CHAPTER VI.

The absolute convergence of trigonometrical series.

6.1. The Lusin-Denjoy theorem. The convergence of the series

$$(1)\qquad \sum_{n=1}^{\infty} (|a_n|+|b_n|)$$

implies the absolute convergence of the series

$$(2)\qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The series (2) may be absolutely convergent at an infinite set of points without (1) being convergent. A simple example is given by the series $\Sigma \sin n!\, x$, whose terms vanish from some place onwards for every x commensurable with π.

If the series (2) *converges absolutely in a set E of positive measure, the series* (1) *converges* [1]). Suppose, for simplicity, that $a_0=0$, and let $a_k \cos kx + b_k \sin kx = \rho_k \cos(kx+x_k)$, where $\rho_k \geqslant 0$, $\rho_k^2 = a_k^2 + b_k^2$. The function

$$(3)\qquad \alpha(x) = \sum_{n=1}^{\infty} \rho_n |\cos(nx+x_n)|$$

is finite at every point of E. Hence there exists a set $\mathcal{E} \subset E$, $|\mathcal{E}|>0$, such that $\alpha(x)$ is bounded on $\mathcal{E}$, $\alpha(x)<M$ say. Since the partial sums $\alpha_n(x)$ of (3) are uniformly bounded on $\mathcal{E}$, the series may be integrated formally over $\mathcal{E}$:

$$(4)\qquad \sum_{n=1}^{\infty} \rho_n \int_{\mathcal{E}} |\cos(nx+x_n)|\, dx \leqslant M|\mathcal{E}|.$$

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1) Lusin [3], Denjoy [1].

To prove the convergence of $\rho_1+\rho_2+\dots$, which is equivalent to our theorem, it is sufficient to show that the integrals I_n on the left in (4) all exceed an $\varepsilon>0$. Let I'_n be the integral analogous to I_n, with $|\cos(nx+x_n)|$ replaced by $\cos^2(nx+x_n)$. Since $I_n>I'_n$, it is sufficient to prove that $I'_n>\varepsilon$. For this purpose we use the formula $2\cos^2(nx+x_n)=1+\cos 2nx\cdot\cos 2x_n-\sin 2nx\cdot\sin 2x_n$. Since the Fourier coefficients of the characteristic function of the set $\mathcal{E}$ tend to 0, we obtain that $I'_n\to\frac{1}{2}|\mathcal{E}|$, which completes the proof, all I'_n being positive.

The set E in the theorem which we have established is of positive measure. This property, while sufficient for the convergence of (1), is not necessary. The problem of necessary and sufficient conditions seems to be unsolved.

6.11. We shall supplement the previous theorem by a few results of the same character. Suppose that, for the series 6.1(2), we have $\rho_1+\rho_2+\dots=\infty$, and let E be the set of points where $\alpha(x)<\infty$. The complementary set H, where the upper limit of the sequence $\{\alpha_n(x)\}$ of continuous functions is equal to $+\infty$, is a product of a sequence of open sets; for if G_N denotes the open set of points where at least one of the functions $\alpha_n(x)$ exceeds N, we have $H=G_1G_2\dots$ It follows that E is the sum of a sequence of closed sets. None of these closed sets contains an interval; for otherwise we should have $\rho_1+\rho_2+\dots<\infty$. It follows that all of them are non-dense, E is of the first category, and therefore, *if* 6.1(2) *converges absolutely in a set of the second category, even if it is of measure* 0, *the series* 6.1(1) *converges*[1]).

6.12. There exist trigonometrical series absolutely convergent in a perfect set but not everywhere (§ 6.6.1). On the other hand, as we shall prove, there exist perfect sets P of measure 0, which, as regards the absolute convergence of trigonometrical series, resemble sets of positive measure: every trigonometrical series absolutely convergent in P is absolutely convergent everywhere. In particular Cantor's well-known set has this property.

A point-set B will be called a *basis*, if every real x can be represented in the form $\alpha_1x_1+\alpha_2x_2+\dots+\alpha_lx_l$, where $\alpha_1,\alpha_2,\dots$ are integers, and $x_1,x_2,\dots$ belong to B. We may also write

[1]) L u s i n [1].

$x=\varepsilon_1x_1+\dots+\varepsilon_mx_m$, where $\varepsilon_j=\pm1$ and the x_j are not necessarily different. We require the following lemma.

Let B be a basis, and let $B^*=B^*_u$ denote the set B translated by a number u. There exists a set S of the second category such that, for every $y\in S$, we have $y=\alpha_1x_1^*+\alpha_2x_2^*+\dots+\alpha_nx_n^*$, with α_j integral and $x_j^*\in B^*$ [1]). To prove this, we observe that for every x we have $x=\alpha_1(x_1^*-u)+\alpha_2(x_2^*-u)+\dots$, i. e. $x+ku=\alpha_1x_1^*+\alpha_2x_2^*+\dots$, where $k=k_x$ is an integer. Let E_n, $-\infty<n<+\infty$, denote the set of x for which $k_x=n$. For any x may exist several k_x; we choose one of them. At least one of these sets, say E_{n_0}, is not of the first category, and we may take for S the set E_{n_0} translated by n_0u. We may say that B^* is a bais for S.

If B is a basis, every trigonometrical series absolutely convergent in B is absolutely convergent everywhere[2]). Suppose first that the trigonometrical series considered contains only sine terms. We prove by induction that $|\sin n(\varepsilon_1x_1+\dots+\varepsilon_mx_m)|\leqslant|\sin nx_1|+|\sin nx_2|+\dots+|\sin nx_m|$, if $\varepsilon_j=\pm1$, and the result follows. In the general case let u be any point of B, and let $x=y+u$. We have $a_n\cos nx+b_n\sin nx=a_n(u)\cos ny+b_n(u)\sin ny$, where $a_n(u)=a_n\cos nu+b_n\sin nu$, $b_n(u)=b_n\cos nu-a_n\sin nu$.

The absolute convergence of the series at the point $y=0$ implies $|a_1(u)|+|a_2(u)|+\dots<\infty$, and therefore the series $b_1(u)\sin y+b_2(u)\sin 2y+\dots$ converges absolutely in a set B^* obtained from B by translating it by $-u$. In virtue of the lemma, B^* is a basis for a set S of the second category. The argument which we applied to sine series shows that $b_1(u)\sin y+b_2(u)\sin 2y+\dots$ is absolutely convergent in S, and consequently, by Theorem 6.11, everywhere. The same may be said of the series with terms $a_n(u)\cos ny+b_n(u)\sin ny=a_n\cos nx+b_n\sin nx$, and the theorem is established.

6.13. To give an example, we shall show that *the Cantor ternary set C constructed on* $(0,1)$ (or on any other interval) *is a basis*. More precisely, we will show that the set of all possible sums $x+y$, with $x\in C$, $y\in C$, fills up the whole interval $(0,1)$[3]). This could be deduced from the fact that the ternary development of any $x\in C$ can be written in the form not containing the digit 1,

[1]) Thence it is not difficult to deduce that B^* is itself a basis (§ 6.6.2), but this is not necessary for our purposes.

[2]) See N i e m y t z k i [1], for the case of sine series.

[3]) S t e i n h a u s [4]. More general results will be found in D e n j o y [2], M i r i m a n o f f [1].

but a geometrical proof is more illuminating. Consider the set Γ of points (x,y) of the plane such that $x \in C$, $y \in C$. The set Γ may be obtained by the following procedure. Divide the square Q_0 with opposite corners at $(0,0)$ and $(1,1)$ into nine equal parts, and, removing the interior of the five squares forming a cross, consider the sum Q_1 of the remaining four corner squares. For any of these corner squares we repeat our procedure, and let Q_2 be the sum of the new corner squares, and so on. Plainly $\Gamma = Q_0 Q_1 Q_2 \dots$ The projection of any Q_i on the diagonal joining the points $(0,0)$ and $(1,1)$ fills up this diagonal. In other words, any straight line L_h with the equation $x+y=h$, $0 \leqslant h \leqslant 1$, meets every Q_i at one point at least. Since the Q_i are closed and form a decreasing sequence, $\Gamma L_h \neq 0$ for $0 \leqslant h \leqslant 1$, and this is just what we wanted to prove.

6.2. Fatou's theorems. The problem of the absolute convergence for sine or cosine series has a very simple solution in the case when the moduli of the coefficients form a decreasing sequence

If the series $a_1 \cos x + a_2 \cos 2x + \dots$, $|a_1| \geqslant |a_2| \geqslant \dots$, *is absolutely convergent at a point* x_0, *then* $|a_1| + |a_2| + \dots < \infty$. *The same is true for the series* $a_1 \sin x + a_2 \sin 2x + \dots$, *provided that* $x_0 \not\equiv 0$ $(\mathrm{mod}\,\pi)$ [1]. To prove the first part of the theorem we may plainly suppose that $0 < x_0 < \pi$. From the hypothesis it follows that $|a_1| \cos^2 x_0 + |a_2| \cos^2 2x_0 + \dots < \infty$. Since $2\cos^2 nx_0 = 1 + \cos ny_0$, where $y_0 = 2x_0$, and since the series $|a_1| \cos y_0 + |a_2| \cos 2y_0 + \dots$ converges (§ 1.23), the result follows. The second part is obtained by a similar argument.

6.21. The set A of points where a trigonometrical series 6.1(2) converges absolutely, possesses curious properties. Let $\bar{A}$ denote the set of points of absolute convergence for the series conjugate to 6.1(2), and let B and $\bar{B}$ be the sets of points where the series 6.1(2) and its conjugate converge, not necessarily absolutely. It will be convenient to place all these sets on the circumference of the unit circle.

Every point of A *is a point of symmetry for the sets* $A, \bar{A}, B, \bar{B}$ [2].

The proof follows from the formulae

$$a_n(x+h) + a_n(x-h) = 2a_n(x)\cos nh, \quad b_n(x+h) - b_n(x-h) = -2a_n(x)\sin nh,$$

[1]) Fatou [2]. The proof of the text is due to Saks.
[2]) Fatou [2].

where the notation is that of § 6.12. From the first of them we deduce that, if $|a_1(x)| + |a_2(x)| + \dots < \infty$, and if the series $a_1(x-h) + a_2(x-h) + \dots$ converges, or converges absolutely, so does the series $a_1(x+h) + a_2(x+h) + \dots$

The theorem remains true if we consider the points of summability, the arcs of uniform convergence, etc.

6.22. *If* A *is infinite, then* B, *and similarly* $\bar{B}$, *is either of measure* 0 *or* 2π [1]. If $x \in A$, $x+h \in A$, then all the points $x+h$, $x+2h$, $x+3h, \dots$ belong to A. Since A is infinite, h may be arbitrarily small, and so A is everywhere dense. Suppose that B and its complement C are both of positive measure, and let x_1 and x_2 be points of density 1 for B and C respectively. There exists an $\varepsilon > 0$ such that, if any interval I, $|I| \leqslant 2\varepsilon$, contains x_1, we have $|IB| > \frac{1}{2}|I|$, and if any interval I', $|I'| \leqslant 2\varepsilon$, contains x_2, then $|I'C| > \frac{1}{2}|I'|$. Let $I = (x_1 - \varepsilon, x_1 + \varepsilon)$, and take an x_0 belonging to A and distant by less than $\frac{1}{2}\varepsilon$ from the middle-point of the arc (x_1, x_2). The set B reflected in x_0 goes into itself, and I into an interval I', $|I'| = 2\varepsilon$, containing x_2. Since the inequalities $|I'B| > \frac{1}{2}|I'|$, $|I'C| > \frac{1}{2}|I'|$ are incompatible, we have a contradiction.

6.3. The absolute convergence of Fourier series. We begin by the following theorem due to S. Bernstein.

If $f \in \mathrm{Lip}\,\alpha$, $\alpha > \frac{1}{2}$, *then* $\mathfrak{S}[f]$ *converges absolutely. For* $\alpha = \frac{1}{2}$ *this is no longer true* [2].

Suppose that 6.1(2) is $\mathfrak{S}[f]$. Then

$$(1) \qquad f(x+h) - f(x-h) \sim 2\sum_{n=1}^{\infty} b_n(x) \sin nh,$$

$$\frac{1}{\pi}\int_0^{2\pi} [f(x+h) - f(x-h)]^2\,dx = 4\sum_{n=1}^{\infty} \rho_n^2 \sin^2 nh,$$

where $\rho_n^2 = a_n^2 + b_n^2$. The left-hand side of the last formula is $\leqslant Ch^{2\alpha}$, where $C, C_1, \dots$ denote constants. On setting $h = \pi/2N$ we obtain two inequalities

$$(2) \qquad \sum_{n=1}^{\infty} \rho_n^2 \sin^2 \frac{\pi n}{2N} \leqslant C N^{-2\alpha}, \qquad \sum_{n=1}^{N} \rho_n^2 \sin^2 \frac{\pi n}{2N} \leqslant C N^{-2\alpha}.$$

[1]) Lusin [1].
[2]) Bernstein [2], [3].

Let us now assume that $N = 2^\nu$, $\nu = 1, 2, \ldots$ Taking into account only the terms with indices n exceeding $\frac{1}{2}N$, we obtain from the last inequality

$$(3) \qquad \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2 < 2C\, 2^{-2\nu\alpha}.$$

Thence, by Schwarz's inequality,

$$(4) \qquad \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leqslant \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2\right)^{1/2} \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1^2\right)^{1/2} < C_1\, 2^{\nu(1/2-\alpha)},$$

and finally

$$(5) \qquad \sum_{n=2}^{\infty} \rho_n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n < C_1 \sum_{\nu=1}^{\infty} 2^{\nu(1/2-\alpha)}.$$

The last series is convergent since $\alpha > \frac{1}{2}$. The proof ot the second part of the theorem we postpone to § 6.33.

6.31. *If $f(x)$ is of bounded variation and belongs to* Lip α *for any positive α, $\mathfrak{S}[f]$ converges absolutely*[1]). That the second condition imposed on f is not superfluous is seen from the example of the series

$$(1) \qquad \sum_{2}^{\infty} \frac{\sin nx}{n \log n},$$

which, being the Fourier series of a function of bounded variation, indeed of an absolutely continuous function (§ 5.12), diverges absolutely (§ 6.2).

Let $\omega(\delta)$ be the modulus of continuity of f, and V the total variation of f over $(0, 2\pi)$. We start from the inequality

$$\sum_{k=1}^{2N} \left[f\left(x + \frac{k\pi}{N}\right) - f\left(x + (k-1)\frac{\pi}{N}\right)\right]^2 \leqslant$$
$$\leqslant \omega\left(\frac{\pi}{N}\right) \sum_{k=1}^{2N} \left| f\left(x + k\frac{\pi}{N}\right) - f\left(x + (k-1)\frac{\pi}{N}\right)\right| \leqslant V\omega\left(\frac{\pi}{N}\right),$$

which we integrate over $(0, 2\pi)$. On account of the periodicity,

[1]) Zygmund [8].

replacing x by $x + \xi$ does not affect the value of the integral, and so all integrals formed from the left-hand side are equal. Hence we have, by turns, with $N = 2^\nu$,

$$2N \int_0^{2\pi} \left[f\left(x + \frac{\pi}{2N}\right) - f\left(x - \frac{\pi}{2N}\right)\right]^2 dx \leqslant 2\pi V \omega\left(\frac{\pi}{N}\right),$$

$$\sum_{n=1}^{\infty} \rho_n^2 \sin^2 \frac{\pi n}{2N} \leqslant C_2 N^{-\alpha-1}, \qquad \sum_{n=1}^{N} \rho_n^2 \sin^2 \frac{\pi n}{2N} \leqslant C_2 N^{-\alpha-1},$$

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2 \leqslant 2C_2\, 2^{-\nu(\alpha+1)}, \qquad \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leqslant C_3\, 2^{-\nu\alpha/2},$$

$$\sum_{n=2}^{\infty} \rho_n \leqslant C_3 \sum_{\nu=1}^{\infty} 2^{-\nu\alpha/2} < \infty.$$

6.32. The problem of the absolute convergence of trigonometrical series may be generalized as follows. Given a series 6.1(2), we ask about the values of the exponent β which makes

$$(1) \qquad \sum_{n=1}^{\infty} (|a_n|^\beta + |b_n|^\beta)$$

convergent. Theorem 6.3 is special a case of the following theorem; it is, in fact, the most important case of it.

If $f \in$ Lip α, $0 < \alpha \leqslant 1$, *the series* (1) *converges for every* $\beta > 2/(2\alpha + 1)$, *but not necessarily for* $\beta = 2/(2\alpha + 1)$[1]).

The proof of the first part resembles the proof of the first part of Theorem 6.3. Let $\gamma = 2/(2\alpha + 1)$. Since $0 < \gamma < 2$, we may also assume that $0 < \beta < 2$. Starting with 6.3(3), and applying Hölders inequality, we obtain

$$\sum_{2^{\nu-1}+1}^{2^\nu} \rho_n^\beta \leqslant \left(\sum_{2^{\nu-1}+1}^{2^\nu} \rho_n^2\right)^{\beta/2} \left(\sum_{2^{\nu-1}+1}^{2^\nu} 1\right)^{1-\beta/2} \leqslant C_4\, 2^{\nu(1-\beta/\gamma)}.$$

Here $1 - \beta/\gamma < 0$, and an argument similar to 6.3(5) yields the convergence of $\rho_2^\beta + \rho_3^\beta + \ldots$ or, what is the same thing, of the series (1). This gives the first part of the theorem.

6.33. The second part of Theorem 6.32, and of Theorem 6.3, is a simple corollary of the results obtained in § 5.3. It was

[1]) Szász [2].

proved there that the real and imaginary components of the first of the series

$$(1)\qquad \sum_{n=2}^{\infty}\frac{e^{in\log n}}{n^{1/2+\alpha}}e^{inx},\qquad 0<\alpha<1,\qquad \sum_{n=2}^{\infty}\frac{e^{in\log n}}{(n\log n)^{3/2}}e^{inx},$$

belong to Lip α, and it is easy to see that, for these components, the series with terms $\rho_n^{2/(2\alpha+1)}$ diverge. The components of the second series in (1) belong to Lip 1 (§ 5.33), and the series with terms $\rho_n^{2/3}$ diverges.

6.34. *If f is of bounded variation and also $f\in$ Lip α, $0<\alpha\leqslant 1$, the series* 6.32(1) *converges for $\beta>2/(2+\alpha)$* [1].

The proof, which is analogous to that of Theorems 6.31 and 6.32, may be left to the reader (see also § 6.6.7).

6.35. Let $F(x)$ be an absolutely continuous and periodic function whose derivative $F'(x)=f(x)$ belongs to L^2.

If a_n, b_n are the Fourier coefficients of f, those of F will be $-b_n/n$, a_n/n. From the inequalities

$$\frac{|a_n|}{n}\leqslant\frac{1}{2}\left(a_n^2+\frac{1}{n^2}\right),\qquad \frac{|b_n|}{n}\leqslant\frac{1}{2}\left(b_n^2+\frac{1}{n^2}\right),$$

we see that $\mathfrak{S}[F]$ converges absolutely. More generally, *if F is absolutely continuous and $F'\in L^p$, $p>1$, then $\mathfrak{S}[F]$ converges absolutely* [2]. The proof remains essentially the same as in the case $p=2$, if, instead Bessel's inequality, we use a more general inequality, due to Young, which will be established in Chapter IX. It is however much simpler to deduce the theorem from Theorem 6.31, observing that, if $F'\in L^p$, $p>1$, then F satisfies a Lipschitz condition of positive order (§ 4.7.3).

The result which we have established is, in turn, contained in the following theorem

6.36. (i) *If $F(x)$ is absolutely continuous, $F'(x)=f(x)$, and $|f|\log^+|f|$ is integrable, then $\mathfrak{S}[F]$ converges absolutely* [3]. It will be convenient to postpone the proof of (i) to Chapter VII, where we shall obtain this theorem as a corollary of the following important result due to Hardy and Littlewood:

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1) Waraszkiewicz [1]; Zygmund [7].
2) Tonelli [2].
3) Zygmund [4].

(ii) *If $\mathfrak{S}[F]$ and $\tilde{\mathfrak{S}}[F]$ are both Fourier series of functions of bounded variation, $\mathfrak{S}[F]$ converges absolutely.*

Here we only observe that the integrability of $|f|(\log^+|f|)^{1-\varepsilon}$, $\varepsilon>0$, would not be sufficient for the truth of (i). For if we take for $\mathfrak{S}[F]$ the series 6.31(1), which converges absolutely only at the points $x\equiv 0\ (\mathrm{mod}\ \pi)$, we have $f(x)\sim 1/x\log^2 x$ as $x\to+0$, (§ 5.221), so that $|f|(\log^+|f|)^{1-\varepsilon}$ is integrable for every $\varepsilon>0$.

6.4. Szidon's theorem on lacunary series. The following theorem on the absolute convergence of Fourier series bears a different character.

If a lacunary trigonometrical series

$$(1)\qquad \sum_{k=1}^{\infty}(a_k\cos n_k x+b_k\sin n_k x),\qquad n_{k+1}/n_k>\lambda>1,$$

is the Fourier series of a bounded function $f(x)$, $|f|\leqslant M$, the series converges absolutely [1].

Taking, instead of $f(x)$, the functions $f(x)\pm f(-x)$, we may restrict ourselves to purely cosine or purely sine series, e. g. to the former. The idea of the proof consists in considering the non-negative polynomials

$$(2)\qquad P_l(x)=\prod_{k=1}^{l}(1+\varepsilon_k\cos m_k x),$$

where $\varepsilon_k=\pm 1$ and the positive integers m_k satisfy a condition $m_{k+1}/m_k>\mu\geqslant 3$. Multiplying out the product P_l we see that it consists of the constant term 1, and of terms $A_\nu\cos\nu x$, where $\nu=\pm m_{k_1}\pm\pm m_{k_2}\pm\ldots\pm m_{k_j}\geqslant 0$, $m_{k_1}<m_{k_2}<\ldots<m_{k_j}\leqslant m_{k_l}$. From the last equation we see that ν is contained between $m_{k_j}(1-\mu^{-1}-\mu^{-2}-\ldots)$ and $m_{k_j}(1+\mu+\mu^2+\ldots)$, i. e. between $m_{k_j}(\mu-2)/(\mu-1)$ and $m_{k_j}\mu/(\mu-1)$. Therefore, since $\mu\geqslant 3$, the numbers $\pm m_{k_1}\pm\ldots\pm m_{k_j}$ corresponding to various sequences $\{k_i\}$ are all different; and, if μ is large enough, $\mu\geqslant\mu_0(\varepsilon)$, the indices ν corresponding to $A_\nu\neq 0$ concentrate in the neighbourhoods $(m_k(1-\varepsilon),\ m_k(1+\varepsilon))$ of the numbers m_k, where $\varepsilon>0$ is arbitrary.

Returning to the series (1), take ε so small that the intervals $(n_k(1-\varepsilon),\ n_k(1+\varepsilon))$, $k=1,2,\ldots$, do not overlap, and an integer r such that $\lambda^r>\mu_0(\varepsilon)$. Put $m_k^{(s)}=n_{kr+s}$, $k=1,2,\ldots$, $0\leqslant s\leqslant r-1$,

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1) Szidon [2]; for a generalization see Zygmund [6].

and let $P_l^{(s)}(x)$ denote the polynomial (2) formed with $\{m_k^{(s)}\}$, $1 \leqslant k \leqslant l$, and $\varepsilon_k = \operatorname{sign} a_{kr+s}$. Since $m_{k+1}^{(s)}/m_k^{(s)} > \lambda^r > \mu_0(\varepsilon)$, we obtain

$$(3) \qquad \sum_{k=1}^{l} |a_{kr+s}| = \frac{1}{\pi}\int_0^{2\pi} f(x)\, P_l^{(s)}(x)\, dx \leqslant \frac{M}{\pi}\int_0^{2\pi} P_l^{(s)}\, dx = 2M,$$

since the constant term of $P_l^{(s)}(x)$ is equal to 1. Making $l \to \infty$, we find that each ot the r partial series into which we have decomposed the series $|a_r| + |a_{r+1}| + |a_{r+2}| + \ldots$ converges. This completes the proof.

If (1) is a pure sine series, we consider, instead of (2), analogous polynomials, with cosines replaced by sines.

6.5. Wiener's theorem. It is obvious that the absolute convergence of $\mathfrak{S}[f]$ at a point x_0 is not a local property but depends on the behaviour of $f(x)$ in the whole interval $(0, 2\pi)$. However, *if to every point x_0 corresponds a neighbourhood I_{x_0} of x_0 and a function $g(x) = g_{x_0}(x)$ such that* (i) $\mathfrak{S}[g]$ *converges absolutely, and* (ii) $g(x) = f(x)$ *in* I_{x_0}, *then* $\mathfrak{S}[f]$ *converges absolutely* [1]).

By the Heine-Borel theorem we can find a finite number of points $x_1, x_2, \ldots x_m$ such that the intervals $I_{x_1}, I_{x_2}, \ldots I_{x_m}$ overlap and cover the whole interval $0 \leqslant x \leqslant 2\pi$. Let $I_{x_k} = (u_k, v_k)$. Without loss of generality we may suppose that $u_k < v_{k-1} < u_{k+1} < v_k$, $k = 1, 2, \ldots, m$, where $(u_{m+1}, v_{m+1}) = (u_1, v_1)$. Let $\lambda_k(x)$ be the periodic and continuous function equal to 1 in (v_{k-1}, u_{k+1}), vanishing outside (u_k, v_k) and linear in the intervals (u_k, v_{k-1}) and (u_{k+1}, v_k). It will be readily seen that $\lambda_1(x) + \lambda_2(x) + \ldots + \lambda_m(x) = 1$. Since λ_k has a derivative of bounded variation, the Fourier coefficients of λ_k are $O(n^{-2})$, so that $\mathfrak{S}[\lambda_k]$ converges absolutely.

Since $\mathfrak{S}[f\lambda_k] = \mathfrak{S}[g_{x_k}\lambda_k] = \mathfrak{S}[g_{x_k}]\,\mathfrak{S}[\lambda_k]$, we obtain that $\mathfrak{S}[f\lambda_k]$ converges absolutely (§ 4.431). To prove the theorem it is sufficient to observe that $\mathfrak{S}[f] = \mathfrak{S}[f\cdot(\lambda_1 + \ldots + \lambda_m)] = \mathfrak{S}[f\lambda_1] + \ldots + \mathfrak{S}[f\lambda_m]$.

6.51 [2]). *Let the Fourier series of a function $f(t)$ be absolutely convergent, and let the values of $f(t)$ belong to an interval (α, β). If $\varphi(z)$ is a function of a complex variable, regular at every point of the interval (α, β), the Fourier series of $\varphi\{f(t)\}$ converges absolutely.*

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1) Wiener [1].

2) Lévy [1], Wiener [1].

Let $f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{int}$, $f^k(t) = \sum_{n=-\infty}^{+\infty} c_n^{(k)} e^{int}$, $k = 0, 1, 2, \ldots$ Since $\mathfrak{S}[f^k]$ is obtained from $\mathfrak{S}[f]$ by formal multiplication, it is easy to see that if $\ldots + |c_{-1}| + |c_0| + |c_1| + \ldots = M$, then $\ldots + |c_{-1}^{(k)}| + |c_0^{(k)}| + |c_1^{(k)}| + \ldots \leqslant M^k$. Suppose that the series $\varphi(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots$ converges for $|z| < r$. In the case $M < r$ the theorem is fairly simple; for the series $\alpha_0 + \alpha_1 f(t) + \alpha_2 f^2(t) + \ldots$ converges uniformly, and, if γ_n are the complex Fourier coefficients of $\varphi\{f\}$, then

$$\gamma_n = \sum_{k=0}^{\infty} \alpha_k c_n^{(k)}, \quad \sum_{n=-\infty}^{+\infty} |\gamma_n| \leqslant \sum_{n=-\infty}^{+\infty}\sum_{k=0}^{\infty} |\alpha_k c_n^{(k)}| = \sum_{k=0}^{\infty} |\alpha_k| \sum_{n=-\infty}^{+\infty} |c_n^{(k)}|,$$

where the sum of the last series is $\leqslant |\alpha_0| + |\alpha_1| M + |\alpha_2| M^2 + \ldots < \infty$.

Let t_0 be an arbitrary point of the interval $0 \leqslant t \leqslant 2\pi$. To prove the theorem in the general case it is sufficient to show that there is a function $g(t)$ such that $\mathfrak{S}[\varphi\{g\}]$ converges absolutely and that $g(t) = f(t)$ in an interval $(t_0 - h, t_0 + h)$. Suppose, for simplicity, that $t_0 = 0$ and let $f(0) = u$. Without real loss of generality we may suppose that $u = 0$, for otherwise we have $\varphi\{f(t)\} = \varphi\{f(t) - u + u\} = \varphi_1\{f_1(t)\}$, where $f_1(t) = f(t) - u$, $\varphi_1(z) = \varphi(z + u)$, and we may consider the functions f_1, φ_1 instead of f, φ.

Let $\varphi(z) = \alpha_0 + \alpha_1 z + \ldots$ be convergent for $|z| < r$. In virtue of the special case already dealt with, it is sufficient to construct a function $g(t)$ with Fourier coefficients c_n', such that $g(t) = f(t)$ in $(-h, h)$ and that $\ldots + |c_{-1}'| + |c_0'| + |c_1'| + \ldots = M' < r$; for then $\mathfrak{S}[\varphi\{g\}]$ will be absolutely convergent.

Let $\lambda(t) = \lambda_\rho(t)$ be a continuous periodic function such that (i) $\lambda(t) = 1$ for $0 \leqslant t \leqslant \rho$, (ii) $\lambda(t) = 0$ for $2\rho \leqslant t \leqslant \pi$, (iii) $\lambda(t)$ is linear in the interval $(\rho, 2\rho)$, (iv) $\lambda(t)$ is even. If $l_n = l_n^\rho$ are the complex Fourier coefficients of $\lambda(t)$, then $l_0 = 3\rho/2\pi$, $l_n = (2\sin{}^1\!/_2\,\rho n \sin{}^3\!/_2\,\rho n)/\pi\rho n^2$, $n \neq 0$. We shall require the following relations

$$(1) \quad \sum_{n=-\infty}^{+\infty} |l_n^\rho| \leqslant A, \qquad (2) \quad \sum_{n=-\infty}^{+\infty} |l_n^\rho - l_{n-1}^\rho| \to 0 \quad \text{as } \rho \to 0,$$

where $A, B, \ldots$ denote constants independent of ρ. To prove (1) we observe that from the inequalities $|\sin u| \leqslant 1$, $|\sin u| \leqslant |u|$, we obtain $|l_n| \leqslant 2/\pi\rho n^2$, $|l_n| \leqslant 3\rho/2\pi$, and so, if $N = [1/\rho] + 1$, the sum in (1) is less than

$$\frac{3\rho}{2\pi} + 2\sum_{n=1}^{N} \frac{3\rho}{2\pi} + 2\sum_{n=N+1}^{\infty} \frac{2}{\pi\rho n^2} < 1 + N\rho + 4/\pi\rho N < A.$$

Now $l_n^\rho - l_{n-1}^\rho$ is the complex Fourier coefficient of the function $\lambda_\rho(t)(1-e^{it})$. Considering the real and imaginary parts of the derivative $\lambda'_\rho(t)(1-e^{it}) - ie^{it}\lambda_\rho(t)$ of this function, we easily find that the total variation of this derivative over $(-\pi, \pi)$ is uniformly bounded, and so, in virtue of the results obtained in § 2.213, we have $|l_n^\rho - l_{n-1}^\rho| \leqslant B/n^2$. If ν is a positive integer, the series in (2) is equal to

$$\sum_{n=-\nu}^{\nu} + \left(\sum_{n=-\infty}^{-\nu-1} + \sum_{n=\nu+1}^{\infty}\right) \leqslant \sum_{n=-\nu}^{\nu} |l_n^\rho - l_{n-1}^\rho| + 2B\sum_{n=\nu+1}^{\infty}\frac{1}{n^2} = P + Q.$$

Taking ν large enough we have $Q < \frac{1}{2}\varepsilon$. If $\rho \to 0$, then $\mathfrak{M}[\lambda_\rho] \to 0$, and so $l_n^\rho \to 0$ for every n. Hence, for fixed ν, $P < \frac{1}{2}\varepsilon$, $P + Q < \varepsilon$ if ρ is small enough, and this proves (2).

Let $q > 0$ be an integer which we shall define in a moment, and let $c_p = u_p + v_p$, where $u_p = c_p$, $v_p = 0$ for $|p| \leqslant q$, and $u_p = 0$, $v_p = c_p$ for $|p| > q$. Since $f(0) = \Sigma c_p = 0$, $\Sigma |c_p| < \infty$ we have

$$\left|\sum_{p=-\infty}^{+\infty} u_p\right| < r/3A, \qquad \sum_{p=-\infty}^{+\infty} |v_p| < r/3A$$

if q is large enough. Denoting by d_n^ρ the Fourier coefficients of the function $f(t)\lambda_\rho(t)$, we have

$$d_n^\rho = \sum_{p=-\infty}^{+\infty} c_p\, l_{n-p}^\rho,$$

$$\sum_{n=-\infty}^{+\infty} |d_n^\rho| \leqslant \sum_{n=-\infty}^{+\infty}\left|\sum_{p=-\infty}^{+\infty} u_p\, l_{n-p}\right| + \sum_{n=-\infty}^{+\infty}\left|\sum_{p=-\infty}^{+\infty} v_p\, l_{n-p}\right| = S + T,$$

$$T \leqslant \sum_{n=-\infty}^{+\infty}\sum_{p=-\infty}^{+\infty} |v_p|\,|l_{n-p}^\rho| = \sum_{p=-\infty}^{+\infty}|v_p| \sum_{n=-\infty}^{+\infty}|l_n^\rho| < \frac{r}{3A}\cdot A = {}^1/_3\, r,$$

$$S = \sum_{n=-\infty}^{+\infty}\left|\sum_{p=-q}^{q} c_p(l_{n-p} - l_n + l_n)\right| \leqslant \sum_{n=-\infty}^{+\infty}\left|\sum_{p=-q}^{q} c_p(l_{n-p} - l_n)\right| +$$

$$+ \sum_{n=-\infty}^{+\infty}|l_n|\left|\sum_{p=-q}^{q} c_p\right| = S_1 + S_2.$$

It it plain that $S_2 < {}^1/_3\, r$. Since $|l_{n-p} - l_n| \leqslant |l_{n-p} - l_{n-p+1}| + \ldots + |l_{n-1} - l_n|$ for $p > 0$, $|l_{n-p} - l_n| \leqslant |l_{n-p} - l_{n-p-1}| + \ldots + |l_{n+1} - l_n|$ for $p < 0$, S_1 is less than a multiple of the series (2) and so tends to 0 with ρ. If $\rho = \rho_0$ is small enough, then $S_1 < {}^1/_3\, r$,

$S + T \leqslant S_1 + S_2 + T < {}^1/_3\, r + {}^1/_3\, r + {}^1/_3\, r = r$. Hence, putting $g(t) = f(t)\lambda_{\rho_0}(t)$, $c'_n = d_n^{\rho_0}$, $h = \rho_0$, we shall have $f(t) = g(t)$ in $(-h, h)$, $\Sigma |c'_n| < r$, and this completes the proof.

As a corollary we obtain that, *if* $\mathfrak{S}[f]$ *converges absolutely and* $f(x) \neq 0$, *then* $\mathfrak{S}[1/f]$ *converges absolutely.*

6.6. Miscellaneous theorems and examples.

1. The set of points where the series $\Sigma\, n^{-1} \sin n!\, x$ converges absolutely contains a perfect subset.

[Consider the graphs of the curves $y = \sin n!\, x$].

2. (i) Every measurable set of positive measure is a basis; (ii) every set of the second category is a basis.

[Let E be an arbitrary set of positive measure, and $x \in E$, $y \in E$. To prove (i) it is sufficient to show that the set of the differences $x - y$ contains an interval. To show this let E_h denote the set E translated by h. Considering the neighbourhood of a point of density 1 for the set E, it is easy to show that $E\,E_h \neq 0$ if h is sufficiently small. This theorem is due to Steinhaus [5]. The proof of (ii) is similar].

3. A necessary and sufficient condition that the Fourier series of a function $h(x)$ should converge absolutely is that there should exist two functions f and g of the class L^2 such that $h(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+t)\,g(t)\,dt$. M. Riesz; see Hardy and Littlewood [8].

[That the condition is sufficient follows from § 2.11. Let c_n be the complex Fourier coefficients of h; to prove that the condition is necessary consider the functions with Fourier coefficients $|c_n|^{1/2}$ and $|c_n|^{1/2}\,\overline{\text{sign } c_n}$].

4. The conditions of Theorems 6.3—6.32 are unnecessarily stringent. Thus Theorems 6.3 and 6.32 remain true, and the proofs unchanged, if we assume that $f \in \mathrm{Lip}(\alpha, 2)$. In Theorem 6.31 we may assume that the function f is of bounded variation and belongs to $\mathrm{Lip}(\alpha, 1)$.

5. Let $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant 2$. If a_n, b_n are the Fourier coefficients of an $f \in \mathrm{Lip}(\alpha, p)$, then $\Sigma(|a_n|^\beta + |b_n|^\beta) < \infty$ for every $\beta < p/\{p(1+\alpha) - 1\}$. Szász [3].

[The proof is similar to that of Theorem 6.32 if, instead of Parseval's relation, we use the inequality of Hausdorff-Young which will be established in Chapter IX].

6. (i) If $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leqslant 1$, then $\Sigma\, n^{\beta - 1/2}(|a_n| + |b_n|) < \infty$ for every $\beta < \alpha$. Hardy [4]. (ii) If f is, in addition, of bounded variation then $\Sigma\, n^{\beta/2}(|a_n| + |b_n|) < \infty$. (iii) If $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant 2$, then $\Sigma\, n^\gamma(|a_n| + |b_n|) < \infty$ for every $\gamma < \alpha - 1/p$.

[To prove the first part of the theorem consider the inequality 6.3(4)].

7. Let $f(x)=\sum_{n=2}^{\infty}\frac{e^{in^{\alpha}}}{n^{\frac{1}{2}\alpha+1}}\frac{e^{inx}}{(\log n)^{\beta}}$, where $0<\alpha<1$, and $\beta-1$ is positive and sufficiently small. Then the real and imaginary parts of f are of bounded variation, belong to Lip α, and yet $\sum \rho_n^k=\infty$ for $k=2/(2+\alpha)$. It follows that Theorem 6.34 cannot be improved. For the proof see Zygmund [7]. See also § 5.7.13.

8. Let a_k, b_k be the Fourier coefficients of a function $f(x)$ and let $t_n=t_n(f)=\frac{1}{2}\rho_1+\rho_2+\ldots+\rho_n$, where $\rho_k\geqslant 0$, $\rho_k^2=a_k^2+b_k^2$.

(i) If $|f(x)|\leqslant 1$, then $t_n\leqslant(2n+1)^{1/2}$. (ii) For every n there is a function $f(x)=f_n(x)$ such that $t_n\geqslant A n^{1/2}$, where A is a positive absolute constant.

See Bernstein [3], where a little more is proved, viz. that for f we may take a trigonometrical polynomial of order n.

[(i) follows from the inequalities of Bessel and Schwarz. To obtain (ii) let $g_t(x)=g_{t,n}(x)=\varphi_1(t)\cos x+\ldots+\varphi_n(t)\cos nx$, where $\varphi_1,\varphi_2,\ldots$ are Rademacher's functions. Then

$$\int_0^1 dt\int_0^{2\pi}|g_t(x)|\,dx=\int_0^{2\pi}dx\int_0^1|g_t(x)|\,dt\geqslant m_1\int_0^{2\pi}(\cos^2 x+\ldots+\cos^2 nx)^{1/2}dx=$$

$$=\tfrac{1}{2}m_1\int_0^{2\pi}\left\{(\cos^2 x+\ldots)^{1/2}+(\sin^2 x+\ldots)^{1/2}\right\}dx\geqslant\tfrac{1}{2}m_1\int_0^{2\pi}\left\{(\cos^2 x+\sin^2 x)+\ldots\right\}^{1/2}dx=\pi m_1 n^{1/2}$$

(§§ 5.7.8, 4.13(3)). Let t_0 be a value of t such that the integral of $|g_{t_0}(x)|$ over $(0,2\pi)$ exceeds $\pi m_1 n^{1/2}$, and let a_k, b_k be the Fourier coefficients of the function $f(x)=\operatorname{sign} g_{t_0}(x)$. Then

$$\sum_{k=1}^{n}(|a_k|+|b_k|)\geqslant\Big|\sum_{k=1}^{n}\varphi_k(t_0)(a_k+b_k)\Big|=\Big|\frac{1}{\pi}\int_0^{2\pi}f(x)\,g_{t_0}(x)\,dx\Big|=$$

$$=\frac{1}{\pi}\int_0^{2\pi}|g_{t_0}(x)|\,dx\geqslant m_1 n^{1/2}.$$

The idea of the proof is taken from Paley [2], where it is applied to another problem. The result may be used to prove the second part of Theorem 6.3].