CHAPTER IV.

Classes of functions and Fourier series.

4.1. Inequalities. We begin by proving a number of inequalities which will be applied in the sequel[1]).

Let $\varphi(u) \geqslant 0$ for $u \geqslant 0$. We say that $f(x)$, $a \leqslant x \leqslant b$, belongs to the class $L_\varphi(a, b)$ if the function $\varphi(|f|)$ is integrable over (a, b). If it is not necessary to specify the interval, we denote the class by L_φ simply. If $\varphi(u) = u^r$, $r > 0$, we write L^r instead of L_φ, L instead of L^1 and put

$$\mathfrak{M}_r[f;\ a, b] = \left(\int_a^b |f|^r dx\right)^{1/r}, \quad \mathfrak{A}_r[f;\ a, b] = \left(\frac{1}{b-a}\int_a^b |f|^r dx\right)^{1/r}.$$

When the interval (a, b) is fixed, we shall write simply $\mathfrak{M}_r[f]$, $\mathfrak{A}_r[f]$. The former expression may have a meaning even when (a, b) is infinite. If $r = 1$ we shall write $\mathfrak{M}$, $\mathfrak{A}$ instead of $\mathfrak{M}_1$, $\mathfrak{A}_1$.

Similarly, given a sequence $a = \{a_n\}$, finite or infinite, we write

$$\mathfrak{N}_r[a] = (\sum |a_n|^r)^{1/r}\ ^{2)}.$$

4.11. Young's inequality. Let $\varphi(u)$, $u \geqslant 0$, $\psi(v)$, $v \geqslant 0$, be two functions, continuous, vanishing at the origin, strictly increasing,

[1]) For a detailled discussion of various inequalites see Hardy, Littlewood and Pólya, *Inequalities.*

[2]) Given a *finite* sequence $a = a_1, a_2, \ldots, a_N$, let $\mathfrak{A}_r[a] = \left(\frac{1}{N}\sum_{n=1}^{N}|a_n|^r\right)^{1/r}$. This expression has properties analogous to those of $\mathfrak{A}_r[f]$, but we shall not consider it here.

tending to ∞, and inverse to each other. Then, for $a, b \geqslant 0$, we have the inequality, due to Young[1]),

$$(1) \qquad ab \leqslant \Phi(a) + \Psi(b), \text{ where } \Phi(x) = \int_0^x \varphi\, du,\ \Psi(y) = \int_0^y \psi\, dv.$$

The geometrical proof is obvious. It is also easy to see that the sign $\leqslant$ can be replaced by $=$ if and only if $b = \varphi(a)$. The functions Φ and Ψ will be called *complementary* functions. If $\varphi(u) = u^\alpha$, $\psi(v) = v^{1/\alpha}$, $\alpha > 0$, $1 + \alpha = r$, $1 + 1/\alpha = r'$, we obtain

$$(2) \qquad ab \leqslant \frac{a^r}{r} + \frac{b^{r'}}{r'},$$

where the 'complementary' exponents r, r' are connected by the relation $1/r + 1/r' = 1$ [2]). This is a generalization of the well-known inequality $2ab \leqslant a^2 + b^2$, to which it reduces if $r = r' = 2$. It is plain that either $r \leqslant 2 \leqslant r'$ or $r' \leqslant 2 \leqslant r$. From (1) we deduce that, if $f(x) \in L_\Phi$, $g(x) \in L_\Psi$, the product fg is integrable. In particular, fg is integrable if $f \in L^r$, $g \in L^{r'}$.

4.12. Hölder's inequalities. Consider non-negative sequences $A = \{A_n\}$, $B = \{B_n\}$, $AB = \{A_n B_n\}$, and suppose that $\mathfrak{N}_r[A] = \mathfrak{N}_{r'}[B] = 1$, $r > 1$. Substituting, in 4.11(2), A_n, B_n for a, b, and adding all the inequalities, we obtain that $\mathfrak{N}[AB] \leqslant 1$. If $\{a_n\}$, $\{b_n\}$ are non-negative and $\mathfrak{N}_r[a]$, $\mathfrak{N}_{r'}[b]$ positive and finite, then, putting $A_n = a_n/\mathfrak{N}_r[a]$, $B_n = b_n/\mathfrak{N}_{r'}[b]$, we have $\mathfrak{N}_r[A] = 1$, $\mathfrak{N}_{r'}[B] = 1$, and from $\mathfrak{N}[AB] \leqslant 1$ we obtain the first of the Hölder inequalities

$$(1) \qquad \mathfrak{N}[ab] \leqslant \mathfrak{N}_r[a]\,\mathfrak{N}_{r'}[b], \qquad \mathfrak{M}[fg] \leqslant \mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[g], \quad r > 1,$$

which is obviously true also if $\mathfrak{N}_r[a] = 0$ or $\mathfrak{N}_{r'}[b] = 0$. The second inequality (1), where $f, g \geqslant 0$, is obtained by the same argument, summation being replaced by integration. In the general case (a, b, f, g complex), we have

$$(2) \qquad |\sum a_n b_n| \leqslant \mathfrak{N}_r[a]\,\mathfrak{N}_{r'}[b], \quad \left|\int_a^b fg\, dx\right| \leqslant \mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[g],$$

[1]) Young [7].

[2]). This notation will be used systematically in this chapter, so that by p' we shall denote the exponent q such that $1/p + 1/q = 1$.

since the left-hand sides in (2) do not exceed $\mathfrak{N}[ab]$, $\mathfrak{M}[fg]$ respectively.

A little attention shows that the first relation (2) degenerates into equality if and only if arg $(a_n b_n)$ and $|a_n|^r/|b_n|^{r'}$ are independent of n (arg 0 and 0/0 denote any numbers we please). For the second relation the conditions are: arg $f(x)\,g(x)$ and $|f(x)|^r/|g(x)|^{r'}$ must be constant almost everywhere.

The number M, finite or infinite, will be called the *essential upper bound* of a function $g(x)$, $a \leqslant x \leqslant b$, if (i) $g(x) \leqslant M$ almost everywhere, (ii) for every $M' < M$ the set of x for which $g(x) > M'$ is of positive measure. If $M < \infty$, we shall call f an *essentially bounded* function. We will prove that *if M is the essential upper bound of $|f(x)|$ in (a, b), then $\mathfrak{M}_r[f; a, b] \to M$ as $r \to \infty$*[1]. In the first place $\mathfrak{M}_r[f] \leqslant M(b-a)^{1/r}$, so that $\lim \mathfrak{M}_r[f] \leqslant M$. Next, if M' is any number $< M$, and E the set of points where $|f(x)| > M'$, then $\mathfrak{M}_r[f] \geqslant |E|^{1/r} M'$, $\lim \mathfrak{M}_r[f] \geqslant M'$, and so $\lim \mathfrak{M}_r[f] = M$. This completes the proof in the case of (a, b) finite, or when (a, b) is infinite and $M = \infty$. Let now (a, b) be infinite and $0 < M < \infty$. We may suppose that $M = 1$. The same agument as before proves that $\underline{\lim}\, \mathfrak{M}_r[f] \geqslant 1$. To show that $\lim \mathfrak{M}_r[f] = 1$ we need only observe that $\mathfrak{M}_r^r[f]$ is a decreasing function of r which, by the preceding remark, is $\geqslant 1$.

In virtue of the result just established, it is natural to define $\mathfrak{M}_\infty[f; a, b]$ as the essential upper bound of $|f|$ in (a, b). By L^∞ we may denote the class of essentially bounded functions. The second inequality (2) has then a meaning (and is obviously true) even when $r = \infty$.

Since any series $a_0 + a_1 + \dots$, $a_n \geqslant 0$, can be represented as the integral, over $(0, \infty)$, of a function $f(x)$, where $f(x) = a_n$ for $n \leqslant x < n+1$, $n = 0, 1, \dots$, the above remarks apply also to series.

4.121. Let $f_i \in L^{r_i}$, $i = 1, 2, \dots, k$, where $r_i > 0$, $1/r_1 + 1/r_2 + \dots + 1/r_k = 1$. An easy induction shows that $\mathfrak{M}[f_1 f_2 \dots f_k] \leqslant \mathfrak{M}_{r_1}[f_1]\,\mathfrak{M}_{r_2}[f_2] \dots \mathfrak{M}_{r_k}[f_k]$. Similarly for series.

4.13. Minkowski's inequality. Let $a = \{a_n\}$, $b = \{b_n\}$ be two sequences, $a + b = \{a_n + b_n\}$. We will now prove Minkowski's inequality

[1]) Hence $\mathfrak{M}_r[f] \to M$ as $r \to \infty$.

(1) $$\mathfrak{N}_r[a+b] \leqslant \mathfrak{N}_r[a] + \mathfrak{N}_r[b], \qquad r \geqslant 1.$$

Writing $(a_n + b_n)^r = (a_n + b_n)^{r-1} a_n + (a_n + b_n)^{r-1} b_n$, and applying Hölder's inequality to the sums of terms $(a_n + b_n)^{r-1} a_n$ and of terms $(a_n + b_n)^{r-1} b_n$, we find that $\mathfrak{N}_r^r[a+b] \leqslant \mathfrak{N}_r^{r-1}[a+b]\,\mathfrak{N}_r[a] + \mathfrak{N}_r^{r-1}[a+b]\,\mathfrak{N}_r[b]$, and (1) follows.

The same argument proves Minkowski's inequality for integrals:

(2) $$\mathfrak{M}_r[f+g] \leqslant \mathfrak{M}_r[f] + \mathfrak{M}_r[g], \qquad r \geqslant 1.$$

If $0 < r < 1$, all these inequalities cease to be true. However we have then

(3) $$\mathfrak{M}_r^r[f+g] \leqslant \mathfrak{M}_r^r[f] + \mathfrak{M}_r^r[g], \quad \mathfrak{N}_r^r[a+b] \leqslant \mathfrak{N}_r^r[a] + \mathfrak{N}_r^r[b], \quad 0 < r < 1,$$

which inequalities are simple corollaries of the inequality $(x+y)^r \leqslant x^r + y^r$, $x \geqslant 0$, $y \geqslant 0$, $0 < r \leqslant 1$, or, what amounts to the same thing, of the inequality $(1+x)^r \leqslant 1 + x^r$. To prove the latter we observe that $(1+x)^r - 1 - x^r$ vanishes for $x = 0$ and has a negative derivative for $x > 0$[1]).

Let $h(x, y)$ be a function defined for $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. An argument similar to that which led to (2) gives the inequality

(4) $$\left\{\int_a^b \left|\int_c^d h(x, y)\,dy\right|^r dx\right\}^{1/r} \leqslant \int_c^d \left\{\int_a^b |h(x, y)|^r\,dx\right\}^{1/r} dy, \qquad r \geqslant 1,$$

which may be considered as the most general form of Minkowski's inequality since it contains the results (1) and (2) as special cases[2]).

4.14. Convex functions and Jensen's inequality. A function $\varphi(x)$, $\alpha \leqslant x \leqslant \beta$, is said to be *convex* if, for any pair of points P_1, P_2 on the curve $y = \varphi(x)$, the points of the arc $P_1 P_2$ are below, or on, the chord $P_1 P_2$. As an example we quote the function x^p, $p \geqslant 1$, which is convex in the interwal $(0, \infty)$.

For any system of positive numbers $p_1, p_2, \dots, p_n$, and any system of points $x_1, x_2, \dots, x_n$ from (α, β), we have the inequality

[1]) From the inequalities (2) and (3) we conclude that, if $f \in L^r$, $g \in L^r$, then $(f+g) \in L^r$, $r > 0$.

[2]) If $(c, d) = (0, 2)$, $h(x, y) = f(x)$ for $0 \leqslant y < 1$, $h(x, y) = g(x)$ for $1 \leqslant y \leqslant 2$, the inequality (4) reduces to (2). If $f(x) = a_n$, $g(x) = b_n$ for $n \leqslant x < n+1$, $n = 0, 1, \dots$, we obtain the inequality (1).

$$(1)\qquad \varphi\left(\frac{p_1 x_1 + p_2 x_2 + \dots + p_n x_n}{p_1 + p_2 + \dots + p_n}\right) \leqslant \frac{p_1 \varphi(x_1) + \dots + p_n \varphi(x_n)}{p_1 + \dots + p_n},$$

due to Jensen[1]. For $n=2$ this is just the definition of convexity, and for $n>2$ it follows by induction.

It is obvious geometrically that, if φ is convex, $\varphi(x+0)$, and similarly $\varphi(x-0)$, must exist. These limits can be neither $+\infty$ nor $-\infty$. Moreover $\varphi(x+0)=\varphi(x-0)=\varphi(x)$, i. e. convex functions are continuous.

Assuming φ continuous, we may take as the definition of convexity that for every arc $P_1 P_2$ there exists a subarc $P_1' P_2'$ lying below or on the chord $P_1' P_2'$. In fact, if there existed an arc $P_1 P_2$ lying, even partially, above the chord $P_1 P_2$, there would exist a subarc $P_1' P_2'$ lying totally above the chord $P_1' P_2'$, so that the two definitions of convexity are equivalent.

It is easy to see that a convex function has no proper maximum in the interior of the interval in which it is defined. Let $\varphi(x)$ be convex in $(0,\infty)$ and let x_0 be a minimum of φ. If $\varphi(x)$ is not constant for $x>x_0$, then $\varphi(x)$ tends to $+\infty$, as $x\to\infty$, at least as rapidly as a multiple of x. This follows from the fact that, if $x_0<x_1<x_2<\dots$, $x_n\to\infty$, the angles which the chords joining $(x_i, \varphi(x_i))$ and $(x_{i+1}, \varphi(x_{i+1}))$ make with the real axis, increase with i. Therefore, if $\varphi(u)$ is convex in $(0,\infty)$, and $\varphi(u)\to\infty$ with u, the relation $f\in L_\varphi$ involves the integrability of f.

Let $f(t)$, $p(t)$ be two functions defined for $a\leqslant t\leqslant b$, and such that $\alpha\leqslant f(t)\leqslant\beta$, $p(t)\geqslant 0$, $p(t)\not\equiv 0$. Let $\varphi(u)$ be a convex function defined for $\alpha\leqslant u\leqslant\beta$. Jensen's inequality for integrals, viz.,

$$(2)\qquad \varphi\left\{\frac{\int_a^b f(t)p(t)\,dt}{\int_a^b p(t)\,dt}\right\} \leqslant \frac{\int_a^b \varphi(f)p(t)\,dt}{\int_a^b p(t)\,dt},$$

is a simple corollary of (1) if $f(t)$ and $p(t)$ are continuous and (a,b) is finite. In fact, if $a=t_0<t_1<\dots<t_n=b$ is a subdivision of (a,b), $\delta_i=t_i-t_{i-1}$, $p_i=p(t_i)\,\delta_i$, $x_i=f(t_i)$, the inequality (1) tends to (2), provided that $\operatorname{Max}\delta_i\to 0$. To prove (2) in the most

[1]) Jensen [1].

general case is not a difficult task, but for the sake of brevity we content ourselves with the case which we shall actually need later, viz. $f\geqslant 0$, $\varphi(u)$ non-negative and increasing with u, (a,b) finite. Since any bounded f is the limit of a uniformly bounded sequence of continuous functions f_n,[1]) we obtain (2) for f and p bounded. Similarly, for f and p integrable, we have $f=\lim f_n$, $p=\lim p_n$, where each f_n and p_n is bounded and $f_n\leqslant f_{n+1}$, $p_n\leqslant p_{n+1}$; an application of Lebesgue's theorem on the integration of monotonic sequences yields the desired result.

4.141. *A necessary and sufficient condition that a function $\chi(x)$ defined at every point of an interval $\alpha\leqslant x\leqslant\beta$, $-\infty<\alpha<\beta<\infty$, should be convex, is that $\chi(x)$ should be the indefinite integral of a function non-decreasing and integrable over (α,β)*, i. e.

$$(1)\qquad \chi(x)=\chi(\alpha)+\int_\alpha^x \xi(t)\,dt,\quad\text{where}\quad \xi(t_1)\leqslant\xi(t_2)\quad\text{for}\quad t_1\leqslant t_2.$$

Suppose first that the condition (1) is satisfied. Since instead of (α,β) we may consider an arbitrary subinterval of (α,β), it is sufficient to show that, if $0<\vartheta<1$, $x=(1-\vartheta)\alpha+\vartheta\beta$, the function χ satisfies the inequality $\chi(x)\leqslant(1-\vartheta)\chi(\alpha)+\vartheta\chi(\beta)$. Without real loss of generality we may assume that $\alpha=0$, $\chi(\alpha)=0$, so that the inequality which we have to prove is

$$\int_0^{\vartheta\beta}\xi(t)\,dt\leqslant\vartheta\int_0^\beta\xi(t)\,dt,\quad\text{or}\quad(1-\vartheta)\int_0^{\vartheta\beta}\xi(t)\,dt\leqslant\vartheta\int_{\vartheta\beta}^\beta\xi(t)\,dt.$$

Now it is sufficient to observe that the left-hand side of the last inequality is at most equal to, and the right-hand side is not less than, the number $\vartheta(1-\vartheta)\beta\,\xi(\vartheta\beta)$.

To prove the second half of the theorem let $R(x,h)$ denote the ratio $[\chi(x+h)-\chi(x)]/h$, $h\neq 0$. From the convexity of χ it follows that

$$(2)\qquad R(x,-k)\leqslant R(x,h),\qquad\qquad (3)\qquad R(x,h)\leqslant R(x,h_1)$$

provided that $0<k$, $0<h<h_1$, and that the points $x, x-k, x+h_1$ belong to (α,β). From (3) we see that $R(x,h)$ tends to a definite limit as $h\to+0$, and, in virtue of (2), this limit, which is the right-hand derivative $D^+\chi(x)$, is finite for $\alpha<x<\beta$. Similarly we prove that $R(x,-h_1)\leqslant R(x,-h)$ for $0<h<h_1$, and that the left-hand derivative $D^-\chi(x)$ exists and is finite for $\alpha<x<\beta$. It follows from (2) that

$$(4)\qquad D^-\chi(x)\leqslant D^+\chi(x).$$

[1]) Let $F(x)$ be the indefinite integral of $f(x)$. We may put e. g. $f_n(x)=n\,[F(x+1/n)-F(x)]$.

Let now $\alpha < x < x_1 < \beta$, and let $h > 0$, $k > 0$, $h + k = x_1 - x$, so that $x + h = x_1 - k$. We have then $D^+\chi(x) \leqslant R(x, h) \leqslant R(x_1, -k) \leqslant D^-\chi(x_1)$. From this and from the inequalities (4) we obtain that, for $x < x_1$,

$$D^-\chi(x) \leqslant D^-\chi(x_1), \quad D^+\chi(x) \leqslant D^+\chi(x_1), \tag{5}$$

i. e. the derivatives $D^-\chi(x)$ and $D^+\chi(x)$ are non-decreasing. Since the set of points where a non-decreasing function is discontinuous is at most enumerable, we infer from (4) and (5) that the set of points where $\chi'(x)$ does not exist is at most enumerable. The derivative $\chi'(x)$ is uniformly bounded in every interval (α', β') completely interior to (α, β). Hence the equation (1) is certainly true if we replace α by α', $\xi(t)$ by $\chi'(t)$ and suppose that $\alpha' \leqslant x \leqslant \beta'$. Making $\alpha' \to \alpha$, $\beta' \to \beta$, and remembering that $\chi(t)$ is continuous, we obtain the formula (1), with $\xi(t) = \chi'(t)$. To show that $\chi'(t)$ is integrable we need only observe that it is of constant sign in the neighbourhood of the points α and β, so that the existence of improper integrals involves the integrability in the sense of Lebesgue. This completes the proof.

4.142. Let now $\varphi(x)$ be an arbitrary function non-negative, non-decreasing, tending to ∞ with x, and vanishing at the origin. The curve $y = \varphi(x)$ may possess discontinuities and stretches of invariability. The inverse function $x = \psi(y)$ has the same properties, and is one valued except for the values which correspond to the stretches of invariability of $\varphi(x)$. If $\varphi(x)$ is constant and has a value y_0 for $\alpha < x < \beta$, we assign to $\psi(y_0)$ any value from the interval (α, β). Since the number of the stretches of invariability is at most enumerable, our choice has no influence upon the values of the integral $\Phi(x)$ of $\varphi(x)$, and it is easy to see that the Young inequality 4.11(1) holds true in this slightly more general case.

From the theorem proved in § 4.141 it follows that every function $\Phi(x)$ which is non-negative, convex, and satisfies the relations $\Phi(0) = 0$ and $\Phi(x)/x \to \infty$ as $x \to \infty$, may be considered as a Young function. More precisely to every such function $\Phi(x)$ corresponds another function $\Psi(x)$ with similar properties, and such that $ab \leqslant \Phi(a) + \Psi(b)$ for every $a \geqslant 0$, $b \geqslant 0$. It is sufficient to take for $\Psi(x)$ the integral of the function $\psi(x)$ inverse to the function $\varphi(x) = \Phi'(x)$. Since $\Phi(x)/x \to \infty$ with x, it is easy to see that $\varphi(x)$ and $\psi(x)$ are unbounded as $x \to \infty$.

4.15. $\mathfrak{M}_\alpha[f]$ and $\mathfrak{A}_\alpha[f]$ as functions of α. A function $\psi(u) \geqslant 0$ will be called a *multiplicatively convex* function if, for every $t_1 \geqslant 0$, $t_2 \geqslant 0$, $t_1 + t_2 = 1$, we have $\psi(t_1 u_1 + t_2 u_2) \leqslant \psi^{t_1}(u_1)\psi^{t_2}(u_2)$. It is the same thing as to say that $\log \psi(u)$ is convex.

Given a function $f(x)$, the expression $\mathfrak{A}_\alpha[f]$ is a non-decreasing function of α. $\mathfrak{A}_\alpha^\alpha[f]$ and $\mathfrak{M}_\alpha^\alpha[f]$ are multiplicatively convex functions of α $(\alpha > 0)$ [1].

[1]) Hausdorff [2].

Substituting $|f|^\alpha$ for f, 1 for g, in the second formula 4.12(1), and dividing both sides by $b - a$, we obtain that $\mathfrak{A}_\alpha[f] \leqslant \mathfrak{A}_{\alpha r}[f]$ for $r > 1$. That the result is not true for $\mathfrak{M}_\alpha$, is easily seen from the example $a = 0$, $b = 2$, $f(x) = 1$.

To prove the second part of the theorem, let $\alpha = \alpha_1 t_1 + \alpha_2 t_2$, $\alpha_i > 0$, $t_i > 0$, $t_1 + t_2 = 1$. Replacing the integrand $|f|^\alpha$ by $|f|^{\alpha_1 t_1} |f|^{\alpha_2 t_2}$, in $\mathfrak{M}_\alpha$, and applying Hölder's inequality with $r = 1/t_1$, $r' = 1/t_2$, we find: $\mathfrak{M}_\alpha^\alpha \leqslant \mathfrak{M}_{\alpha_1}^{\alpha_1 t_1} \mathfrak{M}_{\alpha_2}^{\alpha_2 t_2}$. Dividing both sides by $b - a$, we obtain that $\mathfrak{A}_\alpha^\alpha \leqslant \mathfrak{A}_{\alpha_1}^{\alpha_1 t_1} \mathfrak{A}_{\alpha_2}^{\alpha_2 t_2}$.

4.16. A theorem of Young. Let $f(x)$ and $g(x)$ be two functions of period 2π, belonging to $L^p(0, 2\pi)$ and $L^q(0, 2\pi)$ respectively, and let

$$h(x) = \int_0^{2\pi} f(x + t)\, g(t)\, dt. \tag{1}$$

Then, *if $1/p + 1/q > 1$, and $1/r = 1/p + 1/q - 1$, the function $h(x)$ is of the class L^r and, moreover,*

$$\mathfrak{M}_r[h] \leqslant \mathfrak{M}_p[f]\, \mathfrak{M}_q[g]. \tag{2}$$

We may suppose that $f \geqslant 0$, $g \geqslant 0$. Let λ, μ, ν be any three positive numbers such that $1/\lambda + 1/\mu + 1/\nu = 1$. Writing $f(x+t)\, g(t)$ in the form $f^{p/\lambda} g^{q/\lambda} f^{p(1/p - 1/\lambda)} g^{q(1/q - 1/\lambda)}$, and applying Hölder's inequality with the exponents λ, μ, ν (§ 4.121), we see that $h(x)$ does not exceed

$$\left[\int_0^{2\pi} f^p(x+t)\, g^q(t)\, dt\right]^{1/\lambda} \left[\int_0^{2\pi} f^{p\mu(1/p - 1/\lambda)}(x+t)\, dt\right]^{1/\mu} \left[\int_0^{2\pi} g^{q\nu(1/q - 1/\lambda)}(t)\, dt\right]^{1/\nu}.$$

If we suppose that $1/p - 1/\lambda = 1/\mu$, $1/q - 1/\lambda = 1/\nu$, $\lambda = r$, the condition $1/\lambda + 1/\mu + 1/\nu = 1$ involves $1/p + 1/q - 1/r = 1$. The last two factors in the product are equal to $\mathfrak{M}_p^{p/\mu}[f]\, \mathfrak{M}_q^{q/\nu}[g]$, and the result follows from the formula

$$\int_0^{2\pi} dx \left\{\int_0^{2\pi} f^p(x+t)\, g^q(t)\, dt\right\} = \mathfrak{M}_p^p[f]\, \mathfrak{M}_q^q[g]$$

(§ 2.12). We add two remarks:

(i) The inequality (2) may be stated in a slightly different form. If we put $p = 1/(1-\alpha)$, $q = 1/(1-\beta)$, $0<\alpha<1$, $0<\beta<1$, then $\mathfrak{M}_{\frac{1}{1-\gamma}}[h] \leqslant \mathfrak{M}_{\frac{1}{1-\alpha}}[f]\, \mathfrak{M}_{\frac{1}{1-\beta}}[g]$, where $\gamma = \alpha+\beta<1$.

(ii) Let us change the definition of $h(x)$ slightly, introducing the factor $1/2\pi$ into the right-hand side of (1) (similarly as in § 2.11). We obtain, then, that $\mathfrak{A}_{\frac{1}{1-\gamma}}[h] \leqslant \mathfrak{A}_{\frac{1}{1-\alpha}}[f]\, \mathfrak{A}_{\frac{1}{1-\beta}}[g]$.

4.17. A theorem of Hardy. *Let* $r>1$, $s<r-1$, $f(x) \geqslant 0$, $0 \leqslant x < \infty$, $F(x) = \int_0^x f\,dt$. *If* $f^r(x)\,x^s$ *is integrable over* $(0,\infty)$, *so is* $\{F(x)/x\}^r x^s$, *and*

$$\text{(1)} \qquad \int_0^\infty \left\{\frac{F(x)}{x}\right\}^r x^s\,dx \leqslant \left(\frac{r}{r-s-1}\right)^r \int_0^\infty f^r(x)\,x^s\,dx \text{ [1]}.$$

Since $\int_0^x f\,t^{s/r}\,t^{-s/r}\,dt \leqslant \left(\int_0^x f^r\,t^s\,dt\right)^{1/r}\left(\int_0^x t^{-s/(r-1)}\,dt\right)^{(r-1)/r}$, we see that f is integrable over any finite interval and that $F(x)=o(x^{(r-1-s)/r})$ as $x \to 0$. Applying a similar argument to the integral defining $F(x)-F(\xi)$ we obtain that $F(x)-F(\xi)<\frac{1}{2}\varepsilon x^{(r-1-s)/r}$, if $x>\xi$ and $\xi=\xi(\varepsilon)$ is large enough. Hence $F(x)=[F(x)-F(\xi)]+F(\xi)<\frac{1}{2}\varepsilon x^{(r-1-s)/r}+O(1)<\varepsilon x^{(r-1-s)/r}$ for x large, and, since $\varepsilon>0$ is arbitrary, $F(x)=o(x^{(r-1-s)/r})$ as $x \to \infty$.

Let $0<a<b<\infty$. Integrating by parts, writing $F^{r-1} f x^{s-r+1} = fx^{r/s}\cdot F^{r-1}x^{s-r+1-s/r}$, and applying Hölder's inequality, we obtain

$$\int_a^b \left\{\frac{F}{x}\right\}^r x^s\,dx \leqslant -\left[\frac{F^r x^{s-r+1}}{r-s-1}\right]_a^b + \frac{r}{r-s-1}\left\{\int_a^b f^r x^s\,dx\right\}^{1/r}\left\{\int_a^b \left(\frac{F}{x}\right)^r x^s\,dx\right\}^{1/r'}.$$

Dividing both sides by the last factor on the right, and making $a \to 0$, $b \to \infty$, we obtain (1).

4.2. Mean convergence. Let $f_1(x), f_2(x), \ldots$ be a sequence of functions belonging to a class $L^r(a,b)$, $r>0$. If there exists a function $f(x) \in L^r(a,b)$ such that $\mathfrak{M}_r[f-f_n;\, a,b] \to 0$ as $n \to \infty$, we say that $\{f_n(x)\}$ converges in mean, to $f(x)$, with index r. The following theorem is of fundamental importance.

[1]) See Hardy, Littlewood, and Pólya, *Inequalities*, Chapter IX, where various extensions of this theorem are given.

A necessary and sufficient condition that $\{f_n(x)\}$, $f_n \in L^r(a,b)$, $r \geqslant 1$, *should converge in mean, with index* r, *to a function* $f(x) \in L^r(a,b)$, *is that* $\mathfrak{M}_r[f_m-f_n]$ *should tend to* 0 *as* m *and* n *tend to infinity* [1]).

The necessity of the condition is obvious, since, by Minkowski's inequality, the relations $\mathfrak{M}_r[f-f_m] \to 0$ and $\mathfrak{M}_r[f-f_n] \to 0$ involve $\mathfrak{M}_r[f_m-f_n] \leqslant \mathfrak{M}_r[f-f_m]+\mathfrak{M}_r[f-f_n] \to 0$.

The following remark will be useful in the proof of the sufficiency of the condition.

(i) *If* $\{u_n(x)\}$, $a \leqslant x \leqslant b$, *is a sequence of non-negative functions, and if* $I_1+I_2+\ldots<\infty$, *where* I_n *denotes the integral of* u_n *over* (a,b), *then* $u_1(x)+u_2(x)+\ldots$ *converges almost everywhere to a finite function.*

In fact, if the series diverged to $+\infty$ in a set of positive mesure, then, by Lebesgue's theorem on the integration of monotonic sequences, we should have $I_1+I_2+\ldots=\infty$.

We will now prove that

(ii) *If* $\mathfrak{M}_r[f_m-f_n] \to 0$ *as* $m,n \to \infty$, *we can find a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *which converges almost everywhere to a finite function* $f(x)$.

Let $\varepsilon_i = \operatorname{Max} \mathfrak{M}_r[f_m-f_n]$ for $m \geqslant i$, $n \geqslant i$. Since $\varepsilon_i \to 0$, we have $\varepsilon_{n_1}+\varepsilon_{n_2}+\ldots<\infty$ if $\{n_k\}$ increases sufficiently rapidly. By Hölder's inequality,

$$\text{(1)} \qquad \int_a^b |f_{n_k}-f_{n_{k+1}}|\,dx \leqslant (b-a)^{1/r'}\,\mathfrak{M}_r[f_{n_k}-f_{n_{k+1}}] \leqslant \varepsilon_{n_k}(b-a)^{1/r'},$$

and so, in virtue of (i), the series $|f_{n_1}|+|f_{n_2}-f_{n_1}|+|f_{n_3}-f_{n_2}|+\ldots$ converges almost everywhere. The function $f(x)=f_{n_1}+(f_{n_2}-f_{n_1})+\ldots= \lim f_{n_k}(x)$ exists almost everywhere.

Returning to the proof of the theorem, we observe that, if $n_k>m$, then $\mathfrak{M}_r[f_m-f_{n_k}] \leqslant \varepsilon_m$. Applying Fatou's well-known lemma [2]),

[1]) Fischer [1], F. Riesz [1], [2].

[2]) Fatou's lemma may be formulated as follows: if $g_k(x) \geqslant 0$, $k=1,2,\ldots$, and $g_k(x) \to g(x)$ almost everywhere in (a,b), then $\int_a^b g_k\,dx \leqslant A$, $k=1,2,\ldots$, involves $\int_a^b g\,dx \leqslant A$. In particular, $g(x)$ is integrable over (a,b). See e. g. Saks, *Théorie de l'intégrale*, p. 84.

we obtain that $\mathfrak{M}_r[f_m - f] \leqslant \varepsilon_m$. Thence we conclude that $f \in L^r$ and that $\mathfrak{M}_r[f - f_m] \to 0$ as $m \to \infty$. This completes the proof. We add a few remarks.

(a) In the proof we tacitly assumed that $b - a < \infty$, but the argument holds even when $b - a = \infty$, since (1) subsists if (a, b) is replaced by any finite subinterval (α, β) of (a, b).

(b) The function $f(x)$, the existence of which asserts the theorem, is determined uniquely. In fact, if $\mathfrak{M}_r[f - f_n] \to 0$ and $\mathfrak{M}_r[g - f_n] \to 0$ as $n \to \infty$, then, by Minkowski's inequality, $\mathfrak{M}_r[f-g] \leqslant \mathfrak{M}_r[f-f_n] + \mathfrak{M}_r[f_n - g] \to 0$, i. e. $\mathfrak{M}_r[f-g] = 0$, $f(x) = g(x)$.

(c) We proved the theorem for the case $r \geqslant 1$ because this case is the most interesting in applications, but the result holds also for $0 < r < 1$. In the proof we use, instead of Minkowski's inequality, the first inequality in 4.13(3). In particular, to establish the existence of $f(x)$, we observe that $\{|f_{n_1}| + |f_{n_2} - f_{n_1}| + \ldots\}^r \leqslant |f_{n_1}|^r + |f_{n_2} - f_{n_1}|^r + \ldots$, and that, if we integrate the right-hand side of this inequality over (a, b), we obtain a convergent series, provided that $\varepsilon_{n_1}^r + \varepsilon_{n_2}^r + \ldots < \infty$.

4.21. The Riesz-Fischer theorem. Let $\{\varphi_n(x)\}$ be a system of functions, orthogonal and normal in (a, b). We saw in § 1.61 that, if c_n are the Fourier coefficients of a function $f \in L^2$, with respect to $\{\varphi_n\}$, the series $c_0^2 + c_1^2 + \ldots$ converges. The converse theorem, due to Riesz and Fischer, is one of the most important achievements of the Lebesgue theory of integration.

Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an arbitrary system of functions, orthogonal and normal in (a, b), and let $c_0, c_1, c_2, \ldots$ be an arbitrary sequence of numbers such that $c_0^2 + c_1^2 + c_2^2 + \ldots < \infty$. Then there exists a function $f \in L^2(a, b)$ such that the Fourier coefficient of f with respect to φ_n is c_n, $n = 0, 1, 2, \ldots$, and, moreover,

$$(1) \qquad \int_a^b f^2\,dx = \sum_{n=0}^{\infty} c_n^2, \qquad \int_a^b (f - s_n)^2\,dx \to 0 \quad \text{as } n \to \infty,$$

where s_n denotes the n-th partial sum of the series $c_0\varphi_0 + c_1\varphi_1 + \ldots$ [1].

[1]) Fischer [1], F. Riesz [1]; see also W. H. and G. C. Young [1], where several alternative proofs are given, and Kaczmarz [2].

From the equation

$$\int_a^b (s_{n+k} - s_n)^2\,dx = \sum_{j=n+1}^{n+k} c_j^2$$

we see that $\mathfrak{M}_2[s_m - s_n] \to 0$ as $m, n \to \infty$. In virtue of the last theorem, there is a function $f \in L^2$ such that $\mathfrak{M}_2[f - s_n] \to 0$ as $n \to \infty$. If $n > j$, we have

$$(2) \qquad c_j = \int_a^b s_n \varphi_j\,dx = \int_a^b f\varphi_j\,dx + \int_a^b (s_n - f)\,\varphi_j\,dx.$$

By Hölder's inequality, the last term on the right does not exceed $\mathfrak{M}_2[s_n - f]\,\mathfrak{M}_2[\varphi_j] = \mathfrak{M}_2[s_n - f]$ in absolute value. Hence, making $n \to \infty$, we conclude from (2) that c_j is the Fourier coefficient of f with respect to φ_j, and it remains only to prove the first equation in (1).

In virtue of 4.2(ii), there exists a sequence $\{s_{n_k}(x)\}$ converging to $f(x)$ almost everywhere. Since $\mathfrak{M}_2^2[s_{n_k}] = c_0^2 + c_1^2 + \ldots + c_{n_k}^2 \leqslant c_0^2 + c_1^2 + c_2^2 + \ldots$, an application of Fatou's lemma gives $\mathfrak{M}_2^2[f] \leqslant c_0^2 + c_1^2 + c_2^2 + \ldots$, and this, together with Bessel's inequality $c_0^2 + c_1^2 + c_2^2 + \ldots \leqslant \mathfrak{M}_2^2[f]$, yields the desired result.

4.22. Corollaries. (i) A system $\{\varphi_n(x)\}$, orthogonal and normal in an interval (a, b), is said to be *closed* in this interval if, for any function $f \in L^2(a, b)$, we have the Parseval relation

$$(1) \qquad \int_a^b f^2\,dx = \sum_{n=0}^{\infty} c_n^2,$$

where $c_0, c_1, \ldots$ are the Fourier coefficients of f with respect to $\{\varphi_n\}$. *In the domain of functions of the class L^2 the notions of a closed and of a complete system are equivalent.* That every closed system is complete, is obvious. To prove the converse assertion let $c_0, c_1, \ldots$ be the Fourier coefficients of a function $f \in L^2$. Since $c_0^2 + c_1^2 + \ldots < \infty$, there is, by the Riesz-Fischer theorem, a $g \in L^2$ with Fourier coefficients c_n, and such that $\mathfrak{M}_2^2[g] = c_0^2 + c_1^2 + \ldots$ Since f and g have the same Fourier coefficients, and $\{\varphi_n\}$ is complete, we have $f \equiv g$, and the equation (1) follows.

(ii) We know that the trigonometrical system is complete (§ 1.5). Therefore, if a_m, b_m denote the Fourier coefficients of a function $f \in L^2$, and c_m the complex coefficients of f, we have the Parseval equations

$$(2a)\quad \frac{1}{\pi}\int_0^{2\pi} f^2\,dx = \tfrac12 a_0^2 + \sum_{k=1}^{\infty}(a_k^2+b_k^2), \qquad (2b)\quad \frac{1}{2\pi}\int_0^{2\pi} f^2\,dx = \sum_{m=-\infty}^{+\infty} |c_m|^2$$

which differ only in notation. It may, however, be observed that they can be obtained independently of the Riesz-Fischer theorem. In view of Bessel's inequality, it is only the inverse inequality which demands a proof. Let $\sigma_n(x)$ be the Fejér sums for the function f; σ_n being a trigonometrical polynomial, we have

$$\frac{1}{\pi}\int_0^{2\pi} \sigma_n^2\,dx = \tfrac12 a_0^2 + \sum_{k=1}^{n}(a_k^2+b_k^2)\left(1-\frac{k}{n+1}\right)^2 \leqslant \tfrac12 a_0^2 + \sum_{k=1}^{\infty}(a_k^2+b_k^2),$$

and, since $\sigma_n(x)\to f(x)$ almost everywhere, it is sufficient to apply Fatou's lemma.

If we substitute $|f|^2$ for f^2 in the formula (2b), we obtain a formula which holds also for f complex. To show this, let $f=f_1+if_2$, and let c_n, c_n', c_n'' be the complex Fourier coefficients of f, f_1, f_2. If $2c_n' = a_n' - ib_n'$, $2c_n'' = a_n'' - ib_n''$, then $|f|^2 = |f_1|^2 + |f_2|^2$, $c_n = c_n' + ic_n''$, $|c_n|^2 = |c_n'|^2 + |c_n''|^2 + 2(a_n' b_n'' - a_n'' b_n')$. Since the last term on the right is an odd function of n, we obtain that

$$\sum_{n=-\infty}^{+\infty}|c_n|^2 = \sum_{n=-\infty}^{+\infty}(|c_n'|^2+|c_n''|^2) = \frac{1}{2\pi}\int_0^{2\pi}(|f_1|^2+|f_2|^2)\,dx = \frac{1}{2\pi}\int_0^{2\pi}|f|^2\,dx.$$

(iii) *If $f(x)$ is periodic and belongs to $L^2(0,2\pi)$, the function $\bar f(x)$ defined by the formula*

$$(3)\quad \bar f(x) = -\frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac12 t}\,dt = \lim_{h\to 0}\left\{-\frac{1}{\pi}\int_h^{\pi}\frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac12 t}\,dt\right\}$$

exists almost everywhere and belongs to L^2 [1]. *Moreover* $\bar{\mathfrak{S}}[f] = \mathfrak{S}[\bar f]$. That $\bar{\mathfrak{S}}[f]$ is the Fourier series of a function $g\in L^2$ follows from Bessel's inequality and the Riesz-Fischer theorem. Consequently, the first arithmetic means $\bar\sigma_n(x;f)$ of $\bar{\mathfrak{S}}[f]$ converge almost everywhere. Thence follows the existence of $\bar f(x)$ (§ 3.32), and since, at almost every point, $\bar\sigma_n(x,f)\to g(x)$, $\bar\sigma_n(x;f)\to \bar f(x)$, we obtain that $g\equiv \bar f$. This completes the proof. We may add that, by Parseval's relation,

$$(4)\qquad \frac{1}{\pi}\int_0^{2\pi} f^2\,dx = \tfrac12 a_0^2 + \frac{1}{\pi}\int_0^{2\pi}\bar f^2\,dx.$$

[1]) L u s i n [1].

4.23. The result (iii) obtained in the preceding section will be generalized in Chapter VII, where it will be shown that the integral 4.22(3) exists almost everywhere for any integrable f. Here we will make a few remarks of a different character.

The existence of $\bar f(x)$ is not trivial even when $f(x)$ is continuous. The convergence of this integral is due not to the smallness of $f(x+t)-f(x-t)$ for small t, but to the interference of positive and negative values, for, as we will show, *there exist continuous functions f such that the integral*

$$(1)\qquad \int_0^{\pi}\frac{|f(x+t)-f(x-t)|}{t}\,dt$$

diverges at every point [1]). It will slightly simplify the notation if we consider functions f of period 1 and replace the upper limit of integration π by 1 in the integral (1). We begin by proving the following lemma.

Let $g(x)$, where $|g(x)|\leqslant 1$, $|g'(x)|\leqslant 1$, be a function of period 1, and such that for no value of x the difference $g(x+u)-g(x-u)$ vanishes identically in u [2]). *Then, for $n=2,3,\ldots$, we have*

$$\int_{1/n}^{1}\frac{|g(nx+nt)-g(nx-nt)|}{t}\,dt \geqslant C\log n,\quad \int_0^1\frac{|g(nx+nt)-g(nx-nt)|}{t}\,dt < C_1\log n,$$

where the constants C and C_1 are independent of n.

Let $nx=y$, $nt=u$. In virtue of the periodicity of g, the first integral takes the form

$$\int_0^1 |g(y+u)-g(y-u)|\left(\frac{1}{u+1}+\ldots+\frac{1}{u+n-1}\right)du \geqslant$$
$$\geqslant \left(\frac12+\ldots+\frac1n\right)\int_0^1 |g(y+u)-g(y-u)|\,du.$$

The first factor on the right exceeds a multiple of $\log n$ and the second, as a continuous, periodic, and non-vanishing function of y, is bounded from below by a positive number. This gives the first part of the lemma. Similarly we obtain the second part, observing that the integral of $|g(y+u)-g(y-u)|/u$ over $(0,1)$ does not exceed 2.

Let us now put

$$(2)\qquad f(x) = \sum_{n=1}^{\infty} a_n g(\lambda_n x),$$

where the coefficients $a_n>0$ and the integers $\lambda_1<\lambda_2<\ldots$ will be defined in a moment. The integral of $|f(x+t)-f(x-t)|/t$ over $(1/\lambda_\nu, 1)$ is not less than

[1]) For the divergence almost everywhere of this integral, and of the integrals (4) below, see L u s i n [1], 182, T i t c h m a r s h [2], H a r d y and L i t t l e w o o d [4]. For the general result see K a c z m a r z [3], [4].

[2]) For example, we may take for the curve $y=g(x)$, $0\leqslant x\leqslant 1$ the broken line passing trough the points $(0,0)$, $(^1/_3, ^1/_3)$, $(1,0)$.

$$a_\nu \int_{1/\lambda_\nu}^{1} \frac{|g(\lambda_\nu x+\lambda_\nu t)-g(\lambda_\nu x-\lambda_\nu t)|}{t}\,dt -$$

$$(3)\qquad -\left(\sum_{n=1}^{\nu-1}+\sum_{n=\nu+1}^{\infty}\right) a_n \int_{1/\lambda_\nu}^{1} \frac{|g(\lambda_n x+\lambda_n t)-g(\lambda_n x-\lambda_n t)|}{t}\,dt > C a_\nu \log\lambda_\nu -$$

$$-C_1 \sum_{n=1}^{\nu-1} a_n \log\lambda_n - 2\log\lambda_\nu \sum_{n=\nu+1}^{\infty} a_n,$$

since $|g(\lambda_n x+\lambda_n t)-g(\lambda_n x-\lambda_n t)| \leqslant 2$. If we put $a_n = 1/n!$, $\lambda_n = 2^{(n!)^2}$ the right-hand side of (3) divided by $\nu!$ tends to $C\log 2 > 0$, and this proves that (1) diverges everywhere.

It is interesting to observe that the integrals

$$(4)\qquad \int_0^{\pi} \frac{f(x+t)-f(x)}{t}\,dt, \qquad \int_0^{\pi} \frac{f(x+t)+f(x-t)-2f(x)}{t}\,dt,$$

apparently similar to the integral 4.22(3), may diverge everywhere for f continuous. The proof, although analogous to that given above, is slightly less simple. See also § 3.9.5.

4.3. We have proved that the necessary and sufficient condition that numbers $a_0, a_1, b_1, \ldots$ should be the Fourier coefficients of a function $f \in L^2$ is that $\frac{1}{2}a_0^2 + (a_1^2+b_1^2) + \ldots$ should converge. The question arises if anything so simple can be proved for the classes L^r with $r \neq 2$. The answer is negative and it is just this answer which makes the Riesz-Fischer theorem and the Parseval relation such an exceptional tool of investigation. Postponing to a later chapter the discussion of some partial results which may be obtained in this direction, we will consider here criteria of a different kind, involving the Cesàro or Abel means of the series considered.

Besides the classes L_φ, L^r introduced in § 4.1 we shall consider the classes B of bounded and C of continuous, periodic functions. If a trigonometrical series

$$(1)\qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

is a $\mathfrak{S}[f]$, with f belonging to L_φ, B or C, we shall say that the series (1) itself belongs to L_φ, B, C respectively. By S we shall denote the class of Fourier-Stieltjes series.

The first arithmetic means of the series (1) will be denoted by $\sigma_n(x)$.

4.31. Classes B and C. *A necessary and sufficient condition that the series* 4.3(1) *should belong to C is the uniform convergence of the sequence $\{\sigma_n(x)\}$.* The necessity is nothing else but Fejér's theorem. To prove the sufficiency, we observe that, for $n > |k|$, we have

$$(2)\qquad \left(1-\frac{|k|}{n+1}\right)c_k = \frac{1}{2\pi}\int_0^{2\pi}\sigma_n(x)\,e^{-ikx}\,dx.$$

As $n\to\infty$, the left-hand side tends to c_k, and the expression on the right to the Fourier coefficient of the function $f(x) = \lim\sigma_n(x)$.

A necessary and sufficient condition that 4.3(1) *should belong to B, is the existence of a constant K such that $|\sigma_n(x)| \leqslant K$ for all x and n.* The necessity was proved in § 3.22, with K equal to the essential upper bound of $|f|$. Conversely, if $|\sigma_n| \leqslant K$, we obtain that

$$2K^2 \geqslant \frac{1}{\pi}\int_0^{2\pi}\sigma_n^2\,dx = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{n}(a_k^2+b_k^2)\left(1-\frac{k}{n+1}\right)^2 \geqslant$$
$$\geqslant \tfrac{1}{2}a_0^2 + \sum_{k=1}^{\nu}(a_k^2+b_k^2)\left(1-\frac{k}{n+1}\right)^2,$$

where $\nu > 0$ is any fixed integer less than n. Making $n\to\infty$ we see that $\frac{1}{2}a_0^2 + (a_1^2+b_1^2)+\ldots+(a_\nu^2+b_\nu^2) \leqslant 2K^2$. Since ν is arbitrary, the series $\frac{1}{2}a_0^2+(a_1^2+b_1^2)+\ldots$ converges, and so 4.3(1) is a $\mathfrak{S}[f]$ with $f\in L^2$. Therefore $\sigma_n(x)\to f(x)$ almost everywhere, and the inequalities $|\sigma_n(x)| \leqslant K$, imply that $|f(x)| \leqslant K$ almost everywhere.

4.32. The class S. *A necessary and sufficient condition that the series* 4.3.(1) *should belong to S is that $\mathfrak{M}[\sigma_n] \leqslant V$, where V is a finite constant independent of n* [1]).

If 4.3(1) is a $\mathfrak{S}[dF]$, then

$$(1)\ \sigma_n(x) = \frac{1}{\pi}\int_0^{2\pi}K_n(x-t)\,dF(t), \quad |\sigma_n(x)| \leqslant \frac{1}{\pi}\int_0^{2\pi}K_n(x-t)\,|dF(t)|\,{}^{2}).$$

Integrating this inequality with respect to x, and interchanging the order of integration on the right [3]), we find that

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[1]) Young [8].

[2]) $|dF(t)|$ means the same as $dV(t)$, where $V(t)$ is the total variation of F over $(0, t)$.

[3]) Since $K_n(u)$ is continuous, the justification of this procedure is immediate: we may replace the integral of $|\sigma_n(x)|$ by approximate Riemannian sums and interchange the order of summation and integration.

$$\mathfrak{M}[\sigma_n] \leqslant \frac{1}{\pi}\int_0^{2\pi}|dF(t)|\left|\int_0^{2\pi}K_n(x-t)\,dx\right| = \int_0^{2\pi}|dF(t)| = V,$$

where V is the total variation of F over $(0, 2\pi)$. For the second part of the theorem we need the following important lemma.

4.321. *Given a sequence of functions $\{F_n(x)\}$, $a \leqslant x \leqslant b$, of uniformly bounded variation, either there exists a uniformly bounded subsequence $\{F_{n_k}(x)\}$ converging everywhere to a function $F(x)$ of bounded variation, or $\{|F_n(x)|\}$ diverges uniformly to $+\infty$ as $n \to \infty$*[1].

Suppose first that all the functions F_n are non-negative, non-decreasing and less than a constant V. Let $R = \{r_n\}$ be the sequence consisting of all the rational points from (a, b) and of the points a, b. $\{F_n(r_1)\}$ being bounded, we can find a sequence $(S_1)\ p_1^1, p_2^1, \ldots p_k^1, \ldots$ of indices, such that $\{F_{p_k^1}(r_1)\}$ converges. Rejecting the first term p_1^1, we find from the remaining indices $p_2^1, p_3^1, \ldots$ a subsequence $(S_2)\ p_1^2, p_2^2, \ldots . p_k^2, \ldots$ such that $\{F_{p_k^2}(r_2)\}$ converges. Rejecting p_1^2, we choose among the rest a subsequence $(S_3)\ p_1^3, p_2^3, \ldots$ such that $\{F_{p_k^3}(r_3)\}$ converges and so on. The sequence $p_1^1, p_1^2, p_1^3, \ldots$ being, from some place onwards, a subsequence of every S_i, we see that $\{F_{p_1^k}(x)\}$ converges, at least for rational x, to a limit $F(x)$, non-decreasing over the set where it exists.

For any x interior to (a, b) put $d(x) = \lim\limits_{r \to x+0} F(r) - \lim\limits_{r \to x-0} F(r)$, $r \in R$. Since for any system $x_1, x_2, \ldots, x_n$ we have $d(x_1) + \ldots + d(x_n) \leqslant V$, it follows that the number of the points x where $d(x) \geqslant \varepsilon > 0$ is finite. Let Z be the at most enumerable set of points for which $d(x) > 0$. We will prove that, for any $x \,\bar{\epsilon}\, Z$, $\lim F_{p_1^k}(x)$ exists. In fact, given an arbitrary $\eta > 0$ and an $x \,\bar{\epsilon}\, Z$, $x \neq a, b$, we can find two rational points $r' < x < r''$, such that $0 \leqslant F(r'') - F(r') < \eta$. Since $F_{p_1^k}(r') \leqslant F_{p_1^k}(x) \leqslant F_{p_1^k}(r'')$, where the extreme terms tend to $F(r'), F(r'')$ as $k \to \infty$, we see that the oscillation of $\{F_{p_1^k}(x)\}$ does not exceed η, i. e. the sequence converges.

Let D be the set of points where $\{F_{p_1^k}(x)\}$ diverges; D is at most enumerable. Repeating with D the same argument as with R, we

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[1]) Helly [1].

find a subsequence $\{n_k\}$ of $\{p_1^k\}$ such that $\{F_{n_k}(x)\}$ converges in D, i. e. everywhere in (a, b).

In the general case we put $F_n(x) = F_n(a) + P_n(x) - N_n(x)$, where $P_n(x)$ and $N_n(x)$ denote the positive and negative variations of $F_n(x) - F_n(a)$. Let us suppose that we can find a sequence $\{m_k\}$ such that $\{F_{m_k}(a)\}$ converges to a finite limit. From $\{m_k\}$ we choose a subsequence $\{m_k'\}$ such that $\{P_{m_k'}(x)\}$ converges, and from $\{m_k'\}$ a subsequence $\{n_k\}$ such that $\{N_{n_k}(x)\}$, and therefore $\{F_{n_k}(x)\}$ converges. That $F(x) = \lim F_{n_k}(x)$ is of bounded variation, follows from the fact that $F(x) = \lim F_{n_k}(a) + \lim P_{n_k}(x) - \lim N_{n_k}(x)$, where the last two terms are non-decreasing and bounded functions of x.

If our assumption concerning $\{F_n(a)\}$ does not hold, then $|F_n(a)| \to \infty$. Since the oscillations of the functions $F_n(x)$ are uniformly bounded, it is easy to see that $\{|F_n(x)|\}$ diverges uniformly to $+\infty$ as $n \to \infty$. This completes the proof of the lemma.

The following remark will be useful later. If the total variations $P_n(b) + N_n(b)$ of the functions F_n do not exceed a number W, the same is true for the total variation of F.

4.322. Suppose now, in the case of Theorem 4.32, the condition $\mathfrak{M}[\sigma_n] \leqslant V$ satisfied. Let $F_n(x)$ be the integral of $\sigma_n(t)$ over $(0, x)$. The functions $F_n(x)$ are of uniformly bounded variation over $(0, 2\pi)$. Since $F_n(0) = 0$, $n = 1, 2, \ldots$, $\{|F_n(x)|\}$ cannot diverge to $+\infty$ and so there exists a sequence $\{F_{n_j}(x)\}$ uniformly bounded and converging everywhere to a function $F(x)$ of bounded variation. Let $n_j > |k|$. Integrating by parts, and making $j \to \infty$, we obtain

$$\left(1 - \frac{|k|}{n_j + 1}\right) c_k = \frac{1}{2\pi}\int_0^{2\pi} \sigma_{n_j} e^{-ikx}\,dx = \frac{1}{2\pi} F_{n_j}(2\pi) + \frac{ik}{2\pi}\int_0^{2\pi} F_{n_j} e^{-ikx}\,dx,$$

$$c_k = \frac{1}{2\pi} F(2\pi) + \frac{ik}{2\pi}\int_0^{2\pi} F e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\,dF(x),$$

for $k = 0, \pm 1, \ldots$, so that 4.3(1) is $\mathfrak{S}[dF]$. We complete the theorem by a few remarks.

4.323. *If 4.3(1) is a $\mathfrak{S}[dF]$, where $F(x) = \frac{1}{2}[F(x+0) + F(x-0)]$ for every, x and if the total variation of F over $(0, 2\pi)$ is V, then $\mathfrak{M}[\sigma_n] \to V$ as $n \to \infty$.* It has been proved in § 4.32 that $\overline{\lim}\, \mathfrak{M}[\sigma_n] \leqslant V$, and it remains only to show that the assumption $\underline{\lim}\, \mathfrak{M}[\sigma_n] < W < V$ leads to a contradiction.

In fact, let $\{m_j\}$ be such that $\mathfrak{M}[\sigma_{m_j}] \leqslant W$. The sequence $\{F_{n_j}\}$ considered in the preceding section may, plainly, be chosen from $\{F_{m_j}\}$ and, by the final remark of § 4.321, the total variation of $F_*(x) = \lim F_{n_j}(x)$ would not exceed W. Without loss of generality we may assume that $F_*(x) = \frac{1}{2}[F_*(x+0) + F_*(x-0)]$, for if we replace $F_*(x)$ by $\frac{1}{2}[F_*(x+0)+F_*(x-0)]$ at every point of discontinuity, the total variation of the function will not increase. Since $\mathfrak{S}[dF]$ and $\mathfrak{S}[dF_*]$ have the same coefficients, it follows that the difference $F_1(x) = F(x) - F_*(x)$ is equal to a constant C at almost every point x. On the other hand we have $F_1(x) = \frac{1}{2}[F_1(x+0)+F_1(x-0)]$, so that $F_1(x) = C$ for every x. Hence the total variations of F and F_* over $(0, 2\pi)$ are equal, contrary to what we assumed.

4.324. *A necessary and sufficient condition that* 4.3(1) *should be a* $\mathfrak{S}[dF]$ *with* F *non-decreasing is* $\sigma_n(x) \geqslant 0$, $n = 0, 1, 2, \ldots$

The necessity follows from the first formula 4.32(1) since $K_n \geqslant 0$. Conversely, if $\sigma_n(x) \geqslant 0$, the functions $F_n(x)$ considered in § 4.322 are non-decreasing, and the same is true for $F(x) = \lim F_{n_j}(x)$.

4.325. *A necessary and sufficient condition that* 4.3(1) *should be the Fourier series of a function of bounded variation is that* $\mathfrak{M}[\sigma_n'] = O(1)$. This theorem is equivalent to Theorem 4.32 (§ 2.14).

4.326. Carathéodory's theorem. Let $\{F_k(x)\}$, $0 \leqslant x \leqslant 2\pi$, be a uniformly bounded sequence of functions. If $F_k(x)$ tends almost everywhere to a limit $F(x)$, then $c_n^k \to c_n$ as $k \to \infty$, where c_n^k, c_n, $n = 0, \pm 1, \ldots$ denote the Fourier coefficients of the functions $F_k(x)$, $F(x)$ respectively. Simple examples show that, without additional conditions, the converse theorem is false, and is an important fact that this converse theorem is true when the functions $F_k(x)$ are monotonic. More precisely:

Let $\{F_k(x)\}$, $0 \leqslant x < 2\pi$ *be a sequence of uniformly bounded and non-decreasing functions, and let* c_n^k *be the complex Fourier coefficients of* F_k. *If, for* $n = 0, \pm 1, \pm 2, \ldots$, *we have* $\lim c_n^k = c_n$ *as* $k \to \infty$, *the numbers* c_n *are the Fourier coefficients of a monotonic function* $F(x)$, *and* $F_k(x) \to F(x)$ *at every point* x, $0 < x < 2\pi$, *where* $F(x)$ *is continuous* [1]).

[1]) Carathéodory [2].

In virtue of Theorem 4.321 there is a subsequence $\{F_{k_j}\}$ of $\{F_k\}$ converging everywhere to a non-decreasing function $F(x)$. It is plain that the Fourier coefficients of F are c_n, and we have only to show that $F_n(x) \to F(x)$ except, perhaps, at the set of points where F is discontinuous. Let ξ, $0 < \xi < 2\pi$, be a point of continuity of $F(x)$. Let us suppose that $F_k(\xi)$ does not tend to $F(\xi)$. We can then find a sequence $\{F_{k_i}\}$ such that $\lim F_{k_i}(\xi)$ exists and is $\neq F(\xi)$. To fix ideas let us suppose that $\lim F_{k_i}(\xi) > F(\xi)$. We can find a subsequence $\{F_{l_i}(x)\}$ of $\{F_{k_i}(x)\}$ such that $\lim F_{l_i}(x) = G(x)$ exists everywhere. The Fourier coefficients of G are c_n, and so $F(x) \equiv G(x)$. On the other hand $G(\xi) = \lim F_{l_i}(\xi) = \lim F_{k_i}(\xi) > F(\xi)$, and, since $G(x)$ is non-decreasing and $F(x)$ is continuous for $x = \xi$, we have $G(x) > F(x)$ in an interval $\xi \leqslant x \leqslant \xi + h$, $h > 0$, so that $G(x) \not\equiv F(x)$. This contradiction shows that $F_k(\xi) \to F(\xi)$.

4.33. Classes L_φ [1]). *Let* $\varphi(u)$, $u \geqslant 0$, *be convex, non-negative, and such that* $\varphi(u)/u \to \infty$ *as* $u \to \infty$ [2]). *A necessary and sufficient condition that* 4.3(1) *should belong to* L_φ *is that* $\mathfrak{M}[\varphi|\sigma_n|] \leqslant C$, *where* C *is finite and independent of* n [3]).

We may suppose that $\varphi(u)$ is non-decreasing, for otherwise it is sufficient to consider the function $\varphi^*(u)$ equal to $\varphi(u)$ for $u \geqslant u_0$ and to $\varphi(u_0)$ for $0 \leqslant u \leqslant u_0$, u_0 denoting the point where φ attains its minimum. The classes L_φ and L_{φ^*} are plainly identical.

To prove the necessity of the condition consider the inequality

$$(1) \qquad |\sigma_n(x)| \leqslant \frac{1}{\pi}\int_0^{2\pi} K_n(x-t)\,|f(t)|\,dt.$$

By Jensen's theorem, and taking into account that the integral of the function $p(t) = K_n(x-t)/\pi$ over $(0, 2\pi)$ is equal to 1, we find that

$$(2) \qquad \varphi|\sigma_n(x)| \leqslant \frac{1}{\pi}\int_0^{2\pi} \varphi|f(t)|\,K_n(x-t)\,dt.$$

Integrating this with respect to x and inverting the order of integration, we find the important inequality

[1]) Young [10], see also Zygmund [4].
[2]) It follows that φ is bounded in any finite interval.
[3]) We write $\varphi|\sigma_n|$ instead of $\varphi(|\sigma_n|)$.

(3) $$\mathfrak{M}[\varphi\,|\,\sigma_n\,|] \leqslant \mathfrak{M}[\varphi\,|\,f\,|],$$

which gives the first half of the theorem.

As regards the second half, the Jensen inequality $\varphi(\mathfrak{M}[\sigma_n]/2\pi) \leqslant \mathfrak{M}[\varphi\,|\,\sigma_n\,|]/2\pi \leqslant C/2\pi$ implies that $\mathfrak{M}[\sigma_n] = O(1)$, i. e. the series 4.3(1) is a $\mathfrak{S}[dF]$ (§ 4.32). To prove that $F(x)$ is absolutely continuous, it is sufficient to show that the functions $F_n(x) = \int_0^x \sigma_n(t)\,dt$ are uniformly absolutely continuous, i. e. that, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any finite system S of non-overlapping intervals (a_1, b_1), (a_2, b_2), ..., $(b_1 - a_1) + (b_2 - a_2) + ... < \delta$, we have

(4) $$\sum_i |F_n(b_i) - F_n(a_i)| < \varepsilon, \quad n = 1, 2, \ldots\ ^{1)}.$$

The inequality

$$\varphi\left(\frac{1}{|S|}\int_S |\sigma_n(x)|\,dx\right) \leqslant \frac{\int_0^{2\pi} \varphi\,|\,\sigma_n\,|\,dx}{|S|} \leqslant \frac{C}{|S|}$$

may be written in the form $\varphi(\xi u)/\xi u \leqslant C/\xi$, where $u = 1/|S|$, $\xi = \int_S |\sigma_n|\,dx$. In view of our hypothesis concerning φ, we see that if $u \to \infty$, then $\xi \to 0$, and so if $|S|$ is sufficiently small, then $\xi < \varepsilon$.

Since the left-hand side of (4) does not exceed ξ, the absolute continuity of F follows.

Let $F'(x) = f(x)$. The series 4.3(1) is $\mathfrak{S}[f]$. To show that $f \in L_\varphi$, we observe that $\sigma_n \to f$ almost everywhere, and, applying Fatou's lemma to the inequality $\mathfrak{M}[\varphi\,|\,\sigma_n\,|] \leqslant C$, we find that $\mathfrak{M}[\varphi\,|\,f\,|] \leqslant C$.

As a corollary we obtain that *a necessary and sufficient condition that* 4.3(1) *should belong to* L^r, $r > 1$, *is that* $\mathfrak{M}_r[\sigma_n] = O(1)$[2]. As Theorem 4.32 shows, this result does not hold for $r = 1$.

4.34. *A necessary and sufficient condition that* 4.3(1) *should be a Fourier series is that* $\mathfrak{M}[\sigma_m - \sigma_n] \to 0$ *as* $m, n \to \infty$[3].

[1]) In fact, if, for fixed S, the inequality (4) is satisfied by the functions F_n, it is also satisfied by $F = \lim F_n$.

[2]) W. H. and G. C. Young [1].

[3]) Steinhaus [2], Gross [1].

Let us suppose that 4.3(1) is a $\mathfrak{S}[f]$. Integrating the inequality

(1) $$|\sigma_n(x) - f(x)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|\,K_n(t)\,dt$$

over $(0, 2\pi)$, we find that

(2) $$\mathfrak{M}[\sigma_n - f] \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} \eta(t)\,K_n(t)\,dt, \text{ where } \eta(t) = \int_{-\pi}^{\pi} |f(x+t) - f(x)|\,dx.$$

Since $\eta(t)$ is continuous and vanishes for $t = 0$, and the right-hand side of the last inequality is the n-th Fejér sum of $\mathfrak{S}[\eta]$ at $t = 0$, we see that $\mathfrak{M}[\sigma_n - f] \to 0$, and so $\mathfrak{M}[\sigma_m - \sigma_n] \leqslant \mathfrak{M}[\sigma_m - f] + \mathfrak{M}[\sigma_n - f] \to 0$ as $m, n \to \infty$.

Conversely, the condition $\mathfrak{M}[\sigma_m - \sigma_n] \to 0$ implies $\mathfrak{M}[\sigma_n] = O(1)$, i. e. 4.3(1) is a $\mathfrak{S}[dF]$. To show that F is absolutely continuous, it is enough to prove (as in § 4.33) that $\mathfrak{M}[\sigma_n; S]$[1]) is small with $|S| = (b_1 - a_1) + (b_2 - a_2) + \ldots$, uniformly in n. Now $\mathfrak{M}[\sigma_n; S] \leqslant \mathfrak{M}[\sigma_n - \sigma_\nu; S] + \mathfrak{M}[\sigma_\nu; S] \leqslant \mathfrak{M}[\sigma_n - \sigma_\nu; 0, 2\pi] + \mathfrak{M}[\sigma_\nu; S]$. Let ν be so large that $\mathfrak{M}[\sigma_n - \sigma_\nu] < \frac{1}{2}\varepsilon$ for $n > \nu$. For fixed ν we have $\mathfrak{M}[\sigma_\nu; S] < \frac{1}{2}\varepsilon$ if only $|S| < \delta = \delta(\varepsilon)$. Therefore $\mathfrak{M}[\sigma_n; S] < \varepsilon$ for $n > \nu$, $|S| < \delta$, and this completes the proof.

4.35. Suppose that a convex and non-negative function $\varphi(u)$ satisfies the condition $\varphi(0) = 0$, so that φ is non-decreasing. Assuming that 4.3(1) belongs to L_φ, we may ask under what conditions $\mathfrak{M}[\varphi\,|\,\sigma_n - f\,|] \to 0$. Starting from 4.34(1) and using an argument similar to that of § 4.34, we see that $\mathfrak{M}[\varphi\,|\,\sigma_n - f\,|] \to 0$, if only the function

(1) $$\eta(t) = \int_{-\pi}^{\pi} \varphi\{|f(x+t) - f(x)|\}\,dx.$$

is integrable and tends to 0 with t. This may not be true if φ increases too rapidly, but an insertion of the factor $^1/_4$ into curly brackets saves the situation: if $f \in L_\varphi$, then the function $\mathfrak{M}[\varphi\{^1/_4\,|f(x+t) - f(x)|\}]$ is integrable and tends to 0 with t. In fact, let $f = g + h$, where g is bounded and $\mathfrak{M}[\varphi\,|\,h\,|] < \varepsilon$. By Jensen's inequality we have

[1]) This symbol denotes the integral of $|\sigma_n|$ over S.

$$\mathfrak{M}[\varphi\{{}^1/_4|f(x+t)-f(x)|\}]\leqslant\tfrac{1}{2}\mathfrak{M}[\varphi\{\tfrac{1}{2}|g(x+t)-g(x)|\}]+ +\tfrac{1}{2}\mathfrak{M}[\varphi\{\tfrac{1}{2}|h(x+t)-h(x)|\}].$$

The last term on the right does not exceed ${}^1/_4\,\mathfrak{M}[\varphi|h(x+t)|]+ +{}^1/_4\,\mathfrak{M}[\varphi|h|]<\varepsilon/2$, and, since the preceding term tends to 0, [1] the left-hand side is less than ε for $|t|$ sufficiently small.

At the same time we have proved that, *if the series* 4.3(1) *is a* $\mathfrak{S}[f]$ *with* $f\in L_\varphi$, *where* $\varphi(u)$ *is convex, non-negative, and* $\varphi(0)=0$, *then*

$$\mathfrak{M}[\varphi\{{}^1/_4|f-\sigma_n|\}]\to 0 \text{ as } n\to\infty.$$

In particular, *if* $f\in L^r$, $r\geqslant 1$, *then* $\mathfrak{M}_r[f-\sigma_n]\to 0$ [2]).

4.36. Abel means. So far we have worked with Fejér's kernel. The essential property of this kernel, viz. positiveness, is shared by some other kernels, in particular by Poisson's kernel. Therefore all our results remain true for Abel's method of summation, which, as we know, has a very important function-theoretic significance. Since the proofs are esentially the same as before [3]), we content ourselves with stating the results [4]). By $f(r,x)$ we mean the harmonic function corresponding to the series 4.3(1).

(i) *A necessary and sufficient condition that* 4.3(1) *should belong to C is that* $f(r,x)$ *should converge uniformly as* $r\to 1$; *a necessary and sufficient condition that* 4.3(1) *should belong to B, is that* $f(r,x)$ *should be bounded for* $0\leqslant r<1$, $0\leqslant x\leqslant 2\pi$.

(ii) *A necessary and sufficient condition that* $f(r,x)$ *should satisfy a relation*

$$f(r,x)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-x)+r^2}\,dF(t),$$

where F is of bounded variation, is that $\mathfrak{M}[f(r,x)]=O(1)$ *as* $r\to 1$. *If V is the total variation of F over* $(0,2\pi)$, *and if* $2F(x)=F(x+0)+ +F(x-0)$ *for every* x, *then* $\mathfrak{M}[f(r,x)]\to V$ *as* $r\to 1$. *F is non-decreasing if and only if* $f(r,x)\geqslant 0$.

(iii) Let $\varphi(u)$ satisfy the hypothesis of Theorem 4.33. Then *a necessary and sufficient condition that* 4.3(1) *should belong to* L_φ *is that* $\mathfrak{M}[\varphi|f(r,x)|]=O(1)$ *as* $r\to 1$.

If 4.3(1) *is a* $\mathfrak{S}[f]$ *with* $f\in L_\varphi$, *then* $\mathfrak{M}[\varphi\{{}^1/_4|f(x)-f(r,x)|\}\to 0$ *as* $r\to 1$. *If* $f\in L^r$, $r\geqslant 1$, *then* $\mathfrak{M}_r[f(x)-f(r,x)]\to 0$.

(iv) *The series* 4.3(1) *is a Fourier series if and only if* $\mathfrak{M}[f(r,x)-f(\rho,x)]\to 0$ *as* $r,\rho\to 1$.

4.37. (C,k) **means.** Most of the results remain true, although some inequalities become less precise, for quasi-positive kernels, in particular for the (C,k) kernels, $k>0$. Let $\lambda_n=\lambda_n^{(k)}$ denote the integral of $|K_n^k(u)|/\pi$ over $(0,2\pi)$, and $\lambda=\lambda^{(k)}$ the upper bound of $\{\lambda_n^{(k)}\}$, $n=1,2,\ldots$ We quote the following theorems, the proofs of which follow immediately.

(i) *If* $\mathfrak{M}[\varphi|\sigma_n^k|]=O(1)$, *then the series* 4.3(1) *belongs to* L_φ. *If* 4.3(1) *is a* $\mathfrak{S}[f]$ *with* $f\in L_\varphi$, *then* $\mathfrak{M}[\varphi|\lambda^{-1}\sigma_n^k|]=O(1)$, *and* $\mathfrak{M}[\varphi\{|\sigma_n^k-f|/4\lambda\}]=o(1)$. In particular, *a necessary and sufficient condition that* 4.3(1) *should belong to* L^r, $r>1$, *is that* $\mathfrak{M}_r[\sigma_n^k]=O(1)$. *If* 4.3(1) *is a* $\mathfrak{S}[f]$ *with* $f\in L^r$, $r\geqslant 1$, *then* $\mathfrak{M}_r[f-\sigma_n^k]\to 0$ *as* $n\to\infty$.

(ii) *A necessary and sufficient condition that* 4.3(1) *should belong to S is that* $\mathfrak{M}[\sigma_n^k]=O(1)$.

(iii) *A necessary and sufficient condition that* 4.3(1) *should belong to L is that* $\mathfrak{M}[\sigma_m^k-\sigma_n^k]\to 0$ *as* $m,n\to\infty$.

4.38. Let us replace σ_n by the partial sums s_n in the theorems of §§ 4.31—4.35. The conditions which we obtain remain sufficient (although, as we shall see later, some of them are no longer necessary). The proofs are similar, except at one point: we cannot use the fact that if 4.3(1) is a $\mathfrak{S}[f]$, then $s_n(x)\to f(x)$ almost everywhere, for such a theorem is false. But for our purposes it is sufficient to assume that there exists a *subsequence* $\{s_{n_k}(x)\}$ of $\{s_n(x)\}$ converging to f almost everywhere, and we shall see in § 7.3 that this is certainly true if $\{n_k\}$ increases sufficiently rapidly.

4.39. In the sufficiency-parts of the theorems of §§4.31—4.38 it is enough to assume that the conditions imposed upon $\sigma_n(x)$, $f(r,x)$, or $s_n(x)$, are satisfied not for all indices n, r but only for a sequence of them. The proofs require no changes.

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1) From our hypothesis concerning φ it follows that, in any finite interval $0\leqslant u\leqslant a$, we have $\varphi(u)\leqslant Mu$, with $M=M(a)$.

2) W. H. and G. C. Young [1].

3) That in Abel's method the variable changes continuously is quite immaterial, since we may consider any sequence $\{r_n\}$ tending to 1.

4) See also: Evans, *The logarithmic potential*, Fichtenholz [1].

Thus if, for a sequence $n_1 < n_2 < \dots$, $\{s_{n_k}\}$ or $\{\sigma_{n_k}\}$ converges uniformly, the series 4.3(1) belongs to C. If $\mathfrak{M}[s_{n_k}] = O(1)$, the series belongs to S, etc.

This enables us to state some of the theorems given above in a slightly different form. For example, *a necessary and sufficient condition that* 4.3(1) *should belong to C is that the functions* $\sigma_n(x)$ *should be uniformly continuous.* The necessity follows from the inequality 4.33(1), which, applied to $f(t+h)-f(t)$, shows that $\omega(\delta; \sigma_n) \leqslant \omega(\delta; f)$ (§ 2.2). Conversely, if the functions $\sigma_n(x)$ are uniformly continuous, there exists a sequence $\{\sigma_{n_k}(x)\}$ converging uniformly to a continuous function $f(x)$ [1]), and so the series is $\mathfrak{S}[f]$, $f \in C$.

4.4. Parseval's relations. Let f and g be two functions of the class L^2, with Fourier coefficients a_n, b_n and a'_n, b'_n respectively. Adding the Parseval formulae 4.22(2a) formed for $f+g$ and $f-g$, we obtain

$$(1) \qquad \frac{1}{\pi}\int_0^{2\pi} fg\, dx = \frac{a_0 a'_0}{2} + \sum_{n=1}^{\infty}(a_n a'_n + b_n b'_n),$$

where the series on the right converges absolutely. The formula (1), which is called Parseval's relation for f and g, holds in other cases besides the one in which $f \in L^2$, $g \in L^2$ [2]). Two classes of functions K and K_1 will be called *complementary* classes if (1) holds for every $f \in K$, $g \in K_1$. The series on the right need not be convergent; we shall only suppose that it is summable by some method of summation. It will appear that the Fourier series of functions belonging to complementary classes have, in some cases, much the same, or analogous, properties, and Parseval's formula (1), where f and g enter symmetrically, is just the means to discover these properties in common.

4.41. *The following are pairs of complementary classes:* (i) L_Φ *and* L_Ψ, *where* Φ *and* Ψ *are Young's complementary functions,* (ii) L^r *and* $L^{r'}$ $(r>1)$, (iii) B *and* L, (iv) C *and* S. *In all these cases the series in* 4.4(1) *is summable* $(C, 1)$.

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[1]) We apply here Arzela's well-known theorem on families of uniformly continuous functions. See e. g. Hobson, *Theory of functions* 2, 168.

[2]) The formula is obvious if one of the functions f and g is a trigonometrical polynomial. The series on the right consists then of a finite number of terms.

Part (iv) of the theorem is to be understood in the sense that, if a_n, b_n are the coefficients of a $\mathfrak{S}[f]$, $f \in C$, and a'_n, b'_n are the coefficients of a $\mathfrak{S}[dG] \in S$, then we have the formula 4.4(1) with $fg\,dx$ replaced by $F dG$. Part (iii) is a limiting case $(r=\infty)$ of (ii).

Let $\sigma_n(x)$ be the $(C, 1)$ means of $\mathfrak{S}[f]$, τ_n the $(C, 1)$ means of the series in 4.4(1), and Δ_n the difference between the left-hand side of 4.4(1) and τ_n. We have then

$$(1) \qquad \Delta_n = \frac{1}{\pi}\int_0^{2\pi}(f-\sigma_n)\, g\, dx,$$

and, applying Hölder's inequality, we see that $|\Delta_n|$ does not exceed $\pi^{-1}\mathfrak{M}_r[f-\sigma_n]\,\mathfrak{M}_{r'}[g] \to 0$ as $n\to\infty$. This proves part (ii) of the theorem. To establish part (i), which embraces (ii), we apply Young's inequality to $|\Delta_n|/16$:

$$\pi\,|\Delta_n|/16 \leqslant \mathfrak{M}[\Phi\{{}^1/_4\,|f-\sigma_n|\}] + \mathfrak{M}[\Psi\{{}^1/_4\,|g|\}].$$

From Theorem 4.35, we obtain that $\overline{\lim}\,\Delta_n \leqslant 16\pi^{-1}\mathfrak{M}[\Psi\{{}^1/_4\,|g|\}]$. Let $g = g' + g''$, where g' is a trigonometrical polynomial and $\mathfrak{M}[\Psi\{{}^1/_4\,|g''|\}] < \varepsilon$ [1]). Substituting, in (1), g' and g'' for g, we obtain expressions Δ'_n and Δ''_n, such that $\Delta_n = \Delta'_n + \Delta''_n$. Since g' is only a polynomial, we see from Parseval's formula for f and g' that $\Delta'_n \to 0$. On the other hand, $\overline{\lim}\,\Delta''_n \leqslant 16\pi^{-1}\mathfrak{M}[\Psi\{{}^1/_4\,|g''|\}] < 16\varepsilon/\pi$. Since $\overline{\lim}\,\Delta_n \leqslant \overline{\lim}\,\Delta'_n + \overline{\lim}\,\Delta''_n < 16\varepsilon/\pi$, where ε is arbitrary, we infer that $\Delta_n \to 0$.

If f is bounded, $|f| \leqslant M$, g integrable, then $|f-\sigma_n|\,|g|$ tends to 0 almost everywhere and is majorised by the integrable function $2M|g|$. Applying Lebesgue's theorem on the integration of sequences, we conclude from (1) that $\Delta_n \to 0$.

Finally, to prove (iv), let us replace in (1) $g(x)$ by $dG(x)$. Since $\pi\,|\Delta_n|$ does not exceed Max $|f(x)-\sigma_n(x)|$, $0 \leqslant x \leqslant 2\pi$, multiplied by the total variation of G over $(0, 2\pi)$, we have again $\Delta_n \to 0$, provided that f is continuous.

4.411. Let $g(x)$ be the characteristic function of a set E, and $f(x)$ an arbitrary integrable function. Parseval's formula for f and g may be written in the form

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[1]) We may take for g'' a $(C, 1)$ mean of $\mathfrak{S}[g]$, with index sufficiently large (§ 4.35).

$$\int_E f\,dx = \tfrac{1}{2} a_0 |E| + \sum_{n=1}^{\infty} \int_E (a_n \cos nx + b_n \sin nx)\,dx.$$

Hence $\mathfrak{S}[f]$ *may be integrated term by term over any measurable set and the resulting series is summable* $(C, 1)$ *to the integral of* f *over the set.* As we shall see later, the integrated series converges if $f \in L^r$, $r > 1$. If $f \in L$, this is not necessarily true (§ 4.7.16).

4.42. Applying Parseval's equation 4.4(1) to the functions $f(x+t)$ and $g(x)$, we find the formula

$$(1)\qquad \frac{1}{\pi}\int_0^{2\pi} f(x+t)\,g(x)\,dx = \frac{a_0 a_0'}{2} + \sum_{n=1}^{\infty}\{(a_n a_n' + b_n b_n')\cos nt + (a_n' b_n - a_n b_n')\sin nt\},$$

where the series on the right is uniformly summable $(C, 1)$ in each of the cases considered in Theorem 4.41. Moreover, *given any pair of integrable functions* f, g, *the formula* (1) *holds, in the* $(C, 1)$ *sense, almost everywhere in* t. For the proof it is sufficient to observe that the left-hand side $h(t)$ of (1) is an integrable function and that the series on the right is $\mathfrak{S}[h]$ (§ 2.11).

4.43. Let c_n, c_n' be the complex Fourier coefficients of f, g. The formula 4.4(1) may be written in the form

$$(1)\qquad \frac{1}{2\pi}\int_0^{2\pi} fg\,dx = \sum_{p=-\infty}^{+\infty} c_p c_{-p}' \quad (C, 1).$$

So far we have considered only real functions, but the extension of (1) to the case of f and g complex follows immediately. Substitute $g(x)e^{-inx}$ for $g(x)$ in (1) and let c_p'' denote the Fourier coefficients of $g(x)e^{-inx}$. Since $c_{-p}'' = c_{n-p}'$, we find that

$$(2)\qquad \frac{1}{2\pi}\int_0^{2\pi} fg\,e^{-inx}\,dx = \sum_{p=-\infty}^{+\infty} c_p c_{n-p}' \quad (C, 1), \quad n = 0, \pm 1, \ldots$$

Consequently, *the Fourier series of the product of two functions* f *and* g, $f \in L_\Phi$, $g \in L_\Psi$, *can be obtained by formal multiplication of* $\mathfrak{S}[f]$ *and* $\mathfrak{S}[g]$ *by Laurent's rule. The series defining the coefficients of the product are summable* $(C, 1)$.

The theorem remains valid if $f \in B$, $g \in L$.

4.431. It is obvious that each of the inequalities

$$\sum_{p=-\infty}^{+\infty} |c_p| < \infty, \qquad \sum_{p=-\infty}^{+\infty} |c_p'| < \infty$$

implies the absolute convergence of the series in 4.43(2). If both the inequalities are safisfied, then $\mathfrak{S}[fg]$ converges absolutely.

4.432. In Theorems 4.41, 4.42 and 4.43 we may replace summability $(C, 1)$ by (C, k), $k > 0$. The proofs remain the same if we use the results of § 4.37.

4.44. The problem whether summability (C, k) can be replaced by ordinary convergence is more delicate. In Chapter VII we shall prove that the answer is positive if $f \in L^r$, $g \in L^{r'}$, $1 < r < \infty$. This theorem is rather deep; here we will prove a more elementary result. If s_n denotes the n-th partial sum of $\mathfrak{S}[f]$, the difference δ_n between the integral on the left and n-th partial sum of the series on the right in the formula 4.4(1), may be written in the form

$$(1)\qquad \delta_n = \frac{1}{\pi}\int_0^{2\pi} (f - s_n)\,g\,dx.$$

If the partial sums $s_n(x)$ are uniformly bounded and tend to $f(x)$ almost everywhere, the expression $|f - s_n|\,|g|$ tends to 0 almost everywhere and is majorised by an integrable function. Hence $\delta_n \to 0$, so that the series in 4.4(1) converges to the integral on the left. Hence, reversing the rôle of f and g,

If $f(x)$ *is integrable and* $g(x)$ *is of bounded variation, we have the formula* 4.4(1), *where the series on the right is convergent* [1]).

From this we deduce that, *if* f *is integrable and periodic,* (α, β) *is a finite interval, and* $g(x)$, $\alpha \leqslant x \leqslant \beta$, *is an arbitrary function of bounded variation, not necessarily periodic, then*

$$(1)\qquad \int_\alpha^\beta fg\,dx = \tfrac{1}{2}a_0\int_\alpha^\beta g\,dx + \sum_{n=1}^{\infty}\Big\{a_n\int_\alpha^\beta g\cos nx\,dx + b_n\int_\alpha^\beta g\sin nx\,dx\Big\},$$

i. e. Fourier series may be integrated term by term after having been multiplied by any function of bounded variation [2]). In fact, if $\beta - \alpha = 2\pi$, this is nothing else but the previous theorem. The

[1]) Young [11].
[2]) The case $g(x) = 1$ has been considered in § 2.621.

case $\beta-\alpha<2\pi$ may be reduced to the preceding one, putting $g(x)=0$ for $\beta<x<\alpha+2\pi$. In the general case we break up the interval (α,β) into a finite number of intervals of length $\leqslant 2\pi$.

4.45 [1]). The last result can be extended to the case of an infinite interval. Without loss of generality we may assume that $(\alpha,\beta)=(-\infty,+\infty)$.

The formula

$$(1)\qquad \int_{-\infty}^{+\infty} fg\,dx=\tfrac{1}{2}a_0\int_{-\infty}^{+\infty} g(x)\,dx+\sum_{n=1}^{\infty}\Big\{a_n\int_{-\infty}^{+\infty} g\cos nx\,dx+b_n\int_{-\infty}^{+\infty} g\sin nx\,dx\Big\}$$

holds true for any integrable and periodic function f, provided that $g(x)$ is (i) *integrable and* (ii) *of bounded variation over* $(-\infty,+\infty)$. In fact, let us put

$$(2)\qquad G(x)=\sum_{k=-\infty}^{+\infty} g(x+2k\pi).$$

If the series on the right converges at some point, then it converges uniformly over $(0,2\pi)$, and its sum $G(x)$ is of bounded variation (§ 2.85). On the other hand, since

$$\sum_{k=-\infty}^{+\infty}\int_0^{2\pi}|g(x+2k\pi)|\,dx=\int_{-\infty}^{+\infty}|g(x)|\,dx<\infty,$$

we see that the series in (2) has certainly points of convergence (§ 4.2(i)).

Let $c'_n=\tfrac{1}{2}(a'_n-ib'_n)$ be the the Fourier coefficients of $G(x)$. We have then a formula similar to 4.4(1), with g replaced by G. Observing that uniformly convergent series may be integrated term by term after having been multiplied by any integrable function, and remembering that f is periodic, we obtain from (2) that

$$\int_0^{2\pi} fG\,dx=\int_{-\infty}^{+\infty} fg\,dx,\qquad \int_0^{2\pi} G(x)\,e^{-inx}\,dx=\int_{-\infty}^{+\infty} g(x)\,e^{-inx}\,dx,$$

and the formula just referred to takes the form (1). This completes the proof.

The hypothesis that $g(x)$ is integrable over $(-\infty,\infty)$ is, of course, essential for the truth of the equation (1). However, if $a_0=0$, condition (i) of the previous theorem may be replaced by the condition that $g(x)\to 0$ as $|x|\to\infty$. In fact, let us put $g^*(x)=g(2k\pi)$ for $2k\pi\leqslant x<2(k+1)\pi$, $k=0,\pm1,\dots$, and let v_k be the total variation of $g(x)$ over $(2k\pi, 2(k+1)\pi)$. The function $g^*(x)$ is of bounded variation and, since $\gamma(x)=g(x)-g^*(x)$ does not exceed v_k in absolute value for $2k\pi\leqslant x<2(k+1)\pi$, the function $\gamma(x)$ is integrable and of bounded variation over $(-\infty,\infty)$. Let us apply the formula (1) to the functions f and γ. Since the mean value of f over a period is equal to 0, and $g(x)\to 0$ with $1/x$, it is easy to verify that

[1]) Hardy [7]. An interesting application to the theory of the Riemann ζ function will be found in Hardy [8].

$$\int_{-\infty}^{+\infty} f\gamma\,dx=\int_{-\infty}^{+\infty} fg\,dx,\qquad \int_{-\infty}^{+\infty}\gamma\,e^{-inx}\,dx=\int_{-\infty}^{+\infty} g\,e^{-inx}\,dx,$$

for $n=\pm1,\pm2,\dots$, and the result follows.

4.5. Linear operations.

4.5. Linear operations. We will now prove a series of results on linear operations [1]). These results will find application in the theory of trigonometrical series.

4.51. Linear and metric spaces. A set E of arbitrary elements will be called a *linear* space if

(i) There exists a commutative and associative operation, denoted by $+$, and called *addition*, applicable to every pair x,y of elements of E. If $x\in E$, $y\in E$, then $x+y\in E$.

(ii) There is an element $o\in E$ (*null* element) such that $x+o=x$ for every $x\in E$.

(iii) There exists a distributive and associative operation, denoted by $\cdot$ and called *multiplication*, applicable to every $x\in E$ and any real number α, with the properties that $1\cdot x=x$, $0\cdot x=o$, and that $\alpha\cdot x\in E$.

In most instances it will be convenient to write αx instead of $\alpha\cdot x$. The elements of E will be called points.

E will be called a *metric* space if to every $x\in E$ corresponds a non-negative number $\|x\|$, called the *norm* of x, satisfying the following conditions

$$\|x+y\|\leqslant\|x\|+\|y\|,\quad \|\alpha x\|=|\alpha|\,\|x\|,\quad \|x\|=0 \text{ is equivalent to } x=o.$$

The *distance* $d(x,y)$ of two points x,y is defined as $\|x-y\|$, where $x-y=x+(-1)\cdot y$. We see that $d(x,y)=d(y,x)$, $d(x,y)\leqslant d(x,z)+d(z,y)$, and that $d(x,y)=0$ if and only if $x=y$.

We shall say that a sequence of points x_n tends to the limit x, $x\in E$, and write $\lim x_n=x$, or $x_n\to x$, if $\|x-x_n\|\to 0$ as $n\to\infty$.

Once the distance has been defined, we may introduce various notions familiar to the reader from the elements of the theory of point-sets. First of all we define the *sphere* $S(x_0,\rho)$, with centre x_0 and radius ρ, as the set of points x such that $d(x,x_0)\leqslant\rho$.

[1]) For a more detailed study we refer the reader to Banach's *Opérations linéaires.*

This notion enables us to introduce various sorts of point-sets: open, closed, non-dense, everywhere dense; furthermore we may consider sets of the first category, i. e. sums of sequences of non-dense sets, and sets of the second category, that is sets which are not of the first category.

4.52. Functional operations. Let us consider besides E another space U which is linear and metric. If to every point $x \in E$ corresponds a point $u = u(x)$ belonging to U, we say that $u(x)$ is a functional operation defined in E. The operation $u(x)$ is said to be *additive* if, for any points x_1, x_2 from E, and any numbers λ_1, λ_2, we have $u(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 u(x_1) + \lambda_2 u(x_2)$. If $u(x_n) \to u(x)$ as $x_n \to x$, we say that u is *continuous* at the point x. If an additive operation $u(x)$ is continuous at some point, it is continuous at any other point, i. e. is continuous everywhere. A necessary and sufficient condition that an additive operation $u(x)$ be continuous is the existence of a number M such that

(1) $$\| u(x) \| \leqslant M \| x \|, \text{ for every } x \in E.$$

The sufficiency of the condition is obvious. To prove the necessity, let us suppose that there exists a sequence of points x_n such that $\| u(x_n) \| > n \| x_n \|$. Multiplying x_n by a suitable constant we may assume that $\| x_n \| = 1/n$. Then $x_n \to o$, whereas the last inequality gives $\| u(x_n) \| > 1$, so that u would be discontinuous at the point o.

For the sake of brevity, operations that are continuous and additive will be called *linear* operations. The smallest number M satisfying (1) will be denoted by M_u and called the *modulus* of the linear operation u. M_u may be defined as the upper bound of $\| u(x) \|$ on the unit sphere $\| x \| = 1$. It must be remembered that the norms on the right and on the left in (1) may have quite a different meaning, since the spaces E and U may be different. In the applications which we shall consider in this chapter, the space U will be the set R of all real numbers, and $\| u \|$ will be defined as $| u |$.

4.53. Complete spaces. A linear and metric space is said to be *complete*, if for any sequence of points x_n such that $\| x_m - x_n \| \to 0$ as $m, n \to \infty$, there exists a point x such shat $\| x - x_n \| \to 0$. It is an important property of complete spaces that they are of

the second category, i. e. cannot be represented as sums of sequences of non-dense sets [1]).

4.54. Examples. In the examples which we consider below the points of E are either real numbers or real functions, and in each case addition and multiplication receive their usual interpretation; the null point will be denoted by 0.

(i) If $E = R$, $\| x \| = | x |$, we have a linear, metric, and complete space.

(ii) If E is the set of all functions $x(t)$ defined and continuous in an interval (a, b), and if $\| x \| = \operatorname{Max} | x(t) |$, $a \leqslant t \leqslant b$, then E is a linear, metric, and complete space. The relation $x_n \to x$ means that $x_n(t)$ converges uniformly to $x(t)$.

(iii) If in the previous example we suppose that E is the set of all functions $x(t)$ essentialy bounded on (a, b), and put $\| x \| =$ the essential upper bound of $| x(t) |$, we have again a linear, metric, and complete space; $x_n \to x$ means that $x_n(t)$ converges uniformly to $x(t)$ outside a set T, $| T | = 0$, of values of t.

(iv) Let E be the set of all functions $x(t) \in L^p(a, b)$, $p \geqslant 1$, and let $\| x \| = \| x \|_p = \mathfrak{M}_p[x; a, b]$. The space is linear and metric (§ 4.13). That it is also complete was proved in § 4.2. If $p = \infty$, we obtain, as a special case, the space considered in (iii).

4.541. Classes L^*_Φ. Let Φ and Ψ be a pair of functions complementary in the sense of Young. We ask under what conditions the class $L_\Phi(a, b)$ may be considered as a linear and metric space. First of all we have to define the norm $\| x \|$, and, if the definition is to be useful, the inequality $\| x \| < \infty$ and the integrability of $\Phi[| x(t) |]$ must be, in some degree, equivalent. We might be inclined to put $\| x \| = \Phi_{-1}\left[\int_a^b \Phi(|x|)\, dt\right]$, where Φ_{-1} denotes the function inverse to Φ, but a moment's consideration shows that this definition, which is modelled on the case $\Phi(u) = u^r$, cannot be adopted. First of all the condition $\| x \| = | \alpha | \, \| x \|$ would be satisfied only exceptionally. Moreover, and here lies another difficulty, if $\Phi(u)$ increases very rapidly, the integrability of $\Phi[| x_1(t) |]$ and $\Phi[| x_2(t) |]$ does

[1]) The proofs in the general case and in the case $E = R$ do not differ essentially; see e. g. Hausdorff, *Mengenlehre*, 142.

not involve the integrability of $\Phi[|x_1(t)+x_2(t)|]$. For these reasons we must proceed otherwise [1]).

We shall denote by $L_\Phi^* = L_\Phi^*(a,b)$ the class of all functions $x(t)$, $a \leqslant t \leqslant b$, such that the product $x(t)\,y(t)$ is integrable for every $y(t) \in L_\Psi(a,b)$. If we put

$$\|x\| = \|x\|_\Phi = \operatorname{Sup} \left|\int_a^b x(t)\,y(t)\,dt\right|, \text{ for all } y \text{ with } \rho_y = \int_a^b \Psi|y|\,dt \leqslant 1,$$

then it is easy to verify that L_Φ^* is a linear and metric space.

We assume without proof that $\|x\| < \infty$ for every $x \in L_\Phi^*$. This result will be established in § 4.56.

We shall prove that L_Φ^* is also a complete space. Suppose that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$, so that $\|x_m - x_n\| \leqslant \varepsilon$ for $m, n > \nu = \nu(\varepsilon)$. It follows that

$$(1) \qquad \left|\int_a^b (x_m - x_n)\,y\,dt\right| \leqslant \varepsilon,$$

$$(2) \qquad \int_a^b |x_m - x_n|\,|y|\,dt \leqslant \varepsilon, \text{ if } \rho_y \leqslant 1 \text{ and } m, n \geqslant \nu.$$

Let α be the number such that $(b-a)\,\Psi(\alpha) = 1$. Taking $y(t) = \alpha \operatorname{sign}(x_m - x_n)$, we obtain from (1) that $\mathfrak{M}[x_m - x_n; a, b] \leqslant \varepsilon/\alpha$. Since ε is arbitrary, there exists a sequence $\{x_{m_k}(t)\}$ converging almost everywhere to a function $x(t)$ (§ 4.2(ii)), and, applying Fatou's lemma, we obtain from (2) that $\int_a^b |x - x_n|\,|y|\,dt \leqslant \varepsilon$ if $\rho_y \leqslant 1$, and so $\|x - x_n\| \leqslant \varepsilon$ for $n > \nu$. This completes the proof.

We assumed tacitly that $b - a < \infty$, but the theorem holds true if $b - a = \infty$. In fact, proceeding as before, we show that $\mathfrak{M}[x_m - x_n; a', b'] \to 0$ for every interval (a', b'), $b' - a' < \infty$, contained in (a, b). Thence we infer the existence of a sequence $\{x_{m_k}(t)\}$ converging almost everywhere in (a, b), and the rest of the proof remains unchanged.

It is obvious that, if $x \in L_\Phi$, then $x \in L_\Phi^*$. The converse is false but we shall prove that, if $x \in L_\Phi^*$, there exists a constant $\theta > 0$ such that $\theta x \in L_\Phi$. More precisely, if $x \in L_\Phi^*$, $x \neq 0$, then

1) See Orlicz [1].

$\int_a^b \Phi[x/\|x\|]\,dt < 1$. It is sufficient to prove this for x bounded. We will show first that

$$(3) \qquad \left|\int_a^b xy\,dt\right| \leqslant \left\{\begin{matrix} \|x\| & \text{if } \rho_y \leqslant 1 \\ \|x\|\,\rho_y & \text{if } \rho_y > 1 \end{matrix}\right\}.$$

The first of these inequalities is obvious; to obtain the second let us replace y by y/ρ_y in the integral on the left. The function Ψ is convex (§ 4.141) and so, by Jensen's inequality, we have $\Psi|y/\rho_y| \leqslant \Psi|y|/\rho_y$, so that

$$\int_a^b \Psi|y/\rho_y|\,dt \leqslant 1, \quad \left|\int_a^b x\frac{y}{\rho_y}\,dt\right| \leqslant \|x\|,$$

and this is just the second inequality (3). From (3) we deduce that the integral on the left does not exceed $\|x\|\,\rho_y'$ in absolute value, where $\rho_y' = \operatorname{Max}(\rho_y, 1)$.

We know that Young's inequality may degenerate into equality; in particular we have

$$\left|\int_a^b \frac{x}{\|x\|}\,y\,dt\right| = \int_a^b \Phi\left[\frac{|x|}{\|x\|}\right]dt + \rho_y \leqslant \rho_y',$$

if $y = \varphi[|x|/\|x\|]\operatorname{sign} x$ (§ 4.11). Since ρ_y is finite with (a, b), we see that $\rho_y < \rho_y'$, $\rho_y' = 1$ and the result follows [1]).

It is not difficult to see that a necessary and sufficient condition that $x(t)$ should belong to L_Φ^* is the existence of a constant $\theta > 0$ such that $\theta x \in L_\Phi$. In particular, if $\Phi(u)$ satisfies, for large u, the condition $\Phi(2u)/\Phi(u) < C$, where C is independent of u, and if $b - a < \infty$, the classes L_Φ and L_Φ^* are identical. A simple calculation shows that, if $\Phi(u) = u^r$, where $r > 1$, then $\|x\| = r^{\prime r'}\mathfrak{M}_r[x]$, so that, apart from a numerical factor, we have the same norm as in § 4.54 (iv).

4.55. The Banach-Steinhaus theorem. We begin by proving two lemmas.

(i) Let $\{u_n(x)\}$ be a sequence of linear operations which are defined in a linear and metric space E. If F denotes

1) Here again the result holds true for $b - a = \infty$.

the set of points for which $\overline{\lim}\,\|u_n(x)\| < \infty$, then $F = F_1 + F_2 + \dots$, where the sets F_i are closed and the sequence $\{\|u_n(x)\|\}$ is uniformly bounded on each of them.

Let F_{mn} be the set of points where $\|u_m(x)\| \leqslant n$. Since the operations u_m are continuous, the sets F_{mn} are closed, and so are the products $F_n = F_{1n} F_{2n} \dots$ We have $\|u_m(x)\| \leqslant n$ for $x \in F_n$, $m = 1, 2, \dots$, and $F = F_1 + F_2 + \dots$

(ii) If the space E of the previous lemma is complete, and the set F of the second category (in particular, if $F = E$), then there exists a sphere $S(x_0, \rho)$, $\rho > 0$, and a number K, such that $\|u_m(x)\| \leqslant K$ for $x \in S(x_0, \rho)$ and $m = 1, 2, \dots$

Since $F = F_1 + F_2 + \dots$, and F is of the second category, at least one of the sets $F_1, F_2, \dots$, say F_K, is not non-dense and so there exists a sphere $S(x_0, \rho)$ in which F_K it everywhere dense. Since F_K is closed, we have $S(x_0, \rho) \subset F_K$, and consequently $\|u_m(x)\| \leqslant K$ for $x \in S(x_0, \rho)$, $m = 1, 2, \dots$

Let $\{u_n(x)\}$ be a sequence of linear operations defined in a linear, metric, and complete space E, and let M_{u_n} denote the modulus of the operation u_n (4.52). If $\overline{\lim}\,\|u_n(x)\|$ is finite for every point x belonging to a set F of the second category in E, then the sequence M_{u_n} is bounded. In other words, there is a constant M such that $\|u_m(x)\| \leqslant M\|x\|$, $m = 1, 2, \dots$ [1]).

Let $S(x_0, \rho)$ be the sphere considered in (ii). Since every $x \in S(0, \rho)$ can be writen in the form $x = x_1 - x_0$, where $x_1 \in S(x_0, \rho)$, we see that $\|u_n(x)\| \leqslant 2K$ for $x \in S(0, \rho)$, $n = 1, 2, \dots$ It follows that $\|u_n(x)\|/\|x\| \leqslant 2K/\rho = M$ on the sphere $\|x\| = \rho$, and so $\|u_n(x)\| \leqslant M\|x\|$ for every x and n.

The theorem may also be stated as follows. If the sequence $\|u_n(x)\|$ is unbounded at some point, the set of points where this sequence is bounded is of the first category in E.

4.56. Corollaries. In this section we consider operations of the form

$$(1) \qquad u(x) = \int_a^b x(t)\, y(t)\, dt,$$

[1]) Banach and Steinhaus [1]. The idea of the proof, due to Saks, may be applied to many similar problems.

where x belongs to a linear, metric, and complete space E, and y is a function such that xy is integrable for every $x \in E$.

(i) If the integral (1) is defined for every bounded, or even only continuous, function $x(t)$, then $y \in L(a, b)$. (ii) Conversely, if the integral (1) converges for every $x \in L(a, b)$, then the function y is essentially bounded. (iii) If the integral (1) exists for every $x \in L^*_\Phi(a, b)$, then $y \in L^*_\Psi(a, b)$, where Φ and Ψ are functions complementary in the sense of Young.

To avoid repetition we take these theorems for granted; they can be deduced from more general results which we will now prove.

(iv) *If the sequence*

$$(2) \qquad u_n(x) = \int_a^b x(t)\, y_n(t)\, dt$$

is bounded for every bounded, or even only continuous, function x, then $\mathfrak{M}[y_n; a, b] = O(1)$. (v) *If $\{u_n(x)\}$ is bounded for every $x \in L(a, b)$, then the essential upper bounds of y_n are uniformly bounded.* (vi) *If $\{u_n(x)\}$ is bounded for every $x \in L^*_\Phi$, then $\|y_n\|_\Psi = O(1)$.*

To prove (iv), we observe that, in virtue of (i), each of the functions y_n is integrable, and so $u_n(x)$ is a linear operation defined in the space considered in § 4.54(iv), $r = 1$. Putting $x = \operatorname{sign} y_n$, we see that the modulus M_{u_n} of the operation u_n is equal to $\mathfrak{M}[y_n]$, and it is sufficient to apply the Banach-Steinhaus theorem. The case of continuous functions is not essentially different: we consider the space of § 4.54(iii), and, since the function $\operatorname{sign} y_n(t)$ is the limit of a bounded and almost everywhere convergent sequence of continuous functions, we have $M_{u_n} = \mathfrak{M}[y_n]$ again.

In case (v) we proceed similarly: each of the functions y_n is essentially bounded, and $M_{u_n} =$ the essential upper bound of $|y_n|$.

In case (vi) each of the functions y_n belongs (by (iii)) to L^*_Ψ. In virtue of the inequality $\lambda\,|u_n(x) - u_n(x_0)| \leqslant \|x - x_0\|_\Phi\, \rho'_{\lambda y_n}$ (§ 4.541), where $\lambda > 0$ is a constant so small that $\lambda y_n \in L_\Psi$, we obtain that $u_n(x)$ is a linear operation. Hence, by Theorem 4.55, $|u_n(x)| \leqslant M\|x\|_\Phi$, for $n = 1, 2, \dots$ Now, if the integral of $\Phi(|x|)$ over (a, b) does not exceed 1, then $\|x\| \leqslant 2$, and so the inequality $|u_n(x)| \leqslant M\|x\|_\Phi$ gives $\|y_n\|_\Psi \leqslant 2M$, $n = 1, 2, \dots$, and the theorem is established.

The above proof may be used to establish (i), (ii) and (iii) (proposition (i) in the case of bounded functions, is trivial, since we may put $x = \operatorname{sign} y$). To prove (iii) we put $y_n(t) = y(t)$ whenever $|y| \leqslant n$, and $y_n(t) = 0$ elsewhere. The formula (2) defines a sequence of linear operations, and the inequality $\|y_n\|_{\Psi} = O(1)$ implies $\|y\|_{\Psi} < \infty$.

(vii) *If the sequence* (2) *is bounded for every* $x \in L^r$, *then* $\mathfrak{M}_{r'}[y_n] = O(1)$ [1]. (viii) *If the sequence* (2) *is bounded for every* $x \in L_\Phi$, *then there exists a constant* $\theta > 0$ *such that* $\mathfrak{M}[\Psi|\theta y_n|] = O(1)$ [2].

The first of these propositions is a corollary of (vi). To obtain the second we observe that, if $\|y_n\|_{\Psi} \leqslant M$ for $n = 1, 2, \ldots$, then $\mathfrak{M}[\Psi|y_n/M|] < 1$. (§ 4.541).

The theorems which we have established for integrals have analogues for infinite sums. The proofs remain unchanged [3].

4.6. Transformations of Fourier series. Given a numerical sequence $\lambda_0, \lambda_1, \lambda_2, \ldots$, let us consider, besides the series

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

the following two series

$$\tfrac{1}{2} \lambda_0 + \sum_{n=1}^{\infty} \lambda_n \cos nx, \tag{2}$$

$$\tfrac{1}{2} a_0 \lambda_0 + \sum_{n=1}^{\infty} \lambda_n (a_n \cos nx + b_n \sin nx). \tag{3}$$

Given two classes P, Q of trigonometrical series we shall denote by (P, Q) the class of sequences $\{\lambda_n\}$ transforming P into Q, that is such that, whenever (1) belongs to P, (3) belongs to Q [4].

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1) Hahn [1].
2) Birnbaum and Orlicz [1].
3) See e. g. Banach, *Opérations linéaires*.
4) For the problems discussed in this paragraph see Young [9], Steinhaus [2], [3], Szidon [1], Fekete [1], M. Riesz [3], Zygmund [3], Bochner [1], Verblunsky [1], Kaczmarz [5], Hille and Tamarkin [1_2].

A necessary and sufficient condition for $\{\lambda_n\}$ *to belong to any one of the classes* (B, B), (C, C), (L, L), (S, S) *is that the series* (2) *should be a Fourier-Stieltjes series.*

Let (1) be a $\mathfrak{S}[f]$ and let $\sigma_n(x)$, $l_n(x)$, $\sigma_n^*(x)$ denote the $(C, 1)$ means of the series (1), (2), (3) respectively. We have

$$\sigma_n^*(x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t)\, l_n(t)\, dt. \tag{4}$$

Put $x = 0$. If $\lambda_n \in (C, C)$, or if $\lambda_n \in (B, B)$, the sequence $\{\sigma_n^*(0)\}$ is bounded for every $f \in C$, and, by Theorem 4.56 (iv), we have $\mathfrak{M}[l_n] = O(1)$, i. e. (2) belongs to S. Conversely, if the series (2) is a $\mathfrak{S}[dL] \in S$, the formula (4) may be written in the form

$$\sigma_n^*(x) = \frac{1}{\pi} \int_0^{2\pi} \sigma_n(x+t)\, dL(t). \tag{5}$$

Thence we deduce that the uniform boundedness of $\{\sigma_n(x)\}$ involves that of $\{\sigma_n^*(x)\}$. Similarly, if $\sigma_m(x) - \sigma_n(x)$ tends uniformly to 0 as $m, n \to \infty$, so does $\sigma_m^*(x) - \sigma_n^*(x)$, and this completes the proof of the theorem as regards the classes (B, B) and (C, C).

If $\{\lambda_n\} \in (S, S)$, it transforms, in particular, the series $\frac{1}{2} + \cos x + \cos 2x + \ldots \in S$ into the series (2), which must, therefore, belong to S. Conversely, if the series (2) is a $\mathfrak{S}[dL]$, we obtain from (5) that

$$|\sigma_n^*(x)| \leqslant \frac{1}{\pi} \int_0^{2\pi} |\sigma_n(x+t)|\, |dL(t)|. \tag{6}$$

Integrating this inequality over $(0, 2\pi)$, and inverting the order of integration on the right, we obtain that $\mathfrak{M}[\sigma_n^*] \leqslant (v/\pi)\, \mathfrak{M}[\sigma_n]$, where v is the total variation of $L(t)$ over $(0, 2\pi)$. Hence the series (3) belongs to S.

It remains only to consider the case (L, L). Since

$$|\sigma_m^*(x) - \sigma_n^*(x)| \leqslant \frac{1}{\pi} \int_0^{2\pi} |\sigma_m(x+t) - \sigma_n(x+t)|\, |dL(t)|,$$

$$\mathfrak{M}[\sigma_m^* - \sigma_n^*] \leqslant (v/\pi)\, \mathfrak{M}[\sigma_m - \sigma_n],$$

the sufficiency of the condition is obvious (§ 4.34). To prove the necessity let us consider, for every n, a system $I_n = \{(\alpha_1^n, \beta_1^n), (\alpha_2^n, \beta_2^n), \ldots\}$ of non-overlapping intervals. It follows from (4) that

$$(7)\qquad \int_{I_n} \sigma_n^*(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(t)\left\{\int_{I_n} l_n(t-x)\,dx\right\}dt.$$

Suppose that (2) does not belong to S, so that the indefinite integrals of the functions $l_n(x)$ are not of uniformly bounded variation. We can then find a sequence $I_1, I_2, \ldots$ such that the coefficient of $f(t)$ in (7) is not uniformly bounded. By Theorem 4.56(v), there is an integrable f such that the right-hand side in (7) is unbounded, and, à fortiori, $\mathfrak{M}[\sigma_n] \neq O(1)$. It follows that the series (3) does not belong to S, and, in particular, does not belong to L, although (1) is a Fourier series.

4.61. Let $\bar{P}$ denote the class of trigonometrical series conjugate to those belonging to P. It is plain that if P, and similarly Q, is one of the classes B, C, L, S, then $(\bar{P}, \bar{Q}) = (P, Q)$.

A necessary and sufficient condition that $\{\lambda_n\}$ should belong to any one of the classes $(\bar{B}, B)$, $(\bar{C}, C)$, $(\bar{L}, L)$, $(\bar{S}, S)$ is that the series conjugate to 4.6(2) *should belong to S.*

The proof is similar to that of Theorem 4.6. We need only slightly change the formulae which we have used, so as to introduce conjugate series. In fact, let $\bar{\sigma}_n(x)$ and $\bar{\sigma}_n^*(x)$ denote the first arithmetic means of the series

$$(1)\quad \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx), \qquad (2)\quad \sum_{n=1}^{\infty}\lambda_n(a_n \sin nx - b_n \cos nx)$$

respectively, and let $\bar{l}_n(x)$ be the arithmetic means of the series $\lambda_1 \sin x + \lambda_2 \sin 2x + \ldots$, conjugate to 4.6(2). If the series 4.6(1) is a $\mathfrak{S}[f]$, we have the formula

$$(3)\qquad \bar{\sigma}_n^*(x) = -\frac{1}{\pi}\int_0^{2\pi} f(x+t)\,\bar{l}_n(t)\,dt,$$

analogous to 4.6(4). Considering, for example, the case $(\bar{B}, B)$, we suppose that the series 4.6(1) belongs to B and ask under what conditions (2) is the Fourier series of a bounded function. Arguing as in the preceding section, we obtain that the necessary and sufficient condition is $\mathfrak{M}[\bar{l}_n] = O(1)$. The remaining cases may be left to the reader.

4.62. Let $\chi(u)$, $u \geqslant 0$, be a function non-negative, convex, bounded in any finite interval, and tending to infinity with u.

If the series 4.6(1) *is the Fourier series of a function f such that $\chi(|f|)$ is integrable, and if* 4.6(2) *is a $\mathfrak{S}[dL]$, then* 4.6(3) *is the Fourier series of a function $g(x)$ such that $\chi(|g|\pi/v)$ is integrable, where v denotes the total variation of L over* $(0, 2\pi)$.

Without real loss of generality we may suppose that $\chi(u)$ is non-decreasing. Let $t_i = 2\pi i/N$, $i = 0, 1, \ldots, N$, and let $v(x)$ denote the total variation of L over $(0, x)$, so that $v(2\pi) = v$. Dividing both sides of the inequality 4.6(6) by v, and applying the mean-value theorem in each of the intervals (t_{i-1}, t_i), we obtain that

$$\pi\,|\sigma_n^*(x)|/v \leqslant \sum_{i=1}^{N}\xi_i p_i \Big/ \sum_{i=1}^{N} p_i,$$

where $p_i = v(t_i) - v(t_{i-1})$, $\xi_i = \sigma_n(x + t_i')$, $t_{i-1} \leqslant t_i' \leqslant t_i$. Applying Jensen's inequality, and making $N \to \infty$, we obtain that

$$\chi\left\{\frac{\pi}{v}\,|\sigma_n^*(x)|\right\} \leqslant \sum_{i=1}^{N}\chi(\xi_i)\,p_i \Big/ \sum_{i=1}^{N} p_i,\quad \chi\left\{\frac{\pi}{v}\,|\sigma_n^*(x)|\right\} \leqslant \frac{1}{v}\int_0^{2\pi}\chi|\sigma_n(x+t)|\,|dL(t)|.$$

Now it is sufficient to integrate the last inequality over $(0, 2\pi)$, to invert the order of integration on the right, and to apply Theorem 4.33.

It must be emphasized that the condition which we imposed upon the series 4.6(2) is only sufficient and by no means necessary. This is easily seen in the case $\chi(u) = u^2$, since, by the Riesz-Fischer theorem, a sequence $\{\lambda_n\}$ belongs to the class (L^2, L^2) if and only if $\lambda_n = O(1)$.

The theorem which we have proved may also be stated in the following form. If $\Phi(x)$ is a Young function and the series 4.6(2) belongs to S, the sequence $\{\lambda_n\}$ belongs to the class (L^*_Φ, L^*_Φ). It belongs in particular to every class (L^r, L^r), $r > 1$.

4.63. Let Φ, Ψ and Φ_1, Ψ_1 be two pairs of Young's complementary functions.

The classes $(L^*_\Phi, L^*_{\Phi_1})$ and $(L^*_{\Psi_1}, L^*_\Psi)$ are identical.

The proof will be based on the following lemma. *A necessary and sufficient condition that the series* 4.6(1) *should be a $\mathfrak{S}[f]$ with $f \in L^*_\Phi$ is that, for every $g \in L^*_\Psi$ with Fourier coefficients a_n', b_n', the series*

(1) $$\frac{a_0\, a_0'}{2} + \sum_{n=1}^{\infty} (a_n\, a_n' + b_n\, b_n')$$

should be finite $(C, 1)$[1].

If $f \in L_{\Phi}^*$, $g \in L_{\Psi}^*$, there exist two constants $\lambda > 0$ and $\mu > 0$ such that $\lambda f \in L_{\Phi}$, $\mu g \in L_{\Psi}$, and the necessity of the condition follows from Theorem 4.41(i). To prove that the condition is sufficient let $\sigma_n(x)$ and τ_n denote the first arithmetic means of the series 4.6(1) and (1) respectively. We have then

$$\tau_n = \frac{1}{\pi} \int_0^{2\pi} g(t)\, \sigma_n(t)\, dt.$$

Since the sequence $\{\tau_n\}$ is bounded for every $g \in L_{\Psi}^*$, it follows that $\|\sigma_n\|_{\Phi} = O(1)$, which shows that the series 4.6(1) belongs to L_{Φ}^* (§§ 4.56(vi), 4.33).

Now it is easy to prove the theorem. If $\{\lambda_n\} \in (L_{\Phi}^*, L_{\Phi_1}^*)$ then, for every $f \in L_{\Phi}^*$ with Fourier coefficients a_n, b_n, and every $g \in L_{\Psi_1}^*$ with Fourier coefficients a_n', b_n', the series

(2) $$\tfrac{1}{2} \lambda_0\, a_0\, a_0' + \sum_{n=1}^{\infty} (\lambda_n\, a_n\, a_n' + \lambda_n\, b_n\, b_n')$$

is finite $(C, 1)$.

It means, in virtue of the lemma, that the series with coefficients $\lambda_n a_n'$, $\lambda_n b_n'$ belongs to L_{Ψ}^*, i. e. $\{\lambda_n\} \in (L_{\Psi_1}^*, L_{\Psi}^*)$.

Corollaries. (i) If Φ and Ψ are complementary functions, the classes (L_{Φ}^*, L_{Φ}^*) and (L_{Ψ}^*, L_{Ψ}^*) are identical.

(ii) If $r > 1$, $s > 1$, the classes (L^r, L^s) and $(L^{s'}, L^{r'})$ are identical. In particular $(L^r, L^r) = (L^{r'}, L^{r'})$.

In Ch. IX we shall prove that, if $r < s < r'$, the class (L^r, L^r) is contained in (L^s, L^s).

4.64. *If the series* 4.6(2) *belongs to* L, *then* $\{\lambda_n\} \in (S, L)$, $\{\lambda_n\} \in (B, C)$. Let 4.6(1) be a $\mathfrak{S}[dF]$. From the formula 4.6(4), with $f(x+t)$ replaced by $dF(x+t)$, we find that $\pi\mathfrak{M}[\sigma_m^* - \sigma_n^*]$ does not exceed $\mathfrak{M}[l_m - l_n]$ multiplied by the total variation of F over $(0, 2\pi)$. It follows that $\mathfrak{M}[\sigma_m^* - \sigma_n^*] \to 0$ as $m, n \to \infty$. Thus

[1]) A series $u_0 + u_1 + \ldots$ is said to be finite (C, r), if the r-th Cesàro means of the series forms a bounded sequence.

the series 4.6(3) belongs to L. Similarly we find from 4.6(4) that $\pi\,|\sigma_m^* - \sigma_n^*|$ does not exceed $\mathfrak{M}[l_m - l_n]$. Max $|f|$, i. e. B is transformed into C.

A similar proof shows that, *if the series conjugate to* 4.6(2) *belongs to* L, *then* $\{\lambda_n\} \in (\bar{B}, C)$, $\{\lambda_n\} \in (\bar{S}, L)$.

4.65. The conditions which we imposed upon $\{\lambda_n\}$ in the preceding section are not only sufficient but also necessary. For the first parts of the theorems this follows immediately by considering the series $\frac{1}{2} + \cos x + \cos 2x + \ldots \subset S$ and $\sin x + \sin 2x + \ldots \in \bar{S}$. For the second parts the proof is more difficult and we do not propose to consider it here.

Let $\{\lambda_n\}$ be an arbitrary convex sequence tending to 0, e. g. $\lambda_n = n^{-\alpha}$, $\alpha > 0$, $\lambda_n = 1/\log n$, $\lambda_n = 1/\log\log n$, for n sufficiently large. In § 5.12 we shall prove that the series 4.6(2) with such coefficients belongs to L, i. e. $\{\lambda_n\}$ transforms Fourier-Stieltjes series into Fourier series, bounded functions into continuous.

The sequence $\lambda_n = 1/(\log n)^{1+\varepsilon}$, $\varepsilon > 0$, $n > 1$, belongs to $(\bar{S}, L)$ and $(\bar{B}, C)$. For $\varepsilon = 0$ this is no longer true (§ 5.13).

4.7. Miscellaneous theorems and examples.

1. Let $\varphi(x)$, $x \geqslant 0$, be convex, increasing to ∞ with x, and vanishing at the origin. If $\psi(y)$ is the inverse function, and $a \geqslant 0$, $b \geqslant 0$, then $ab \leqslant a\varphi(a) + b\psi(b)$.

2. Given a function $F \in L^r(a,b)$, $r > 1$, let $I_G = \left|\int_a^b FG\, dx\right|$, where $G \in L^{r'}$. Show that $\mathfrak{M}_r[F] = \operatorname{Sup} I_G$ for all G with $\mathfrak{M}_{r'}[G] \leqslant 1$.

[Since $\mathfrak{M}_r[F] < \infty$, we may suppose that $\mathfrak{M}_r[F] = 1$. By Young's inequality we have $I_G \leqslant \mathfrak{M}_r^r[F]/r + \mathfrak{M}_{r'}^{r'}[G]/r' \leqslant 1$, and for a special function G, viz. when $G = |F|^{r-1} \operatorname{sign} F$, we have $I_G = 1$. It is easy to see that the theorem holds true when $\mathfrak{M}_r[F] = \infty$.

We add that, if $(a, b) = (0, 2\pi)$, it is sufficient to take for the functions G only trigonometrical polynomials, since for any $G \in L^{r'}$ we can find a trigonometrical polynomial g such that $\mathfrak{M}_{r'}[G - g] < \varepsilon$ and so, by Minkowski's inequality, $|\mathfrak{M}_{r'}[G] - \mathfrak{M}_{r'}[g]| < \varepsilon$].

3. Let $\chi(x)$, $x \geqslant 0$, be convex and strictly increasing, $\chi(0) = 0$. Let $f(x)$ be integrable and periodic, and $F(x)$ the indefinite integral of $f(x)$. If $\mathfrak{M}[\chi|f|; 0, 2\pi] \leqslant C$, and $0 < h \leqslant 2\pi$, then $|F(x+h) - F(x)| \leqslant h\chi_{-1}(C/h)$, where χ_{-1} is the function inverse to χ. If $f \in L^r$, $r \geqslant 1$, then $\omega(F; \delta) = o(\delta^{1/r'})$. Young [3].

[Apply Jensen's inequality].

4. If $f \log^+ |f|$ is integrable over $(-\pi, \pi)$ [1]), so is $f \log 1/|x|$.
[Apply Young's inequality to the product $2|f| \cdot \frac{1}{2} \log 1/|x|$].

5. If a_n are the cosine coefficients of $f(x)$, and $f(x) \log 1/|x|$ is integrable over $(-\pi, \pi)$, the series $a_1 + a_2/2 + a_3/3 + \dots$ converges and has the sum $-\frac{1}{\pi}\int\limits_0^{2\pi} f(x) \log (2 \sin \frac{1}{2} x)\, dx$. Hardy and Littlewood [7].

[Express the partial sums of the series as an integral. The partial sums of the series $\cos x + \frac{1}{2} \cos 2x + \dots$ are $O(\log 1 (|x|)$ uniformly in n. This follows from the first formula 1.12(3) and from the general theorem that, if $u_n = O(1/n)$, $f(r) = u_0 + u_1 r + u_2 r^2 + \dots$, $s_n = u_0 + \dots + u_n$, then $f(r) - s_n = O(1)$ as $r = 1 - 1/n \to 1$. To prove the latter fact we observe that, if $|u_k| \leqslant A/k$, then $|f(r) - s_n| \leqslant (1-r)[|a_1| + 2|a_2| + \dots + n|a_n|] + A/n(1-r) = O(1)$].

6. Let $\omega_p(\delta) = \omega_p(\delta; f) = \operatorname{Max} \mathfrak{M}_p[f(x+h) - f(x); 0, 2\pi]$ for $0 \leqslant h \leqslant \delta$. The function f is said to belong to $\operatorname{Lip}(\alpha, p)$, if $\omega_p(\delta) = O(\delta^\alpha)$. Show that (i) if $f \in \operatorname{Lip}(\alpha, p)$, then $f \in \operatorname{Lip}(\alpha, p_1)$, $0 < p_1 < p$, (ii) if f is continuous and $p \to \infty$, then $\omega_p(\delta) \to \omega(\delta)$, (iii) if $f \in \operatorname{Lip}(\alpha, p)$, then $f \in L^p$.

[To prove (iii), integrate the inequality $\mathfrak{M}_p^p[f(x+h) - f(x)] \leqslant C$ with respect to h, invert the order of integration, and consider a value of x for which the function $[f(x+h) - f(x)]^p$ is integrable with respect to h. Tamarkin, *Fourier Series*, p. 49].

7. A necessary and sufficient condition that $f(x)$ should belong to $\operatorname{Lip}(1,1)$ is that there should exist a function $g(x)$ of bounded variation, equivalent to $f(x)$. Hardy and Littlewood [6_1].

[To prove that the condition is sufficient, let $\sigma_n(x)$ be the first arithmetic means of $\mathfrak{S}[f]$. Then $\mathfrak{M}[\sigma_n(x+h) - \sigma_n(x)] \leqslant \mathfrak{M}[f(x+h) - f(x)] \leqslant Ch$, $\mathfrak{M}[\sigma_n'(x)] \leqslant C$, and it is sufficient to apply Theorem 4.325. To prove that the condition is necessary it is enough to suppose that $f(x)$ is non-decreasing.
For a more elementary proof see the paper referred to above].

8. A necessary and sufficient condition that $f(x)$ should belong to $\operatorname{Lip}(1, p)$, $p > 1$, is that f should be equivalent to the indefinite integral of a function belonging to L^p. Hardy and Littlewood [6_1].
[The condition is necessary since

$$\mathfrak{M}_p^p[f(x+h) - f(x)] \leqslant \int\limits_0^{2\pi} \Big\{ \int\limits_x^{x+h} |f'(t)|\, dt \Big\}^p dx \leqslant h^p \int\limits_0^{2\pi} |f'|^p\, dt.$$

To show that the condition is sufficient we prove that $\mathfrak{M}_p[\sigma_n'(x)] = O(1)$].

9. Let $s_n(x)$ and $\sigma_n(x)$ be the partial sums and the $(C, 1)$ means of $\mathfrak{S}[f]$. Then (i) a necessary and sufficient condition that f should belong to $\operatorname{Lip} \alpha$, $0 < \alpha \leqslant 1$, is that the σ_n should belong to $\operatorname{Lip} \alpha$ uniformly in n, (ii) if $f \in \operatorname{Lip} \alpha$, then $\omega(\delta; s_n) = O(\delta^\alpha \log 1/\delta)$ uniformly in n.

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[1]) If u is real, u^+ denotes the number $\operatorname{Max}(u, 0)$.

10. Let $\sigma_n(x)$ be the first arithmetic means of a trigonometrical series. A necessary and sufficient condition that the series should be a Fourier series is that there should exist a function $\varphi(u) \geqslant 0$, $\varphi(u)/u \to \infty$ with u, and such that $\mathfrak{M}[\varphi|\sigma_n|] = O(1)$. de la Vallée-Poussin [2].
[If $\varphi(u) \geqslant 0$, $\varphi(u)/u \to \infty$, there exists a convex function $\varphi_1(u) \leqslant \varphi(u)$, satisfying the same conditions. If $f \in L$, then there exists a function $\varphi(u)$, $\varphi(u)/u \to \infty$, such that $\varphi(|f|) \in L$].

11. Let $f(r, x) = \frac{1}{2} a_0 + (a_1 \cos x + b_1 \sin x) r + \dots$ A necessary and sufficient condition that $f(r, x)$ should be a difference of two non-negative harmonic functions is that $\mathfrak{M}[|f(r, x)|] = O(1)$ as $r \to 1$.

12. Let $\varphi(u)$ be convex, non-negative, and increasing, and let $\frac{1}{2} a_0 + (a_1 \cos x + b_1 \sin x) + \dots$ be a $\mathfrak{S}[dF]$. A necessary and sufficient condition that the positive variation $P(x)$ of $F(x)$ should be absolutely continuous, and that $P'(x)$ should belong to L_φ is: $\mathfrak{M}[\varphi\{f^+(r, x)\}] = O(1)$ as $r \to 1$, where $f(r, x)$ has the same meaning as in the previous theorem.

13. If $f \in L^2$, and c_n are the complex Fourier coefficients of f, then the function $h(x) = \frac{1}{2\pi}\int\limits_0^{2\pi} f(x+t) f(t)\, dt$ is continuous, and $h(x) \sim \sum\limits_{n=-\infty}^{+\infty} |c_n|^2 e^{inx}$ (§ 2.12). Show that Parseval's theorem is a simple corollary of this result.

14. Let $\sigma_n^r(x)$, $r > 0$, be the r-th Cesàro means of a trigonometrical series. A necessary and sufficient condition that the series should belong to L_Φ^* is $\|\sigma_n^r\|_\Phi = O(1)$. If the series is a $\mathfrak{S}[f] \in L_\Phi^*$ then $\|f - \sigma_n^r\|_\Phi \to 0$ as $n \to \infty$.

15. Let X be the set of all functions $x(t)$ which are the characteristic functions of measurable sets contained in $(0, 2\pi)$. If the sequence 4.56(2), where $(a, b) = (0, 2\pi)$, is bounded for every $x \in X$, then $\mathfrak{M}[y_n] = O(1)$. Saks [1].
[The proof runs on the same lines as that of § 4.55. If we put $\|x_1 - x_2\| = \int\limits_0^{2\pi} |x_1(t) - x_2(t)|\, dt$, X becomes a metric and complete space. X is not a linear space but it has the following property which may in most cases be used instead of linearity: let $S(u, \rho)$, $\rho > 0$, be an arbitrary sphere; for any $x \in S(0, \rho)$ there exist two points x_1 and x_2 belonging to $S(u, \rho)$ such that $x = x_1 - x_2$. It suffices to put $x_1(t) = u(t) + x(t)[1 - u(t)]$, $x_2(t) = u(t)[1 - x(t)]$].

16. There exists a function $f \in L$ and a measurable set E such that $\mathfrak{S}[f]$ integrated formally over E diverges.
[This follows from the previous theorem and from the results of § 5.12].