CHAPTER V.

Properties of some special series.

5.1. In this chapter we intend to study some particular series, which are not only interesting in themselves, but provide examples illuminating many points of the general theory. The latter consideration will be decisive in our choice of material.

5.11. Series with coefficients monotonically tending to zero. In § 1.23 we have proved that if a sequence $\{a_n\}$ decreases monotonically to 0, or, more generally, if $\{a_n\}$ tends to 0 and is of bounded variation, both series

$$(1) \qquad a)\ \tfrac12 a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \qquad b)\ \sum_{n=1}^{\infty} a_n \sin nx$$

converge uniformly, except in arbitrarily small neighbourhoods of the points $x \equiv 0 \pmod{2\pi}$. We will now prove some further theorems on the behaviour of these series.

It is obvious that, if $a_n \geqslant 0$, a necessary and sufficient condition for the uniform convergence of the series (1a) is the convergence of $a_0 + a_1 + \dots$ For the series (1b) the situation is less trivial.

If $a_n \geqslant a_{n+1} \to 0$, a necessary and sufficient condition for the uniform convergence of the series (1b) *is* $na_n \to 0$ [1]).

We shall consider only the values $0 < x \leqslant \frac14\pi$. To prove the sufficiency we denote by $r_M(x)$ the M-th remainder $a_M \cos Mx + \dots$ of the series (1b), and put $\varepsilon_n = \operatorname{Max} ka_k$ for $k \geqslant n$, $N = N_x = [1/x] + 1$, so that $N > 1$, $1/N < x \leqslant 1/(N-1)$. For any x we put $r_M(x) =$

[1]) Chaundy and Jollife [1].

$= r'_M(x) + r''_M(x)$, where r''_M denotes the sum of all the terms belonging to r_M with indices $\geqslant N$. If $N \leqslant M$, we have $r'_M(x) = 0$. If $N > M$, then

$$|r'_M(x)| \leqslant x \sum_{k=M}^{N-1} ka_k \leqslant \frac{1}{(N-1)}\varepsilon_M(N-M) \leqslant \varepsilon_M.$$

It follows that $|r'_M(x)| \leqslant \varepsilon_M$ for every $M > 0$. Applying Abel's transformation to r''_M, $M < N$, we obtain

$$|r''_M(x)| \leqslant \sum_{k=N}^{\infty}(a_k - a_{k+1})|\bar{D}_k(x)| + a_N|\bar{D}_{N-1}(x)| \leqslant \frac{8a_N}{x} \leqslant 8Na_N \leqslant 8\varepsilon_M,$$

since $|\bar{D}_k(x)| = |\sin x + \dots + \sin kx| \leqslant 1/\sin\frac12 x \leqslant \pi/x \leqslant 4/x$. Similarly, if $N \leqslant M$, then $|r''_M(x)| \leqslant 8\, a_M/x \leqslant 8\, Ma_M \leqslant 8\,\varepsilon_M$. Hence $|r_M(x)| \leqslant$ $\leqslant |r'_M(x)| + |r''_M(x)| \leqslant 9\varepsilon_M$ for $0 < x \leqslant \frac14\pi$. Since this inequality is obvious for $x = 0$, the uniformity of convergence follows.

Conversely, assuming that the series (1b) converges uniformly, and putting $x = \pi/2N$, $N \to \infty$, we deduce from the inequality

$$\sum_{[\frac12 N]+1}^{N} a_n \sin nx \geqslant \sin\frac{\pi}{4}\cdot a_N \sum_{[\frac12 N]+1}^{N} 1 \geqslant \sin\frac{\pi}{4}\cdot\tfrac12 N a_N$$

that $Na_N \to 0$. This completes the proof.

If na_n is bounded, the above argument shows that the partial sums $s_n(x)$ of (1b) are uniformly bounded, but, as is seen from the series $\sin x + \frac12 \sin 2x + \dots$, the sequence $\{s_n(x)\}$ need not be uniformly convergent.

5.12 [1]). (i) *If $a_n \to 0$ and $\{a_n\}$ is quasi-convex, the series* 5.11(1a) *converges, save for $x = 0$, to an integrable function $f(x)$, and is the Fourier series of $f(x)$. If $\{a_n\}$ is convex, $f(x)$ is non-negative.*

Applying Abel's transformation twice, we obtain the expression for the n-th partial sum of the series 5.11(1a)

$$(1) \quad s_n(x) = \sum_{m=0}^{n}(m+1)\Delta^2 a_m K_m(x) + K_n(x)(n+1)\Delta a_{n+1} + D_n(x)a_{n+1},$$

where D_m and K_m denote Dirichlet's and Fejér's kernels. If $x \neq 0$, the last two terms on the right tend to 0 with $1/n$, and therefore $s_n(x) \to f(x) = \Delta^2 a_0 K_0(x) + 2\Delta^2 a_1 K_1(x) + \dots$, which is non-negative for $\{a_n\}$ convex. Since $|f(x)| \leqslant |\Delta^2 a_0| K_0(x) + 2|\Delta^2 a_1| K_1(x) + \dots$,

[1]) Young [9], Kolmogoroff [1].

and the last series integrated over $(-\pi, \pi)$ gives the finite value $\pi(|\Delta^2 a_0| + 2|\Delta^2 a_1| + \dots)$, $f(x)$ is integrable.

The problem of the series 5.11(1a) being a Fourier series is slightly more delicate, and we shall see in a moment why it is so.

From the expression for $f(x)$ and $s_n(x)$ we easily find that $|f(x) - s_n(x)|$ is contained between the expressions

$$\pm \{\sum_{m=n+1}^{\infty} (m+1) |\Delta^2 a_m| K_m(x) + K_n(x)(n+1)|\Delta a_{n+1}|\} + a_{n+1}|D_n(x)|.$$

Integrating this over $(-\pi, \pi)$ we find that $\mathfrak{M}[f - s_n] = o(1) + 2a_{n+1} L_n$, where L_n denotes the integral of $|D_n(x)|$ over $(0, \pi)$. Now it is not difficult to prove that $L_n \sim \log n$ (see Ch. VIII). Hence

(ii) *Let $s_n(x)$ denote the partial sums of the series* 5.11(1a). *If $a_n \to 0$ and $\{a_n\}$ is quasi-convex, the relation $\mathfrak{M}[f - s_n] \to 0$ holds if and only if $a_n = o(1/\log n)$.*

If $a_n \log n \to \infty$, e. g. if $a_n = (\log n)^{-1/2}$, $n > 1$, then $\mathfrak{M}[f - s_n] \to \infty$, $\mathfrak{M}[s_n] \to \infty$. The series

$$\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}, \tag{2}$$

which plays an important part in some problems, is a limiting case, since here the sequence $\mathfrak{M}[f - s_n]$ is bounded and yet it does not tend to 0.

To complete the proof of (i), we observe that the series 5.11(1a) is certainly $\mathfrak{S}[f]$ if $\mathfrak{M}[f - s_n] \to 0$ (and in particular if $a_n \log n \to 0$). When this condition is not satisfied we must proceed otherwise and two ways are open for us. The first of them consists in proving that $\mathfrak{M}[f - \sigma_n] \to 0$ as $n \to \infty$, or that $\mathfrak{M}[f(x) - f(r, x)] \to 0$ as $r \to 1$, where $\sigma_n(x)$ and $f(r, x)$ denote respectively Fejér's and Abel's means of the series considered. We prefer to base the proof of (i) on the following theorem, which will be established in Chapter XI: *If a trigonometrical series converges, except at one point, to an integrable function f, the series is $\mathfrak{S}[f]$.*

Remarks. (a) Given an arbitrary sequence of positive numbers $\varepsilon_n \to 0$, we can easily construct, e. g. geometrically, a convex sequence $\{a_n\}$ such that $a_n \geqslant \varepsilon_n$, $a_n \to 0$. Thus *there exist Fourier series with coefficients tending to* 0 *arbitrarily slowly* (see also § 2.9.2).

(b) If a_n, b_n are the Fourier coefficients of an integrable function, the series $\Sigma b_n/n$ converges (§ 2.621). The example of the Fourier series (2) shows that the series $\Sigma a_n/n$ may be divergent.

5.121. In the preceding section we proved that, if $a_n \to 0$, $\Delta^2 a_n \geqslant 0$, the series 5.11(1a) is a Fourier series. We will now show that the condition $\Delta^2 a_n \geqslant 0$ cannot be replaced by $\Delta a_n \geqslant 0$. More precisely, *there exists a cosine series with coefficients monotonicaly decreasing to* 0 *and yet the sum $f(x)$ of this series is not integrable*[1]. In fact, let us suppose that there exists a sequence of integers $0 = \lambda_1 < \lambda_2 < \dots$ such that a_k is constant for $\lambda_n < k \leqslant \lambda_{n+1}$, $n = 1, 2, \dots$ Making Abel's transformation, we obtain for $f(x)$ the formula

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x) = \sum_{n=1}^{\infty} \alpha_n D_{\lambda_n}(x), \tag{1}$$

where $\alpha_n = \Delta a_{\lambda_n}$. We require the following two inequalities

$$\int_{1/n}^{\pi} |D_n(x)|\, dx > C \log n, \quad L_n = \int_0^{\pi} |D_n(x)|\, dx \leqslant C_1 \log n, \quad n = 2, 3, \dots, \tag{2}$$

where C and C_1 are positive constants. The second inequality is a corollary of the relation $L_n \sim \log n$, which will be proved in Chapter VIII. On the other hand, since $D_n(x) = O(n)$, the integral of $|D_n(x)|$ over $(0, 1/n)$ is $O(1)$, and the first inequality (2) is also a corollary of the relation $L_n \sim \log n$. From (1), (2), and the inequality $|D_n(x)| \leqslant 2/x$, $0 < x \leqslant \pi$, we see that

$$\int_{1/\lambda_\nu}^{\pi} |f|\, dx \geqslant C \alpha_\nu \log \lambda_\nu - C_1 \sum_{n=1}^{\nu-1} \alpha_n \log \lambda_n - 2 \log(\pi \lambda_\nu) \sum_{n=\nu+1}^{\infty} \alpha_n. \tag{3}$$

Putting $\alpha_n = 1/n!$, $\lambda_n = 2^{(n!)^2}$, and arguing as in § 4.23, we obtain that the left-hand side of (3) is unbounded as $\nu \to \infty$.

5.13[2]. Next we shall consider the partial sums $\bar{s}_n(x)$ of the series 5.11(1b) with coefficients monotonically tending to 0. Let $\bar{D}_n(x) = \sin x + \dots + \sin nx = [\cos \frac{1}{2} x - \cos(n + \frac{1}{2}) x]/2 \sin \frac{1}{2} x$, $\tilde{D}_n(x) = [1 - \cos(n + \frac{1}{2}) x]/2 \sin \frac{1}{2} x \geqslant 0$, $0 \leqslant x \leqslant \pi$. We have

$$\bar{s}_n(x) = \sum_{m=1}^{n} \Delta a_m \bar{D}_m(x) + a_{n+1} \bar{D}_n(x) \to \sum_{m=1}^{\infty} \Delta a_m \bar{D}_m(x) = \bar{f}(x). \tag{1}$$

Substituting $\tilde{D}_m$ for $\bar{D}_m$ in the last series we obtain a function $\tilde{f}(x)$ differing from $\bar{f}(x)$ by $\frac{1}{2} a_1 \operatorname{tg} {}^1/_4 x$. The series defining $\tilde{f}(x)$ has non-negative terms and, since the integrals of $\tilde{D}_n$ over $(0, \pi)$ are exactly of order $\log n$ (§ 2.631), we conclude that $\tilde{f}(x)$, and

[1]) Szidon [1].

[2]) Young [9], Szidon [1], Hille and Tamarkin [1₂].

therefore $\bar{f}(x)$, is integrable if and only if the series with terms $\Delta a_n \cdot \log n$ converges.

As in § 5.12, we see that $\mathfrak{M}[\bar{f} - \bar{s}_n] \to 0$, provided that $\Delta a_2 \log 2 + \Delta a_3 \log 3 + \ldots < \infty$. (Observe that $a_n \log n \leqslant \Delta a_n \log n + \Delta a_{n+1} \log (n+1) + \ldots = o(1)$).

Since $a_n \to 0$, a simple calculation shows that

$$2\bar{f}(x) \sin x = a_1 + a_2 \cos x + \sum_{m=2}^{\infty} (a_{m+1} - a_{m-1}) \cos mx.$$

The series on the right, which is uniformly convergent, is $\mathfrak{S}[2\bar{f} \sin x]$. Writing the Fourier formulae for the coefficients $a_1, a_2, a_3 - a_1, \ldots$ of the last series, we obtain, by addition of some of these formulae, that

$$(2) \qquad a_n = \frac{2}{\pi} \int_0^{\pi} \bar{f}(x) \sin nx \, dx, \quad n = 1, 2, \ldots$$

Collecting the results we may enounce the following theorem.

If $a_n \geqslant a_{n+1} \to 0$, the sum $\bar{f}(x)$ of the series 5.11(1b) is bounded below in the interval $(0, \pi)$, and we have the formula (2), where $\bar{f} \sin nx$ is continuous[1]. *A necessary and sufficient condition for the integrability of $\bar{f}$ is the convergence of the series $\Delta a_2 \log 2 + \Delta a_3 \log 3 + \ldots$ If this condition is satisfied then $\mathfrak{M}[\bar{f} - \bar{s}_n] \to 0$.*

If $a_n \geqslant a_{n+1} \to 0$, the convergence of the series $\Delta a_2 \log 2 + \ldots$ implies that of $a_1 + {}^1/_2\, a_2 + {}^1/_3\, a_3 + \ldots$ and vice versa. The first part of this proposition follows from Abel's transformation, if we observe that $\log n - \log (n-1) \simeq 1/n$. For the second part we must use the fact that, if $a_1 + \frac{1}{2} a_2 + \ldots < \infty$, then $a_1 + a_2/2 + \ldots + a_n/n \geqslant a_n(1 + \ldots + 1/n)$ and so $a_n = O(1/\log n)$.

5.2. Approximate expressions for certain series[2].

It is important in some cases to know the behaviour of the series 5.11(1) in the neighbourhood of the point $x = 0$, and we intend to give approximate expressions for their sums, which we shall denote by $f(x)$, $\bar{f}(x)$ respectively.

5.21. We suppose that the coefficients a_n in 5.11(1b) form a sequence decreasing monotonically to 0 and convex. Put $x_p = \pi/2p$.

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[1]) The continuity of $\bar{f} \sin nx$ follows from that of $\bar{f} \sin x$.

[2]) Salem [1]. Less precise results had been obtained previously by Young [3].

A simple computation shows that $\bar{f}(x_p) = b_1 \sin x_p + b_2 \sin 2x_p + \ldots + b_p \sin p\, x_p$, where $b_j = b_j^{(p)}$, $j=1, 2, \ldots, p-1$, may be written in either of the forms

$$b_j = a_j + (a_{2p-j} - a_{2p+j}) - (a_{4p-j} - a_{4p+j}) + \ldots,$$

$$b_j = (a_j + a_{2p-j}) - (a_{2p+j} + a_{4p-j}) + (a_{4p+j} + a_{6p-j}) - \ldots,$$

and $b_p = b_p^{(p)} = a_p - a_{3p} + a_{5p} \ldots$

Since a_n and Δa_n decrease, the expressions in brackets also decrease, and we find that $a_j \leqslant b_j \leqslant a_j + a_{2p-j}$, i. e. $a_j \leqslant b_j \leqslant 2a_j$, $j = 1, 2, \ldots p-1$. Observing that $u \geqslant \sin u \geqslant 2u/\pi$ for $0 \leqslant u \leqslant \pi/2$, we find that the ratio of $\bar{f}(x_p) - b_p$ to $[a_1 + 2a_2 + \ldots + (p-1)\, a_{p-1}]/p$ is contained betwen 1 and π.

To find a simpler expression for $\bar{f}(x_p)$ we shall make an additional asumption about $\{a_n\}$, viz. that *na_n is non-decreasing*. To elucidate this hypothesis we observe that in all the series 5.11(1) that occur in practice and have coefficients steadily decreasing to 0, na_n is monotonic, at least for n sufficiently large. Moreover, if na_n is non-increasing, the function $\bar{f}(x)$ is continuous, or has a simple discontinuity, at the point $x = 0$ (§ 5.11).

If a_n is non-increasing and na_n non-decreasing, then $[a_1 + 2a_2 + \ldots + (p-1)\, a_{p-1}]/p$ is contained between $\frac{1}{2}(p-1)\, a_{p-1}$ and pa_p or, à fortiori, between $\frac{1}{2} pa_p - \frac{1}{2} a_p$ and pa_p. Since pa_p is bounded below by a positive number, and $0 < b_p < a_p$ we find, finally, that $\bar{f}(x_p) \sim pa_p$. To find a formula for an arbitrary $x \to 0$ we require the following lemma.

If x_p' is an arbitrary point in the interval $\pi/2p \leqslant x \leqslant \pi/2(p-1)$, then $f(x_p') - f(x_p) = o(pa_p)$ as $p \to \infty$.

In the formula 5.13(1) we break up the sum defining $\bar{f}(x)$, into two parts $P(x)$ and $Q(x)$, P consisting of terms with indices $\leqslant rp$, where r is a fixed but large integer. Since $|\bar{D}_k'(x)| \leqslant 1 + 2 + \ldots + k \leqslant k^2$, we find, by the mean-value theorem, that

$$|P(x_p') - P(x_p)| \leqslant (x_p' - x_p)\, [\Delta a_1 \cdot 1^2 + \ldots + \Delta a_{pr} \cdot (pr)^2] \to 0,$$

since $(x_p - x_p') \leqslant \pi/2p\,(p-1)$, $k \Delta a_k \to 0$, and so $k^2 \Delta a_k = o(k)$.

Remembering that $\bar{D}(x) = [\frac{1}{2} \operatorname{ctg} \frac{1}{2} x - \cos (n + \frac{1}{2})\, x]/2 \sin \frac{1}{2} x$, we put accordingly $Q = Q_1 + Q_2$, where $Q_1 = \frac{1}{2} \operatorname{ctg} \frac{1}{2} x \cdot (\Delta a_{pr+1} + \ldots) = a_{pr+1} \cdot \frac{1}{2} \operatorname{ctg} \frac{1}{2} x$. It is easy to see that $Q_1(x_p) - Q_1(x_p') = o(1)$ as $p \to \infty$. Since Δa_n is non-increasing, we find that $|Q_2(x_p)|$ and

$|Q_2(x_p')|$ do not exceed $Cp^2 \Delta a_{pr}$, where C is an absolute constant (§ 1.22). Now the inequality $na_n \leqslant (n+1)a_{n+1}$ involves $n \Delta a_n \leqslant a_n$ and therefore

$$Cp^2 \Delta a_{pr} = C(p/r)\, pr\, \Delta a_{pr} \leqslant C(p/r)\, a_{pr} \leqslant (C/r)\, pa_p \leqslant \varepsilon pa_p,$$

if r is sufficiently large. Collecting our inequalities together, we obtain ultimately $|f(x_p')-f(x_p)| \leqslant |P(x_p)-P(x_p')| + |Q_1(x_p')-Q(x_p)| + |Q_2(x_p)| + |Q_2(x_p')| \leqslant o(1)+o(1)+2\varepsilon pa_p < 3\varepsilon pa_p$ for p large. Since ε is arbitrary, the lemma follows.

From what we have proved it follows that $\bar{f}(x) \sim pa_p$, *where the integer p is defined by the condition $\pi/2p \leqslant x < \pi/2(p-1)$.* It is however preferable to state this result in a slightly different form. We may always suppose that $a_n = a(n)$, where $a(x)$ is a convex and decreasing function of x. Indeed in most cases a_n is just given as $a(n)$, but even if it be not so we can, for example, define $a(x)$ by the condition of continuity and that of being linear in every interval $(n, n+1)$.

5.211. *Let $a(x)$, $x \geqslant 0$, be a function decreasing to 0, convex, and such that $na(n)$ is non-decreasing. If $a(n) = a_n$, the sum of the series* 5.11(1b) *satisfies the relation $\bar{f}(x) \sim x^{-1} a(x^{-1})$ as $x \to 0$.*

In fact, if $p = p_x = [\pi/2x]+1$, then $\pi/2p \leqslant x \leqslant \pi/2(p-1)$ and, by the previous result, $\bar{f}(x) \sim pa(p) \sim x^{-1}a(p)$. It remains only to show that $a(p) \sim a(x^{-1})$. For small x we have $x^{-1} \leqslant p \leqslant 2x^{-1}$. From the first inequality we see that $a(x^{-1}) \geqslant a(p)$. From the second, assuming p even, we deduce that $a(x^{-1}) \leqslant a(\frac{1}{2}p) = (2/p)(p/2)a(\frac{1}{2}p) \leqslant (2/p)\, pa(p) = 2a(p)$. Using the inequality $p+1 \leqslant 2x^{-1}$, which is true for small x, we find that $a(x^{-1}) \leqslant 2a(p)$ for p odd, and so in any case $a(p) \leqslant a(x^{-1}) \leqslant 2a(p)$. This completes the proof.

5.22. Supposing the sequence $a_0, a_1, \ldots$ convex and decreasing to 0, we find for the series 5.11(1a) the estimates

$$(1) \qquad f(x_p) \leqslant \tfrac{1}{2}a_0 + \sum_{k=1}^{p-1} (a_k - a_{2p-k}) \cos kx_p,$$

$$(2) \qquad f(x_p) \geqslant \tfrac{1}{2}a_0 + \sum_{k=1}^{p-1} [(a_k - a_{2p-k}) - (a_{2p+k} - a_{4p-k})] \cos kx_p.$$

Replacing in (1) a_0 by $(a_0-a_1)+\ldots+(a_{2p-1}-a_{2p})+a_{2p}$, and $a_k - a_{2p-k}$ by $(a_k - a_{k+1}) + \ldots + (a_{2p-k-1} - a_{2p-k})$, we find that

$$(3) \qquad f(x_p) \leqslant 2\,[\tfrac{1}{2}\Delta a_0 + \sum_{k=1}^{p-1} \Delta a_k D_k(x_p)] + \tfrac{1}{2}a_{2p},$$

where D_k denotes Dirichlet's kernel. To obtain a lower bound for $f(x)$, we shall make an additional hypothesis concerning $\{a_k\}$, viz. that $k(a_k - a_{k+1})$ is a non-increasing function of k (from the convexity of $\{a_k\}$ we only have $k\Delta a_k \to 0$). From this assumption we deduce that $(a_{2p+k} - a_{4p-k}) \leqslant \frac{1}{2}(a_k - a_{2p-k})$, $k = 1, 2, \ldots, p-1$, and therefore, using (2), that

$$(4) \qquad f(x_p) \geqslant \tfrac{1}{2}[\tfrac{1}{2}\Delta a_0 + \sum_{k=1}^{p-1} \Delta a_k D_k(x_p)].$$

It is natural to suppose that $a_1 + a_2 + \ldots = \infty$. Thence it follows that $\Delta a_1 + 2\Delta a_2 + \ldots + (p-1)\Delta a_{p-1} = (a_1 - a_p) + (a_2 - a_p) + \ldots + (a_{p-1} - a_p) \to \infty$, and from (3), (4) we conclude that $f(x_p) \sim \Delta a_1 + 2\Delta a_2 + \ldots + (p-1)\Delta a_{p-1} \sim \Delta a_1 + 2\Delta a_2 + \ldots + p\Delta a_p$.

Now let x_p' be any point in the interval $(\pi/2p, \pi/2(p-1))$. We find, as previously, that $|f(x_p) - f(x_p')| \leqslant o(1) + o(p^2 \Delta a_p)$. This, together with the inequality $p^2 \Delta a_p \leqslant \Delta a_1 + 2\Delta a_2 + \ldots + p\Delta a_p$, yields the final result: $f(x) \sim \Delta a_1 + 2\Delta a_2 + \ldots + p\Delta a_p$, *where p satisfies the condition $\pi/2p \leqslant x < \pi/2(p-1)$.*

5.221. *If $a(x)$, $x \geqslant 0$, is a positive and convex function, tending to 0, then for the sum $f(x)$ of the series* 5.11(1a), *with $a_n = a(n)$, $n(a_n - a_{n+1})$ non-increasing, and $a_0 + a_1 + \ldots = \infty$, we have the formulae*

$$(1) \qquad f(x) \sim \int_1^{1/x} t\,[a(t) - a(t+1)]\,dt \sim \int_0^{1/x} t\,|a'(t)|\,dt.$$

To prove the first formula let us put $g_k = \Delta a_1 + 2\Delta a_2 + \ldots + k\Delta a_k$, and let $F(x)$ be the first integral in (1). We have to prove that $F(x) \sim g_p$, where $p > 1/x$ has the same meaning as in § 5.211. Let q be the largest integer $\leqslant 1/x$. Since $a(t)$ is convex, $a(t) - a(t+1)$ is non-increasing, and it is easy to see that $F(x) \geqslant g_q - a_1$. Similarly we find that $F(x) \leqslant F(1/p) \leqslant g_p + a_1$. From the inequalities $g_q \leqslant g_p = g_q + (g_p - g_q) \leqslant g_q + (p-q)\, q\Delta a_q = g_q + O(q^2 \Delta a_q) = g_q + O(g_q) = O(g_q)$, we see that $g_p \sim g_q$, and so $F(x) \sim g_p$.

Let $H(x)$ be the second integral in (1). To prove the second formula in (1) it is sufficient to show that $F(x) \sim H(x)$. This, and even a stronger result, viz. $F(x) \simeq H(x)$, follows from the inequalities $|a'(t)| \geqslant |a(t) - a(t-1)| \geqslant |a'(t+1)|$. The details of the proof may be left to the reader.

In the above proof we assumed tacitly that $a'(t)$ exists. The existence of $a'(t)$ follows, except for a set of t which is at most enumerable and has no influence upon the integral, from the mere convexity of $a(t)$ (§ 4.141). Let us assume now that $a''(x)$ exists. The inequality $n\Delta a_n \geqslant (n+1)\Delta a_{n+1}$ will certainly be satisfied if only (*) $a'(x) + (x-1)a''(x) \geqslant 0$. This test may be proved as follows. Let $\alpha(x) = x[a(x) - a(x+1)]$; then $\alpha'(x) = a(x) - a(x+1) + x[a'(x) - a'(x+1)]$. By the mean value theorem we shall have $\alpha'(x) \leqslant 0$ provided that $a'(x+\theta)/a''(x+\theta) + x \geqslant 0$, where θ is a number contained between 0 and 1, and the latter inequality is a consequence of (*). Of course it is sufficient for (*) to be satisfied for x large.

Examples. If $a_n = n^{-\alpha}$, $0 < \alpha < 1$, $n \geqslant 1$, then $f(x)$ and $\bar{f}(x)$ are of order $x^{\alpha-1}$ as $x \to +0$. If $a_n = 1/\log n$, $n \geqslant 2$, then $f(x) \sim 1/x(\log x)^2$, $\bar{f}(x) \sim 1/x\,|\log x|$, as $x \to 0$. In particular the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}, \tag{2}$$

which converges everywhere, is not a Fourier series. This follows also from the fact that the series (2) integrated term by term diverges at the point 0 (§ 2.621).

5.3. A power series. We shall now consider the power series

$$\sum_{n=1}^{\infty} e^{icn\log n} \frac{z^n}{n^{1/2+\alpha}}, \tag{1}$$

where α and $c \neq 0$ are real constants, $z = e^{ix}$, $0 \leqslant x \leqslant 2\pi$. The series (1), which was first studied by Hardy and Littlewood, possesses many interesting properties.

If $0 < \alpha < 1$, *the series* (1) *converges uniformly in the interval* $0 \leqslant x \leqslant 2\pi$ *to a function* $\varphi_\alpha(x) \in \mathrm{Lip}\,\alpha$ [1]).

The theorem is a corollary of certain lemmas, which are interesting in themselves and have wider applications.

5.31. van der Corput's lemmas. Given a real function $f(u)$ and numbers $a < b$, we put

$$F(u) = e^{2\pi i f(u)}, \quad I(F; a, b) = \int_a^b F(u)\,du, \quad S(F; a, b) = \sum_{a<n\leqslant b} F(n).$$

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[1]) Hardy and Littlewood [9]. Following Hille [1], we base our proof on van der Corput's lemmas. See van der Corput [1].

(i) *If* $f(u)$, $a \leqslant x \leqslant b$, *has an increasing derivative* $f'(u)$, *and if* $f''(u) \geqslant \rho > 0$, *then* $|I(F; a, b)| < 4\rho^{-1/2}$.

Suppose that there exists a $\lambda > 0$ such that $f'(u) \geqslant \lambda$, or $f'(u) \leqslant -\lambda$, throughout (a, b). Since $2\pi i F(u)\,du = dF(u)/f'(u)$, an application of the second mean-value theorem to the real and imaginary parts of $I(F; a, b)$ shows that $|I| \leqslant 2/\pi\lambda < 1/\lambda$.

Assuming that the conditions of the lemma are satisfied, suppose for the moment that $f'(u)$ is of constant sign, say $f' \geqslant 0$, in (a, b). If $a < c < b$, then $f'(u) \geqslant (c-a)\rho$ in the interval $c \leqslant u \leqslant b$. Therefore $|I(F; a, b)| \leqslant |I(F; a, c)| + |I(F; c, b)| < (c-a) + 1/(c-a)\rho$. Choosing c so as to make the last expression a minimum, we find that $|I(F; a, b)| < 2\rho^{-1/2}$. In the general case (a, b) is a sum of two intervals in each of which $f'(u)$ is of constant sign, and the result follows by the addition of the inequalities for these intervals.

(ii) *Let* $D(F; a, b) = I(F; a, b) - S(F; a, b)$. *If* $f'(u)$ *is monotonic and* $|f'(u)| \leqslant \frac{1}{2}$, *then* $|D(F; a, b)| \leqslant A$, *where* A *is an absolute constant.*

Suppose first that a and b are not integers. S may be written as the Stieltjes integral of $F(u)\,d\xi(u)$ over (a, b), where $\xi(u)$ is any function which is constant in the intervals $n < u < n+1$ and has jumps equal to 1 at the points n. It will be convenient to put $\xi(u) = [u] + \frac{1}{2}$ for $u \neq 0, \pm 1, \ldots$, $2\xi(u) = \xi(u+0) + \xi(u-0)$. It follows that

$$D(F; a, b) = \int_a^b F(u)\,d\chi(u), \quad \text{where } \chi(u) = u - [u] - \tfrac{1}{2}.$$

The function $\chi(u)$ is of period 1. Integrating by parts, we find that $D(F; a, b) = -I(F'\chi; a, b) + R$, where $|R| \leqslant 1$. The terms of $\mathfrak{S}[\chi]$ are $-\sin 2\pi nu/\pi n$ and the partial sums are uniformly bounded. Multiplying $\mathfrak{S}[\chi]$ by F' and integrating over (a, b), we see that $D(F; a, b) - R$ is equal to the sum of the expressions

$$\frac{1}{2\pi i n}\left[\int_a^b \frac{f'(u)}{f'(u)+n}\, de^{2\pi i[f(u)+nu]} - \int_a^b \frac{f'(u)}{f'(u)-n}\, de^{2\pi i[f(u)-nu]}\right] \tag{1}$$

for $n = 1, 2, \ldots$. The factors $f'/(f' \pm n)$ are monotonic and of constant sign. The second mean-value theorem shows that (1) does not exceed $2/\pi n\,(n-\frac{1}{2})$ in absolute value, and so the series of expressions (1) converges absolutely. This completes the proof in the case when a and b are not integers. If a or b, or both, are integers, it is sufficient to observe that $D(F; a, b)$ differs from $\lim_{\varepsilon\to+0} D(F; a+\varepsilon, b-\varepsilon)$ by at most 1.

(iii) *Under the conditions of* (i) *we have*

$$|S(F; a, b)| \leqslant [f'(b) - f'(a) + 2]\,(4\rho^{-1/2} + A).$$

Put $\beta_p = p - \frac{1}{2}$, $p = 0, \pm 1, \dots$, and let $\beta_p = f'(\alpha_p)$, $F_p(u) = \exp 2\pi i\,[f(u) - pu]$. It is obvious that $|f'(u) - p| \leqslant \frac{1}{2}$ in the interval (α_p, α_{p+1}). Let $\alpha_r, \alpha_{r+1}, \dots, \alpha_{r+s}$ be all the points α, if such exist, belonging to the interval $a \leqslant u \leqslant b$. Using (i) and (ii), we see that the expressions $S(F; \alpha_p, \alpha_{p+1}) = S(F_p; \alpha_p, \alpha_{p+1}) = I(F_p; \alpha_p, \alpha_{p+1}) - D(F_p; \alpha_p, \alpha_{p+1})$ do not exceed $4\rho^{-1/2} + A$ in absolute value. The same may be said of $S(F; a, \alpha_r)$ and $S(F; \alpha_{r+s}, b)$. Since $S(F; a, b)$ is a sum of analogous expressions formed for the intervals (a, α_r), $(\alpha_r, \alpha_{r+1}), \dots, (\alpha_{r+s}, b)$, the number of which is $s + 2 = f'(\alpha_{r+s}) - f'(\alpha_r) + 2 \leqslant f'(b) - f'(a) + 2$, the result follows.

5.32. *The partial sums* $s_N(x)$ *of the series* 5.3(1) *with* $\alpha = -\frac{1}{2}$, *are* $O(N^{1/2})$ *uniformly in* x.

The function $f(u) = (2\pi)^{-1}(cu \log u + ux)$ has an increasing derivative. If $\nu \geqslant 0$ is an integer, $a = 2^\nu$, $b = 2^{\nu+1}$, we conclude from § 5.31(iii) that $|S(F; a, b)| \leqslant C 2^{\nu/2}$ with C depending only on c. The same is true if $2^\nu = a < b < 2^{\nu+1}$. Let $2^n < N \leqslant 2^{n+1}$. Then $|s_N(x)| \leqslant 1 + |S(F; 1, 2)| + |S(F; 2, 4)| + \dots + |S(F; 2^n, N)| \leqslant 1 + C\{2^{1/2} + \dots + 2^{n/2}\} \leqslant C_1 2^{n/2} \leqslant C_1 N^{1/2}$, with C_1 depending only on c.

We can now easily prove Theorem 5.3. Using Abel's transformation we obtain for the N-th partial sum of the series 5.3(1) the expression

$$(1)\qquad \sum_{\nu=1}^{N-1} s_\nu(x)\,\Delta\nu^{-1/2-\alpha} + s_N(x)\,N^{-1/2-\alpha}.$$

Since $\Delta\nu^{-1/2-\alpha} = O(\nu^{-3/2-\alpha})$, we conclude from (1) and from the relation $s_\nu(x) = O(\nu^{1/2})$ that the partial sums of 5.3(1) are (a) uniformly convergent if $\alpha > 0$, (b) uniformly $O(\log N)$ if $\alpha = 0$, (c) uniformly $O(N^{-\alpha})$ if $\alpha < 0$. Take $0 < \alpha < 1$. Making $N \to \infty$ in (1) we obtain

$$\varphi_\alpha(x+h) - \varphi_\alpha(x) = \sum_{\nu=1}^{\infty}\{s_\nu(x+h) - s_\nu(x)\}\,\Delta\nu^{-1/2-\alpha} = \sum_{\nu=1}^{N} + \sum_{\nu=N+1}^{\infty} = P + Q,$$

where $h > 0$, $N = [1/h]$. The terms in Q are $O(\nu^{1/2})\,\Delta\nu^{-1/2-\alpha} = O(\nu^{-1-\alpha})$, so that $Q = O(N^{-\alpha}) = O(h^\alpha)$. On the other hand, since $s'_\nu(x)$, apart from a constant factor, is the partial sum of the series 5.3(1) with $\alpha = -{}^3/_2$, we have (see case (c) above) that $s'_\nu(x) = O(\nu^{3/2})$. Applying the mean-value theorem to $s_\nu(x+h) - s_\nu(x)$, we find that the terms of P are $O(h\nu^{3/2})\,\Delta\nu^{-1/2-\alpha} = O(h\nu^{-\alpha})$, and so $P = O(hN^{1-\alpha}) = O(h^\alpha)$. Therefore $|\varphi_\alpha(x+h) - \varphi_\alpha(x)| \leqslant |P| + |Q| = O(h^\alpha)$ and the theorem follows.

5.33. Theorem 5.3 ceases to be true when $\alpha = 0$ (and so when $\alpha = 1$). In this case much more can be said: if $\alpha = 0$, the series 5.3(1) is nowhere summable A, and, à fortiori, is not a Fourier series [1]). However, if $\beta > 1$, $c \lesseqgtr 0$, the series

$$(1)\qquad \sum_{n=2}^{\infty} \frac{e^{icn\log n}}{n^{1/2}(\log n)^\beta} z^n, \quad z = e^{ix},$$

converges uniformly for $0 \leqslant x \leqslant 2\pi$. For the proof we replace $\Delta\nu^{-1/2-\alpha}$ by $\Delta\nu^{-1/2}\log^{-\beta}\nu = O(\nu^{-3/2}\log^{-\beta}\nu)$ [2]) in 5.32(1), $N^{-1/2-\alpha}$ by $N^{-1/2}\log^{-\beta}N$, and observe that the series with terms $O(\nu^{-1}\log^{-\beta}\nu)$ converges.

5.34. *There exists a continuous function* $f(x)$ *such that, if* a_n, b_n *are the Fourier coefficients of* f, *the series* $\Sigma(|a_n|^{2-\varepsilon} + |b_n|^{2-\varepsilon})$ *diverges for every* $\varepsilon > 0$ [3]). For, if $f(x)$ is the real, or imaginary, part of the function 5.33(1), where $\beta > 1$, and $\rho_n^2 = a_n^2 + b_n^2$, $\rho_n \geqslant 0$, then $\Sigma\rho_n^{2-\varepsilon} = \infty$ for $\varepsilon > 0$, and this is equivalent to our theorem.

5.4. Lacunary series. We now pass to the lacunary trigonometrical series, that is to series where the terms different from 0 are 'very sparse'. Such series may be written in the form

$$(1)\qquad \sum_{k=1}^{\infty}(a_k \cos n_k x + b_k \sin n_k x),$$

assuming, for simplicity, that the constant term also vanishes. When speaking on lacunary series, we shall suppose throughout

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[1]) Hardy and Littlewood [9].

[2]) This inequality follows from the mean-value theorem applied to the difference $\alpha(n) - \alpha(n+1)$, where $\alpha(x) = x^{-1/2}\log^{-\beta}x$.

[3]) The first example of a continuous function having this property was given by Carleman [1].

that the indices n_k satisfy an inequality $n_{k+1}/n_k > \lambda > 1$, i. e. increase at least as rapidly as a geometrical progression with ratio greater than 1.

Given a lacunary series (1) consider the sum

$$\sum_{k=1}^{\infty}(a_k^2+b_k^2). \tag{2}$$

In Chapter X we shall learn that, if (2) is finite, the series (1) converges almost everywhere. Here we shall prove the converse. *If the series* (1) *converges in a set of positive measure, the series* (2) *converges.* We shall prove even a more general theorem. Let T^* be any linear method of summation satisfying the two first conditions of Toeplitz (§ 3.1); the third condition need not be satisfied. All methods of summation used in Analysis are T^*-methods.

If a series of the form (1) *is summable* T^* *in a set* E *of positive measure, the series* (2) *converges* [1]).

It must be observed that, when we speak of the summability of the series (1), we understand that the vacant terms are replaced by zeros. Consequently, the q-th partial sum $s_q(x)$ of (1) consists of the terms $a_k \cos n_k x + b_k \sin n_k x$ with $n_k \leqslant q$. If β_{pq} denotes an element of the matrix T^* considered, the hypothesis of the last theorem may be stated as follows: for every $x \in E$ the series

$$\sum_{q=0}^{\infty}\beta_{pq}\, s_q(x) = \sigma_p(x), \qquad p=0,1,2,\dots, \tag{3}$$

converge, and lim $\sigma_p(x)$ exists and is finite.

To avoid unnecessary complications we begin by the case when each line of the matrix $\{\beta_{pq}\}$ possesses only a finite number of terms different from 0. It will be convenient to consider the series (1) in the complex form, putting $2c_k = a_k - ib_k$, $c_{-k} = \bar{c}_k$, $c_0 = 0$, $n_{-k} = -n_k$, $k = 1, 2, \dots$ Let, moreover, $\beta_{p,q} + \beta_{p,q+1} + \dots = R_p(q)$. It is easy to see that

$$\sigma_p(x) = \sum_{k=-\infty}^{+\infty} c_k e^{in_k x} R_p(|n_k|),$$

the sum on the right being in reality finite. Since $\{\sigma_p(x)\}$ converges in E, we can find a subset $\mathcal{E}$ of E, $|\mathcal{E}| > 0$, and a number

[1]) Zygmund [5]; see also Kolmogoroff [2].

M, such that $|\sigma_p(x)| \leqslant M$ for $p = 0, 1, \dots, x \in \mathcal{E}$. In fact, we have $E = E_1 + E_2 + \dots$, where E_n is the set of x such that $|\sigma_p(x)| \leqslant n$ for $p = 0, 1, 2, \dots$ At least one of the sets E_i, say E_M, is of positive measure and may be taken for $\mathcal{E}$. It follows that

$$\begin{aligned} M^2|\mathcal{E}| \geqslant \int_{\mathcal{E}} \sigma_p^2(x)\,dx = |\mathcal{E}|\sum_{k=-\infty}^{+\infty}|c_k|^2 R_p^2(|n_k|) + \\ + \sum_{\substack{j,k=-\infty \\ j\neq k}}^{+\infty} c_j \bar{c}_k R_p(|n_j|) R_p(|n_k|) \int_{\mathcal{E}} e^{i(n_j-n_k)x}\,dx. \end{aligned} \tag{4}$$

Let us denote the last integral by $2\pi b_{j,k}$. The numbers $b_{j,k}$ are the complex Fourier coefficients of a function $\chi(x)$ which is equal to 1 in $\mathcal{E}$ and to 0 elsewhere. Applying Schwarz's inequality to the second sum on the right, we see that it does not exceed

$$\begin{aligned} 2\pi\Big\{\sum_{j,k=-\infty}^{+\infty}|c_j|^2|c_k|^2 R_p^2(|n_j|)R_p^2(|n_k|)\Big\}^{1/2}\Big\{\sum_{\substack{j,k=-\infty\\ j\neq k}}^{+\infty}|b_{j,k}|^2\Big\}^{1/2} = \\ = 2\pi\sum_{k=-\infty}^{+\infty}|c_k|^2 R_p^2(|n_k|)\cdot\Big\{\sum_{\substack{j,k=-\infty\\ j\neq k}}^{+\infty}|b_{j,k}|^2\Big\}^{1/2} \end{aligned} \tag{5}$$

in absolute value.

From the condition $n_{k+1}/n_k > \lambda > 1$ it follows that a number $\Delta = \Delta(\lambda)$ exists such that every integer m can be represented no more than Δ times in the form $n_j \pm n_k$, $j > 0$, $k > 0$. In fact, assume that $m = n_j + n_k$, $j \geqslant k$. Then $m > n_j \geqslant m/2$, and the number of n_j satisfying this inequality is less than the smallest integer y such that $\lambda^y > 2$. Similarly, if $m = n_j - n_k > 0$, then $n_j > m$. As $n_j/n_k > \lambda$, we have $n_j - n_j/\lambda < m$, i. e. $n_j < m\lambda/(\lambda-1)$, and the number of n_j in the interval $(m, m\lambda/(\lambda-1))$ is also bounded. We add that the property of $\{n_j\}$ just established is the only thing which we use in the proof, and that it may sometimes hold even if $n_{j+1}/n_j \to 1$ as $j \to \infty$.

Now it is not difficult to see that the last factor on the right in (5) does not exceed $\{\Delta(\dots+|\gamma_{-1}|^2+|\gamma_0|^2+|\gamma_1|^2+\dots)\}^{1/2} < \infty$, where γ_m denote the complex Fourier coefficients of χ. Thence, for ν sufficiently large, we have

$$2\pi\Big(\sum_{\substack{|j|,|k|>\nu\\ j\neq k}}|b_{j,k}|^2\Big)^{1/2} < \tfrac{1}{2}|\mathcal{E}|. \tag{6}$$

In the series (1) we may omit the terms $a_k \cos n_k x + b_k \sin n_k x$ for $1 \leqslant k \leqslant \nu$, replacing them by zeros. It does no damage to the summability T^* of the series considered and can only change the value of M. Assuming the inequality (6), we deduce from (4) and (5) that

$$M^2 |\mathcal{E}| \geqslant \tfrac{1}{2} |\mathcal{E}| \sum_{k=-\infty}^{+\infty} |c_k|^2 R_p^2(|n_k|).$$

Let $K > 0$ be any fixed integer. Since $\lim_p R_p(|n_k|) = 1$, $k = 1, 2, \ldots$, we conclude that

$$\sum_{k=-K}^{K} |c_k|^2 R_p^2(|n_k|) \leqslant 2M^2, \qquad \sum_{k=-K}^{K} |c_k|^2 \leqslant 2M^2,$$

and, since the last inequality holds for any K, the convergence of (2) follows.

To remove the condition imposed upon $\{\beta_{pq}\}$ we proceed as follows. Let $\sigma_p^*(x)$ be an expression analogous to $\sigma_p(x)$ (see (3)), except that the upper limit of summation in the sum defining σ_p^* is not ∞ but a number $Q = Q(p)$. We take Q very large, so as to satisfy the two following conditions (i) $|\sigma_p(x) - \sigma_p^*(x)| \leqslant 1/p$ for $x \in E - E^p$, where the set E^p is of measure $\leqslant 2^{-p-1}|E|$, $p = 1, 2, \ldots$ (ii) $\lim_p (\beta_{p0} + \beta_{p1} + \ldots + \beta_{pQ}) = 1$. Putting $E^* = E^1 + E^2 + \ldots$, so that $|E^*| \leqslant \frac{1}{2}|E|$, we see that in the set $E - E^*$ of positive measure the expressions $\sigma_p^*(x)$ tend to a finite limit. But condition (ii) ensures that the σ_p^* are also T^*-means, corresponding to a matrix with only a finite number of terms different from 0 in each row, and, in virtue of the special case already dealt with, the theorem is established completely.

This theorem shows that, if the series (2) is infinite, the series (1) is practically non-summable by any method of summation. Considering, in particular, the method $(C, 1)$, we obtain: *if the series* (2) *diverges*, (1) *is not a Fourier series.*

5.5. Rademacher's series. Several properties of lacunary trigonometrical series are shared by Rademacher's series

$$(1) \qquad \sum_{k=0}^{\infty} c_k \varphi_k(t), \quad 0 \leqslant t \leqslant 1,$$

(§ 1.32). This is not surprising since Rademacher's functions form a lacunary subsequence of a complete orthogonal system (§ 1.8.5).

(i) *The series* (1) *converges almost everywhere if* $c_0^2 + c_1^2 + \ldots < \infty$ [1]. (ii) *If* $c_0^2 + c_1^2 + \ldots = \infty$, *the series* (1) *is almost everywhere non-summable by any method* T^* [2].

The proof of (ii) follows exactly the same line as that of Theorem 5.4 and may be left to the reader. We need only observe that the system of functions $\varphi_{j,k}(t) = \varphi_j(t)\,\varphi_k(t)$, $0 \leqslant j < k$, $0 \leqslant k < \infty$, is orthogonal and normal in (0, 1).

Under the hypothesis of (i), the series (1), whose partial sums we denote by $s_n(t)$, is the Fourier series of a function $f(t) \in L^2$ (§ 4.21) and moreover we have

$$\int_0^1 (f - s_n)^2\, dt \to 0, \quad \int_0^1 |f - s_n|\, dt \to 0, \quad \int_a^b (s_n - f)\, dt \to 0,$$

where $0 \leqslant a < b \leqslant 1$. The third relation, which holds uniformly in a, b, is a consequence of the second, and the second follows from the first by an application of Schwarz's inequality.

Let us denote by $F(t)$ the indefinite integral of $f(t)$, and by E, $|E| = 1$, the set of points where $F'(t)$ exists and is finite. We have proved that, whatever the interval I, the integral of s_n over I tends to the corresponding integral of f. Therefore, the integral of $s_n - s_{k-1}$ over I tends, as $n \to \infty$, to the integral of $f - s_{k-1}$. Let I be of the form $(l 2^{-k}, (l+1) 2^{-k})$, $l = 0, 1, \ldots, 2^k - 1$. Since the integral of $\varphi_j(t)$, over I vanishes for $j \geqslant k$, the integral of $s_n(t) - s_{k-1}(t)$ over I is equal to 0, provided that $n \geqslant k$. Hence, if I is of the form $(l 2^{-k}, (l+1) 2^{-k})$, the integral of $f(t)$ over I is equal to the integral of $s_{k-1}(t)$ over I. Now let $t_0 \neq p/2^q$, $t_0 \in E$, and let $t_0 \in I_k = (l 2^{-k}, (l+1) 2^{-k})$. Since $s_{k-1}(t)$ is constant over I_k, we have

$$s_{k-1}(t_0) = \frac{1}{|I_k|} \int_{I_k} s_{k-1}(t)\, dt = \frac{1}{|I_k|} \int_{I_k} f(t)\, dt \to F'(t_0) \text{ as } k \to \infty.$$

5.51. (i) *If the series* 5.5(2) *is convergent, the sum* $f(t)$ *of the series* 5.5(1) *belongs to* L^q *for every* $q > 0$ [3]. It is sufficient to prove the theorem for $q = 2, 4, 6, \ldots$ We shall show that

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1) Rademacher [1], see also Paley and Zygmund [1], and Kolmogoroff [3], where a very simple proof is given.

2) Khintchine and Kolmogoroff [1] (for the case of convergence), Zygmund [5].

3) Khintchine [1], Paley and Zygmund [1].

$$(1) \qquad \int_0^1 f^{2k}(t)\,dt \leqslant M_k \Big(\sum_{n=0}^{\infty} c_n^2\Big)^k, \quad k = 1, 2, \dots,$$

where M_k is a constant depending only on k.

Denoting by $s_n(t)$ the partial sums of the series 5.5(1), we have

$$(2) \qquad \int_0^1 s_n^{2k}(t)\,dt = \sum A_{\alpha_1\,\alpha_2\,\dots\,\alpha_r}\, c_{m_1}^{\alpha_1} \dots c_{m_r}^{\alpha_r} \int_0^1 \varphi_{m_1}^{\alpha_1} \dots \varphi_{m_r}^{\alpha_r}\,dt,$$

where $A_{\alpha_1, \alpha_2, \dots, \alpha_r} = (\alpha_1 + \alpha_2 + \dots + \alpha_r)!/\alpha_1!\,\alpha_2! \dots \alpha_r!$, and the summation on the right is taken over the set of $m_1, m_2, \dots, m_r, \alpha_1, \alpha_2, \dots, \alpha_r$ defined by the relations.

$$0 \leqslant m_i \leqslant n,\ 0 \leqslant \alpha_i \leqslant 2k,\ i = 1, 2, \dots, r,\ 1 \leqslant r \leqslant 2k,\ \alpha_1 + \alpha_2 + \dots + \alpha_r = 2k.$$

Now is it easily verified that the integrals on the right vanish unless $\alpha_1, \alpha_2, \dots, \alpha_r$ are all even, in which case they are equal to 1. Thus the right-hand side of (2) may be written $\sum A_{2\beta_1, \dots 2\beta_r}\, c_{n_1}^{2\beta_1} \dots c_{n_r}^{2\beta_r}$. Observing that

$$\sum A_{\beta_1, \beta_2, \dots, \beta_r}\, c_{m_1}^{2\beta_1} c_{m_2}^{2\beta_2} \dots c_{m_r}^{2\beta_r} = (c_0^2 + c_1^2 + \dots + c_n^2)^k,$$

we obtain (2) with $f(t)$ replaced by $s_n(t)$, M_k being now the upper bound of the ratio $A_{2\beta_1, \dots, 2\beta_r}/A_{\beta_1, \dots, \beta_r}$. Since $s_n(t) \to f(t)$ for almost every t, an appeal to Fatou's lemma completes the proof.

It is easy to see that $M_k \leqslant (2k)!/2^k\,k! = (k+1) \dots 2k/2^k \leqslant k^k$. This enables us to strengthen the theorem which we have just proved and to show that

(ii) *The function* $\exp \mu f^2(t)$ *is integrable for every* $\mu > 0$.

Let $C = c_0^2 + c_1^2 + \dots$ Integrating the equation $\exp \mu f^2 = 1 + \mu f^2/1! + \mu^2 f^4/2! + \dots$ over the interval $0 \leqslant t \leqslant 1$, and using the inequalities (1) with $M_k = k^k$, $k = 0, 1, \dots$ we obtain that

$$(3) \qquad \int_0^1 \exp \mu f^2\,dt \leqslant \sum_{k=0}^{\infty} \frac{k^k}{k!} (\mu C)^k.$$

In virtue of Stirling's formula $k! \simeq 2\pi\, e^{-k} k^{k+1/2}$, the series on the right is certainly convergent if $e\mu C < 1$, that is if C is small enough. It follows that, for every $\mu > 0$, the function $\exp \mu (f - s_n)^2$ is integrable if only $n = n(\mu)$ is large enough. Since $f^2 \leqslant 2[(f - s_n)^2 + s_n^2]$, and $s_n(t)$ is bounded, the integrability of $\exp \mu f^2$ follows.

5.6. Applications of Rademacher's functions [1]. The theorems established in the preceding paragraph enable us to prove some results about the series

$$(1) \qquad \pm \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \pm (a_n \cos nx + b_n \sin nx),$$

which we obtain from the standard series

$$(2) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

by changing the signs of terms of the latter in a quite arbitrary manner. Let $\frac{1}{2} a_0 = A_0(x)$, $a_n \cos nx + b_n \sin nx = A_n(x)$, $n = 1, 2, \dots$ Neglecting the sequences $\{\pm 1\}$ containing only a finite number of $+1$ or of -1, we may present the series (1) in the form

$$(3) \qquad \sum_{n=0}^{\infty} A_n(x)\,\varphi_n(t),$$

where φ_n are Rademacher's functions and the parameter t, $t \neq p/2^q$, runs through the interval $(0, 1)$. If the values of t for which the series (3) possess a property P form a set of measure 1, we shall say that almost all the series (1) possess the property P.

(i) *If the series*

$$(4) \qquad {}^1\!/_4\, a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

converges, then almost all the series (1) *converge almost everywhere in the interval* $0 \leqslant x \leqslant 2\pi$. (ii) *If the series* (4) *diverges, then, whatever method* T^* *of summation we consider, almost all the series* (1) *are almost everywhere non-summable* T^*.

Let $S_t(x)$ denote the series (3), and, if the series converges, let $S_t(x)$ also denote the sum. Let E be the set of points (x, t) in the rectangle $0 \leqslant x \leqslant 2\pi$, $0 \leqslant t \leqslant 1$, where the series converges. Assuming that the series (4) converges, we obtain from Theorem 5.5 (i) that the intersection of E with every line $x = x_0$, $0 \leqslant x_0 \leqslant 2\pi$, is of measure 1. Since the set E is measurable, its plane measure is 2π, and therefore the intersection of E with almost every line $t = t_0$, $0 \leqslant t_0 \leqslant 1$, is of measure 2π; this is just the first part of the theorem. The second part is proved by the same argument provided we can show that the divergence of (4) implies the divergence of

[1] Paley and Zygmund [1].

$$(5) \qquad A_0^2(x) + A_1^2(x) + \dots + A_n^2(x) + \dots$$

for almost every x.

To establish the latter proposition suppose that the series (5) converges in a set of positive measure. Then there exists a set $H, |H| > 0$, and a constant M such that the sum of the series (5) does not exceed M for $x \in H$. Put $A_n(x) = \rho_n \cos(nx + x_n)$, $\rho_n \geqslant 0$. The series (5) may be integrated over H and we have

$$\sum_{n=1}^{\infty} \rho_n^2 \int_H \cos^2(nx + x_n)\, dx \leqslant M\,|H|.$$

The coefficients of ρ_n^2 in this inequality tend to $\frac{1}{2}|H|$ and so all of them exceed an $\varepsilon > 0$. Thence we conclude that the series $\rho_1^2 + \rho_2^2 + \dots$, i. e. the series (4), converges, contrary to our hypothesis.

The following proposition is an immediate corollary of (ii).

If the series (3) *diverges, almost all the series* (1) *are not Fourier series.*

The theorem of Riesz-Fischer asserts that, if (4) is finite, the series (2) is a Fourier series. Now we see that the Riesz-Fischer theorem is, in a way, the best possible: *no condition on the moduli of a sequence* $\{a_n, b_n\}$ *which permits* (4) *to diverge can possibly be a sufficient condition for* (2) *to be a Fourier series* [1]).

(iii) *If* (4) *is finite, then almost all the series* (1) *belong to* L^q *for every* $q > 0$. *More generally, for almost every* t *the function* $\exp \mu S_t^2(x)$ *is integrable over the interval* $0 \leqslant x \leqslant 2\pi$, *however large* μ *may be.*

Let C denote the sum of the series (4), and let μ be so small that the series in 5.51(3) converges. If $K = K(\mu, C)$ is the sum of the latter series, we have, as in 5.51(3),

$$\int_0^1 \exp \mu S_t^2(x)\, dt \leqslant K.$$

Integrating this inequality over the range $0 \leqslant x \leqslant 2\pi$ and interchanging the order of integration, we find that

$$(6) \qquad \int_0^{2\pi} dx \int_0^1 \exp \mu S_t^2(x)\, dt = \int_0^1 dt \int_0^{2\pi} \exp \mu S_t^2(x)\, dx \leqslant 2\pi K.$$

1) Littlewood [1], [2].

The interchanging of the order of integration is legitimate since the integrand is positive.

From (6) we conclude that the integral $\int_0^{2\pi} \exp \mu S_t^2(x)\, dx$ is finite for almost every t. This establishes the theorem for μ positive and sufficiently small. To establish the general result we argue as in the proof of Theorem 5.51(ii).

5.61. Let 5.6(4) be finite. In this case it is natural to ask whether the functions $S_t(x)$ are continuous functions of x for almost all t. But this is not so. In Chapter VI we shall prove that if a lacunary trigonometrical series is the Fourier series of a bounded function, the series of coefficients converges absolutely. Thus for no sequence of signs is the series

$$(1) \qquad \pm \sin 10\, x \pm \frac{1}{2} \sin 10^2 x + \dots \pm \frac{1}{n} \sin 10^n x \pm \dots$$

the Fourier series of a bounded function.

If the series

$$(2) \qquad \sum_{k=2}^{\infty} (a_k^2 + b_k^2) \log^{1+\varepsilon} k$$

converges for an $\varepsilon > 0$, *then almost all the series* 5.6(1) *are Fourier series of continuous functions.*

As the series (1) shows, the theorem is not true for $\varepsilon = 0$.

We require two lemmas.

(i) *Let* $\sigma_{n,t}(x)$ *denote the* $(C, 1)$ *means of the series* 5.6(3). *If the series* 5.6(4) *is finite, then, for almost every* t, *we have* $\sigma_{n,t}(x) = o(\sqrt{\log n})$, *uniformly in* x.

Let us put $\Phi(x) = \exp \mu x^2 - 1$, $\mu \geqslant 1$, $\varphi(x) = \Phi'(x) = 2\mu x \exp \mu x^2$. We will obtain an inequality for the function $\Psi(x)$ complementary to $\Phi(x)$ (§ 4.11). Let $x = \psi(y)$ be the function inverse to $y = \varphi(x)$. Since $\log \varphi(x) = \log 2\mu x + \mu x^2 \geqslant \mu x^2$ for $x \geqslant 1$, we see that $x = \psi(y) \leqslant \mu^{-1/2} \sqrt{\log y}$ whenever $x \geqslant 1$. Let y_0 be the root of the equation $\psi(y_0) = 1$. It follows that $\psi(y) \leqslant 1$ for $0 \leqslant y \leqslant y_0$, and $1 \leqslant \psi(y) \leqslant \mu^{-1/2} \sqrt{\log y}$ for $y \geqslant y_0$. Thence we deduce that $\Psi(y) \leqslant y$ for $y \leqslant y_0$, and $\Psi(y) \leqslant \mu^{-1/2} y \sqrt{\log y}$ for $y \geqslant y_0$, i. e. $\Psi(y) \leqslant y \chi(y)$, where $\chi(y) = \operatorname{Max}(1, \mu^{-1/2} \sqrt{|\log y|})$.

Applying Young's inequality to the integral defining $\sigma_{n,t}(x)$, we see that

$$(3) \qquad \pi\, |\sigma_{n,t}(x)| \leqslant \int_0^{2\pi} \Phi\, |S_t(u)|\, du + \int_0^{2\pi} \Psi\{K_n(u - x)\}\, du,$$

where K_n denotes Fejér's kernel. Since $K_n \leqslant n$, the second integral on the right is less than

$$\left(\frac{\log n}{\mu}\right)^{1/2} \int_0^{2\pi} K_n(u - x)\, du = \pi \left(\frac{\log n}{\mu}\right)^{1/2},$$

provided that $\log n > \mu$. Taking t such that $\exp \mu S_t^2(u)$ is integrable for every value of μ, we see that the first integral on the right in (2) is finite, and so the left-hand side of (3) is certainly less than $2\pi \mu^{-1/2}\sqrt{\log n}$ if n is large enough. Since we may take μ as large as we please, the lemma follows.

(ii) *If the first arithmetic means for the series* 5.6(2) *are* $O(\log n)^{1/2}$, *the series*

$$(4)\qquad \sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{(\log n)^{1/2+\varepsilon}}, \qquad \varepsilon > 0,$$

is uniformly summable $(C, 1)$.

Let us put $c_0 = c_1 = 0$, $c_\nu = (\log \nu)^{-1/2-\varepsilon}$ for $\nu \geq 2$, $h_\nu = h_\nu^{(n)} = (n+1-\nu)/(n+1)$, $C_\nu = c_\nu h_\nu$, and let $\sigma_n(x)$, $\tau_n(x)$ denote the first arithmetic means for the series 5.6(2) and (4) respectively. Applying Abel's transformation twice we obtain

$$(5)\qquad \tau_n(x) = \sum_{\nu=0}^{n-1} (\nu+1)\,\sigma_\nu(x)\,\Delta^2 C_\nu + (n+1)\,\sigma_n(x)\,\Delta C_n.$$

Since $\Delta C_n = C_n$, the last term on the right is $o(1)$ uniformly in x.

The reader will have no difficulty in proving the formula $\Delta^2 C_\nu = h_\nu \Delta^2 c_\nu + 2\Delta h_\nu\, \Delta c_{\nu+1} + \Delta^2 h_\nu\, c_{\nu+2}$, which is analogous to the formula for the second derivative of the product of two functions. In our case $\Delta^2 h_\nu = 0$ and so, by (5),

$$(6)\qquad \tau_n(x) = \sum_{\nu=0}^{n-1} h_\nu^{(n)} (\nu+1)\,\sigma_\nu(x)\,\Delta^2 c_\nu + \frac{2}{n+1}\sum_{\nu=0}^{n-1} (\nu+1)\,\sigma_\nu(x)\,\Delta c_{\nu+1} + o(1).$$

Given any function $\lambda(x)$, let us put $\alpha(u) = \alpha_x(u) = \lambda(x) - \lambda(x+u)$, $\beta(u) = \beta_x(u) = \lambda(x) - 2\lambda(x+u) + \lambda(x+2u)$. Since $\alpha(0) = \beta(0) = \beta'(0) = 0$ we obtain, by Taylor's formula, that $\alpha(u) = -\lambda'(x+\theta u)$, $\beta(u) = \tfrac12 \lambda''(x+\theta_1 u)$, where $0<\theta<1$, $0<\theta_1<1$. Taking $\lambda(x) = (\log x)^{-1/2-\varepsilon}$, $u=1$, we obtain that $\alpha_\nu(1) = \Delta c_\nu = O(\nu^{-1}\log^{-3/2-\varepsilon}\nu)$, $\beta_\nu(1) = \Delta^2 c_\nu = O(\nu^{-2}\log^{-3/2-\varepsilon}\nu)$. Thence we see that $(\nu+1)\,\sigma_\nu(x)\,\Delta c_{\nu+1} \to 0$, and, by (6),

$$(7)\qquad \tau_n(x) = \sum_{\nu=0}^{n-1} h_\nu^{(n)} (\nu+1)\,\sigma_n(x)\,\Delta^2 c_\nu + o(1) = \sum_{\nu=0}^{n} h_\nu^{(n)} (\nu+1)\,\sigma_\nu(x)\,\Delta^2 c_\nu + o(1).$$

Since the partial sums of the series with terms $(\nu+1)\,\sigma_\nu(x)\,\Delta^2 c_\nu$ are uniformly convergent, the same is true for the first Cesàro means, so that the last sum in (7) converges uniformly, and the lemma is established.

To complete the proof of the theorem let $a_n' = a_n(\log n)^{\frac{1+\varepsilon}{2}}$, $b_n' = b_n(\log n)^{\frac{1+\varepsilon}{2}}$. In virtue of (i), the first arithmetic means of almost all series with terms $\pm(a_n' \cos nx + b_n' \sin nx)$ are $o\sqrt{\log n}$, so that, by (ii), almost all series with terms $\pm(a_n \cos nx + b_n \sin nx)$ are uniformly summable $(C, 1)$, i. e. belong to the class C.

We add that this theorem can be generalized, viz. *if* (2) *is finite, almost all the series* 5.6(1) *converge uniformly over* $(0, 2\pi)$ [1].

[1]) Paley and Zygmund [1]; see also Salem [2].

5.7. Miscellaneous theorems and examples.

1. Let $\{a_n\}$ be a sequence tending to 0. A necessary and sufficient condition that $\{a_n\}$ should be quasi-convex is that it should be a difference of two convex sequences tending to 0.

If $\{a_n\}$ tends to 0 and is quasi-convex, then the sequences $\{a_n\}$ and $\{n\Delta a_n\}$ are of bounded variation.

2. If we put $f(x) = \Sigma n^{-\alpha} \cos nx$, $g(x) = \Sigma n^{-\alpha} \sin nx$, $0<\alpha<1$, then $f(x) \simeq x^{\alpha-1} \sin \tfrac12 \pi\alpha\, \Gamma(1-\alpha)$, $g(x) \simeq x^{\alpha-1} \cos \tfrac12 \pi\alpha\, \Gamma(1-\alpha)$ as $x \to +0$.

[This follows from the first formula in 3.11(1) and from the fact that $A_n^\beta\, \Gamma(\beta+1)/n^\beta = 1 + O(1/n)$ (§ 3.12)].

3. Let $g_k(x) = \tfrac12 + \sum_{n=1}^{\infty} \left(\frac{\sin nh}{nh}\right)^k \cos nx$, $k = 1, 2, \dots$, $0 < kh \leqslant \pi$. The function $g_k(x)$ vanishes in the interval (kh, π) and is equal to a polynomial of order $k-1$ in each of the intervals $((k-2)\,h, kh)$, $((k-4)\,h, (k-2)\,h)$, ...

[Consider the function $f_k(x) = \sum_{m=-\infty}^{+\infty}{}' \frac{e^{imx}}{i^k m^k}$ of § 2.15 and the expression $g_k(x+kh) - \binom{k}{1} g_k(x+(k-2)\,h) + \dots \pm f_k(x-kh)$.

The result may also be obtained by repeated application of Theorem 2.11 to the function $f_1(x)$ (§ 1.8.2i)].

4. If $a_n \geqslant a_{n+1} \to 0$, the series $\Sigma a_n \cos nx$ is a Fourier-Riemann series. Szidon [1].

5. If $a_n \geqslant a_{n+1} \to 0$ and $\Sigma a_n \sin nx \in L$, then $\Sigma a_n \cos nx \in L$.

6. (i) If $\{a_n\}$, $a_n \to 0$, is convex, the functions $f(x) = \Sigma a_n \cos nx$ and $\bar f(x) = \Sigma a_n \sin nx$ have continuous derivatives in any interval $(\varepsilon, \pi-\varepsilon)$, $\varepsilon > 0$. (ii) If $\{a_n\}$ is only monotonic, this is not necessarily true, and the functions may be almost everywhere non-differentiable.

[(i) follows from the fact that the series differentiated term by term are uniformly summable $(C, 1)$ in $(\varepsilon, \pi-\varepsilon)$. To prove (ii) observe that the second series in 5.121(1) behaves like a lacunary series if $\lambda_{n+1}/\lambda_n > \lambda > 1$ and apply the following theorem].

7. Let the series 5.4(1) be a $\mathfrak{S}[f]$. If $f'(x)$ exists and is finite in a set E of positive measure, then $\Sigma n_k^2 (a_k^2 + b_k^2) < \infty$.

[This follows from Theorem 5.4 since the differentiated series is summable in E by a method T^*].

8. Let $\varphi_0(t), \varphi_1(t), \dots$ be Rademacher's functions and let $\Sigma c_n^2 < \infty$, $f(t) = \Sigma c_n \varphi_n(t)$, $0 \leqslant t \leqslant 1$. Then $m_\alpha \mathfrak{N}_2[c] \leqslant \mathfrak{M}_\alpha[f] \leqslant M_\alpha \mathfrak{N}_2[c]$, $\alpha > 0$, where the constants m_α and M_α depend only on α.

[The second inequality follows from Theorem 5.51 and from the fact that $\mathfrak{M}_\alpha[f; 0, 1] = \mathfrak{A}_\alpha[f]$ is a non-decreasing function of α. To prove the first inequality for $0<\alpha<2$ observe that $\mathfrak{M}_\alpha^\alpha$ is a multiplicatively convex function of α].

9. Let 5.4(1) be a $\mathfrak{S}[f]$ and let $n_{k+1}/n_k > \lambda > 1$. Then we have the inequalities $m_{\alpha,\lambda}\, \mathfrak{N}_2[\rho] \leqslant \mathfrak{M}_\alpha[f; 0, 2\pi] \leqslant M_{\alpha,\lambda}\, \mathfrak{N}_2[\rho]$, where $\rho_n^2 = a_n^2 + b_n^2$ and the constants m and M depend only on α and λ.

[It is sufficient to prove, for lacunary series, a theorem analogous to Theorem 5.51(i). The proof is similar if, for fixed α, λ is sufficiently large. In the general case we split up the series considered into a finite number of series for each of which the number λ is large].

10. If the series 5.4(1), with $n_{k+1}/n_k > \lambda > 3$, converges in an interval (a, b), then the series converges absolutely. Fatou [1].

[Let $a_k \cos n_k x + b_k \sin n_k x = \rho_k \cos (n_k x + x_k)$. It is easily seen geometrically that there is a point x^* in (a, b) such that $\cos (n_k x^* + x_k) > \varepsilon > 0$ for k sufficiently large. The theorem holds for $\lambda > 1$. See Zygmund [6]].

11. The points of convergence and those of divergence for the series $\sum (\sin 10^n x)/n$ are everywhere dense in the interval $(0, 2\pi)$.

12. Let $0 < \alpha < 1$ and $0 < \beta$. The series $\sum_{n=1}^{\infty} n^{-\beta} e^{in^\alpha} e^{inx}$ converges everywhere if $\alpha + \beta > 1$; the convergence is uniform if $\frac{1}{2}\alpha + \beta > 1$. Hardy [1]

13. If $1 \leqslant \frac{1}{2}\alpha + \beta \leqslant 2$, the sum of the previous series belongs to $\mathrm{Lip}\,(\frac{1}{2}\alpha + \beta - 1)$. Hardy [1], Zygmund [7].

[Apply van der Corput's lemmas and an argument similar to that of § 5.32].

14. The function $F(x) = -x + \lim_{m\to\infty} \int_0^x \prod_{p=1}^{m} (1 + \cos 4^p t)\, dt$ is a continuous function of bounded variation with Fourier coefficients $\neq o(1/n)$. F. Riesz [5].

[The product $p_m = (1 + \cos 4t) \dots (1 + \cos 4^m t)$ is a trigonometrical polynomial of order $\alpha_m = 4^m + 4^{m-1} + \dots + 4$. Since the lowest term of the polynomial $p_{m+1} - p_m = p_m \cos 4^{m+1} t$ is of order $\beta_{m+1} = 4^{m+1} - 4^m - \dots - 4 > \alpha_m$, p_m is a partial sum of p_{m+1}, i. e. $\{p_m\}$ is a subsequence of the sequence of partial sums of a trigonometrical series (*) $1 + a_1 \cos x + a_2 \cos 2x + \dots$ Let $P_m(x)$ be the integral of p_m over the interval $(0, x)$, and let γ_m be the number of non-vanishing terms in p_m; it is easy to see that $\gamma_{m+1} = 3\gamma_m - 1$, i. e. $\gamma_{m+1} - \gamma_m = 3(\gamma_m - \gamma_{m-1})$, $\gamma_{m+1} - \gamma_m = 3^m$. Since $p_{m+1} - p_m$ consists of 3^m terms each of which does not exceed 1 in absolute value, we have $|P_{m+1} - P_m| \leqslant 3^m/\beta_{m+1} = O(3^m/4^m)$ and so the function $P(x) = \lim P_m(x) = P_1 + (P_2 - P_1) + \dots$ is continuous. $P_m(x)$ is non-decreasing and so is its limit. It follows that the function $F(x) = -x + P(x)$ is continuous and of bounded variation. To obtain $\mathfrak{S}[F]$ we reject the linear term from the series (*) integrated term by term. Since $a_{4^m} = 1$, the coefficients of $\mathfrak{S}[F]$ are not $o(1/n)$].