CHAPTER III.

Summability of Fourier series.

3.1. Toeplitz matrices. An infinite matrix

$$\mathfrak{A} = \left\{ \begin{matrix} a_{00}, a_{01}, \dots a_{0n}, \dots \\ a_{10}, a_{11}, \dots a_{1n}, \dots \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ a_{n0}, a_{n1}, \dots a_{nn}, \dots \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{matrix} \right\}$$

is called a *Toeplitz* matrix, or *T*-matrix, if the following three conditions are satisfied (i) $\lim_i a_{ik} = 0$, $k = 0, 1, \dots$, (ii) $\lim_i A_i = 1$, (iii) $N_i \leqslant C$, $i = 0, 1, \dots$, where $A_i = a_{i0} + a_{i1} + \dots$, $N_i = |a_{i0}| + |a_{i1}| + \dots$ and C is independent of i. Given a sequence $\{s_n\}$, we 'transform' it by the matrix $\mathfrak{A}$, i. e. consider the sequence $\sigma_n = a_{n0} s_0 + a_{n1} s_1 + \dots$, provided that the series on the right converge. If $\sigma_n \to s$, we say that the sequence $\{s_n\}$, or the series with partial sums s_n, is summable $\mathfrak{A}$ to the value s. The expressions σ_n are called *T*-means.

If $\mathfrak{A}$ is a T-matrix and if $s_n \to s$, where s is finite, then $\sigma_n \to s$ [1]). In fact, if $s_k = s + \varepsilon_k$, $\varepsilon_k \to 0$, then $\sigma_n = \sigma'_n + \sigma''_n$, where $\sigma'_n = A_n s \to s$ (by (ii)). Given any $\varepsilon > 0$, suppose that $|\varepsilon_k| < \varepsilon/2C$ for $k > k_0$. Since $|\sigma''_n| \leqslant (|a_{n0}|\,|\varepsilon_0| + \dots + |a_{nk_0}|\,|\varepsilon_{k_0}|) + (|a_{n\,k_0+1}|\,|\varepsilon_{k_0+1}| + \dots)$, where the second sum on the right is less than $C \cdot \varepsilon/2C = \varepsilon/2$, and the first sum tends to 0 (by (i)), it follows that $|\sigma''_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for n large, i. e. $\sigma''_n \to 0$, $\sigma_n \to s$.

It is useful to note that, if $s = 0$, condition (ii) is not necessary in the proof. If s_n depends on a parameter and if $s_n \to s$ uniformly, then $\sigma_n \to s$ also uniformly.

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1) Toeplitz [1].

3.101. Condition (iii) is a consequence of (ii) if $\mathfrak{A}$ is *positive* i. e. if all a_{ik} are non-negative. For such matrices we can prove the following more general result:

$$\underline{\lim}\, s_n \leqslant \underline{\lim}\, \sigma_n \leqslant \overline{\lim}\, \sigma_n \leqslant \overline{\lim}\, s_n.$$

To prove e. g. the first inequality we may plainly suppose that $\underline{\lim}\, s_n = \underline{s} > -\infty$. Let α be any number $< \underline{s}$. Then $s_k > \alpha$ for $k > k_0$, and so, by (i), we have the inequality $\sigma_n \geqslant o(1) + (a_{n\,k_0+1} + \dots)\,\alpha = o(1) + \alpha\,[A_n + o(1)]$, and therefore $\underline{\lim}\, \sigma_n \geqslant \alpha$, $\underline{\lim}\, \sigma_n \geqslant \underline{s}$. In particular if $s_n \to \infty$, then $\sigma_n \to \infty$.

If $\mathfrak{A}$ is not positive the result is not necessarily true. A moment's consideration shows that, if $\underline{\lim}\, s_n = \underline{s}$, $\overline{\lim}\, s_n = \bar{s}$, $\overline{\lim}\, N_i = C$, then $\underline{\lim}\, \sigma_n$ and $\overline{\lim}\, \sigma_n$ are both contained in the interval $[\frac{1}{2}(\underline{s} + \bar{s}) - C \cdot \frac{1}{2}(\bar{s} - \underline{s}), \frac{1}{2}(\underline{s} + \bar{s}) + C \cdot \frac{1}{2}(\bar{s} - \underline{s})]$. In fact, we may put $s_n = s'_n + s''_n$, where $s'_n = \frac{1}{2}(\underline{s} + \bar{s})$, $\overline{\lim}\, |s''_n| = \frac{1}{2}(\bar{s} - \underline{s})$. Then $\sigma_n = \sigma'_n + \sigma''_n$, where $\sigma'_n \to \frac{1}{2}(\underline{s} + \bar{s})$ and $\overline{\lim}\, \sigma''_n \leqslant C \cdot \frac{1}{2}(\bar{s} - \underline{s})$.

3.102. Let $\{p_n\}$, $\{q_n\}$ be two sequences of numbers, and let $P_n = p_0 + \dots + p_n$, $Q_n = q_0 + \dots + q_n$, $q_n > 0$, $Q_n \to \infty$. *If $s_n = p_n/q_n \to s$, then $\sigma_n = P_n/Q_n \to s$.* In fact, $\sigma_n = (q_0 s_0 + q_1 s_1 + \dots + q_n s_n)/Q_n$, so that we have here a positive *T*-matrix. In particular, if $q_n = 1$ for $n = 0, 1, \dots$, we obtain the classical result of Cauchy: *if $s_n \to s$, then $(s_0 + s_1 + \dots + s_n)/(n+1) \to s$.*

3.11. Cesàro means. Given a sequence $\{s_n\}$ we put, for $n = 0, 1, \dots$, $s_n^0 = s_n$, $s_n^1 = s_0^0 + s_1^0 + \dots + s_n^0$, ..., $s_n^k = s_0^{k-1} + s_1^{k-1} + \dots + s_n^{k-1}$, ... Similarly, let $A_n^0 = 1$ $(n = 0, 1, \dots)$, $A_n^1 = A_0^0 + A_1^0 + \dots + A_n^0$, ..., $A_n^k = A_0^{k-1} + A_1^{k-1} + \dots + A_n^{k-1}$, ... We shall say that the sequence $\{s_n\}$ is summable by the k-th Cesàro mean, or summable (C, k), $k = 0, 1, \dots$, to limit s, if $s_n^k/A_n^k \to s$ as $n \to \infty$. It follows from § 3.102 that summability (C, k) of a sequence involves summability $(C, k+1)$ to the same limit [1]). To find the numerical values of A_n^k it is con-

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1) Les us define, for every $k = 0, 1, \dots$, the sequence $h_n^k = (h_0^{k-1} + \dots + h_n^{k-1})/(n+1)$, $n = 0, 1, \dots$, $h_n^0 = s_n$. $\{s_n\}$ is said to be summable by the k-th Hölder mean, or summable (H, k), if $h_n^k \to s$ as $n \to \infty$. The methods (C, k) and (H, k) are known to be equivalent. Although the latter is logically simpler, it is less useful in applications and its extension to the case of k non-integral much less easy. See Hausdorff [1].

venient to use the following proposition, which is easily proved by means of Abel's transformation: If $A_n = a_0 + a_1 + \dots + a_n$, then

$$\sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{\infty} A_n x^n,$$

provided that the series on the right is convergent. This permits us to restate our definition as follows:

A sequence $\{s_n\}$, or a series $u_0 + u_1 + \dots$ with partial sums s_n, is summable (C, α) to the value s if $\sigma_n^\alpha = s_n^\alpha / A_n^\alpha \to s$, s_n^α and A_n^α being given by the relations

$$(1)\qquad \sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1}, \quad \sum_{n=0}^{\infty} s_n^\alpha x^n = \frac{\sum_{n=0}^{\infty} s_n x^n}{(1-x)^\alpha} = \frac{\sum_{n=0}^{\infty} u_n x^n}{(1-x)^{\alpha+1}}.$$

In this definition $\alpha\ (\neq -1, -2, \dots)$ is no longer a positive integer. However it will appear soon that only the case $\alpha > -1$ is interesting. The following relations are consequences of (1):

$$(2)\qquad A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)\dots(\alpha+n)}{n!} \sim \frac{n^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \neq -1, -2, \dots$$

$$(3)\qquad A_n^{\alpha+\beta+1} = \sum_{k=0}^{n} A_k^\alpha A_{n-k}^\beta, \qquad (4)\qquad s_n^{\alpha+\beta+1} = \sum_{k=0}^{n} A_{n-k}^\beta s_k^\alpha,$$

$$(5)\qquad s_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k = \sum_{k=0}^{n} A_{n-k}^\alpha u_k, \quad (6)\qquad A_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1}, \ A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1},$$

$$(7)\qquad s_n^\alpha = \sum_{k=0}^{n} s_k^{\alpha-1}, \ s_n^\alpha - s_{n-1}^\alpha = s_n^{\alpha-1}, \quad (8)\qquad \sum_{k=0}^{\infty} |A_k^\alpha| < \infty, \ \alpha < -1, \text{ [1])}$$

(9) A_n^α is positive for $\alpha > -1$, increasing for $\alpha > 0$, and decreasing for $0 > \alpha > -1$. If $\alpha < -1$, A_n^α is of constant sign for n sufficiently large.

3.12. The Gamma-function. In 3.11(2) Γ is the Euler Gamma-function. Except in Chapter IX, the reader is not expected to be acquainted with the theory of this function, and may take the relation 3.11(2) just as a definition (Gauss's definition) of Γ. It remains, then, only to show that $\lim A_n^\alpha / n^\alpha$ exists and is different from 0. For this purpose we write

1) See (2).

$$\log A_n^\alpha = \sum_{k=1}^{n} \log\left(1+\frac{\alpha}{k}\right) = \sum_{k=0}^{n} \left\{\frac{\alpha}{k} + O(k^{-2})\right\} = \alpha(\log n + C + \varepsilon_n) + (C' + \eta_n),$$

where C is Euler's constant (§ 1.74), $\varepsilon_n, \eta_n \to 0$, and C' is the sum of all the terms $O(k^{-2})$. This completes the proof.

3.13. *If* $\sigma_n^\alpha \to s$, $\alpha > -1$, $h > 0$, *then* $\sigma_n^{\alpha+h} \to s$. We obtain from 3.11(4) that $\sigma_n^{\alpha+h} = \left(\sum_{k=0}^{n} A_{n-k}^{h-1} A_k^\alpha \sigma_k^\alpha\right) / A_n^{\alpha+h}$. This is a positive T-matrix, and so the result follows. Also, more generally, the limits of indetermination of $\sigma_n^{\alpha+h}$ are contained between those of σ_n^α.

If $u_0 + u_1 + \dots$ *is summable* (C, α), *and if* $\alpha > -1$, *then* $u_n = o(n^\alpha)$. We have $u_n / A_n^\alpha = \left(\sum_{k=0}^{n} A_{n-k}^{-\alpha-2} A_k^\alpha \sigma_k^\alpha\right) / A_n^\alpha$ (§ 3.11(4), $\beta = -\alpha - 2$). Suppose, as we may, that $\sigma_k^\alpha \to 0$. We need only show that conditions (i) and (iii) of Toeplitz are satisfied (§ 3.1). The former is obviously satisfied. As regards the latter, let us suppose first $\alpha \geqslant 0$. Then, A_k^α being non-decreasing, we have $N_n \leqslant \sum_{k=0}^{n} |A_k^{-\alpha-2}| = O(1)$. If $-1 < \alpha < 0$, we obtain from 3.11(3) that $N_n = 2$, since $A_k^{-\alpha-2}$ is negative for $k > 0$.

It is often useful to consider the difference

$$(1)\qquad s_n - \sigma_n^1 = (u_1 + 2u_2 + \dots + nu_n)/(n+1).$$

If it tends to 0, in particular if $u_n = o(1/n)$, the $(C, 1)$ summability of $u_0 + u_1 + \dots$ involves the convergence of this series.

3.14. Abel's method of summation. The series $u_0 + u_1 + \dots$ is said to be summable by Abel's method (some say Poisson's), or summable A, to sum s, if $u_0 + u_1 x + u_2 x^2 + \dots$ is convergent for $|x| < 1$, and

$$(1)\qquad \lim_{x \to 1} \sum_{k=0}^{\infty} u_k x^k = \lim_{x \to 1} (1-x) \sum_{k=0}^{\infty} s_k x^k = s,$$

where x tends to 1 along the real axis.

If $u_0 + u_1 + \dots$ *is summable* (C, α), $\alpha > -1$, *to* s, *then* (1) *holds as* $x \to 1$ *along any path* L *lying between two chords of the unit circle which pass through* $x = 1$. Such paths L will be spoken of as not touching the circle. They are characterized by inequalities $|1-x|/(1-|x|) < \text{const.}$, $x \in L$.

To avoid the difficulty that the variable x in (1) changes continuously, we consider an arbitrary sequence of points x_n lying on L and tending to 1. Since

$$\sum_{k=0}^{\infty} u_k x_n^k = (1-x_n)^{\alpha+1} \sum_{k=0}^{\infty} s_k^{\alpha} x_n^k = (1-x_n)^{\alpha+1} \sum_{k=0}^{\infty} \sigma_k^{\alpha} A_k^{\alpha} x_n^k,$$

we need only show that the matrix $\mathfrak{A}$ with $a_{nk} = A_k^{\alpha}(1-x_n)^{\alpha+1} x_n^k$ is a T-matrix. If $x_n \to 1$ along the real axis, the matrix is positive, so that the limits of indetermination by the method A are contained between those by the method (C, α).

3.2. As we shall see in Chapter VIII, there exist continuous functions with Fourier series divergent at some points. It is therefore natural to consider the summability of Fourier series. Although some older results, e. g. those of Poisson, in the theory of trigonometrical series can now be recognized as applications of methods of summability, the first deliberate step in this direction was made by Fejér (1902). The results proved in this chapter, together with the examples of Chapter VIII, show that, if we do not restrict ourselves to functions with rather special differential properties, it is rather the summability than the ordinary convergence which is important in the theory of representation of functions by means of their Fourier series.

3.201. Let $s_n(x)$ be the n-th partial sum of $\mathfrak{S}[f]$:

$$(1)\qquad f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

and let $\sigma_n(x) = \sigma_n(x; f)$ be the first arithmetic means of $\{s_n\}$.

Using the formulae 2.3(2), we see that

$$(2)\qquad \begin{aligned} \sigma_n(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, K_n(t)\, dt, \\ \sigma_n(x) - f(x) &= \frac{1}{\pi}\int_{0}^{\pi} \varphi(t)\, K_n(t)\, dt, \end{aligned}$$

where, as usual, $\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$, and $K_n = (D_0 + D_1 + \ldots + D_n)/(n+1)$. Multiplying the numerator and the denominator of $D_k(t)$ by $2\sin\frac{1}{2}t$, and replacing the products of sines by differences of cosines, we find that

$$(3)\qquad (n+1)K_n(t) = \sum_{k=0}^{n} \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}t} = \frac{1}{2}\left(\frac{\sin(n+1)\frac{1}{2}t}{\sin\frac{1}{2}t}\right)^2.$$

It is customary, in general, in the theory of Fourier series to call the Toeplitz means of the series $\frac{1}{2} + \cos t + \cos 2t + \ldots$ *kernels.* The expression $K_n(t)$ is called 'Fejér's kernel' and has the following properties:

(i) $K_n(t) \geqslant 0$, (ii) $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} K_n(t)\, dt = 1$, (iii) $M_n(\delta) \to 0$ as $n \to \infty$, for every $\delta > 0$, where $M_n(\delta) = \operatorname{Max}|K_n(t)| = \operatorname{Max} K_n(t)$ for $\delta \leqslant t \leqslant \pi$, $n = 0, 1, \ldots$

Condition (ii) follows from the analogous property of D_n, and (iii) from the inequality $M_n(\delta) \leqslant 1/2\,(n+1)\sin^2\frac{1}{2}\delta$. Kernels with such properties are called *positive* kernels. Kernels satisfying, besides (ii), (iii), the condition (i') $\int_{-\pi}^{\pi} |K_n(t)|\, dt \leqslant C$ will be called 'quasi-positive'. Condition (i') follows from (ii) if (i) is satisfied.

3.21. Fejér's theorem [1]. *If the limits $f(x \pm 0)$ exist, $\mathfrak{S}[f]$ is summable $(C, 1)$ at the point x to the value $\frac{1}{2}[f(x+0) + f(x-0)]$. In particular, if f is continuous at x, $\mathfrak{S}[f]$ is summable there to the value $f(x)$. If f is continuous at every point of an interval $I = (a, b)$* [2], *$\mathfrak{S}[f]$ is uniformly summable in I.*

The proof will be based only on the properties (i), (ii), (iii) of K_n. We may assume that $2f(x) = f(x+0) + f(x-0)$, so that $|\varphi_x(t)| < \varepsilon$ for $0 \leqslant t \leqslant \delta = \delta(\varepsilon)$. From 3.201(2) we see that $|\sigma_n(x) - f(x)|$ does not exceed

$$(1)\qquad \frac{1}{\pi}\int_0^{\pi} |\varphi(t)|\, K_n(t)\, dt = \frac{1}{\pi}\left(\int_0^{\delta} + \int_{\delta}^{\pi}\right) \leqslant \frac{\varepsilon}{\pi}\int_0^{\pi} K_n\, dt + \frac{M_n(\delta)}{\pi}\int_0^{\pi} |\varphi|\, dt.$$

Let us denote the last two terms by P, Q. We have $P = \varepsilon/2$ (cond. (ii)), $Q \to 0$ (cond. (iii)), so that $P + Q < \varepsilon$ for $n > n_0 = n_0(\varepsilon)$, and, ε being arbitrary, the first part of the theorem follows.

[1]) Fejér [1].

[2]) We mean by this that f is continuous also at the points a, b.

If f is continuous at every point of I, we can find a δ such that $|\varphi_x(t)| < \varepsilon$ for $0 \leqslant t \leqslant \delta$, $x \in I$, and so (1) holds for any $x \in I$. The integral in Q does not exceed

$$\int_0^\pi |(f(x+t)| + |f(x-t)| + 2|f(x)|)\,dt = \int_{-\pi}^{\pi} |f(t)|\,dt + 2\pi|f(x)|.$$

Hence $Q \to 0$ uniformly in I, so that $P+Q < \varepsilon$ for $n > n_0$, $x \in I$.

If, in particular, (a, b) coincides with $(0, 2\pi)$, $\sigma_n(x)$ converges uniformly to $f(x)$.

3.211. The theorem would be true even if K_n were only quasi-positive. In fact, K_n in 3.21(1) should then be replaced by $|K_n|$. We should have $P = C\varepsilon/2$, $Q \to 0$, i. e. $P + Q < C\varepsilon$ for $n > n_0$.

3.22. *If* $m \leqslant f(x) \leqslant M$ *in* $(0, 2\pi)$, *then* $m \leqslant \sigma_n(x) \leqslant M$ [1]), i. e. the Fejér means are contained in the same range as the function f. (In particular $\sigma_n \geqslant 0$ if $f \geqslant 0$). This follows from the first formula 3.201(2) if we replace $f(x+t)$ first by m, and then by M, and take into account conditions (i), (ii).

If $m \leqslant f(x) \leqslant M$ *for* $x \in I = (a, b)$, *then, for every* $\delta > 0$, *there exists an integer* $n_0 = n_0(\delta)$ *such that*

(1) $\quad m - \delta \leqslant \sigma_n(x) \leqslant M + \delta$, *for* $x \in I_\delta = (a + \delta, b - \delta)$, $n > n_0$.

Break up the first integral 3.201(2) into three, extended over $(-\pi, -\delta)$, $(-\delta, \delta)$, (δ, π), and denote them by σ_n', σ_n'', σ_n'''. If $x \in I_\delta$, $|t| < \delta$, then $x + t \in I$, and σ_n'' is contained between m and M, multiplied by the integral of $K_n(t)/\pi$ over $(-\delta, \delta)$. In virtue of conditions (ii) and (iii) this last integral tends to 1. Since $|\sigma_n'|$ and $|\sigma_n''|$ do not exceed $M_n(\delta)/\pi$ multiplied by the integral of $|f(t)|$ over $(-\pi, \pi)$, and so tend to 0, a moment's consideration shows that (1) is valid.

From (1) we obtain in particular that $m \leqslant \underline{\lim}\, \sigma_n(x) \leqslant \overline{\lim}\, \sigma_n(x) \leqslant M$, for every $a < x < b$.

Given a function $f(x)$ let $M(a, b)$ and $m(a, b)$ denote the upper and lower bound respectively of f in (a, b). For every x let $M(x) = \lim M(x-h, x+h)$, $m(x) = \lim m(x-h, x+h)$ as $h \to 0$. The numbers $M(x)$, $m(x)$ are called the maximum and minimum respectively of f at the point x. From the last remark it follows

[1]) More precisely $m < \sigma_n(x) < M$, unless $f = \text{const}$.

that, for every x, $m(x) \leqslant \underline{\lim}\, \sigma_n(x) \leqslant \overline{\lim}\, \sigma_n(x) \leqslant M(x)$. If in particular $m(x) = M(x) = \infty$, then $\sigma_n(x) \to \infty$.

3.23. Corollaries of Fejér's theorem. (i) *If* $\mathfrak{S}[f]$ *converges at a point where* f *is continuous, or has a simple discontinuity, then it converges to* $\frac{1}{2}[f(x+0) + f(x-0)]$. In fact, if a series converges to s, it is summable $(C, 1)$ to the same value.

More generally, if x is a point of continuity of f, the interval of oscillation of the partial sums $s_n(x)$ contains $f(x)$.

(ii) *If* f *is of bounded variation, the partial sums of* $\mathfrak{S}[f]$ *are uniformly bounded.* Since the $\sigma_n(x; f)$ are uniformly bounded, it is sufficient to observe that the Fourier coefficients of f are $O(1/n)$ (§ 2.213) and to use the formula 3.13(1).

(iii) *If* f *is continuous and of period* 2π, *there exists, for every* $\varepsilon > 0$, *a trigonometrical polynomial* $T(x)$ *such that* $|f(x) - T(x)| < \varepsilon$ *everywhere.* We may take for $T(x)$ the expressions $\sigma_n(x; f)$ with n sufficiently large.

(iv) *The trigonometrical system is complete* (§ 1.5). If all the Fourier coefficients of a continuous function f vanish, $f(x)$, as the limit of Fejér's means, vanishes identically. For the case of discontinuous f see the argument in § 1.5.

(v) Hardy observed that *Dirichlet's Theorem* (§ 2.6) *can be deduced from Fejér's* by means of the following theorem from the general theory of series: *If* $u_0 + u_1 + \ldots$ *is summable* $(C, 1)$ *to a sum* s *and* $|u_n| \leqslant A/n$, $n = 1, 2, \ldots$, *where* A *is a constant, the series is convergent* [1]).

Without loss of generality we may assume that $s = 0$, $A = 1$. Let p, $p < n$, be a function of n tending to $+\infty$ which we shall define presently. Since $\sigma_p^1 \to 0$, the relation

$$\sigma_n^1 = \frac{s_0 + \ldots + s_p}{n+1} + \frac{s_{p+1} + \ldots + s_n}{n+1} \to 0 \text{ involves } \frac{s_{p+1} + \ldots + s_n}{n+1} \to 0.$$

If $k < n$, then $|s_n - s_k| \leqslant |u_{k+1} + \ldots + u_n| < 1/(k+1) + \ldots + 1/n < (n-k)/k$ and so the last relation may be written in the form

$$(1) \qquad \frac{n-p}{n+1} s_n + \theta \cdot \frac{(n-p)\,(n-p+1)}{2p\,(n+1)} \to 0,$$

[1]) Hardy [5].

where $\theta = \theta(n, p)$ does not exceed 1 in absolute value. Put now $n-p=[\varepsilon n]$, i. e. $p = n-[\varepsilon n]$, where $0<\varepsilon<1/2$ is arbitrary but fixed. Dividing both sides of (1) by $(n-p)/(n+1)$, we see that $\overline{\lim}\,|s_n| \leqslant \varepsilon/2(1-\varepsilon) < \varepsilon$, that is $s_n \to 0$.

Although the above argument is, on the whole, not simpler than the direct proof of Dirichlet's theorem, it is interesting as an application of the theory of summability to the convergence of Fourier series.

3.3. Summability (C, r) of Fourier series. *Fejér's theorem remains true if we replace summability $(C, 1)$ by (C, r), $r>0$* [1]. Denoting the (C, r) means of $\mathfrak{S}[f]$ by $\sigma_n^r(x)$, we find from 3.11(5), 2.3(2) the formulae

$$\sigma_n^r(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, K_n^r(t)\, dt, \quad \sigma_n^r(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)\, K_n^r(t)\, dt,$$

$$(1) \qquad K_n^r(t) = \sum_{k=0}^{n} A_{n-k}^{r-1} D_k(t)/A_n^r,$$

and it is sufficient to show that the kernel K_n^r is quasi-positive. We may suppose that $0<r<1$. Condition (ii) of § 3.201 is obviously satisfied. Conditions (i') and (iii) follow from the inequalities

$$(2) \qquad |K_n^r(t)| \leqslant 2n, \quad |K_n^r(t)| \leqslant Cn^{-r}t^{-r-1} \quad \text{for } 1/n \leqslant t \leqslant \pi,$$

which we will now prove; C is a constant independent of n. From the formula defining K_n^r we obtain

$$K_n^r(t) = \frac{1}{2A_n^r \sin\frac{1}{2}t}\,\mathfrak{I}\sum_{k=0}^{n} A_{n-k}^{r-1} e^{i(k+\frac{1}{2})t} = \mathfrak{I}\left[\frac{e^{i(n+\frac{1}{2})t}}{2A_n^r \sin\frac{1}{2}t}\sum_{k=0}^{n} A_k^{r-1} e^{-ikt}\right] =$$

$$(3) \qquad = \mathfrak{I}\left\{\frac{e^{i(n+\frac{1}{2})t}}{2A_n^r \sin\frac{1}{2}t}\left[(1-e^{-it})^{-r} - \sum_{k=n+1}^{\infty} A_k^{r-1} e^{-ikt}\right]\right\}.$$

Since A_n^{r-1} decreases steadily to 0, the last series converges for $t \neq 0$ and its sum does not exceed $4A_{n+1}^{r-1}/|1-e^{-it}|$ in absolute value (§ 1.23). And since $|\mathfrak{I}(z)| \leqslant |z|$, we have that, for $0<t\leqslant\pi$, $|K_n^r(t)|$ does not exceed

$$\{(2\sin\tfrac{1}{2}t)^{-r-1} + 4A_{n+1}^{r-1}(2\sin\tfrac{1}{2}t)^{-2}\}/A_n^r \leqslant \tfrac{1}{2}C\,(n^{-r}t^{-r-1} + n^{-1}t^{-2}).$$

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[1]) M. Riesz [1], [2]; Chapman [1].

Taking into account that $nt^2 = n^r t^{r+1}(nt)^{1-r} \geqslant n^r t^{r+1}$ for $nt \geqslant 1$, we obtain the second inequality (2). To prove the first, we note that $|D_k(t)| \leqslant \frac{1}{2}+1+\ldots+1 = k+\frac{1}{2} \leqslant n+1$ for $0 \leqslant k \leqslant n$, and so, applying 3.11(6), we obtain from (1) that $|K_n^r(t)| < n+1 \leqslant 2n$ $(n>0)$.

It is of some interest to note that for $r=1$ the formulae (2) are consequences of 3.201(3).

3.31. $\mathfrak{S}[f]$ *is summable (C, r), $r>0$, to the value $f(x)$ at every point x where $\Phi_x(t) = o(t)$* [1]) *and so, in particular, almost everywhere* (§ 2.703). This theorem is a simple consequence of 3.3(2). In fact

$$\pi\,|\sigma_n^r(x) - f(x)| \leqslant \left(\int_0^{1/n} + \int_{1/n}^{\pi}\right)|\varphi_x(t)|\,|K_n^r(t)|\,dt = P + Q.$$

From the first inequality in 3.3(2) we see that $P \leqslant 2n\,\Phi_x(1/n) \to 0$.

Integrating by parts we find that $Q \leqslant Cn^{-r}[\Phi_x(t)\,t^{-r-1}]_{1/n}^{\pi} +$

$$+ C(1+r)\,n^{-r}\int_{1/n}^{\pi}\Phi_x(t)\,t^{-r-2}\,dt = o(1) + C(1+r)\,n^{-r}\int_{1/n}^{\pi} o(t^{-1-r})\,dt = o(1)$$

(§ 1.71).

3.32. Summability (C, r) of conjugate series. Let $\bar{\sigma}_n^r$ denote the (C, r) means of $\bar{\mathfrak{S}}[f]$.

For almost every x the difference

$$(1) \qquad \bar{\sigma}_n^r(x) - \left(-\frac{1}{\pi}\int_{1/n}^{\pi}\psi_x(t)\,\tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2}t\,dt\right), \quad 0<r\leqslant 1,$$

where $\psi_x(t) = f(x+t) - f(x-t)$, tends to 0 as $n\to\infty$. This is in particular true for every x where $\Psi_x(t) = o(t)$ (§ 2.703)[2]). The proof is, roughly, the same as in Theorem 3.31. We have

$$(2) \qquad \begin{aligned} \bar{\sigma}_n^r(x) &= -\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\,\bar{K}_n^r(t)\,dt = -\frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\pi}\right) = A + B, \\ \bar{K}_n^r(t) &= \frac{1}{A_n^r}\sum_{k=0}^{n} A_{n-k}^{r-1}\bar{D}_k(t) = \tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2}t - \frac{1}{A_n^r}\sum_{k=0}^{n} A_{n-k}^{r-1}\frac{\cos(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}. \end{aligned}$$

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[1]) See Lebesgue [3] for $r=1$, Hardy [2] for the general case.

[2]) See Privaloff [2], Plessner [2] for $r=1$, Hardy and Littlewood [4], Zygmund [2] for the general case.

Exactly in the same way as in § 3.3 we show that $|K_n^r(t)| < 2n$, so that $A \to 0$, and the difference (1) is equal to

$$(3)\qquad \frac{1}{\pi}\int_{1/n}^{\pi} \psi_x(t)\, H_n^r(t)\, dt + o(1),$$

where $H_n^r(t)$ denotes the last sum in (2). For $H_n^r(t)$ we obtain the expression 3.3(3) with $\mathfrak{I}$ replaced by $\mathfrak{R}$. It follows that $H_n^r(t)$ satisfies the second inequality in 3.3(2), which, as we have shown in § 3.31, is sufficient to prove that (3) tends to 0.

3.321. The result of the preceding section shows that, for almost every x, the summability (C, r), $r > 0$, of $\bar{\mathfrak{S}}[f]$ is equivalent to the existence of the integral

$$(1)\qquad -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\, \tfrac{1}{2}\operatorname{ctg} \tfrac{1}{2} t\, dt = \lim_{h\to 0}\left(-\frac{1}{\pi}\int_h^{\pi} \psi_x(t)\, \tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2} t\, dt\right)^{1)}.$$

The problem of the existence of this integral is very delicate. We shall show in Chapter VII that it exists almost everywhere, for every integrable f. Taking this result here for granted, we obtain that $\bar{\mathfrak{S}}[f]$ is summable (C, r), $r > 0$, almost everywhere, to the value $\bar{f}(x)$ given in (1).

3.4. Abel's summability. In connection with 3.201(1) we put, for $0 \leqslant r < 1$,

$$f(r, x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\, r^n,$$

$$\bar{f}(r, x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)\, r^n.$$

Taking into account 1.12(1) and 1.12(2), we easily find that

$$f(r, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, P_r(t)\, dt,\quad f(r, x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)\, P_r(t)\, dt,$$

$$(1)\qquad \bar{f}(r, x) = -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\, Q_r(t)\, dt.$$

---

1) If $\bar{f}(x, h)$ denotes the second integral in (1), and if $1/(n+1) \leqslant h \leqslant 1/n$, then $\pi\,|\bar{f}(x, h) - \bar{f}(x, 1/n)| \leqslant (n+1)\,\Psi_x(1/n) \to 0$, as $n \to \infty$.

The functions

$$(2)\qquad P_r(t) = \tfrac{1}{2}(1-r^2)/\Delta_r(t),\quad Q_r(t) = r \sin t/\Delta_r(t),$$
$$\text{where } \Delta_r(t) = 1 - 2r\cos t + r^2,\ 0 \leqslant r < 1,$$

are called, for historical reasons, *Poisson's kernel* and *Poisson's conjugate kernel*. The expression on the right in the first formula (1) is called *Poisson's integral*. It is not difficult to see that $P_r(t)$ is a positive kernel, i. e. satisfies the conditions (i), (ii), (iii) of § 3.201. That r, which now plays the rôle of the index n, is a continuous variable, is irrelevant. Condition (i) follows from the inequality $\Delta_r(t) > 0$. Condition (ii) may be obtained integrating both sides of 1.12(1) over the range $(-\pi, \pi)$. Since $\Delta_r(t) = (1-r)^2 + 4r\sin^2 \tfrac{1}{2} t$, we see that $M_r(\delta) = \operatorname{Max} P_r(t)$ for $0 < \delta \leqslant t \leqslant \pi$ is $\leqslant (1-r^2)/8r\sin^2\tfrac{1}{2}\delta \to 0$ as $r \to 1$, so that condition (iii) is also fulfilled. Hence

Theorem 3.21 *remains true if we replace summability* $(C, 1)$ *by summability A.* The reader has, no doubt, noticed, that this theorem is a consequence of Fejér's theorem and of Theorem 3.14, but a direct study of Poisson's kernel is interesting in itself.

3.41. The functions $f(r, x)$, $\bar{f}(r, x)$, as the real and imaginary parts of a function analytic inside the unit circle (§ 1.12), are harmonic, that is, when treated as functions of rectangular coordinates ξ, η, they satisfy Laplace's equation $\partial^2 u/\partial\xi^2 + \partial^2 u/\partial\eta^2 = 0$. Let us denote the polar coordinates of points in the unit circle by r, x $(0 \leqslant r < 1,\ 0 \leqslant x < 2\pi)$, and let $f(x)$ be a continuous and periodic function of x. The function $f(r, x)$ defined by Poisson's integral tends uniformly to $f(x)$ as $r \to 1$. In other words, Poisson's integral gives a solution (or rather, as it is well-known, *the* solution) for the case of the unit circle of the following very famous problem ('Dirichlet's problem'): Given (1) a plane region G, whose boundary is a simple closed curve L, (2) a function $f(p)$, defined and continuous at the points $p \in L$, to find a function $F(p)$, harmonic in G, continuous in $G + L$, and coinciding with $f(p)$ on L. However, in this special case of the unit circle, Poisson's integral gives a solution of a more general Dirichlet's problem, viz. when the limit function is an arbitrary integrable function (§ 3.442).

3.42. *If* $m \leqslant f(x) \leqslant M$, *then* $m \leqslant f(r, x) \leqslant M$. *If* $m \leqslant f(x) \leqslant M$ *for* $x \in I = (a, b)$, *then, for every* $\delta > 0$, *there exists a number* r_0

such that $m-\delta \leqslant f(r,x) \leqslant M+\delta$ *for* $x \in (a+\delta, b-\delta)$, $r_0 < r < 1$. The proof is essentially the same as in § 3.22.

If $M(x_0)$ *and* $m(x_0)$ *are the maximum and minimum of* f *at a point* x_0 (§ 3.22), *and if* L *is an arbitrary path leading from inside the unit circle to the point* $(1, x_0)$, *the limits of indetermination of* $f(r,x)$, *as the point* (r,x) *approaches* $(1,x_0)$ *along* L, *are contained between* $m(x_0)$ *and* $M(x_0)$. In fact, given an $\varepsilon > 0$, there exists an h such that $m(x_0)-\varepsilon \leqslant f(x) \leqslant M(x_0)+\varepsilon$ for $|x-x_0| < h$. Supposing, as we may, that $h<\varepsilon$, let us apply the preceding theorem with $(a,b)=(x_0-h,\ x_0+h)$, $\delta=h/2$. Then, if (r,x) belongs to the curvilinear quadrangle (Q) $r_0<r<1$, $|x-x_0|<h/2$, $f(r,x)$ is contained between $m(x_0)-\varepsilon-h/2$ and $M(x_0)+\varepsilon+h/2$, and à fortiori between $m(x_0)-3\varepsilon/2$ and $M(x_0)+3\varepsilon/2$. Since, from some point onwards, L lies entirely in Q, and ε is arbitrary, the theorem follows[1]. In particular, if f is continuous at x_0, $\lim f(r,x)$ along L exists and is equal to $f(x_0)$.

3.43. Let x_0 be a point of simple discontinuity for f. To determine the behaviour of $f(r,x)$ in the neighbourhood of $(1,x_0)$, suppose that $x_0=0$, $2f(0)=f(+0)+f(-0)$, $d=f(+0)-f(-0)\neq 0$. Let $\delta(x)$ denote the periodic function equal to $(\pi-x)/2$ for $0<x<2\pi$. The difference $g(x)=f(x)-\delta(x)\,d/\pi$ is continuous at $x=0$, and $g(0)=f(0)$. If $g(r,x)$ and $\delta(r,x)$ are Poisson's integrals for g and δ, then $f(r,x)=g(r,x)+\delta(r,x)\,d/\pi$. Let α be the angle at which a path L meets the real axis at the point $(1,0)$, that is $\alpha=\lim\beta$, where β is the angle of the vector $(1,0)\,(r,x)$ with the real axis. Since $g(r,x)\to g(0)=f(0)$, and $\delta(r,x)=\operatorname{arctg}\{r\sin x/(1-r\cos x)\}$ (§ 1.12(3)), we see that $f(r,x)$ tends to $f(0)+\alpha\,d/\pi$, i. e. *the limit is a linear function of the angle at which* L *meets the radius at the point* $(1,x_0)$. It is plain that if $\alpha=\lim\beta$ does not exist, $f(r,x)$ oscillates finitely as $(r,x)\to(1,x_0)$ along L.

3.44. Fatou's theorems[2]. Let $F(x)$ be a function with Fourier coefficients A_n, B_n. *If* $[F(x+t)-F(x-t)]/2t \to l$ *as* $t\to 0$, *where* l *is not necessarily finite, then* $\mathfrak{S}'[F]$ *is summable A at the point* x *to the value* l, *i. e.*

[1]) The corresponding result for Fejér's means is as follows: for every $\{h_n\}\to 0$, the limits of indetermination of $\{\sigma_n(x_0+h_n)\}$ are contained between $m(x_0)$ and $M(x_0)$.

[2]) Fatou [1]. See also Grosz [1].

$$(1)\qquad \sum_{n=1}^{\infty} n\,(B_n\cos nx - A_n\sin nx)\,r^n = \frac{\partial F(r,x)}{\partial x} \to l \ \text{ as } r\to 1.$$

More generally, if $l_1 \leqslant l_2$ are the limits of indetermination of the ratio $[F(x+t)-F(x-t)]/2t$, as $t\to 0$, the limits of indetermination of the expression in (1) are contained between l_1 and l_2[1]). We have

$$(2)\qquad \begin{aligned} F(r,x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\,P_r(t-x)\,dt,\\ \frac{\partial F(r,x)}{\partial x} &= -\frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\,P_r'(t-x)\,dt, \end{aligned}$$

where $'$ denotes differentiation with respect to t; and, since P_r' is odd,

$$\frac{\partial F(r,x)}{\partial x} = -\frac{1}{\pi}\int_0^{\pi} \chi(t)\,2\sin t\,P_r'(t)\,dt,$$

where $\chi(t)=[F(x+t)-F(x-t)]/2\sin t$. Then, in order to prove the theorem, it is sufficient to show that the even function $-\sin t\,P_r'(t)/r = (1-r^2)\sin^2 t/\Delta_r^2(t)$ possesses the properties of positive kernels. Conditions (i) and (iii) of § 3.201 are obviously satisfied, and we verify (ii) by substituting $x=0$, $F(t)=\sin t$, i. e. $\chi(t)=1$.

3.441. *If* $F'(x_0)$ *exists and is finite, then* $\partial F(r,x)/\partial x \to F'(x_0)$ *when* $(r,x)\to(1,x_0)$ *along any path* L *not touching the circle*. Suppose, for simplicity, that $x_0=0$, $F(0)=0$, and let $r=r(u)$, $x=x(u)$, $0\leqslant u\leqslant 1$, $r(1)=1$, be a parametric equation of L. Put $-\sin t\,P_r'(t-x) = A_u(t)$ for $(r,x)\in L$. The theorem will be proved, when we show that $A_u(t)$ satisfies the following conditions

$$\text{(i)}\quad \int_{-\pi}^{\pi}|A_u(t)|\,dt = O(1), \qquad \text{(ii)}\quad \frac{1}{\pi}\int_{-\pi}^{\pi} A_u(t)\,dt \to 1,$$

(iii) $M_u(\delta)=\operatorname{Max}|A_u(t)|\,(0<\delta\leqslant t\leqslant\pi)$ tends to 0, as $u\to 1$, i. e. that $A_u(t)$ is, essentially, a quasi-positive kernel. In fact, put-

[1]) l_1 and l_2 are contained between the smallest and the largest of the four derivates of F at the point x.

ting $F(t)/\sin t - F'(0) = G(t)$, and denoting by $\theta(u)$ the left-hand side of (ii), we deduce that

$$\frac{\partial F(r,x)}{\partial x} - \theta(u)F'(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} G(t)\,A_u(t)\,dt = \int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}.$$

The last two integrals on the right tend to 0 for fixed δ (cond. (iii)), and the preceding term is small with δ (cond. (i)).

Now relation (ii) follows from the second formula 3.44(2) if we put $F(t) = \sin t$. The left-hand side of (i) is equal to

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin(t+x)\,P_r'(t)|\,dt \leqslant 2|\sin x|\int_0^{\pi}|P_r'(t)|\,dt + 2\int_0^{\pi}\sin t\,|P_r'(t)|\,dt.$$

Since $P_r'(t) \leqslant 0$ in $(0,\pi)$, the first term on the right is less than $2x\,P_r(0) \leqslant 2x/(1-r) = O(1)$, if $(r,x)\in L$. The last term on the right is also bounded, $-2\sin t\,P_r'(t)/r$ being a positive kernel. Condition (iii) is obvious.

3.442. Corollary. Let F be an integral of f. *For any x_0 where $f(x_0)$ is finite and equal to $F'(x_0)$, we have $f(r,x)\to f(x_0)$, as $(r,x)\to(1,x_0)$ along any path not touching the circle.* In fact, supposing for simplicity that the constant term of $\mathfrak{S}[f]$ vanishes, we have $\mathfrak{S}[f] = \mathfrak{S}'[F]$, and the result follows from Theorem 3.441.

3.45. *At any point x where f is finite and is the differential coefficient of its integral F, we have*

$$(1)\qquad \bar f(r,x) - \left(-\frac{1}{\pi}\int_{\eta}^{\pi}\frac{f(x+t)-f(x-t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt\right)\to 0,\ \ as\ r\to 1\ ^{1)},$$

where the number $\eta = \eta(r)$, $0<\eta<\pi/2$, is the root of the equation $\cos x = 2r/(1+r^2)$. It is plain that $\eta\to 0$ as $r\to 1$. More precisely, from the formula $\sin\eta = (1-r)(1+r)/(1+r^2)$ we find that $\eta\simeq 1-r^{2)}$.

The last formula in 3.4(1) gives us $\pi\bar f(r,x) = -\int_0^{\eta}\psi_x(t)\,Q_r(t)\,dt\ +$

$$-\int_{\eta}^{\pi}\psi_x(t)\,[Q_r(t)-Q_1(t)]\,dt - \int_{\eta}^{\pi}\psi_x(t)\,Q_1(t)\,dt,$$ and we have only

[1]) Privaloff [2], Plessner [2]. See also Fatou [1].

[2]) The theorem holds true if we replace η by $1-r$ in (1), but this is irrelevant for our purposes.

to show that the first two terms on the right are $o(1)$. Let $B(h) = o(h)$ be the integral of $\psi_x(t)$ over the range $0\leqslant t\leqslant h$. From the formula for $Q_r'(t)$ we see that $Q_r(t)$ is monotonically increasing in $(0,\eta)$. Hence, applying the second mean-value theorem to the first term, we find that it is equal to $Q_r(\eta)\,[B(\eta)-B(\tau)] = o(1)$, since $0<\tau<r$ and $Q_r(\eta) < r + r^2 + \ldots < 1/(1-r)$. It is easy to verify that $Q_r(t) - Q_1(t) = -(1-r)^2\,Q_r(t)/2\,(1-\cos t)$. Applying the same mean-value theorem to the second term in question, we find that it is equal to the expression $(1-r)^2\,Q_r(\eta)/2 = O(1-r)$ multiplied by the integral

$$(2)\qquad \int_{\eta}^{\xi}\frac{\psi_x(t)}{1-\cos t}\,dt = \left[\frac{B(t)}{1-\cos t}\right]_{\eta}^{\xi} + \int_{\eta}^{\xi}\frac{B(t)\sin t}{(1-\cos t)^2}\,dt\quad (\eta<\xi<\pi).$$

Since $B(\xi)/(1-\cos\xi) = o(\xi^{-1}) = o(\eta^{-1})$, and the last integrand is $o(t^{-2})$, the left-hand side of (2) is $o(\eta^{-1}) = o(1-r)^{-1}$ and this completes the proof.

Since $\eta(r)$ tends continuously to 0 as $r\to 1$, we see that *a necessary and sufficient condition for the summability A of $\bar{\mathfrak{S}}[f]$ at the point x, is the existence of the integral* 3.321(1), *which represents then the sum of $\bar{\mathfrak{S}}[f]$.*

3.5. The Cesàro summation of differentiated series. According to Theorem 3.442, $\mathfrak{S}[f]$ is summable A at any point x where f is the finite derivative of its integral, whereas to prove the summability $(C,1)$ we used a somewhat stronger condition, viz. $\Phi_x(t) = o(t)$. Indeed it may be shown that the former condition does not ensure the summability $(C,1)$ of $\mathfrak{S}[f]$. We will now prove that.

(i) *At every point x where $F'(x) = \lim\,[F(x+h)-F(x-h)]/2h$ exists and is finite, $\mathfrak{S}'(F)$ is summable (C,r), $r>1$, to the value $F'(x)$* (ii) *At every point x where f is finite and is the differential coefficient of its integral, $\mathfrak{S}[f]$ is summable (C,r), $r>1$, to the value $f(x)$* [1]).

To prove (i), of which (ii) is a corollary, it is sufficient to show that $L_n^r(t) = \sin t\,[K_n^r(t)]'$ is a quasi-positive kernel if $r>1$ [2]). This will be a consequence of the inequalities

[1]) Lebesgue [3] for $r=2$, Privaloff [2], Young [6] in the general case.

[2]) The situation is the same as in § 3.44, except that $\sin t\,P_r'(t)$ is a positive kernel.

(1) $|L_n^r(t)| \leqslant n$ for $0 \leqslant t \leqslant 1/n$, (2) $|L_n^r(t)| \leqslant C/n^{r-1}t^r$ for $1/n \leqslant t \leqslant \pi$,

valid for $1<r<2$. C is a constant independent of n and t.

Let D_k be Dirichlet's kernel. Since $|D_k'| \leqslant 1+2+\dots+k \leqslant n^2$ for $0 \leqslant k \leqslant n$, we find that $|[K_n^r]'| \leqslant n^2$, i. e. $|L_n^r(t)| \leqslant n^2 t \leqslant n$ for $0 \leqslant t \leqslant 1/n$, and the inequality (1) is established.

Using Abel's transformation, we verify the formula $\sum_{k=0}^{n} A_k^\alpha e^{ikt} = \left[-A_n^\alpha e^{i(n+1)t} + \sum_{k=0}^{n} A_k^{\alpha-1} e^{ikt}\right] \Big/ (1-e^{it})$. Applying this formula twice to the last expression but one in 3.3(3), we find that

$$K_n^r(t) = C_n(2\sin\tfrac{1}{2}t)^{-2} + \Im\left\{\frac{e^{i(n+1/2)t}}{2A_n^r \sin\frac{1}{2}t}\left[(1-e^{-it})^{-r} - \frac{\sum_{k=n+1}^{\infty} A_k^{r-3} e^{-ikt}}{(1-e^{-it})^2}\right]\right\} =$$

$$= C_n(2\sin\tfrac{1}{2}t)^{-2} + \frac{\sin\left[(n+\frac{1}{2}+\frac{1}{2}r)t - \frac{\pi}{2}r\right]}{A_n^r(2\sin\frac{1}{2}t)^{r+1}} + \Im\left[\frac{e^{i(n+3/2)t}}{A_n^r(2\sin\frac{1}{2}t)^3}\sum_{k=n+1}^{\infty} A_k^{r-3} e^{-ikt}\right],$$

where $C_n = A_n^{r-1}/A_n^r + A^{r-2}/2A_n^r = O(1/n)$. Let P, Q, R denote the three terms in the last formula for K_n^r. Then $P_n' = O(1/nt^3)$, $Q_n' = O(1/n^r t^{r+2}) + O(1/n^{r-1}t^{r+1}) = O(1/n^{r-1}t^{r+1})$ if $nt \geqslant 1$. Let $\alpha(t) = \exp i(n+3/2)t/(2\sin\frac{1}{2}t)^3$ and let $\beta(t)$ be the sum following $\alpha(t)$ in R. Using Theorem 1.22, we see that $|\beta(t)| \leqslant 4|A_{n+1}^{r-3}|/|1-e^{-it}| = O(n^{r-3}/t)$, $|\beta'(t)|$ [1] $\leqslant 4(n+1)|A_{n+1}^{r-3}|/|1-e^{-it}| = O(n^{r-2}/t)$. Since, on the other hand, $\alpha(t) = O(t^{-3})$, $\alpha'(t) = O(n/t^3) + O(1/t^4) = O(n/t^3)$ if $nt \geqslant 1$, we find that $|R_n'| \leqslant |\alpha'\beta + \beta'\alpha|/A_n^r = O(n^{-2}t^{-4})$.

Collecting the results, we obtain that $[K_n^r(t)]' = O(1/nt^3) + O(1/n^{r-1}t^{r+1}) + O(n^{-2}t^{-4}) = O(1/n^{r-1}t^{r+1})$ if $nt \geqslant 1$. Thence we have $L_n^r(t) = O(n^{1-r}t^{-r})$ if $t \geqslant 1/n$, $1<r<2$, and this completes the proof.

Let $G(h)$ be the integral of $\varphi(t) = f(x+t) + f(x-t) - 2f(x)$ over the interval $0 \leqslant t \leqslant h$. Applying (ii) to $\mathfrak{S}[\varphi]$, we see that *$\mathfrak{S}[f]$ is summable (C, r), $r>1$ at the point x and has the sum $f(x)$, if $G(h) = o(h)$ as $h \to 0$.*

Essentially the same proof shows that *under the hypothesis of Theorem* (ii), *we have the relation* 3.32(1), *for* $1 \leqslant r \leqslant 2$.

[1]) The series defining β' converges for $t \neq 0$ if $r<2$.

3.6. Fourier sine series. Let $f(x)$ be an odd function. From the first formula 3.4(1) we deduce that

$$(1) \qquad f(r,x) = \frac{1}{\pi}\int_0^\pi f(t)\,[P_r(x-t) - P_r(x+t)]\,dt.$$

(i) *If $f(x) \not\equiv 0$ is odd and non-negative in $(0,\pi)$, the function $f(r,x)$ is positive for $0<x<\pi$. More generally, if $f(x) \not\equiv$ const. satisfies an inequality $m \leqslant f(x) \leqslant M$ for $0<x<\pi$, then*

$$(2) \qquad m\,\mu(r,x) < f(r,x) < M\,\mu(r,x) \quad \textit{for } 0<x<\pi, \quad 0<r<1,$$

where $\mu(r,x)$, which is positive for $0<x<\pi$, is the Poisson integral for the function $\mu(x) = \operatorname{sign} x$ $(|x|<\pi)$.

The first part of the theorem follows from (1) if we note that $P_r(x-t) > P_r(x+t)$ for $0<x<\pi$, $0<t<\pi$. For this reason we have also

$$\frac{m}{\pi}\int_0^\pi [P_r(x-t) - P_r(x+t)]\,dt < f(r,x) < \frac{M}{\pi}\int_0^\pi [P_r(x-t) - P_r(x+t)]\,dt,$$

which is just (2).

(ii) *Theorem* (i) *remains true if we replace summability A by summability* $(C, 3)$ [1]). In particular, the inequality (2) should be replaced by $m\mu_n^3(x) < \sigma_n^3(x) < M\mu_n^3(x)$, where σ_n^3 and μ_n^3 denote the $(C, 3)$ means of $\mathfrak{S}[f]$ and $\mathfrak{S}[\mu]$.

For the proof it is sufficient to show that the kernel $K_n^3(t)$ is a strictly decreasing function in $(0,\pi)$, or, $K_n^3(t)$ being a trigonometrical polynomial, that $[K_n^3(t)]' \leqslant 0$ in $(0,\pi)$. The last expression is the Cesàro mean $s_n^3(t)/A_n^3$ of the series $\frac{1}{2} + \cos t + \cos 2t + \dots$ differentiated term by term. Thus from 3.11(1) we conclude that

$$\sum_{n=0}^{\infty} s_n^3(t)\,r^n = -\left[\frac{1-r^2}{2(1-r)^2\Delta_r(t)}\right]^2 \cdot \frac{4r\sin t}{1-r^2},$$

where $\Delta_r(t) = 1 - 2r\cos t + r^2$. Using the formulae 3.11(1) again, we see that the expression in square brackets is the power series $K_0(t) + 2K_1(t)r + \dots + (n+1)K_n(t)r^n + \dots$, where the coefficients $K_n(t) \geqslant 0$ are Fejér's kernels. Since $r/(1-r^2) = r + r^3 + \dots$ has also

[1]) F e j é r [4].

non-negative coefficients, we see that $s_n^3(t) < 0$ [1]) in $(0, \pi)$, and this completes the proof.

3.7. Convergence factors. A sequence $\lambda_0, \lambda_1, \ldots$ is said to be *convex* if $\Delta^2 \lambda_n \geqslant 0$, $n = 0, 1, \ldots$, where $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$, $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$. Suppose, in addition, that $\{\lambda_n\}$ is bounded. Since, for $\{\lambda_n\}$ convex, $\Delta \lambda_n$ is non-increasing and cannot be negative for any value of n (for otherwise we should have $\lambda_n \to \infty$), we have $\Delta \lambda_n \geqslant 0$, i. e. $\lambda_n \geqslant \lambda_{n+1} \to \lambda > -\infty$. In the equation $\lambda_0 - \lambda = \Delta \lambda_0 + \Delta \lambda_1 + \ldots$ the terms on the right are steadily decreasing, and so, by a well-known theorem of Abel, $n \Delta \lambda_n \to 0$. Taking this into account, and applying to the series $1.\Delta \lambda_0 + 1.\Delta \lambda_1 + \ldots$ Abel's transformation, we obtain: *If $\{\lambda_n\}$ is convex and bounded, then $\{\lambda_n\}$ decreases, $n\Delta\lambda_n \to 0$ and the series*

$$(1) \qquad \sum_{n=0}^{\infty} (n+1)\, \Delta^2 \lambda_n$$

converges to the sum $\lambda_0 - \lim \lambda_n$.

If a function $\lambda(x)$ is twice differentiable and $\lambda''(x) \geqslant 0$, the sequence $\{\lambda_n\} = \{\lambda(n)\}$ is convex. In fact, by the mean-value theorem, $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1} = -\lambda'(\theta_n) + \lambda'(\theta_{n+1}) \geqslant 0$, where $n < \theta_n < n+1$. In particular, if we put $\lambda_n = 1/\log n$ for $n = 2, 3, \ldots$, and choose for λ_0, λ_1 suitable values, $\{\lambda_n\}$ will be convex.

We need the following lemma:

Let s_n and σ_n denote the partial sums and the first aritmetic means of a series $u_0 + u_1 + \ldots$ If $\{\sigma_n\}$ converges and $s_n = o(1/\mu_n)$, where $\{\mu_n\}$ is convex and tends to 0, the series $u_0\mu_0 + u_1\mu_1 + \ldots$ converges. Applying twice Abel's transformation to the partial sum t_n of the last series, we find that it is equal to

$$\sum_{k=0}^{n-2} (k+1)\, \sigma_k \Delta^2 \mu_k + n\sigma_{n-1} \Delta\mu_{n-1} + s_n \mu_n \to \sum_{k=0}^{\infty} (k+1)\, \sigma_k\, \Delta^2 \mu_k.$$

Remark. A sequence $\{\lambda_n\}$ will be called a *quasi-convex* sequence if the series (1) converges absolutely. The lemma will subsist for quasi-convex $\{\mu_n\}$ if we prove that $n\Delta\mu_{n-1} \to 0$. But

$$|\Delta \mu_{n-1}| = \lim_{N\to\infty} \left| \sum_{k=n-1}^{N} \Delta^2 \mu_k \right| \leqslant n^{-1} \sum_{k=n-1}^{\infty} (k+1)\, |\Delta^2 \mu_k| = o(n^{-1}).$$

[1]) The first two coefficients of the series $[K_0 + 2K_1 r + \ldots]^2 = {}^1/_4 + 2K_1 r + \ldots$ are positive for $0 < t < \pi$, and this shows that $s_n^3(t) < 0$ for $0 < t < \pi$.

As a corollary we have the following theorem.

3.71. *If a_n, b_n are the Fourier coefficients of a function f, the series*

$$\sum_{k=2}^{\infty} \frac{a_k \cos kx + b_k \sin kx}{\log k}, \qquad \sum_{k=2}^{\infty} \frac{a_k \sin kx - b_k \cos kx}{\log k}$$

converge almost everywhere [1]) (§§ 2.73, 3.31, 3.321).

It is not difficult to deduce that *if f is continuous in (a, b), the first series converges uniformly in every interval $(a+\delta, b-\delta)$.*

3.8. Summability of Fourier-Stieltjes series [2]). Let $F(x)$, $0 \leqslant x \leqslant 2\pi$, be a function of bounded variation. From Theorems 2.13 and 3.5 we see that $\mathfrak{S}[dF]$ is summable (C, r), $r > 1$, at almost every point and has the sum $F'(x)$. We will now prove a stronger result, viz.

Let $\sigma_n^r(x)$ and $\bar{\sigma}_n^r(x)$ denote the r-th Cesàro means of $\mathfrak{S}[dF]$ and $\bar{\mathfrak{S}}[dF]$. If $0 < r \leqslant 1$, then

$$(1a) \quad \sigma_n(x) \to F'(x), \qquad (1b) \quad \bar{\sigma}_n^r(x) - \left\{ -\frac{1}{\pi} \int_{1/n}^{\pi} \frac{F(x+t) + F(x-t) - 2F(x)}{4 \sin^2 \frac{1}{2} t} \right\} dt \to 0,$$

for almost every x.

We shall only sketch the proof, which is similar to that of Theorems 3.31 and 3.32. First of all we need the following lemma, analogous to the result of § 2.703. *Let*

$$F_x^*(t) = F(x+t) - F(x-t) - 2tF'(x), \quad G_x^*(t) = F(x+t) + F(x-t) - 2F(x),$$

and let $\Phi_x^*(h)$, $\Psi_x^*(h)$ be the total variations of the functions $F_x^*(t)$, $G_x^*(t)$ over the interval $0 \leqslant t \leqslant h$. Then, for almost every x we have $\Phi_x^*(h) = o(h)$, $\Psi_x^*(h) = o(h)$.

Let α be an arbitrary number, and let $V_\alpha(t)$ be the total variation of the function $F(t) - \alpha t$. For almost every x we have $V'_\alpha(x) = |F'(x) - \alpha|$, i. e.

$$\frac{1}{h} \int_0^h |\, d_t\{F(x \pm t) - \alpha t\} \,| \to |\, F'(x) - \alpha \,| \quad \text{as } h \to +0,$$

where the suffix t indicates that the variation is taken with respect to the variable t. Considering rational values of α and arguing as in § 2.703, we prove that, for almost every x, we have

$$(2) \qquad \int_0^h |\, d_t \{F(x \pm t) - t\,F'(x)\} \,| = o(h),$$

and hence

$$\int_0^h |\, d_t\, F_x^*(t) \,| = o(h), \qquad \int_0^h |\, d_t\, G_x^*(t) \,| = o(h).$$

[1]) For the first part see Hardy [2], for the second Plessner [2].
[2]) Young [3], M. Riesz [2], Plessner [2].

Now it is easy to prove the theorem. From the formulae 1.47(1) we obtain that

$$\sigma_n^r = \frac{1}{\pi}\int_{-\pi}^{\pi} K_n^r(x-t)\,dF(t) = \frac{1}{\pi}\int_0^{\pi} K_n^r(t)\,d_t\{F(x+t)-F(x-t)\},$$

$$\sigma_n^r(x)-F'(x) = \frac{1}{\pi}\int_0^{\pi} K_n^r(t)\,d_t F_x^*(t);\ |\sigma_n^r(x)-F(x)| \leq \frac{1}{\pi}\int_0^{\pi} |K_n^r(t)|\,|d_t F_x^*(t)| =$$

$$= \frac{1}{\pi}\int_0^{1/n} + \frac{1}{\pi}\int_{1/n}^{\pi}.$$

Supposing that $\Phi_x^*(h)=o(h)$, we obtain, in virtue of the inequalities 3.3(2), that the first term in the last sum is $\leq 2n\Phi_x^*(1/n)/\pi = o(1)$. Integrating by parts, we find that the second term does not exceed

$$\frac{C}{\pi n^r}[\Phi_x^*(t)\,t^{-r-1}]_{1/n}^{\pi} + \frac{C(1+r)}{\pi n^r}\int_{1/n}^{\pi}\Phi_x^*(t)\,t^{-r-2}\,dt = o(1),$$

and this gives the first part of the theorem. To obtain the second we observe that

$$\overline{\sigma}_n^r(x) = -\frac{1}{\pi}\int_0^{\pi}\overline{K}_n^r(t)\,d_t[F(x+t)+F(x-t)] = -\frac{1}{\pi}\int_0^{\pi}\overline{K}_n^r(t)\,d_t\,G_x(t),$$

$$\overline{\sigma}_n^r(x) - \left(-\frac{1}{\pi}\int_{1/n}^{\pi}\frac{d_t\,G_x(t)}{2\operatorname{tg}\frac{1}{2}t}\right) = -\frac{1}{\pi}\int_0^{1/n}\overline{K}_n^r(t)\,d_t\,G_x(t) + \frac{1}{\pi}\int_{1/n}^{\pi} H_n^r(t)\,d_t\,G_x(t)$$

(§ 3.32). From the lemma we easily deduce that each of the terms on the right is $o(1)$. Integrating by parts we verify that the left-hand side of the last equation differs from the left-hand side of (1b) by a term tending to 0 as $n\to\infty$. This completes the proof.

3.81. The lemma proved in the preceding section is of fundamental importance for the theory of Fourier-Stieltjes series. From it we deduce that the partial sums of $\mathfrak{S}[dF]$ and $\overline{\mathfrak{S}}[dF]$ are $o(\log n)$ at almost every point. Similarly, taking for granted the result that $\overline{\mathfrak{S}}[dF]$ is summable $(C,1)$ almost everywhere, we verify that Theorem 3.71 holds true for Fourier-Stieltjes series.

3.9. Miscellaneous theorems and examples.

1. Let (L) $x=\varphi(t)$, $y=\psi(t)$, $0\leq t\leq 2\pi$, be a closed and convex curve. If $\varphi_n(t)$ and $\psi_n(t)$ are the Fejér means of $\mathfrak{S}[\varphi]$ and $\mathfrak{S}[\psi]$, the curves $x=\varphi_n(t)$, $y=\psi_n(t)$, $n=0,1,\ldots$, lie in the interior of the region limited by L. Fejér [5].

[If A, B, C are constants, and $A\varphi(t)+B\psi(t)+C>0$, then $A\varphi_n(t)+B\psi_n(t)+C>0$].

2. Let $f_n(r,x)$ be the n-th partial sum of the series $f(r,x)$ (§ 3.4). If $m\leq f(x)\leq M$, $0\leq x\leq 2\pi$, then $m\leq f_n(r,x)\leq M$ for $0\leq r\leq\frac{1}{2}$, but not necessarily for $r>\frac{1}{2}$. Fejér [2].

[The expression $\frac{1}{2}+r\cos x+\ldots+r^n\cos nx = \dfrac{1-r^2-2r^{n+1}[\cos(n+1)x - r\cos nx]}{2(1-2r\cos x+r^2)}$ is non-negative for $r\leq 1/2$. The sum $\frac{1}{2}+\cos x$ is negative for $x=\pi$, if $r>\frac{1}{2}$].

3. Let $F_x(h)$ and $\Phi_x(h)$ denote the integrals of $\varphi_x(t)$ and $|\varphi_x(t)|$ over the interval $0\leq t\leq h$. Neither of the conditions (i) $F_x(h)=o(h)$, (ii) $\Phi_x(h)=O(h)$ necessitates the summability $(C,1)$ of $\mathfrak{S}[f]$ at the point x. Show that if *both* of them are satisfied, then $\mathfrak{S}[f]$ is summable $(C,1)$ at the point x, to the value $f(x)$.

[The argument is analogous to that of § 3.31, except that now we consider the integrals of $\varphi(t)K_n(t)$ over intervals $(0,k/n)$, $(k/n,\pi)$, where k is large but fixed. In virtue of (ii), the second integral is small with $1/k$. The Fejér kernel has a bounded number of maxima and minima in $(0,k/n)$, and so, employing the second mean-value theorem[1]) and the relation (i), we obtain that the first integral tends to 0.

This generalization of Theorem 3.31 is typical and many other theorems may be generalized in the same way. The theorem is due to Hardy and Littlewood [5]].

4. Let $\{\alpha_n\}$ be an arbitrary sequence of numbers such that $\alpha_n=O(1/n)$, and let $\sigma_n^r(x)$ be the r-th Cesàro means of $\mathfrak{S}[f]$, $r>0$. At any point x where $\Phi_x(h)=o(h)$, we have $\sigma_n^r(x+\alpha_n)\to f(x)$.

[This is an analogue of Theorem 3.441. The proof is similar to that of Theorem 3.3].

5. Let $s_n^*(x)$ be the modified partial sums of $\mathfrak{S}[f]$ (§ 2.3). A necessary and sufficient condition for the convergence of the series (S) $\sum_{k=1}^{\infty}\dfrac{s_k^*-f}{k}$ at a point x where $\Phi_x(h)=o(h)$, is the existence of the integral $\displaystyle\int_0^{\pi}\frac{\varphi_x(t)}{2\sin\frac{1}{2}t}\,dt$.

[Let $u_n(x)$ be the n-th partial sum of the series $\sin x+\frac{1}{2}\sin 2x+\ldots = (\pi-x)/2$, $r_n(x)=(\pi-x)/2-u_n(x)$. It is plain that $|u_n(x)|\leq nx$, and making Abel's transformation we obtain that $r_n(x)=O(1/nx)$. Let $S_n(x)$ be the n-th partial sum of S. We have

$$S_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{\varphi_x(t)}{2\sin\frac{1}{2}t}u_n(t)\,dt = \frac{1}{\pi}\int_0^{1/n} + \frac{1}{\pi}\int_{1/n}^{\pi} = A+B.$$

Now $A\to 0$, and, in virtue of the inequality for r_n, we obtain that $S_n(x) - \dfrac{1}{\pi}\displaystyle\int_{1/n}^{\pi}\frac{\varphi_x(t)}{2\sin\frac{1}{2}t}\,\frac{\pi-t}{2}\,dt\to 0$. See also Hardy and Littlewood [4]].

6. Let $s_n(x)$ be the n-th partial sum of $\mathfrak{S}[f]$. If $f\in\mathrm{Lip}\,\alpha$, $0<\alpha\leq 1$, then $|s_n(x)-f(x)| = O(n^{-\alpha}\log n)$, (Lebesgue [1]).

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[1]) Instead of this we may integrate by parts. The latter argument holds true for the method (C,r), $r>0$.

[See the expression 2.701(1), where the last term is now $O(n^{-\alpha-1})$. It has been shown by Lebesgue (l. c.) that the logarithm in the term $O(n^{-\alpha}\log n)$ cannot be omitted].

7. Let $\sigma_n(x)$ be the first arithmetic means of $\mathfrak{S}[f]$. If $f \in \mathrm{Lip}\,\alpha$, $0<\alpha<1$, then $\sigma_n(x)-f(x)=O(n^{-\alpha})$. If $\alpha=1$, then $\sigma_n(x)-f(x)=O(\log n)/n$. S. Bernstein [1]

$$[\pi\,|\sigma_n-f|\leqslant\int_0^\pi|\varphi_x(t)|\,K_n(t)\,dt\leqslant n\int_0^{1/n}O(t^\alpha)\,dt+\frac{1}{n}\int_{1/n}^\pi O(t^{\alpha-2})\,dt].$$

8. That the previous theorem cannot be strenghtened for $\alpha=1$, may be seen from the following result. If at a point x the right-hand side and the left-hand side derivatives exist, and $f'(x+0)-f'(x-0)=g$, then we have $\sigma_n(x)-f(x)\simeq 2g(\log n)/\pi n$. Szász [1], Alexits [1], Jacob [1].

[Let $g=1$. We have then $\varphi(t)=4(1+\varepsilon(t))\sin\frac{1}{2}t$, where $\varepsilon(t)=o(1)$.

$$\sigma_n(x)-f(x)=\frac{1}{\pi(n+1)}\int_0^\pi\frac{1-\cos(n+1)t}{\sin\frac{1}{2}t}\,dt+\frac{1}{\pi(n+1)}\int_0^\pi\varepsilon(t)\frac{1-\cos(n+1)t}{\sin\frac{1}{2}t}\,dt.$$

The first term on the right is $\simeq 2(\log n)/\pi n$, and the second is $o(\log n)/n$ (§ 2.631)].

9. If f is integrable in the sense of Denjoy-Perron, then, for almost every x, $\mathfrak{S}[F]$ is summable (C,r), $r>1$, to the value $f(x)$. Privaloff [1].

[This is a corollary of Theorem 3.5].

10. If $l=f(x+0)-f(x-0)$ exists and is finite, the sequence $nb_n(x)=n(b_n\cos nx-a_n\sin nx)$ is summable (C,r), $r>1$, to the value l/π. If f is of bounded variation, the theorem holds true for $r>0$. Fejér [3].

[The proof of the first part is similar to that of Theorem 3.5].

11. The sequence $\{s_\nu\}$ is said to be summable by the first logarithmic mean, to the value s, if $\tau_n=(s_1+s_2/2+\dots+s_n/n)/\log n\to s$ as $n\to\infty$. If $\{s_\nu\}$ is summable $(C,1)$ to s, then $\tau_n\to s$.

[For the theory of the logarithmic means see Hardy and Riesz, *Dirichlet's series*].

12. The method considered in the previous problem may be sometimes effective if the sequence is summable $(C,1+\varepsilon)$ for any $\varepsilon>0$, but not for $\varepsilon=0$. An instance in point is Theorem 2.631, which may be interpreted in the sense that the sequence $n(a_n\sin nx-b_n\cos nx)$ is summable by the first logarithmic mean (see also § 3.9.10). Theorem 3.5 may be completed in the same way: If $F'(x)=\lim[F(x+h)-F(x-h)]/2h$ exists and is finite, then $\mathfrak{S}'[F]$ is summable at the point x by the first logarithmic mean and has the sum $F'(x)$. Zygmund [1], Hardy [6].

13. Let f be integrable, $\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$. The summability of $\bar{\mathfrak{S}}'[f]$ is closely connected with the existence of $\lim_{h\to 0}I(x,h)$, where $I(x,h)=-\frac{1}{4\pi}\int_h^\pi\frac{\varphi_x(t)}{\sin^2\frac{1}{2}t}\,dt$. More precisely, at any point x where (*) $\int_0^h\varphi_x(t)\,dt=o(h^2)$ (in particular at any point where $f'(x)$ exists and is finite) we have relation $\frac{\partial\bar f(r,x)}{\partial x}-I(x,1-r)\to 0$ as $r\to 1$ Plessner [2].

14. A result analogous to the previous theorem holds for Cesàro means of order $r>1$, or for the first logarithmic mean[1]). The proof is similar to that of Theorem 3.5.

15. In Theorem 3.6 (ii), summability $(C,3)$ cannot be replaced by summability $(C,2)$. Fejér [4].

$[\{K_n^2(t)\}'$ is positive, if $\sin(n+3/2)t=0$, $\cos(n+3/2)t=-1$, $\cos\frac{1}{2}t<\frac{1}{2}]$.

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[1]) In the condition (*), $\varphi_x(t)$ must be replaced by $|\varphi_x(t)|$.