Its main feature is the use of Green's formula

$$(1)\qquad \int\limits_C\frac{\partial w}{\partial r}\,ds=\iint\limits_S\Delta w\,d\sigma.$$

Here S is the circle $\xi^2+\eta^2\leqslant r^2$, C its circumference, and w a function of rectangular variables ξ,η, which, together with its first and second derivatives, is continuous in S; $\partial w/\partial r$ denotes the derivative in the direction of the radius vector, and Δw the expression $\partial^2 w/\partial\xi^2+\partial^2 w/\partial\eta^2$.

As we already know, it is sufficient to prove Theorem 7.21 for the case $1<p\leqslant 2$, $u(z)>0$. Consider $u(z)$, $v(z)$, $|F(z)|=(u^2+v^2)^{\frac{1}{2}}$ as functions of ξ,η. A simple calculation shows that

$$(2)\qquad \Delta u^p=p\,(p-1)\,u^{p-2}\,|F'|^2,\quad \Delta|F|^p=p^2\,|F|^{p-2}\,|F'|^2,$$

so that, since $p\leqslant 2$, $|f|\geqslant u$, we find $\Delta|F|^p\leqslant p'\,\Delta u^p$. Let $\mathfrak{M}_p^p[u(re^{ix})]=\lambda(r)$, $\mathfrak{M}_p^p[F(re^{ix})]=\mu(r)$. We shall apply the formula (1) to the functions $w=u^p$ and $w=|F|^p$. Since $ds=r\,dx$, the left-hand sides will represent $r\,d\lambda/dr$ and $r\,d\mu/dr$ respectively, and, in virtue of the inequality $\Delta|F|^p\leqslant p'\Delta u^p$, we obtain $\mu'(r)\leqslant p'\lambda'(r)$. Integrating this inequality with respect to r, and taking into account that $\lambda(0)=\mu(0)$, $p'>1$, we find $\mu(r)\leqslant p'\lambda(r)$. This is just the inequality 7.21(1), with $A_p=p'^{1/p}$, $1<p\leqslant 2$. If u is no longer positive, the value of A_p is increased by the factor 2. It follows that $A_p\leqslant 2p^{1/p'}<2p$ for $p\geqslant 2$. For better estimates we refer the reader to the original paper.

1) Stein [1].

7.24. Theorem 7.21 ceases to be true when $p=1$, since the sum $\bar{f}(x)$ of $\bar{\mathfrak{S}}[f]$ is not necessarily integrable (§ 5.221). It follows, in particular, that the proper, i. e. the best possible, value of A_p is unbounded when p tends to 1 or to ∞. The place of Theorem 7.21 is taken by two other theorems. We shall prove the first of them by M. Riesz's method, whereas for the second the method developed in the preceding section will be more convenient.

(i) *If $f(x)$ is integrable, so is $|\bar{f}(x)|^p$, for any $0<p<1$. Moreover there is a constant B_p depending only on p and such that $\mathfrak{M}_p[\bar{f}] \leqslant B_p \mathfrak{M}[f]$, $0<p<1$* [1]).

(ii) *If $|f|\log^+|f|$ is integrable, then $\bar{f}$ is integrable and $\bar{\mathfrak{S}}[f]=\mathfrak{S}[\bar{f}]$. There exist two absolute constants A and B such that*

$$(1)\qquad \int_0^{2\pi} |\bar{f}|\,dx \leqslant A\int_0^{2\pi} |f|\log^+|f|\,dx + B.$$

As regards (i) it is, as before, enough to prove that, if $F(z)=u+iv$ is regular for $|z|<1$, then $\mathfrak{M}_p[v] \leqslant B_p \mathfrak{M}_p[u]$. Suppose first that $u>0$. Taking the real parts in 7.21(2), we have, since $|\arg(u+iv)^p| \leqslant \frac{1}{2}p\pi$,

$$\frac{\cos\frac{1}{2}p\pi}{2\pi}\int_0^{2\pi}(u^2+v^2)^{p/2}\,dx \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} u\,dx\right)^p.$$

This inequality holds à fortiori if we omit the term u^2 on the left, but then we obtain just what we wanted to prove, with $B_p^p=(2\pi)^{1-p}\sec\frac{1}{2}\pi p$. To remove the assumption $u>0$, we proceed as in § 7.21, but, since Minkowski's inequality does not work for $p<1$, we apply the inequality $|\bar{\varphi}|^p = |\bar{\varphi}_1+\bar{\varphi}_2|^p \leqslant |\bar{\varphi}_1|^p + |\bar{\varphi}_2|^p$ (§ 4.13) and the value of B_p is increased by the factor $2^{1/p}$.

To establish (ii) it is again sufficient to prove the inequality (1) with $f, \bar{f}$ replaced by u, v. Suppose first that $u>e$. We verify that $\Delta(u\log u) = |F'(z)|^2/u$, $\Delta|F| = |F'|^2/|F| \leqslant \Delta(u\log u)$. Denoting by $\lambda(r)$ and $\mu(r)$ the integrals of $u\log u$ and $|F|$ over

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1) Kolmogoroff [4]; Littlewood [3], Hardy [9], Tamarkin [1].
2) Zygmund [4]; Titchmarsh [3], Littlewood [4], Stein [1]; $\log^+ x$ denotes the function which is equal to $\log x$ for $x>1$ and to 0 elsewhere.

the interval $0\leqslant x\leqslant 2\pi$, we find that $\mu'(r)\leqslant\lambda'(r)$, and hence $\mu(r)\leqslant\lambda(r)$, since $|F(0)|=u(0)\leqslant u(0)\log u(0)$.

In the general case we proceed as in § 7.21, viz. put $\varphi(x) = \varphi_1(x)+\varphi_2(x)+\varphi_3(x)$, where $\varphi_1=\operatorname{Max}\{\varphi(x), e\}$, $\varphi_3=\operatorname{Min}\{\varphi(x), -e\}$, so that $|\varphi_2(x)|\leqslant e$. Since $\mathfrak{A}[\bar{\varphi}_2(x)] \leqslant \mathfrak{A}_2[\bar{\varphi}_2(x)] \leqslant \mathfrak{A}_2[\varphi_2(x)] \leqslant e$, (§ 4.15) the inequality (1) follows, with $A=2$, $B=2\pi e$.

That $\bar{\mathfrak{S}}[f]=\mathfrak{S}[\bar{f}]$ is a corollary of the relation $\mathfrak{M}[\bar{f}-\bar{s}_n]\to 0$ which will be established in § 7.3 ($\bar{s}_n$ denote the partial sums of $\bar{\mathfrak{S}}[f]$).

7.25. It is important to observe that the integrability of $|f|\log^+|f|$ is essential for that of $\bar{f}$, and cannot be replaced by anything less stringent. This follows from the following result, which is, in some respects, a converse of Theorem 7.24(ii).

If f is non-negative and $\bar{f}$ integrable, then $f\log^+ f$ is integrable [1]).

Suppose, as we may, that $f\geqslant 1$, and let $u(z)$, $v(z)$ denote the Poisson integral of f and the conjugate harmonic function. Putting $F(z)=u+iv$ we consider the integral of the function $z^{-1}F(z)\log F(z)$, taken round the circle $|z|=r$. Applying Cauchy's theorem and taking the real parts on both sides of the equation, we have

$$(1)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left\{u\log\sqrt{u^2+v^2} - v\operatorname{arctg}\frac{v}{u}\right\}dx = u(0)\log u(0).$$

In virtue of the inequality $0\leqslant v\operatorname{arctg}(v/u)\leqslant\frac{1}{2}\pi|v|$, we obtain

$$(2)\qquad \frac{1}{2\pi}\int_0^{2\pi} u\log u\,dx \leqslant \frac{1}{4}\int_0^{2\pi}|v|\,dx + u(0)\log u(0).$$

In § 7.56 (see also § 7.26(iii)) we shall learn that, if $\bar{f}$ is integrable, then $\bar{\mathfrak{S}}[f]=\mathfrak{S}[\bar{f}]$, so that the integral on the right in (2) is bounded and the result follows by an application of Fatou's lemma.

7.26. Integral B. There exist, as we have already mentioned, functions $f\in L$ such that $\bar{f}$ is not integrable. It is interesting to observe that, with a suitable definition of an integral, more general than that of Lebesgue, the function $\bar{f}$ is integrable.

Given any function $f(x)$, $a\leqslant x<b$, we repeat it periodically in the intervals $a+kh\leqslant x<a+(k+1)h$, $k=\pm1,\pm2,\ldots$, where $h=b-a$. Let $a=x_0<x_1<\ldots<x_n=b$ be any subdivision of (a,b), ξ_i an arbitrary point from (x_{i-1},x_i), and $\rho=\operatorname{Max}(x_i-x_{i-1})$. Consider the expression

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1) The theorem is due to M. Riesz.

(1) $$I(t)=\sum_{i=1}^{n} f(\xi_i+t)\,(x_i-x_{i-1}), \qquad 0\leqslant t<b-a,$$

and suppose that there exists a number I with the following property: for any $\varepsilon>0$ we can find a $\delta=\delta(\varepsilon)$ such that $|I(t)-I|<\varepsilon$, except for t belonging to a set of measure less than ε, provided that $\rho\leqslant\delta$ (independently of the values of x_i,ξ_i). We shall say, then, that $f(x)$ is integrable B over (a,b) and that I is the value of the integral [1]). It is easy to grasp the meaning of the above definition if we proceed as follows: besides the function $f(x)$ we consider a whole family of functions $f_t(x)=f(t+x)$, depending on a parameter t, and for each of them we form Riemannian sums. If $f(x)$ is not integrable R, no $f_t(x)$ is, but it may happen that 'on the whole' those sums approach I. If this happens, f is integrable B; we could also say that f is integrable R 'in measure'.

(i) *If f is integrable L over (a,b), it is also integrable B, both integrals having the same value.*

Put $f=f_1+f_2$, and correspondingly $I(t)=I_1(t)+I_2(t)$, where f_1 is continuous and the integral of $|f_2|$ over (a,b) is less than $\frac{1}{3}\varepsilon^2/(b-a)$. The integral of $|I_2(t)|$ over (a,b) is less than $\frac{1}{3}\varepsilon^2$, so that the set T of t where $|I_2(t)|>\frac{1}{3}\varepsilon$ is of measure $<\varepsilon$. If I,I_1,I_2 are the integrals of f,f_1,f_2 over (a,b), then $|I(t)-I|\leqslant|I_1(t)-I_1|+|I_2(t)|+|I_2|$. The first term on the right is less than $\frac{1}{3}\varepsilon$ if only $\rho\leqslant\delta=\delta(\varepsilon)$. The second is less than $\frac{1}{3}\varepsilon$ for $t\,\overline{\epsilon}\,T$. The third is less than $\frac{1}{3}\varepsilon^2(b-a)<\frac{1}{3}\varepsilon$, assuming, as we may, that $\varepsilon(b-a)<1$. Hence $|I(t)-I|<\varepsilon$ for $t\,\overline{\epsilon}\,T$, $|T|<\varepsilon$, if only $\rho\leqslant\delta$, and the theorem follows.

(ii) *For every $f\in L$, $\bar{f}$ is integrable B over $(0,2\pi)$, and $\overline{\mathfrak{S}}[f]=\mathfrak{S}[\bar{f}]$* [2]).

Substituting $\bar{f}$ for f in the expression (1), we obtain a function $\bar{I}(t)$, conjugate to $I(t)$. By Theorem 7.24(i), we have $\mathfrak{M}_{1/2}[\bar{I}(t)]\leqslant B_{1/2}\,\mathfrak{M}[I(t)]\leqslant\leqslant 2\pi B_{1/2}\,\mathfrak{M}[f]$. It follows that $|\bar{I}(t)|<\frac{1}{2}\varepsilon$, for $t\,\overline{\epsilon}\,T$, $|T|<\varepsilon$, if only the integral of $|f|$ over $(0,2\pi)$ is less than $\eta=\eta(\varepsilon)$. In the general case we put $f=f_1+f_2$, where f_1 is a trigonometrical polynomial and the integral of $|f_2|$ is less than η. We find then that $|\bar{I}(t)|<\varepsilon$ for $t\,\overline{\epsilon}\,T$, $|T|<\varepsilon$, provided that $\rho\leqslant\delta=\delta(\varepsilon)$. Thus the integral B of $\bar{f}$ over $(0,2\pi)$ exists and has the value 0.

We shall now show that the products $\bar{f}\cos kx$, $\bar{f}\sin kx$ are integrable B over $(0,2\pi)$, to the values $-\pi b_k$, πa_k, $k=1,2,\ldots$ We may suppose that $a_0=a_1=\ldots=a_k=b_1=\ldots=b_k=0$. We have then

(2a) $\overline{f\cos kx}=\bar{f}\cos kx$, (2b) $\overline{f\sin kx}=\bar{f}\sin kx$.

This is easy to verify when f is a trigonometrical polynomial. Hence (2)

[1]) Integral B is one of several definitions of an integral propounded by Denjoy; see Denjoy [3], Boks [1]. Proposition (i) (see below) belongs also to Denjoy, but the proof of the text, which is much simpler, has been given by Saks.

[2]) Kolmogoroff [5]. The example of the series conjugate to 5.12(2) (or simply that of the odd function equal to $1/x\log(x/2\pi)$ in the interval $0<x\leqslant\pi$) shows that a function may be integrable B over $(-\pi,\pi)$ without being integrable B over $(0,\pi)$.

is true if we replace $f,\bar{f}$ by $\sigma_n,\bar{\sigma}_n$, where $\sigma_n,\bar{\sigma}_n$ denote the $(C,1)$ means of $\mathfrak{S}[f]$, $\overline{\mathfrak{S}}[f]$ respectively. If $n\to\infty$, then $\bar{\sigma}_n\cos kx\to\bar{f}\cos kx$ and, to prove (2a), it is sufficient to show that $\overline{(\sigma_{n_i}-f)\cos kx}\to 0$ for a sequence $\{n_i\}\to\infty$. This follows from the relations $\mathfrak{M}_{1/2}[\overline{(\sigma_n-f)\cos kx}]\leqslant B_{1/2}\,\mathfrak{M}[\sigma_n-f]\to 0$ (§ 4.2). Similarly we prove (2b). The formulae (2) show that the products $\bar{f}\cos kx$ and $\bar{f}\sin kx$ are integrable B over $(0,2\pi)$, the value of the integrals being 0. This completes the proof of (ii). As a corollary we obtain the following theorem.

(iii) *If $\bar{f}$ is integrable L, then $\overline{\mathfrak{S}}[f]$ is the Fourier-Lebesgue series of $\bar{f}$* [1]).

7.3. Mean convergence of Fourier series [2]).

The theorems on conjugate functions which we proved in the preceding paragraph enable us to obtain some results for the partial sums $s_n,\bar{s}_n$ of $\mathfrak{S}[f],\overline{\mathfrak{S}}[f]$.

(i) *If $f\in L^p$, $p>1$, then $\mathfrak{M}_p[f-s_n]\to 0$.*

(ii) *If f is integrable, then $\mathfrak{M}_p[f-s_n]\to 0$, $\mathfrak{M}_p[\bar{f}-\bar{s}_n]\to 0$ for every $0<p<1$.*

(iii) *If $|f|\log^+|f|$ is integrable, then $\mathfrak{M}[f-s_n]\to 0$, $\mathfrak{M}[\bar{f}-\bar{s}_n]\to 0$.*

Let $s_n^*,\bar{s}_n^*$ denote the modified partial sums $s_n,\bar{s}_n$ (§ 2.3). Since $s_n-s_n^*$ and $\bar{s}_n-\bar{s}_n^*$ tend uniformly to 0, it is sufficient to prove the theorems for $s_n^*,\bar{s}_n^*$ instead of $s_n,\bar{s}_n$. From the formula

$$s_n^*(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\sin nt}{2\operatorname{tg}\frac{1}{2}t}\,dt,$$

replacing $\sin nt$ by $\sin n(t+x)\cos nx-\cos n(t+x)\sin nx$, we see that $|s_n^*(x)|\leqslant|g_1(x)|+|g_2(x)|$, where g_1 is conjugate to $f(x)\sin nx$, g_2 to $f(x)\cos nx$. Theorem 7.21 and Minkowski's inequality give

(1) $$\mathfrak{M}_p[s_n^*]\leqslant 2A_p\,\mathfrak{M}_p[f],$$

an inequality important in itself. Now put $f=f'+f''$, where f' is a trigonometrical polynomial and $\mathfrak{M}_p[f'']<\varepsilon$. Similarly we have $s_n^*=s_n^{*\prime}+s_n^{*\prime\prime}$, $f-s_n^*=(f'-s_n^{*\prime})+(f''-s_n^{*\prime\prime})$ and so, if $p>1$,

$$\mathfrak{M}_p[f-s_n^*]\leqslant\mathfrak{M}_p[f'-s_n^{*\prime}]+\mathfrak{M}_p[f'']+\mathfrak{M}_p[s_n^{*\prime\prime}]=\mathfrak{M}_p[f'']+\mathfrak{M}_p[s_n^{*\prime\prime}]$$

for n large. By (1), the right-hand side does not exceed $(2A_p+1)\,\varepsilon$, and the first part of the theorem follows.

If $|f|\log^+|f|$ is integrable, then

[1]) See also Titchmarsh [4], Smirnoff [1].

[2]) See the papers referred to in the preceding paragraph.

$$\mathfrak{M}[s_n^*] \leqslant 2A\int_0^{2\pi}|f|\log^+|f|\,dx + 2B \tag{2}$$

(§ 7.24). Let us apply this result to the function kf, where k is a positive constant. It follows that

$$\mathfrak{M}[s_n^*] \leqslant 2A\int_0^{2\pi}|f|\log^+|kf|\,dx+\frac{2B}{k}<\varepsilon,$$

if $2B/k=\frac{1}{2}\varepsilon$ and the integral of $2A|f|\log^+|kf|$ over $(0, 2\pi)$ does not exceed $\frac{1}{2}\varepsilon$. To obtain that $\mathfrak{M}[f-s_n^*]\to 0$, we again write $f=f'+f''$, where f' is a polynomial, and the integral of $|f''|\log^+|kf''|$ is small, and proceed as before.

From the formula defining $\bar{s}_n^*$, we conclude that $|\bar{s}_n^*(x)-\bar{f}(x)|\leqslant \leqslant |g_1(x)|+|g_2(x)|$, g_1 and g_2 having the previous meaning. Thus $\mathfrak{M}[\bar{s}_n^*]$ satisfies an inequality analogous to (2), with $2A$, $2B$ replaced by $3A$, $3B$, and again $\mathfrak{M}[\bar{f}-\bar{s}_n^*]\to\infty$.

Theorem (iii) is proved in the same way, except that for $p<1$ we use the inequality $\mathfrak{M}_p^p[f'+f'']\leqslant\mathfrak{M}_p^p[f']+\mathfrak{M}_p^p[f'']$.

As corollaries of the above theorems we obtain the following results, the first of which is a generalization of Theorem 4.41(ii).

(iv) *If the Fourier coefficients of a function $f\in L^p$, $p>1$, are a_n, b_n, those of a $g\in L^{p'}$ are a_n', b_n', we have the Parseval formula*

$$\frac{1}{\pi}\int_0^{2\pi} f g\,dx=\tfrac{1}{2}a_0a_0'+\sum_{n=1}^{\infty}(a_na_n'+b_nb_n'), \tag{3}$$

the series on the right being convergent

(v) *The formula* (3) *holds also if $|f|\log^+|f|$ is integrable and g bounded.*

The proofs are similar to those of Theorems 4.41(ii) and 4.41(iii), if we take into account that $\mathfrak{M}_p[f-s_n]\to 0$ in case (iv) and $\mathfrak{M}[f-s_n]\to 0$ in case (v).

(vi) *For any integrable f there is a sequence of indices n_k such that $s_{n_k}(x)$ converges almost everywhere to $f(x)$; similarly we can find a sequence $\{m_k\}$ such that $\bar{s}_{m_k}(x)$ tends almost everywhere to $\bar{f}(x)$.* This follows from (ii) and Theorem 4.2(ii).

We add that for $\{n_k\}$ and $\{m_k\}$ we may take any sequences increasing sufficiently rapidly and, consequently, $\{n_k\}$ and $\{m_k\}$

may be subsequences of arbitrary sequences of integers tending to $+\infty$.

7.31. Theorem 7.3(i) ceases to be true for $p=1$ or $p=\infty$: $\mathfrak{M}[f-s_n]$ does not necessarily tend to 0 for f integrable, nor does s_n tend uniformly to f for f continuous. It is interesting to observe that if f and $\bar{f}$ are both integrable, or both continuous, then $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ behave much in the same way, as is seen from the following theorems [1]).

(i) *If f and $\bar{f}$ are both continuous, and $\mathfrak{S}[f]$ converges uniformly, so does $\bar{\mathfrak{S}}[f]$. If $\bar{f}$ and f are both bounded and $\mathfrak{S}[f]$ has partial sums uniformly bounded, so has $\bar{\mathfrak{S}}[f]$.*

(ii) *If $\bar{\mathfrak{S}}[f]$ is a Fourier series and $\mathfrak{M}[s_n]$ is bounded, so is $\mathfrak{M}[\bar{s}_n]$; and if $\mathfrak{M}[f-s_n]\to 0$, so does $\mathfrak{M}[\bar{f}-\bar{s}_n]$.*

The proofs are based on the following two lemmas, the first of which may be considered as the limiting case, for $p=\infty$, of the second [2]).

(a) *If $t_n(x)$ is a trigonometrical polynomial of order n, and $|t_n(x)|\leqslant M$, then $|t'_n(x)|\leqslant 2nM$.*

(b) *If $\mathfrak{M}_p[t_n(x)]\leqslant M$, $p\geqslant 1$, then $\mathfrak{M}_p[t'_n(x)]\leqslant 2nM$.*

The proofs are very simple. In the formula

$$t'_n(x)=\frac{1}{\pi}\int_0^{2\pi}t_n(x+u)\,[\sin u+2\sin 2u+\ldots+n\sin nu]\,du$$

we add to the expression in brackets the sum $(n-1)\sin(n+1)u+ +(n-2)\sin(n+2)u+\ldots+\sin(2n-1)u$, which, since t_n is a polynomial of order n, does not change the value of the integral. Adding together the terms $k\sin ku$ and $k\sin(2n-k)u$, we obtain the formula

$$t'_n(x)=\frac{2}{\pi}\int_0^{2\pi}t_n(x+u)\sin nu\,K_{n-1}(u)\,du, \tag{1}$$

K_{n-1} denoting the Fejér kernel. It follows that $|t'_n(x)|$ does not exceed the $(n-1)$-st Fejér mean of the function $2|t_n(x)|$, and it remains to appeal to Theorem 3.22 and the formula 4.33(3).

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[1]) F e j é r [6], Z y g m u n d [9].

[2]) The first is due to S. B e r n s t e i n [1]. For the second see Z y g m u n d [9] and F. R i e s z [3]. The factor 2 on the right may be made to disappear, but this makes no difference to us.

Let σ_n and $\bar{\sigma}_n$ denote the first arithmetic means of $\mathfrak{S}[f]$ and $\tilde{\mathfrak{S}}[f]$. Suppose that $\mathfrak{S}[f]$ converges uniformly. The formula 3.13(1) for the difference $\bar{s}_n - \bar{\sigma}_n$ now takes the form: $\bar{\sigma}_n - \bar{s}_n = s_n'/(n+1) = (s_n' - s_{n_0}')/(n+1) + s_{n_0}'/(n+1)$, where n_0 is fixed and so large that $|s_n - s_{n_0}| < \frac{1}{4}\varepsilon$, uniformly in x, for any $n \geqslant n_0$. From (a) we see that $|s_n' - s_{n_0}'| \leqslant \frac{1}{2}\varepsilon n$. Since for $n \geqslant n_1 \geqslant n_0$ we have $|s_{n_0}'|/(n+1) < \frac{1}{2}\varepsilon$, it follows that $|\bar{\sigma}_n - \bar{s}_n| < \varepsilon$ for $n \geqslant n_1$, i. e. $\bar{\sigma}_n - \bar{s}_n \to 0$. But, $\bar{f}$ being continuous, we have $\bar{\sigma}_n \to \bar{f}$, and so $\bar{s}_n \to \bar{f}$, uniformly in x. This gives the first part of (i). The proof of the second part is still simpler and may be left to the reader.

We prove (ii) by the same method, using (b) for $p = 1$. Considering, for example, the second part of (ii), we observe that $\mathfrak{M}[s_n - s_{n_0}] < \frac{1}{4}\varepsilon$ if n_0 is large enough and $n > n_0$. Thence, arguing as before, we obtain that $\mathfrak{M}[\bar{\sigma}_n - \bar{s}_n] = \mathfrak{M}[s_n'/(n+1)] \to 0$. This and the relation $\mathfrak{M}[\bar{f} - \bar{\sigma}_n] \to 0$, give $\mathfrak{M}[\bar{f} - \bar{s}_n] \to 0$, and the theorem is established.

We shall complete (i) by the following remark. The relation $\bar{\sigma}_n - \bar{s}_n \to 0$ was established under the sole hypothesis that $\mathfrak{S}[f]$ converges uniformly. We have then $\tilde{\mathfrak{S}}[f] = \mathfrak{S}[\bar{f}]$, where $\bar{f} \in L^2$, and so $\bar{\sigma}_n$ converges almost everywhere. Therefore[1], *if $\mathfrak{S}[f]$ converges uniformly, $\tilde{\mathfrak{S}}[f]$ converges almost everywhere. If the partials sums of $\mathfrak{S}[f]$ are uniformly bounded, the partial sums of $\tilde{\mathfrak{S}}[f]$ are bounded at almost every point.*

7.4. Privaloff's theorem. Theorem 7.21 teaches us that, except in limiting cases, the functions f and $\bar{f}$ have, so to speak, the same integrability. It is therefore natural to ask if anything similar is true for continuity. The answer is given by the following theorem due to Privaloff.

If $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$, then $\bar{f} \in \mathrm{Lip}\,\alpha$[2].

Consider the formulae

$$\bar{f}(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+t) - f(x)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt,$$

(1)

$$\bar{f}(x+h) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+t) - f(x+h)}{2\,\mathrm{tg}\,\frac{1}{2}(t-h)}\,dt,$$

[1]) Fejér [6]; see also Privaloff [4], Zygmund [7].
[2]) Privaloff [3].

where $h > 0$. They differ slightly from 7.1(1), but, since $\mathrm{tg}\,\frac{1}{2}t$ is an odd function of t, the additional terms vanish. The integrands are $O(|t|^{\alpha-1})$, $O(|t-h|^{\alpha-1})$ respectively. Consequently, if we cut the interval $(-2h, 2h)$ out of the interval of integration $(-\pi, \pi)$ in (1), we commit an error $R_1 = O(h^\alpha)$ in the first formula and an error $R_2 = O(h^\alpha)$ in the second. Hence the difference $\bar{f}(x+h) - \bar{f}(x)$ is equal to

$$(2)\quad -\frac{1}{\pi}\left(\int_{-\pi}^{-2h} + \int_{2h}^{\pi}\right)[f(x+t) - f(x)]\,[\mathrm{ctg}\,\tfrac{1}{2}(t-h) - \mathrm{ctg}\,\tfrac{1}{2}t]\,dt + R_2 - R_1 + R,$$

where $R = [f(x+h) - f(x)]\int_{2h}^{\pi}[\mathrm{ctg}\,\frac{1}{2}(t-h) - \mathrm{ctg}\,\frac{1}{2}(t+h)]\,dt =$

$$= O(h^\alpha)\left[2\log\frac{\sin\frac{1}{2}(t-h)}{\sin\frac{1}{2}(t+h)}\right]_{2h}^{\pi} = O(h^\alpha).$$

Since $\mathrm{ctg}\,\frac{1}{2}(t-h) - \mathrm{ctg}\,\frac{1}{2}t = \sin\frac{1}{2}h/\sin\frac{1}{2}(t-h)\sin\frac{1}{2}t$, the function under the integral sign in (2) is $O(|t|^\alpha)\cdot O(h|t|^{-2})$, hence the integral itself is $O(h^\alpha)$. Collecting the terms, we find that $\bar{f}(x+h) - \bar{f}(x) = O(h^\alpha)$ uniformly in x, and the theorem is established.

The theorem fails for $\alpha = 0$ and $\alpha = 1$. The function conjugate to $\sin x + \frac{1}{2}\sin 2x + \ldots = \frac{1}{2}(\pi - x)$, $0 < x < 2\pi$, is not bounded. Integrating the last series formally, we obtain a function which is Lip 1, and whose conjugate is not. Repeating the previous argument we find that, *if $f \in$ Lip 1, then* $\omega(\delta; \bar{f}) = O(\delta \log 1/\delta)$.

7.5. Power series of bounded variation. We conclude this chapter by a few theorems on Fourier series of functions which, together with their conjugate, are of bounded variation. It will be more convenient to state these theorems in the form bearing on power series. We shall say that a power series

$$(1)\quad a_0 + a_1 z + a_2 z^2 + \ldots = F(z)$$

is of bounded variation, if its real and imaginary components, for $z = e^{ix}$, are both Fourier series of functions of bounded variation. We know (§ 2.631) that $F(e^{ix})$ is then continuous; consequently the series (1) converges uniformly for $|z| = 1$, and hence converges uniformly for $|z| \leqslant 1$. The theorems we aim at are as follows.

(i) *If the power series* (1) *is of bounded variation, it converges absolutely on the circle* $|z|=1$ [1]).

(ii) *If the power series* (1) *is of bounded variation, the function* $F(e^{ix})$ *is absolutely continuous* [2]).

We shall base the proofs on a number of lemmas which are interesting and important in themselves.

7.51. A function $F(z)$, regular for $|z|<1$, is said to belong to the class H^p, $p>0$, if the expression

$$\mu_p(r)=\mu_p(r;F)=\frac{1}{2\pi}\int_0^{2\pi}|F(re^{ix})|^p\,dx$$

is bounded as $r\to 1$ [3]). We shall write H instead of H^1, and μ instead of μ_1. If $p>1$, H^p coincides with the class of power series whose real parts are Poisson's integrals of functions belonging to L^p. The real and imaginary parts of a function belonging to H are represented by Poisson-Stieltjes integrals (§ 4.36).

In virtue of Theorems 2.13 and 4.36(ii), a necessary and sufficient condition that the series 7.5(1) should be of bounded variation is that the function $G(z)=zF'(z)=a_1z+2a_2z^2+\dots$ should belong to H. It is familiar that $2\pi\mu(r;zF')$ represents the length of the curve $w=F(z)$, $|z|=r$.

The first lemma we need is as follows.

If $G(z)=G_1(z)\,G_2(z)=\alpha_0+\alpha_1z+\dots$, *and* $\mu_2(r;G_1)\leqslant A_1^2$, $\mu_2(r;G_2)\leqslant A_2^2$, *where* $A_1\geqslant 0, A_2\geqslant 0$, *the series* $|\alpha_1|+|\alpha_2|/2+|\alpha_n|/n+\dots$ *converges to a sum* $\leqslant \pi A_1A_2$.

Put $G_k(z)=\alpha_0^{(k)}+\alpha_1^{(k)}z+\dots$, $G_k^*(z)=|\alpha_0^{(k)}|+|\alpha_1^{(k)}|z+\dots$, $k=1,2$, $G^*(z)=G_1^*(z)\,G_2^*(z)=\alpha_0^*+\alpha_1^*z+\dots$ In virtue of Parseval's relation we have $\mu_2(r;G_k)=\mu_2(r;G_k^*)$, and it is easy to see that $|\alpha_n|\leqslant\alpha_n^*$, $n=0,1,\dots$ Moreover, by Schwarz's inequality, we have $\mu(r;G^*)\leqslant\mu_2^{1/2}(r;G_1^*)\,\mu_2^{1/2}(r;G_2^*)=\mu_2^{1/2}(r;G_1)\,\mu_2^{1/2}(r;G_2)\leqslant A_1A_2$.

Let us fix a value of r and consider the absolutely convergent series $\alpha_1^*r\sin x+\alpha_2^*r^2\sin 2x+\dots=\Im\{G^*(re^{ix})\}$. Multiplying both sides of the equation by $\frac12(\pi-x)$, integrating the result

[1]) Hardy and Littlewood [10]. See also Fejér [9].
[2]) F. and M. Riesz [1].
[3]) Hardy [10].

over $(0,2\pi)$, and taking into account that then n-th sine coefficient of $\frac12(\pi-x)$ is $1/n$, we obtain

$$(1)\qquad \sum_{n=1}^{\infty}\frac{\alpha_n^*}{n}r^n=\frac{1}{\pi}\int_0^{2\pi}\Im\,G^*(re^{ix})\,\tfrac12(\pi-x)\,dx\leqslant\frac12\int_0^{2\pi}|G^*(re^{ix})|\,dx.$$

The last integral does not exceed πA_1A_2. Making $r\to1$, we find that $\alpha_1^*+\frac12\alpha_2^*+\dots\leqslant\pi A_1A_2$, and, since $|\alpha_n|\leqslant\alpha_n^*$, the lemma follows.

7.52. In virtue of this lemma, to prove Theorem 7.5(i) it would be sufficient to show that the function $G(z)=zF'(z)=a_1z+2a_2z^2+\dots$ is a product of two functions G_1, G_2 of the class H^2. This proposition will be established later, but for our actual purposes a less strong result will do. Suppose namely that $G(z)$ has only a finite number of zeros $\zeta_1,\zeta_2,\dots\zeta_k$ in the circle $|z|<1$. Put $b_h(z)=z$ if $\zeta_h=0$; if $\zeta_h\neq0$, let $b_h(z)=(z-\zeta_h)/(1-z\bar\zeta_h)$, $B(z)=b_1(z)\,b_2(z)\dots b_k(z)$. Each function $b_h(z)$ is regular for $|z|\leqslant1$, has a simple zero at ζ_h and only there, and $|b_h(z)|=1$ for $|z|=1$ [1]). Therefore the function $H(z)=G(z)/B(z)$ is regular for $|z|<1$, and, as $r\to1$, $\overline{\lim}\,\mu(r;H)=\overline{\lim}\,\mu(r;G)$.

Let $A=\overline{\lim}\,\mu(r;G)$. The function $H(z)$ has no zeros for $|z|<1$, and so $\sqrt{H(z)}$ is regular for $|z|<1$. Put $G_1(z)=\sqrt{H(z)}$, $G_2(z)=\sqrt{H(z)}\,B(z)$, so that $G_1G_2=G$. It follows that $\mu_2[r;G_1]=\mu[r;H]$, $\mu_2[r;G_2]\leqslant\mu[r;H]$; $\overline{\lim}\,\mu_2[r;G_k]\leqslant A$ as $r\to1$, $k=1,2$. Now, as it is seen from Parseval's relation, $\mu_2(r)$ increases with r, so that we have $\mu_2(r;G_k)\leqslant A$ for $r<1$. An appeal to the lemma of § 7.51 gives the following result. *If $zF'(z)$ has only a finite number of zeros in* $|z|<1$, *and if* $\overline{\lim}\,\mu(r;zF')\leqslant A$, *then* $|a_1|+|a_2|+\dots\leqslant\pi A$.

Now it is easy to complete the proof of Theorem 7.5(i). Let μ now denote the upper bound of $\mu(r,zF')$ for $0\leqslant r<1$. (It will be proved in § 7.53(i) that $\mu(r)$ is a non-decreasing function of r, so that $\mu=A$, but this result is not required here). If $0<\rho<1$, the function $\rho zF'(z\rho)$ has only a finite number of zeros

[1]) This last fact, familiar to anyone acquainted with the elements of conformal representation, may be proved as follows: $|b_h(e^{ix})|=|e^{ix}-\zeta_h|/|1-e^{ix}\bar\zeta_h|=|e^{ix}-\zeta_h|/|e^{-ix}-\bar\zeta_h|=|e^{ix}-\zeta_h|/|\overline{e^{ix}-\zeta_h}|=1$. It follows that $|b_h(z)|<1$ for $|z|<1$.

for $|z|<1$. Thus $|a_1|\rho+|a_2|\rho^2+\ldots\leqslant\pi\mu$; making $\rho\to 1$, we find that $|a_1|+|a_2|+\ldots\leqslant\pi\mu$ and the theorem follows.

7.521. As a corollary of Theorems 7.5(i) and 7.24(ii) we obtain: *If $F(x)$ is absolutely continuous and $|F'(x)|\log^+|F'(x)|\in L$, then $\mathfrak{S}[F]$ converges absolutely* (§ 6.36).

7.53. Passing to the proof of Theorem 7.5(ii), we shall again require a few lemmas

(i) *If $F(z)$ is regular for $|z|<1$, $\mu_p(r;F)$ is a non-decreasing function of r.* It is not difficult to deduce this from the following proposition which we shall prove first.

(ii) *If $f_1(z), f_2(z), \ldots, f_n(z)$ are regular inside and on the boundary of a plane region R, and $\varphi(z)=|f_1(z)|^p+\ldots+|f_n(z)|^p$, $p>0$, the function $\varphi(z)$ cannot attain a proper maximum inside B.*

Suppose, on the contrary, that $\varphi(z)$ does attain such a maximum at a point z_0 interior to R. Let C be a circle $|z-z_0|\leqslant r$ contained in R and such that (a) if $f_k(z_0)\neq 0$, then $f_k(z)\neq 0$ in C, $k=1,2,\ldots n$, (b) at a point z_1, $|z_1-z_0|=r$, $\varphi(z)$ takes a value smaller than $\varphi(z_0)$. Let $\psi(z)$ be the sum of terms $\varepsilon_k f_k^p(z)$ extended over the values of k for which $f_k(z_0)\neq 0$. The unit factors ε_k are so chosen that the function $\psi(z)$, which is regular in C, takes the value $\varphi(z_0)$ at the point z_0. For every z, $|z-z_0|\leqslant r$, we have

$$|\psi(z)|\leqslant|f_1(z)|^p+\ldots+|f_n(z)|^p=\varphi(z)\leqslant\varphi(z_0)=\psi(z_0)=|\psi(z_0)|,$$

and for $z=z_1$ we actually have $\varphi(z)<\varphi(z_0)$, i. e. $|\psi(z)|<|\psi(z_0)|$. This is in contradiction with the principle of maximum and (ii) is established.

Consider now the function $\varphi_n(z)=\{|F(\eta_1 z)|^p+\ldots+|F(\eta_n z)|^p\}/n$, where $\eta_1, \eta_2, \ldots, \eta_n$ are the n-th unit roots. It is obvious that, for every $0<r<1$, $\varphi_n(re^{ix})\to\mu_p(r;F)$ uniformly in x. Let $0<\rho<r<1$ and let $\operatorname{Max}|\varphi_n(z)|$ for $|z|\leqslant r$ be attained at a point $z=re^{ix}$. We have then $\varphi_n(\rho e^{ix})\leqslant\varphi_n(re^{ix})$, and, making $n\to\infty$, $\mu(\rho)\leqslant\mu(r)$.

(iii) *Let $\zeta_1,\zeta_2,\ldots$ be a sequence of points such that $0<|\zeta_n|<1$, and that the product $|\zeta_1|.|\zeta_2|\ldots$ converges. If $\zeta_n^*=1/\bar{\zeta}_n$, the product*

$$\prod_{n=1}^{\infty}\frac{z-\zeta_n}{z-\zeta_n^*}\cdot\frac{1}{|\zeta_n|} \tag{1}$$

converges absolutely and uniformly in every circle $|z|\leqslant r<1$, to a function $B(z)$ vanishing at the points ζ_n and only there.

The terms of (1) differ only by constant unit factors from the expressions $b_h(z)$ considered in § 7.52. If $|z|\leqslant r$, the difference $1-(z-\zeta_n)/(z-\zeta_n^*)=(\zeta_n-\zeta_n^*)/(z-\zeta_n^*)$ does not exceed $(1-|\zeta_n|^2)/(1-r)<2(1-|\zeta_n|)/(1-r)$ in absolute value; and since, by hypothesis, the series $(1-|\zeta_1|)+(1-|\zeta_2|)+\ldots$ converges, the product with factors $(z-\zeta_n)/(z-\zeta_n^*)$ converges absolutely and uniformly for $|z|\leqslant r$. So does the product (1). Since the terms of (1) are less than 1 in absolute value, we obtain that $|B(z)|<1$ for $|z|<1$ and the lemma is established.

(iv) *If $\zeta_1,\zeta_2,\ldots$ are all the zeros, different from the origin, of a function $F(z)\in H^p$, $|z|<1$, each counted according to its multiplicity, the product $|\zeta_1|\cdot|\zeta_2|\ldots$ converges* [1]. Let $B_n(z)$ denote the n-th partial product of (1) multiplied by z^k, if $F(z)$ has a zero of order k at the origin. The relation $\mu_p(r;F)\to\mu$ as $r\to 1$ implies $\mu_p(r;F/B_n)\to\mu$, $(n=1,2,\ldots)$ and so, by (i), $\mu_p(r;F/B_n)\leqslant\mu$. Making $r=0$ we find $|\zeta_1\zeta_2\ldots\zeta_n|\geqslant\mu^{-1/p}|F(z)/z^k|_{z=0}$ and the lemma follows.

(v) *If $\mu_p(r;F)\leqslant\mu$, $0\leqslant r<1$, we have $F(z)=G(z)B(z)$, where $|B(z)|\leqslant 1$, $G(z)$ is regular and different from 0, and $\mu_p(r;G)\leqslant\mu$* [2].

This lemma, which is fundamental for the whole theory, now follows immediately. If $F(z)\neq 0$ for $|z|<1$, we may put $B(z)=1$, $G(z)=F(z)$. If $\zeta_1,\zeta_2,\ldots,B_n(z)$ have the same meaning as in (iv), we put $B(z)=\lim B_n(z)$. From the formula $\mu_p(r;F/B_n)\leqslant\mu$, we deduce that $\mu_p(r;G)\leqslant\mu$, where the function $G=F/B$ has no zero for $|z|<1$. Since $|B|<1$, the lemma is established.

(vi) *If $F\in H$, then $F=F_1F_2$ with F_1 and F_2 belonging to H^2.* If $F=GB$, where G and B have the same meaning as in (v), we put $F_1=\sqrt{G}$, $F_2=\sqrt{G}B$. Since $\mu_2(r;F_k)\leqslant\mu(r;G)$, $k=1,2$, the lemma follows.

7.55. Now we are in a position to prove Theorem 7.5(ii), which we state in the following equivalent form. *If the power*

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[1]) Lemma (iv), as well as some other results of this section, is known to be true for a more general class of functions, viz. for functions F such that $\mathfrak{M}[\log^+F(re^{ix})]=O(1)$. The latter class, although very important in the general theory of analytic functions, has less applications to the theory of trigonometrical series.

[2]) F. Riesz [4].

series 7.5(1) *belongs to* H, *the real and imaginary parts of the series on the unit circle are Fourier series*. It is sufficient to show that $\mathfrak{M}\,[F(re^{ix})-F(\rho e^{ix})]\to 0$ as $r,\rho\to 1$ (§ 4.36). Using the last lemma of the previous section and applying Schwarz's inequality, we easily obtain

$$\mathfrak{M}\,[F(re^{ix})-F(\rho e^{ix})]\leqslant \mathfrak{M}_2[F_1(re^{ix})]\,\mathfrak{M}_2[F_2(re^{ix})-F_2(\rho e^{ix})]+$$
$$+\,\mathfrak{M}_2[F_2(\rho e^{ix})]\,\mathfrak{M}_2[F_1(re^{ix})-F_1(\rho e^{ix})].$$

Since the second factor in each term on the right tends to 0 as $r,\ \rho\to 1$, the result follows.

7.56. From the lemmas established in the preceding sections we shall deduce a number of interesting consequences.

(i) *If* $F(z)\,\epsilon\, H^p$, *then, for almost every* $z_0=e^{ix_0}$, $F(e^{ix_0})=\lim F(z)$ *exists and is finite as* $z\to z_0$ *along any path not touching the circle* [1]). This theorem is only novel in the case $p<1$. With the notation of § 7.54(v) put $F_1(z)=G^{p/2}(z)$, $F_2(z)=B(z)$. F_1 and F_2 belong to H^2. Since for each of them our theorem is true, it is also true for $F=F_1^{2/p}F_2$.

(ii) *The function* $|F(e^{ix})|^p$ *of* (i) *is integrable.* This is a consequence of Fatou's lemma.

(iii) *If* $F(z)\,\epsilon\, H^p$, *then* $\mathfrak{M}_p[F(re^{ix})-F(e^{ix})]\to 0$ *as* $r\to 1$ [2]). This theorem is known to us for $p>1$ (§ 4.36). Let $p\leqslant 1$, $0<r<\rho<1$. If F_1 and F_2 have the same meaning as in (i), then, applying the first inequality of 4.13(3), we obtain

$$|F(re^{ix})-F(\rho e^{ix})|^p\leqslant |F_1(\rho e^{ix})|^2\,|F_2(re^{ix})-F_2(\rho e^{ix})|^p+$$
$$+\,|F_2(re^{ix})|^p\,|F_1^{2/p}(re^{ix})-F_1^{2/p}(\rho e^{ix})|^p.$$

Making $\rho\to 1$ and integrating over $(0,2\pi)$, we find

1) F. Riesz [4]. The theorem is false for harmonic functions: there is a harmonic function $u(z)$, $|z|<1$, such that $\mu_p(r;u)=O(1)$ for every $0<p<1$, while $\lim\limits_{r\to 1}u(re^{ix})$ exists only in a set of measure 0. See Hardy and Littlewood [12].

2) F. Riesz [4].

$$\int_0^{2\pi}|F(re^{ix})-F(e^{ix})|^p\,dx\leqslant\int_0^{2\pi}|F_1(e^{ix})|^2\,|F_2(re^{ix})-F_2(e^{ix})|^p\,dx+$$
$$+\int_0^{2\pi}|F_2(re^{ix})|^p\,|F_1^{2/p}(re^{ix})-F_1^{2/p}(e^{ix})|^p\,dx.$$

The first integral on the right tends to 0 with $1-r$ since the product $|F_1(e^{ix})|^2\,|F_2(re^{ix})-F_2(e^{ix})|^p$ is less than the integrable function $2^p\,|F_1(e^{ix})|^2$ and tends to 0 almost everywhere. Let $F_1^{1/p}(z)=L(z)$; $L(z)\,\epsilon\,H^{2p}$. Since $|F_2|\leqslant 1$, the second integral does not exceed

$$(1)\qquad\left[\int_0^{2\pi}|L(re^{ix})-L(e^{ix})|^{2p}dx\right]^{1/2}\left[\int_0^{2\pi}|L(re^{ix})+L(e^{ix})|^{2p}dx\right]^{1/2}.$$

The first factor here tends to 0 if $2p>1$, the second is bounded, and the result follows for $p>1/2$. Assuming this, we obtain, from (1), the result for $p>1/4$, and so on.

(iv) *If* $F(z)\,\epsilon\,H^\alpha$, *and* $|F(e^{ix})|^\beta$ *is integrable for* $\beta>\alpha$, *then* $F(z)\,\epsilon\,H^\beta$ [1]). The theorem is obvious if $\alpha>1$. It is also simple if $F(z)\neq 0$ for $|z|<1$; for if $G(z)=F^{\alpha/2}(z)$, then $G(z)\,\epsilon\,H^2$ and $G(e^{ix})\,\epsilon\,L^{2\beta/\alpha}$ so that $G(z)\,\epsilon\,H^{2\beta/\alpha}$, $F(z)\,\epsilon\,H^\beta$.

In the general case we have $F=GB$, where $G(z)\neq 0$, $G\,\epsilon\,H^\alpha$, and the function B is a product of certain rational functions (§ 7.54(v)). Since $|B(z)|<1$, the function $B(e^{ix})$ exists for almost every x and $|B(e^{ix})|\leqslant 1$. We shall show that $|B(e^{ix})|=1$ for almost all x. Taking this result for granted, we can easily prove our theorem. For if $F(e^{ix})\,\epsilon\,L^\beta$, $|B(e^{ix})|=1$, then $G(e^{ix})\,\epsilon\,L^\beta$ and, since $G(z)\,\epsilon\,H^\alpha$, $G(z)\neq 0$, we obtain that $G(z)\,\epsilon\,H^\beta$, in virtue of the case already dealt with. Since $F(z)=B(z)\,H(z)$, $F(z)\,\epsilon\,H^\beta$ and the theorem is established.

Using Theorem 7.24(i), we obtain, as a corollary, the following proposition.

(v) *If the function* $\bar{f}$ *conjugate to an integrable function* f *is integrable, then* $\bar{\mathfrak{S}}[f]=\mathfrak{S}[\bar{f}]$.

We have still to prove that $|B(e^{ix})|=1$ for almost every x. We may obviously assume that the number of zeros $\zeta_1,\zeta_2,\dots$ is infin-

1) Smirnoff [1].

ite and that $F(0)\neq 0$. Since $|B(z)|\leqslant 1$, it is sufficient to show that $\mu(r;B)\to 1$ as $r\to 1$. Now $\mu(0;B)=|\zeta_1|\cdot|\zeta_2|\ldots$ and, since $\mu(r)$ is a non-decreasing function of r, $\lim\mu(r;B)\geqslant|\zeta_1|\cdot|\zeta_2|\ldots$ Let B_N denote the N-th partial product of 7.53(1) and R_N the product of the remaining terms, so that $B=B_N R_N$. Then we have $\lim\limits_{r\to 1}\mu(r;R_N)\geqslant|\zeta_{N+1}||\zeta_{N+2}|\ldots$ and, since $|B_N(z)|$ tends uniformly to 1 as $|z|\to 1$, we obtain that $\lim\mu(r;B)\geqslant|\zeta_{N+1}||\zeta_{N+2}|\ldots$ Taking N arbitrarily large, we see that $\lim\mu(r;B)\geqslant 1$, i. e. $\lim\mu(r;B)=1$.

7.6. Miscellaneous theorems and examples.

1. The formula 7.1(1) may be written $\bar f(x)=-\dfrac{1}{\pi}\displaystyle\int_0^\infty\frac{f(x+t)-f(x-t)}{t}\,dt$ (§ 2.9.8).

2. There is an integrable $f(x)$ such that $\bar f(x)$ is non-integrable in every interval. L u s i n [1].

[Take $f\geqslant 1$ such that $f\log f$ is nowhere integrable, and apply Theorems 7.25, 2.531].

3. (i) If $|f(x)|\leqslant 1$, then $\exp\lambda|\bar f|$ is integrable for every $\lambda<\frac{1}{2}\pi$. (ii) If f is continuous, $\exp\lambda|\bar f|$ is integrable for every λ. (iii) If s_n, $\bar s_n$ denote the partial sums of $\mathfrak{S}[f]$, $\bar{\mathfrak{S}}[f]$ respectively, then $\mathfrak{M}[\exp\lambda|f-s_n|;\,0,\,2\pi]\to 2\pi$, $\mathfrak{M}[\exp\lambda|\bar f-\bar s_n|]\to 2\pi$, for $\lambda<\frac{1}{4}\pi$ if $|f|\leqslant 1$, and for any λ if f is continuous. Z y g m u n d [4]; see also W a r s c h a w s k i [1].

[To prove (i) let F, u, v have the same meaning as in § 7.21. Then

$$\frac{1}{2\pi i}\int\limits_{|z|=r} z^{-1}\exp\{\pm i\lambda F(z)\}\,dz=\exp\{\pm i\lambda F(0)\},\qquad \int_0^{2\pi}\cos\lambda u\,\exp(\pm\lambda v)\,dx=\text{const.}].$$

4. If $F(z)=u(z)+iv(z)$ is an arbitrary function regular for $|z|<1$ and such that $u>0$, $v>0$, then $u(e^{ix})\in L^{2-\varepsilon}$, $v(e^{ix})\in L^{2-\varepsilon}$ for every $\varepsilon>0$ but not necessarily for $\varepsilon=0$.

[Let $F_1=F\exp(-\pi i/4)=u_1+iv_1$, where $|v_1|\leqslant u_1$. Apply to F_1 an argument similar to that of Theorem 7.24(i)].

5. Let Φ,Ψ and Φ_1,Ψ_1 be two pairs of Young's complementary functions. If, for any $f\in L^*_\Phi$, (i) the conjugate function $\bar f$ belongs to $L^*_{\Phi_1}$ and (ii) there exists a constant A independent of f and such that $\|\bar f\|_{\Phi_1}\leqslant A\|f\|_\Phi$, then, for any $g\in L^*_{\Psi_1}$, we have $\bar g\in L^*_\Psi$ and, moreover, $\|\bar g\|_\Psi\leqslant 2A\|g\|_{\Psi_1}$.

[It is sufficient to prove that, if $\|v\|_{\Phi_1}\leqslant A\|u\|_\Phi$ for any function $u+iv$ regular for $|z|<1$ and such that $v(0)=0$, then $\|v\|_\Psi\leqslant 2A\|u\|_{\Psi_1}$. Denoting by h an arbitrary polynomial such that $\mathfrak{M}[\Phi|h|;\,0,2\pi]\leqslant 1$, we have

$$\|v\|_\Psi=\operatorname*{Sup}_h\left|\int_0^{2\pi}v\,h\,dx\right|=\operatorname*{Sup}_h\left|\int_0^{2\pi}u\,\bar h\,dx\right|\leqslant 2A\sigma\|u\|_{\Psi_1},$$

where $\sigma=\operatorname{Max}\{1,\operatorname{Sup}\mathfrak{M}[\Phi_1|\bar h/2A|]\}$ (§ 4.541). On the other hand, since $\mathfrak{M}[\Phi|h|]\leqslant 1$, we have $\|h\|_\Phi\leqslant 2$, and so, by (ii), $\|\bar h\|_{\Phi_1}\leqslant A\|h\|_\Phi\leqslant 2A$. Hence (§ 4.541) $\mathfrak{M}[\Phi_1|\bar h/2A|]<1$, $\sigma=1$, and $\|v\|_\Psi\leqslant 2A\|u\|_{\Psi_1}$.

6. Let $s(x)$, $x\geqslant 0$, be a function which is *concave* (i. e. $-s$ is convex), non-negative, has a continuous derivative for $x>0$, and tends to $+\infty$ with x, and let $S(x)$ be the indefinite integral of $s(x)$. Let $R(x)$, $x\geqslant 0$, be a function which is non-negative, convex, tends to $+\infty$ with x, and has the first and second derivatives continuous for $x>0$. Suppose in addition that there is a constant $C>0$ such that $S''(x)+S'(x)/x\leqslant CR''(x)$. Under these conditions, *if* $f\in L_R$, *then* $\bar f\in L_S$.

[The proof is substantially the same as that of § 7.23. Observe that $S(2x)\leqslant C_1 S(x)>0$ with C_1 independent of x. If $S(x)\leqslant R(x)$, then we have $\mathfrak{M}[S|\bar f|]\leqslant C_2\,\mathfrak{M}[R|f|]$, where C_2 is independent of f].

7. (i) If $|f|(\log^+|f|)^\alpha\in L$, $\alpha>0$, then $|\bar f|\log^{\alpha-1}(2+|\bar f|)\in L$, and there are two constants $A=A_\alpha$, $B=B_\alpha$ such that

$$\int_0^{2\pi}|\bar f|\log^{\alpha-1}(2+|\bar f|)\,dx\leqslant A\int_0^{2\pi}|f|(\log^+|f|)^\alpha\,dx+B.$$

(ii) If the integral of $\exp|f|^\alpha$, $\alpha>0$, over $(0,2\pi)$ is $\leqslant 1$, then the function $\exp\lambda|\bar f|^\beta$ is integrable for $\beta=\alpha/(\alpha+1)$ and $\lambda<\lambda_0=\lambda_0(\alpha)$.

(iii) Theorem (i) is not true for $\alpha=0$.

8. Let σ and $\bar\sigma_n$ denote the k-th arithmetic means, $k>0$, for $\mathfrak{S}[dF]$ and $\bar{\mathfrak{S}}[dF]$ respectively, where F is a function of bounded variation. If $f=F'$, and g denotes the function defined by 7.11(1), then $\mathfrak{M}_p[\sigma_n-f]\to 0$, $\mathfrak{M}_p[\bar\sigma_n-g]\to 0$ for every $0<p<1$.

9. The constant A_p of Theorem 7.21 satisfies an inequality $A_p>Ap$, where A is a positive absolute constant. T i t c h m a r s h [5].

[Consider the function $f(x)=(\pi-x)/2$, $0<x<2\pi$, and observe that $\bar f(x)\sim\log 1/x$ as $x\to+0$].

10. Let $P_n(z)=(1+z+z^2+\ldots+z^n)^2/(n+1)=(1+2z+3z^2+\ldots+z^{2n})/(n+1)$, $Q_n(z)=(1+2z+3z^2+\ldots+(n+1)z^n)/(n+1)$. If $\sum|\alpha_k|<\infty$, $m_k+2n_k<m_{k+1}$, $k=1,2,\ldots$, the real and imaginary parts of the power series $\sum\alpha_k z^{m_k}P_{n_k}(z)$, $z=e^{ix}$, are Fourier series. If in addition $\alpha_k\log n_k\to\infty$, the partial sums t_ν of the power series satisfy the relation $\overline{\lim}\,\mathfrak{M}[t_\nu(e^{ix})]=\infty$. The example is due to F. R i e s z; see Z y g m u n d [9].

[The point of this example is that the phenomenon observed in § 5.12 for Fourier series subsists for power series. Use the relations $\mathfrak{M}[P_n(e^{ix})]=2\pi$, $\mathfrak{M}[Q_n(e^{ix})]>C\log n$, where $C>0$ is an absolute constant].

11. Let $F(z)=u(z)+iv(z)$ be a function regular for $|z|<1$. If, for any point $x_0\in E$, $|E|>0$, $\lim u(z)$ exists and is finite as $z\to e^{ix_0}$ along any path not touching the circle, the same is true for the function $v(z)$ and a l m o s t e v e r y point $x_0\in E$. P r i v a l o f f [2]; see also P l e s s n e r [3].

For the proof, which is rather deep, the reader is referred to the original papers.

12. If $F(x)$ is integrable and $F'(x)$ exists and is finite for $x \in E, |E| > 0$, the integral (*) $-\frac{1}{\pi}\int\limits_0^\pi \frac{F(x+t)+F(x-t)-2F(x)}{4\sin^2 \frac{1}{2} t}\,dt$ exists for almost every $x \in E$. Plessner [3].

[This follows from the previous theorem and Theorem 3.9.13].

13. If the conditions of the previous theorem are satisfied, then, for almost every $x \in E$, $\overline{\mathfrak{S}}'[F]$ is summable (C, k), $k > 1$, to the value (*).

14. If $f(x)$ is integrable in the sense of Denjoy-Perron, the function $\bar{f}(x)$ defined by 7.1(1) exists for almost every x. Plessner [3].

15. If either (i) $0 < \alpha < 1$, $p \geqslant 1$, or (ii) $\alpha = 1$, $p > 1$, and if f belongs to Lip (α, p), so does $\bar{f}$. The theorem is false for $\alpha = 1$, $p = 1$. Hardy and Littlewood [13].

[Using Minkowski's inequality 4.13(4), the proof of (i) is similar to that of Theorem 7.4; (ii) is equivalent to Theorem 7.21 (§ 4.7.6)].