CHAPTER II.

Fourier coefficients. Tests for the convergence of Fourier series.

2.1. Operations on Fourier series. We begin by proving a few theorems which show that certain formal operations on Fourier series are legitimate.

If $f(x) \sim \sum_{-\infty}^{+\infty} c_m e^{imx}$ *and* u *is constant, then we have*

$$f(x+u) \sim \sum_{-\infty}^{+\infty} c_m e^{imu} e^{imx} = \tfrac{1}{2} a_0(u) + \sum_{m=1}^{\infty} (a_m(u) \cos mx + b_m(u) \sin mx),$$

where $a_m(u) = a_m \cos mu + b_m \sin mu$, $b_m(u) = b_m \cos mu - a_m \sin mu$.

In fact, $\frac{1}{2\pi}\int_0^{2\pi} f(x+u) e^{-imx} dx = \frac{e^{imu}}{2\pi}\int_0^{2\pi} f(x+u) e^{-im(x+u)} dx = e^{imu} c_m$.

2.11. *Let* $f(x) \sim \sum_{-\infty}^{+\infty} c_m e^{imx}$, $g(x) \sim \sum_{-\infty}^{+\infty} d_m e^{imx}$. *Then*

$$(1) \qquad h(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+t)\, g(t)\, dt \sim \sum_{-\infty}^{+\infty} c_m d_{-m} e^{imx}.$$

More precisely: under the conditions of the theorem, (i) the function $h(x)$ exists for almost all x and is integrable, (ii) its Fourier coefficients are $c_m d_{-m}$ [1]. The formulae in (1) are obtained by term-by-term integration of the product of the Laurent series for $\mathfrak{S}[f(x+t)]$ and $\mathfrak{S}[g]$.

To prove (i) it is sufficient to suppose that $f \geqslant 0$, $g \geqslant 0$. Let $f_n(x) = \text{Min}(f(x), n)$, $g_n(x) = \text{Min}(g(x), n)$, and let $h_n(x)$ be the function obtained from f_n, g_n by means of (1). Using Fubini's

[1] W. H. Young [1].

well known theorem on the inversion of the order of integration, we have

$$(2) \qquad \int_0^{2\pi} h_n\, dx = \int_0^{2\pi} dx \int_0^{2\pi} f_n(x+t)\, g_n(t)\, dt = \int_0^{2\pi} g_n(t) \left[\int_0^{2\pi} f_n(x+t)\, dx\right] dt = \int_0^{2\pi} f_n(x)\, dx \int_0^{2\pi} g_n(x)\, dx.$$

Since $\{f_n(t)\, g_n(x+t)\}$ is increasing and tends to $f(t)\, g(x+t)$, it follows that $\{h_n(x)\}$ is also increasing and tends to $h(x)$. Hence, making $n \to \infty$, we find from (2) that $h(x)$ is integrable, and, in particular, finite almost everywhere.

Using Fubini's theorem again, we have

$$\frac{1}{2\pi}\int_0^{2\pi} h(x)\, e^{-imx} dx = \frac{1}{4\pi^2}\int_0^{2\pi} g(t)\, e^{imt}\left[\int_0^{2\pi} f(x+t)\, e^{-im(x+t)} dx\right] dt = c_m d_{-m}.$$

We leave it to the reader to rearrange $\mathfrak{S}[h]$ in the form with real coefficients.

2.12. Differentiation of Fourier series. Suppose that $f(x)$ is an integral, i. e. is absolutely continuous. Integrating by parts, we have, for $m \neq 0$,

$$(1) \qquad c_m = \frac{1}{2\pi}\int_0^{2\pi} f e^{-imx} dx = \frac{1}{2\pi i m}\int_0^{2\pi} f' e^{-imx} dx = \frac{c'_m}{im},$$

or $c'_m = imc_m$, c'_m being the Fourier coefficient of f'. Since f is periodic, we find that $c'_0 = 0$. In other words, if $\mathfrak{S}'[f]$ denotes the result of differentiating $\mathfrak{S}[f]$ term by term, we have $\mathfrak{S}'[f] = \mathfrak{S}[f']$:

$$f' \sim i \sum_{-\infty}^{+\infty} mc_m e^{imx} = \sum_{m=1}^{\infty} m(b_m \cos mx - a_m \sin mx).$$

If f *is a* k*-th integral, then* $\mathfrak{S}^{(k)}[f] = \mathfrak{S}[f^{(k)}]$.

2.13. Suppose that f has a number of simple discontinuities at points $0 \leqslant x_1 < x_2 < \ldots < x_k < 2\pi$ and that it is absolutely continuous in the interior of each interval (x_i, x_{i+1}). *Let us put*

$d_i = [f(x_i+0) - f(x_i-0)]/\pi$. *Then* $\mathfrak{S}'[f] - \mathfrak{S}[f'] = d_1 D(x - x_1) + \dots + d_k D(x - x_k)$, *where* $D(x) = \frac{1}{2} + \cos x + \cos 2x + \dots$ [1].

Let $\varphi(x)$ be periodic and equal to $(\pi - x)/2$ for $0 < x < 2\pi$, $\varphi(0) = \varphi(2\pi) = 0$. Since $d_i \varphi(x - x_i)$ has at x_i the same jump as $f(x)$, the difference $g(x) = f(x) - \Phi(x)$, where $\Phi(x) = d_1 \varphi(x - x_1) + \dots + d_k \varphi(x - x_k)$, is everywhere continuous, indeed absolutely continuous. Moreover, except at the points x_i, $g' - f' = (d_1 + \dots + d_k)/2 = C$. Now $\mathfrak{S}'[f] = \mathfrak{S}'[\Phi] + \mathfrak{S}'[g] = \mathfrak{S}'[\Phi] + \mathfrak{S}[g'] = \mathfrak{S}'[\Phi] + \mathfrak{S}[f' + C] = \mathfrak{S}[f'] + \mathfrak{S}'[\Phi] + C$. Taking into account the particular form of C and $\mathfrak{S}'[\Phi]$ (§ 1.8, 2 (iv)), the result follows.

2.14. Let $F(x)$ be a function of bounded variation, so that, if c_m are the complex coefficients of $\mathfrak{S}[dF]$, the difference $F - c_0 x$ is periodic (§ 1.45). Let C_m be the Fourier coefficients of the latter function. Then, for $m \neq 0$,

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} (F - c_0 x)\, e^{-imx}\, dx = \frac{1}{2\pi\, im}\int_0^{2\pi} e^{-imx}\, d(F - c_0 x) = \frac{c_m}{im}.$$

Let us agree to write

$$F(x) \sim c_0 x + C_0 + \sum_{-\infty}^{+\infty}{}' \frac{c_m}{im} e^{imx}, \text{ instead of } F(x) - c_0 x \sim C_0 + \sum_{-\infty}^{+\infty}{}' \frac{c_m}{im} e^{imx},$$

where $'$ denotes that the term for which $m = 0$ is omitted, i. e. we represent F as the sum of a linear and a periodic function. Then $\mathfrak{S}[dF]$ is obtained by formal differentiation of the former series, that is, *the class of Fourier-Stieltjes series, and that of formally differentiated Fourier series of functions of bounded variation are identical.*

2.15. Integration of Fourier series. Let f be periodic and F an integral of f. Since $F(x + 2\pi) - F(x)$ is equal to the integral of f over $(x, x + 2\pi)$, or, what is the same thing, over $(0, 2\pi)$, a necessary and sufficient condition for the periodicity of F is that the constant term of $\mathfrak{S}[f]$ should vanish. Suppose this condition satisfied. Then $\mathfrak{S}[f]$ is obtained by formal differentiation of $\mathfrak{S}[F]$, i. e.

$$(1)\quad F(x) \sim C + \sum_{-\infty}^{+\infty}{}' \frac{c_m}{im} e^{imx} = C + \sum_{m=1}^{\infty} \frac{a_m \sin mx - b_m \cos mx}{m}.$$

Here C is the constant of integration and depends on the choice of F. If $c_0 \neq 0$, the periodic function $F - c_0 x$ is an integral of $f - c_0$ and the series in (1) is $\mathfrak{S}[F - c_0 x]$.

Example. Let $f_0(x), f_1(x), \dots f_k(x), \dots$, $0 < x < 2\pi$, be the functions defined by the conditions (i) $f_0(x) = -1$, (ii) $f'_k(x) = f_{k-1}(x)$, (iii) the integral of f_k over $(0, 2\pi)$ vanishes, $k = 1, 2, \dots$ The reader will easily verify that $f_k(x) \sim \sum_{m=-\infty}^{+\infty}{}' \frac{(\operatorname{sign} m)\, e^{imx}}{i^{k+1} m^k}$. In the interval $(0, 2\pi)$ the function $f_k(x)$ a is polynomial of order k.

2.2. Modulus of continuity. Let $f(x)$ be a function defined for $a \leqslant x \leqslant b$; let $\omega(\delta) = \omega(\delta; f) = \operatorname{Max} |f(x_1) - f(x_2)|$ for all x_1, x_2 belonging to (a, b) and such that $|x_1 - x_2| \leqslant \delta$. The function $\omega(\delta)$ is called the *modulus of continuity* of f [1]) and this notion is very useful in the theory of Fourier series. The function f is continuous if and only if $\omega(\delta) \to 0$ with δ. If $\omega(\delta) < C\delta^{\alpha}$, where $0 < \alpha \leqslant 1$ and C denotes a number independent of δ, we say that f satisfies the *Lipschitz condition* of order α, or $f \in \operatorname{Lip} \alpha$, in (a, b). The restriction $\alpha \leqslant 1$ is quite natural, since if $\omega(\delta)/\delta \to 0$ with δ, $f'(x)$ exists and is equal to 0 everywhere, so that $f = \text{const.}$

Suppose now for simplicity that (a, b) coincides with $(0, 2\pi)$ and consider a periodic and integrable function f, not necessarily continuous. Let $\omega_1(\delta) = \omega_1(\delta; f) = \operatorname{Max} \int_0^{2\pi} |f(x + h) - f(x)|\, dx$ for all $0 < h \leqslant \delta$. The function $\omega_1(\delta)$ will be called the *integral modulus of continuity* of f.

2.201. *For every integrable* f, $\lim \omega_1(\delta; f) = 0$ *as* $\delta \to 0$. Given a function g, let $I(g)$ denote the integral of $|g|$ over $(0, 2\pi)$. If for any $\varepsilon > 0$ we have $f = f_1 + f_2$, where $\omega_1(\delta; f_1) \to 0$ with δ, and $I(f_2) < \varepsilon$, then $\omega_1(\delta; f) \to 0$. In fact: $\omega_1(\delta; f) \leqslant \omega_1(\delta; f_1) + \omega_1(\delta; f_2) \leqslant \omega_1(\delta; f_1) + 2\, I(f_2) < 3\varepsilon$, if $0 < \delta \leqslant \delta_0(\varepsilon)$. Now the theorem is

[1]) The series $D(x)$, which is very important in the theory of Fourier series, diverges everywhere. However, it is summable to 0, for example by Abel's method, if $x \neq 0 \pmod{2\pi}$.

[1]) Lebesgue [1]

certainly true when E is the characteristic function of a set E[1]) consisting of a finite number of intervals, hence it is true also when E is an arbitrary open set, and consequently when E is measurable. It follows that the theorem holds when f assumes only a finite number of values, hence when f is bounded, and finally when f is integrable.

2.21. *If c_m are the complex Fourier coefficients of a function f, then*

$$(1)\qquad |c_m| \leqslant \tfrac{1}{2}\,\omega\left(\frac{\pi}{m}\right), \quad |c_m| \leqslant \frac{1}{4\pi}\,\omega_1\left(\frac{\pi}{m}\right).$$

Replacing x by $x+\pi/m$ in the integral defining c_m, we have that $2\pi c_m$ is equal to

$$\int_0^{2\pi} f(x)\,e^{-imx}\,dx = -\int_0^{2\pi} f\left(x+\frac{\pi}{m}\right)e^{-imx}\,dx = \tfrac{1}{2}\int_0^{2\pi}\left[f(x)-f\left(x+\frac{\pi}{m}\right)\right]e^{-imx}\,dx$$

and the last integral does not exceed either $\pi\omega(\pi/m)$ or $\frac{1}{2}\,\omega_1(\pi/m)$ in absolute value.

2.211. The Riemann-Lebesgue theorem. *The Fourier coefficients of integrable functions tend to* 0. This follows from Theorem 2.201 and the second formula 2.21(1). A slightly simpler proof runs as follows: $f=f_1+f_2$, where f_1 is bounded and $I(f_2)<\varepsilon$. ($I(f)$ has the same meaning as in § 2.201). Correspondingly, $c_m = c'_m + c''_m$, where $|c''_m| \leqslant I(f_2)/2\pi < \varepsilon/2\pi$ and $c'_m \to 0$ (§ 1.61, Corollary). Hence $|c_m| \leqslant |c'_m| + |c''_m| < \varepsilon$ for $m > m_0$.

2.212. *If $f \in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqslant 1)$, then $c_m = O(m^{-\alpha})$* [2]). Here O cannot be replaced by o (§ 2.9.3), except in the case $\alpha=1$. In this case, since f is absolutely continuous, the differentiated $\mathfrak{S}[f]$ is still a Fourier series, so that $c_m = o(m^{-1})$.

2.213. *If f is of bounded variation, then $|a_m| \leqslant V/m$, $|b_m| \leqslant V/m$, $m=1,2,\ldots$, where V denotes the total variation of f over* $(0, 2\pi)$. Suppose first that f is non-decreasing and $\geqslant 0$. Using the second mean-value theorem we have

$$\pi a_m = \int_0^{2\pi} f(x)\cos mx\,dx = f(2\pi)\int_\xi^{2\pi}\cos mx\,dx, \quad 0<\xi<2\pi,$$

[1]) The function equal to 1 in a set E and to 0 elsewhere is called the characteristic function of E.

[2]) Lebesgue [1].

and so $|a_m| \leqslant 2f(2\pi)/\pi m$. In the general case, since $f=f_1-f_2$, where f_1, f_2 are respectively the positive and negative variations of f, we find that $|a_m|$, and similarly $|b_m|$, does not exceed $2\,[f_1(2\pi)+f_2(2\pi)]/\pi m = 2V/\pi m \leqslant V/m$.

The result can also be stated in the following form: *the coefficients of $\mathfrak{S}[df]$ form a bounded sequence.* The simplest examples show that this result cannot be improved (§ 1.8.2(v)). The fact that it cannot be improved even when f is of bounded variation *and* continuous lies much deeper. We state without proof the following result, which will be established in Ch. XI. Let C be the well-known ternary set of Cantor constructed on $(0, 2\pi)$. *If $f(x)$ is any function constant in each of the intervals complementary to C, but not equivalent to a constant in $(0, 2\pi)$, the Fourier coefficients of f are not $o(1/n)$.*

Taking f continuous and of bounded variation we obtain the required example.

2.22. Fourier-Riemann coefficients. Theorem 2.211 is no longer true for Fourier-Riemann series. Let

$$f(x) = \frac{d}{dx}(x^\nu \cos 1/x) \quad 0<\nu<\tfrac{1}{2}, \quad S(x) = \sum_{n=1}^{\infty} e^{in^\alpha} n^\beta e^{inx}.$$

It was shown by Riemann [1]) that the Fourier coefficients of the function f, which is integrable R, are not necessarily $o(1)$, and not even $o(n^{(1-2\nu)/4})$. It can also be proved that the real and imaginary parts of the series $S(x)$ are both Fourier-Riemann series, if only $0<\alpha<1$, $\beta<\alpha/2$ [2]). We will give here a stronger example, based on the fact that the integral of $\sin^2 nx$ over (a, b) tends to $(b-a)/2$ as $n\to\infty$.

2.221. *Given an arbitrary sequence of numbers $\lambda_n\to\infty$, $\lambda_n = o(n)$, there exists a function f integrable R, whose sine coefficients b_n exceed λ_n for infinitely many n* [3]).

Let $\lambda_n = \varepsilon_n n$, $\varepsilon_n \to 0$. We shall define a sequence of non-overlapping intervals $I_k = (\alpha_k/2, \alpha_k)$, $k=1,2,\ldots$, approaching the point 0 from the right. Let $f(x) = c_k \sin n_k x$ in I_k, and $f(x)=0$ elsewhere in $(-\pi, \pi)$. The positive coefficients c_k and the integers $n_1 < n_2 < \ldots$ satisfy a series of relations; in particular (1) $n_k\alpha_k$ are multiples of 4π, so that f is continuous for $x \neq 0$ and the integral of f over I_k vanishes; (2) $c_k/n_k = 1/k \to 0$, which implies that f is integrable R over $(0, \pi)$. Let $n_1 = 4$, $c_1 = 4$, $I_1 = (\pi/2, \pi)$ and suppose we have defined n_i, c_i, I_i for $i=1,2,\ldots,k-1$ and consequently $f(x)$ for $\alpha_{k-1}/2 \leqslant x \leqslant \pi$. Put

[1]) Riemann [1].

[2]) This is implicitly contained in Hardy [1].

[3]) Titchmarsh [1].

$\alpha_k = 4\pi/p$, p being the smallest integer such that (3) $\alpha_k \leqslant 1/n_{k-1}$. A little attention shows that (3_1) $\alpha_k \geqslant \frac{1}{2} n_{k-1}$. Let n_k divisible by p be so large that (4) the integral of $\sin^2 n_k x$ over I_k exceeds $\alpha_k/8$, (5) the integral of $f \sin n_k x$ over (α_k, π) is less than 1 in absolute value, and, finally, (6) $4\varepsilon_{n_k} < 1/16\, k n_{k-1}$. To investigate the behaviour of the integral, extended over $(0, \pi)$, of the product $f(x) \sin n_k x$, we break up this integral into three, extended over $(0, \alpha_{k+1})$, $(\alpha_k/2, \alpha_k)$, (α_k, π), and denote them by A_k, B_k, C_k. We have $|C_k| \leqslant 1$ (cond. (5)), and, since $\sin n_k x$ is monotonic in $(0, \alpha_{k+1})$ (cond. (3)), the second mean-value theorem shows that $A_k \to 0$. In virtue of conditions (4), (2), (3_1), (6) we have $B_k > c_k \alpha_k/8 = n_k \alpha_k / 8k \geqslant n_k 1/16\, k n_{k-1} > 4\varepsilon_{n_k} n_k = 4\lambda_{n_k}$. Therefore we have $\pi b_{n_k} = A_k + B_k + C_k > 4\lambda_{n_k} - 1 - o(1)$, i. e. $b_{n_k} > \lambda_{n_k}$ for k large, and the result follows.

2.222. Since integration by parts subsists for Denjoy's integrals, both special and general [1]), the argument of § 2.11 proves that Fourier-Denjoy series, which are obtained by term-by-term differentiation of $\mathfrak{S}[F]$, with F continuous, have coefficients $o(n)$. This result cannot be improved, as Theorem 2.221 shows.

2.3. Formulae for partial sums. The object of the rest of this chapter is to establish some conditions for the convergence of Fourier series and of the conjugate series. It will be convenient to treat these two problems side by side. If

$$(1)\qquad \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \qquad \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)$$

are $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ respectively, the n-th partial sums, $s_n(x) = s_n(x; f)$ and $\bar{s}_n(x) = \bar{s}_n(x; f)$, of these series can be written in the following forms

$$(2)\qquad \begin{aligned} s_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt + \\ &+ \frac{1}{\pi} \sum_{k=1}^{n} \Big(\cos kx \int_{-\pi}^{\pi} f(t) \cos kt\, dt + \sin kx \int_{-\pi}^{\pi} f(t) \sin kt\, dt\Big) = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \Big(\tfrac{1}{2} + \sum_{k=1}^{n} \cos k(t-x)\Big)\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t-x)\, dt = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) D_n(t)\, dt, \end{aligned}$$

[1]) For the theory of these integrals we refer the reader to Saks's *Théorie de l'intégrale*, Ch. X.

$$\bar{s}_n(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \Big(\sum_{k=1}^{n} \sin k(t-x)\Big)\, dt = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) \bar{D}_n(t)\, dt,$$

where $D_n(u) = \dfrac{\sin(n+\frac{1}{2})u}{2 \sin \frac{1}{2} u}$, $\bar{D}_n(u) = \dfrac{\cos \frac{1}{2} u - \cos(n+\frac{1}{2})u}{2 \sin \frac{1}{2} u}$ (§ 1.12).

The functions D_n and $\bar{D}_n$ are called 'Dirichlet's kernel', and 'Dirichlet's conjugate kernel' respectively. However, instead of considering s_n and $\bar{s}_n$, it will be slightly more convenient to consider the expressions $s_n^*(x) = s_n(x) - (a_n \cos nx + b_n \sin nx)/2$, $\bar{s}_n^*(x) = \bar{s}_n(x) - (a_n \sin nx - b_n \cos nx)/2$. Since the differences $s_n - s_n^*$ and $\bar{s}_n - \bar{s}_n^*$ tend uniformly to 0, this is completely justified. Putting

$$D_n^*(u) = D_n(u) - \tfrac{1}{2} \cos nu = \frac{\sin nu}{2 \operatorname{tg} \frac{1}{2} u},$$

$$\bar{D}_n^*(u) = \bar{D}_n(u) - \tfrac{1}{2} \sin nu = \frac{1 - \cos nu}{2 \operatorname{tg} \frac{1}{2} u},$$

and arguing as before, we have

$$(3)\qquad s_n^*(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_n^*(t)\, dt, \quad \bar{s}_n^*(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \bar{D}_n^*(t)\, dt.$$

If $f \equiv 1$, then $s_n^*(x) = 1$ for $n > 0$. Since $D_n^*(t)$ is even, $\bar{D}_n^*(t)$ odd, we have

$$(4)\qquad \begin{aligned} s_n^*(x) - f(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) D_n^*(t)\, dt - \frac{f(x)}{\pi} \int_{-\pi}^{\pi} D_n^*(t)\, dt = \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{\varphi(t)}{2 \operatorname{tg} \frac{1}{2} t} \sin nt\, dt, \\ \bar{s}_n^*(x) &= -\frac{1}{\pi} \int_0^{\pi} \frac{\psi(t)}{2 \operatorname{tg} \frac{1}{2} t} (1 - \cos nt)\, dt, \end{aligned}$$

where $\varphi(t) = \varphi_x(t) = \varphi_x(t; f) = f(x+t) + f(x-t) - 2f(x)$, $\psi(t) = \psi_x(t) = \psi_x(t; f) = f(x+t) - f(x-t)$.

2.4. Dini's test. *If the first of the integrals*

$$(1)\qquad \int_0^{\pi} \frac{|\varphi_x(t)|}{2 \operatorname{tg} \frac{1}{2} t}\, dt, \qquad \int_0^{\pi} \frac{|\psi_x(t)|}{2 \operatorname{tg} \frac{1}{2} t}\, dt$$

is finite, then $\mathfrak{S}[f]$ *converges at* x *to the sum* $f(x)$. *If the second integral is finite,* $\tilde{\mathfrak{S}}[f]$ *converges at the point* x *to the value which we shall denote by* $\bar{f}(x)$,

$$(2) \qquad \bar{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \frac{\psi_x(t)}{2\operatorname{tg}\frac{1}{2}t}\,dt.$$

For the proof it is sufficient to observe that, in virtue of 2.3(4), the differences $s_n^*(x) - f(x)$ and $\bar{s}_n^*(x) - \bar{f}(x)$ are respectively Fourier sine and cosine coefficients of integrable functions.

Since $2\operatorname{tg}\frac{1}{2}t \simeq t$ as $t \to 0$, the denominators in (1) may be replaced by t.

The integrals (1) converge if $\varphi_x(t) = O(t^\alpha)$, $\psi_x(t) = O(t^\alpha)$, $\alpha > 0$, as $t \to 0$; in particular if $f'(x)$ exists and is finite. However, the first of these integrals converges even when f is discontinuous at x, provided that $\frac{1}{2}\varphi_x(t) = \frac{1}{2}[f(x+t) + f(x-t)] - f(x)$ tends sufficiently rapidly to 0 with t. The second is divergent if only $f(x+0) \neq f(x-0)$ and, as we shall see later, $\tilde{\mathfrak{S}}[f]$ will certainly diverge at such points.

If $f \in \operatorname{Lip}\alpha$, $\alpha > 0$, $\mathfrak{S}[f]$ *and* $\tilde{\mathfrak{S}}[f]$ *converge everywhere.* It is easy to show that the convergence is uniform, but this theorem is contained in the more general result of § 2.71.

2.5. Theorems on localization. *If* f *vanishes in an interval* $I = (a, b)$, $\mathfrak{S}[f]$ *and* $\tilde{\mathfrak{S}}[f]$ *converge uniformly in any interval* $I' = (a + \varepsilon, b - \varepsilon)$ *interior to* I, *and the sum of* $\mathfrak{S}[f]$ *is* 0 [1]. If the word 'uniformly' is omitted, the theorem becomes a simple corollary of Theorem 2.4, since, if $x \in I'$, $\varphi_x(t)$ and $\psi_x(t)$ vanish for small t and the integrals 2.4(1) are finite. We need the following lemma.

2.501. *Let* f *be integrable,* g *bounded* ($|g| < A$), *both periodic. The Fourier coefficients of the function* $\chi(t) = f(x+t)\,g(t)$, *depending on the parameter* x, *tend uniformly to* 0 [2].

It is sufficient to show that $\omega_1(\delta; \chi) \to 0$ with δ, uniformly in x. We have

[1]) Riemann [1], Lebesgue, *Leçons sur les séries trigonométriques*, 60, Hobson [1].

[2]) Hobson [1]; Plessner [1].

$$\int_{-\pi}^{\pi} |\chi(t+h) - \chi(t)|\,dt \leqslant \int_{-\pi}^{\pi} |f(x+t+h) - f(x+t)|\,|g(t+h)|\,dt + \int_{-\pi}^{\pi} |f(x+t)|\,|g(t+h) - g(t)|\,dt.$$

If $|h| \leqslant \delta$, the first term on the right is less than $A\omega_1(\delta; f) \to 0$. To prove that the second term tends uniformly to 0, we put $|f| = f_1 + f_2$, where f_1 is bounded $(0 \leqslant f_1 < B)$ and the integral of f_2 over $(-\pi, \pi)$ is less than $\varepsilon/4A$. The term considered is, obviously, less than $B\omega_1(\delta; g) + 2A \cdot \varepsilon/4A < \varepsilon$, for δ sufficiently small, and the lemma follows.

2.502. From the conditions of Theorem 2.5 we see that $f(x+t) = 0$ for $x \in I'$, $|t| < \varepsilon$. Let $\lambda(t)$ be equal to 0 for $|t| < \varepsilon$ and to 1 elsewhere. Using 2.3(3) we find that $s_n^*(x)$ is equal to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\lambda(t)}{2\operatorname{tg}\frac{1}{2}t}\,\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,g(t)\sin nt\,dt,$$

where $g = \lambda/2\operatorname{tg}\frac{1}{2}t$. In virtue of Theorem 2.501, $s_n^*(x)$ tends uniformly to 0 if $x \in I'$. Similarly, if $\bar{f}(x)$ is given by 2.4(2), and $x \in I'$, $\bar{s}_n^*(x) - \bar{f}(x)$ tends uniformly to 0.

2.51. The results of the preceding paragraph may also be stated in a slightly different form. Two series $u_0 + u_1 + \ldots$ and $v_0 + v_1 + \ldots$ will be called *equiconvergent* if their difference $(u_0 - v_0) + (u_1 - v_1) + \ldots$ converges and has the sum 0 [1]. If the difference converges but not necessarily to 0, the series in question will be called *equiconvergent in the wider sense.*

If two functions f_1 *and* f_2 *are equal in an interval* I, *then* $\mathfrak{S}[f_1]$ *and* $\mathfrak{S}[f_2]$ *are uniformly equiconvergent in any interval* I' *interior to* I; $\tilde{\mathfrak{S}}[f_1]$ *and* $\tilde{\mathfrak{S}}[f_2]$ *are uniformly equiconvergent in* I' *but in the wider sense.*

For the proof we consider the difference $f = f_1 - f_2$.

Considering, for simplicity, convergence at a point, we may also put our results in the following form: The convergence of $\mathfrak{S}[f]$, $\tilde{\mathfrak{S}}[f]$ and the sum of $\mathfrak{S}[f]$ (but *not* of $\tilde{\mathfrak{S}}[f]$) at a point x, depend only on the behaviour of f in an arbitraily small neighbourhood of x. ('Riemann's principle of localization').

[1]) Szegö [1].

2.52. Approximate formulae for s_n. It is sometimes convenient to use the approximate formulae

(1)
$$s_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\sin nt}{t}\,dt+o(1),$$
$$s_n(x)-f(x)=\frac{1}{\pi}\int_{0}^{\pi}\varphi_x(t)\,\frac{\sin nt}{t}\,dt+o(1).$$

In the first of them the error tends uniformly to 0, in the second it tends to 0 for every x, and uniformly in any interval where f is bounded. For the proof of the first result we observe that the difference of the integral on the right and the integral defining s_n^* is the Fourier coefficient of the function $f(x+t)\,g(t)$, where $g=1/t-\frac{1}{2}\operatorname{ctg}\frac{1}{2}t$ is bounded in $(-\pi,\pi)$. In the second case we encounter the Fourier coefficients of the function equal to $[f(x+t)-f(x)]\,g(t)$[1].

2.53. A theorem of Steinhaus[2]. *If at a point x_0 the derivates of a bounded function $\rho(x)$ are all finite, the series $\mathfrak{S}[\rho f]$ and $\rho(x_0)\,\mathfrak{S}[f]$ are equiconvergent at x_0.* In fact, the difference of the n-th partial sums of these series is equal to $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x_0+t)\,g(t)\sin nt\,dt$, where $g(t)=g_{x_0}(t)=[\rho(x_0+t)-\rho(x_0)]/2\sin\frac{1}{2}t$, and tends to 0, because it is the Fourier coefficient of an integrable function.

Suppose $\rho(x_0)=1$. The theorem shows that 'slight' modifications of f in the neigbourhood of x_0 that leave $f(x_0)$ unaltered, have no influence either upon the convergence or the sum of $\mathfrak{S}[f]$ at x_0. More generally

2.531. *If $\rho(x)$ is periodic and satisfies the Lipschitz condition of order 1, the series $\mathfrak{S}[\rho f]$ and $\rho(x_0)\,\mathfrak{S}[f]$ are uniformly equiconvergent for all x_0. Similarly $\bar{\mathfrak{S}}[\rho f]$ and $\rho(x_0)\,\bar{\mathfrak{S}}[\rho f]$ are uniformly equiconvergent in the wider sense.*

We need only prove that $\omega_1(\delta;\chi)\to 0$ uniformly in x, where $\chi(t)=\chi_x(t)=f(x+t)\,g_x(t)$. Arguing as in the proof of Theorem

[1]) If we replace $\sin nt$ by $\cos nt-1$ in the first integral (1), we obtain an approximate expression for $\bar{s}_n(t)$, where the error tends uniformly to a continuous function.

[2]) Steinhaus [1].

2.501, it remains to show, since $g_x(t)$ is uniformly bounded, $|g_x(t)|\leqslant M$, that $\displaystyle\int_{-\pi}^{\pi}|g_x(t+h)-g_x(t)|\,dt$ tends uniformly to 0 with h. Break up this integral into two, the first extended over $(-\varepsilon/8M,\varepsilon/8M)$. Since $g_x(t)$ is uniformly continuous outside this interval, the second integral tends uniformly to 0, and the first is less than $2.2M.\,\varepsilon/8M=\varepsilon/2$, so that the whole is less than ε, for h sufficiently small.

2.6. Functions of bounded variation. *If f is of bounded variation, $\mathfrak{S}[f]$ converges at every point x to the value $[f(x+0)+f(x-0)]/2$. If f is in addition continuous at every point of an interval $I=(a,b)$, $\mathfrak{S}[f]$ is uniformly convergent in I.* This theorem, due essentially to Dirichlet, is the first, chronologically, in the theory of Fourier series[1]. Its proof is elementary and uses only the results of § 2.213. We may suppose that at any point of simple discontinuity we have $f(x)=[f(x+0)+f(x-0)]/2$[2], so that the first part of the theorem asserts that $\mathfrak{S}[f]$ converges everywhere to $f(x)$. From 2.3(4) we have

$$(1)\qquad s_n^*(x)-f(x)=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^{\eta}+\int_{\eta}^{\pi}\right]\frac{\varphi_x(t)}{2\operatorname{tg}\frac{1}{2}t}\sin nt\,dt=P+Q+R,$$

where η will be defined in a moment. Since $|\sin nt|\leqslant nt\leqslant 2n\operatorname{tg}\frac{1}{2}t$, we see that $|P|\leqslant \operatorname{Max}|\varphi_x(t)|$ $(0\leqslant t\leqslant 1/n)$ and so tends to 0. For fixed η, R is the Fourier coefficient of a function of bounded variation and hence is $O(1/n)=o(1)$. By the second mean-value theorem

$$(2)\qquad Q=\tfrac{1}{2}\operatorname{ctg}\frac{\pi}{2n}\cdot\frac{1}{\pi}\int_{\pi/n}^{\eta'}\varphi_x(t)\sin nt\,dt,\qquad \pi/n<\eta'<\eta.$$

Since $\varphi_x(t)$ is continuous for $t=0$, and $\varphi_x(0)=0$, the total variation

[1]) Dirichlet himself considered only functions having a finite number of maxima and minima, and in particular monotonic functions. Since, however, functions of bounded variation are differences of such functions, it is natural to associate Dirichlet's name with this theorem, which is only more general in appearance.

[2]) The set of points where a monotonic function, and so a function of bounded variation, is discontinuous, is at most enumerable.

of the function equal to $\varphi_x(t)$ in $(\pi/n, \eta')$ and to 0 elsewhere, is less than ε, if η is sufficiently small [1]). In virtue of Theorem 2.213, the second factor on the right in (2) is less than ε/n in absolute value, whence $|Q| < \varepsilon/\pi$. Therefore $|s_n^*(x) - f(x)| < o(1) + \varepsilon/\pi + o(1) < \varepsilon$ for $n > n_0$, i. e. $s_n^*(x) \to f(x)$.

If f is continuous in I, then, for $x \in I$, $\varphi_x(t)$ is uniformly small for small t and hence $P \to 0$ uniformly. For fixed η, the total variation of the function $\varphi_x(t)/2\operatorname{tg}\frac{1}{2}t$ over (η, π) is uniformly bounded [2]), and so again $R \to 0$ uniformly. If $x \in I$, $|t| < \delta$, the total variation of $f(x+t)$, and hence that of $\varphi_x(t)$, in a small intervall will be small [3]), and this gives as before that $|Q| < \varepsilon/\pi$ for η small but fixed. This completes the proof [4]).

2.601. A sequence of functions $f_n(x)$ convergent to $f(x)$ in a neighbourhood of a point x_0 is said to converge uniformly *at* the point x_0 if, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$ and a $p = p(\varepsilon)$ such that $|f(x) - f_n(x)| < \varepsilon$ for $|x - x_0| < \delta$, $n > p$.

If f is of bounded variation, $\mathfrak{S}[f]$ converges uniformly at every point x_0 where f is continuous. In fact, repeating the argument of § 2.6 it is easy to see that, if $|x - x_0|$ is small enough, the expression $|P| + |Q| + |R|$ is uniformly small.

2.61. Young's theorem. *If f is of bounded variation, a necessary and sufficient condition for the convergence of $\mathfrak{S}[f]$ at a point x is the existence of the integral*

$$(1) \qquad \bar{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{\psi_x(t)}{2\operatorname{tg}\frac{1}{2}t}\,dt = \lim_{h\to 0}\left[-\frac{1}{\pi}\int_h^\pi \frac{\psi_x(t)}{2\operatorname{tg}\frac{1}{2}t}\,dt\right],$$

1) The total variation of $\varphi(t)$ in an interval $\alpha \leqslant t \leqslant \alpha'$, $0 < \alpha < \alpha'$, tends to 0 if $\alpha' \to 0$, for otherwise there would exist a sequence of non-overlapping intervals (α_n, α'_n) tending to 0, on which the total variation of φ would exceed a $\delta > 0$, and so φ would not be of bounded variation.

2) This follows e. g. from the obvious fact that if V_i, M_i denote respectively the total variation of g_i and $\operatorname{Max}|g_i|$, the total variation of $g_1 g_2$ is $\leqslant M_1 V_2 + M_2 V_1$.

3) The total variation is continuous wherever the function is continuous.

4) The decomposition of $s_n^* - f$ into *three* parts P, Q, R was not necessary, since it was not difficult to prove that $P + Q$ is small for small η (see the usual proof of Dirichlet's theorem in textbooks). However, the argument of the text can be applied to some other theorems.

which represents then the sum of $\mathfrak{S}[f]$ [1]). In virtue of Theorem 2.63 it is sufficient to consider only the points of continuity of f. Let $\bar{f}(x, h)$ denote the value of the integral (1) with the lower limit h instead of 0. Using the formula 2.3(4) we see that $\bar{s}_n^*(x) - \bar{f}(x, \pi/n)$ may be represented as the sum of three terms. Two of them are analogous to Q, R from the preceding section, with $\varphi_x(t)\sin nt$ replaced by $\psi_x(t)\cos nt$. The same proof as before shows that they tend to 0. The absolute value of the third is less than

$$\frac{1}{\pi}\int_0^{\pi/n} |\psi_x(t)|\,\frac{1-\cos nt}{2\operatorname{tg}\frac{1}{2}t}\,dt \leqslant \frac{n^2}{2\pi}\int_0^{\pi/n} |\psi_x(t)|\,t\,dt = o(1).$$

It follows that $\bar{s}_n^*(x) - \bar{f}(x, \pi/n) \to 0$. In order to complete the proof it is enough to show that $\bar{f}(x, h) - \bar{f}(x, \pi/n) \to 0$ as $n \to \infty$, if $\pi/(n+1) < h < \pi/n$. But $|\bar{f}(x, h) - \bar{f}(x, \pi/n)| \leqslant [\pi/n - \pi/(n+1)] \cdot \frac{1}{2}\operatorname{ctg}\frac{1}{2}h \cdot \operatorname{Max}|\psi_x(t)|\,(0 < t \leqslant \pi/n) = o(1/n) = o(1)$.

2.62. Corollaries. *Let f be of bounded variation in an interval $I = (a, b)$. Then* (i) *$\mathfrak{S}[f]$ converges to $[f(x+0) + f(x-0)]/2$ at any point interior to I. If, besides that, f is continuous in I, $\mathfrak{S}[f]$ converges uniformly in every interval $(a+\delta, b-\delta)$,* (ii) *a necessary and sufficient condition for the convergence of $\bar{\mathfrak{S}}[f]$ at a point x interior to I, is the existence of the integral* 2.61(1), *which represents the sum of $\bar{\mathfrak{S}}[f]$.*

This follows immediately from Theorems 2.6, 2.61 and 2.51. Proposition (i) is known as 'Jordan's test'.

2.621. Integrated Fourier series. *Let F be the indefinite integral of f and let the first series in* 2.3(1) *be $\mathfrak{S}[f]$. Then we have, for $-\infty < x < \infty$,*

$$(1) \qquad F(x) = \frac{a_0 x}{2} + C + \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)/n,$$

the series on the right being uniformly convergent [2]). For the proof it is sufficient to observe that the series on the right, without its linear term, is the Fourier series of the function $F - a_0 x/2$, which is continuous and of bounded variation. If follows also that for every α, β we have

1) Young [2].

2) Lebesgue, *Leçons*, 102.

$\int_\alpha^\beta f\,dx = \left[\frac{a_0 x}{2}\right]_\alpha^\beta + \sum_{n=1}^{\infty}\left[\frac{a_n \sin nx - \cos nx}{n}\right]_\alpha^\beta$, i. e. Fourier series may be integrated term by term over any interval (α, β). From (1) we have:

If the first series in 2.3(1) *is a* $\mathfrak{S}[f]$, *the series* $b_1/1 + b_2/2 + \dots$ *converges*. This may be false for the series $a_1/1 + a_2/2 + \dots$ (See Chapter V).

2.622. *If f is of bounded variation, the partial sums of* $\mathfrak{S}[f]$ *are uniformly bounded.* We postpone the proof till § 3.23.

2.63. Conjugate series at points of discontinuity. We have seen that simple discontinuites are, in principle, no obstacles for the convergence of $\mathfrak{S}[f]$. For the conjugate series the situation is different: *If* $f(x+0) - f(x-0) = l > 0$, *then* $\bar{\mathfrak{S}}[f]$ *diverges at x to* $-\infty$ [1]).

This is contained in the following, more precise, result [2]).

2.631. *If* $f(x+0) - f(x-0) = l$, *then* $\bar{s}_n(x)/\log n \to -l/\pi$.

Since $f(x+t) - f(x-t) = l + \varepsilon(t)$, $\varepsilon(t) \to 0$, we may write

$$(1)\qquad \bar{s}_n^*(x) = -\frac{l}{\pi}\int_0^\pi \bar{D}_n^*(t)\,dt - \frac{1}{\pi}\int_0^\pi \varepsilon(t)\,\bar{D}_n^*(t)\,dt.$$

To find the first of the integrals on the right, let us denote them by I_n, I'_n and consider the function $f(t) = (\pi - t)/2$ $(0 < t < 2\pi)$. Here $l = f(+0) - f(-0) = \pi$, $\varepsilon(t) = -t$, $\bar{s}_n^*(0) = -1 - 1/2 - \dots - 1/(n-1) - 1/2n = -\log n + O(1)$. Substituting this in (1) we find that $I_n = \log n + O(1) \simeq \log n$. Now we will show that $I'_n = o(\log n)$. We break up this integral into two, the first of which is extended over $(0, \delta)$, where δ is so small that $|\varepsilon(t)| < \eta/2$ for $0 \leqslant t \leqslant \delta$. Since $\bar{D}^* \geqslant 0$, the first term is less than $\eta I_n/2$. The second term is bounded, and so less than $\eta I_n/2$ in absolute value for $n > n_0$. It follows that $|I'_n| < \eta I_n$ $(n > n_0)$, i. e. $I'_n = o(I_n) = o(\log n)$. This completes the proof.

This theorem gives us a means of determining the simple discontinuities of functions from their Fourier series.

2.632. Corollaries. (i) If the Fourier coefficients of a function f are $o(1/n)$, f cannot possess simple discontinuities. In fact, for such functions $\bar{s}_n^*(x) = o(\log n)$ uniformly in x (§§ 1.72, 1.74). In particular, *if the Fourier coefficients of a function f of bounded variation are $o(1/n)$, f is continuous.*

(ii) If f is continuous at a point x, then $\bar{s}_n(x) = o(\log n)$. If f is continuous in an interval (a, b), then $\bar{s}_n(x) = o(\log n)$, uniformly in every interval $(a + \delta, b - \delta)$.

1) Pringsheim [1].

2) Lukács [1].

2.7. Lebesgue's test. Let $\varphi(t) = \varphi_x(t)$, $\chi(t) = \varphi(t)/2 \operatorname{tg} \frac{1}{2}t$, $\eta = \pi/n$.

We begin by proving the following lemma.

2.701. *For every x,* $\pi\,|s_n^*(x) - f(x)|$ *is less than*

$$(1)\qquad \int_\eta^\pi \frac{|\varphi(t) - \varphi(t+\eta)|}{t}\,dt + A\eta\int_\eta^\pi \frac{|\varphi(t)|}{t^2}\,dt + 2n\int_0^{2\eta} |\varphi(t)|\,dt + o(1),$$

where A is an absolute constant. The last term on the right tends to 0 *uniformly in any interval where f is bounded.* Applying the device of § 2.21, we see that $\pi[s_n^*(x) - f(x)]$ is equal to

$$\int_0^\pi \chi(t)\sin nt\,dt - \int_{-\eta}^{\pi-\eta} \chi(t+\eta)\sin nt\,dt = \int_\eta^{\pi-\eta} [\chi(t) - \chi(t+\eta)]\sin nt\,dt +$$
$$+ \int_{\pi-\eta}^{\pi} \chi(t)\sin nt\,dt + \int_0^\eta \chi(t)\sin nt\,dt + \int_0^{2\eta} \chi(t)\sin nt\,dt.$$

Let us denote the integrals on the right by I_1, I_2, I_3, I_4 respectively. The sum $|I_3| + |I_4|$ is less than the third term in (1). We may assume that $|\chi(t)\sin nt| \leqslant |\varphi(t)| \leqslant |f(x+t)| + |f(x-t)| + |2f(x)|$ for $t \in (\pi - \eta, \pi)$ and, since an indefinite integral is a continuous function, we see that $I_2 \to 0$. Finally, $|I_1|$ is less than

$$\int_\eta^{\pi-\eta} \frac{|\varphi(t) - \varphi(t+\eta)|}{2\operatorname{tg}\frac{1}{2}(t+\eta)}\,dt + \int_\eta^{\pi-\eta} |\varphi(t)|\left[\frac{1}{2\operatorname{tg}\frac{1}{2}t} - \frac{1}{2\operatorname{tg}\frac{1}{2}(t+\eta)}\right]dt.$$

The difference in square brackets is equal to the expression $\sin\frac{1}{2}\eta/\sin\frac{1}{2}t\,\sin\frac{1}{2}(t+\eta) \leqslant A\eta/t^2$.

This completes the proof.

2.702. Let $\Phi(h) = \Phi_x(h)$ be the integral of $|\varphi_x(t)|$ over $(0, h)$. Lebesgue's test may be formulated as follows: $\mathfrak{S}[f]$ *converges to the value $f(x)$ at every point x at which*

$$(1) \qquad \Phi_x(h) = o\,(h), \quad \int_\eta^\pi \frac{|\varphi(t) - \varphi(t+\eta)|}{t}\,dt \to 0$$ [1]

as $\eta \to 0$. Using Lemma 2.701, it remains to show that the second term in 2.701(1) is $o\,(1)$. Integrating by parts, we find for it the value $A\eta\,\{[\Phi(t)\,t^{-2}]_\eta^\pi + 2\int_\eta^\pi \Phi(t)\,t^{-3}\,dt\} = o\,(1)$, since $\Phi(t) = o\,(t)$, (§ 1.71).

2.703. An important discovery of Lebesgue is that the first condition in 2.702(1) is satisfied almost everywhere. The result may be stated in the following form.

Let $F_x(h) = \int_0^h |f(x+t) - f(x)|\,dt$. *Then, for almost every* x, *we have* $F_x(h) = o\,(h)$ *as* $h \to \pm 0$. This proposition represents a generalization of the well-known theorem on the differentiability of an integral, to which it reduces if we omit the sign of absolute value in the definition of F. Les us denote by E_α, where α is rational, the set of x for which the relation $\frac{1}{h}\int_0^h |f(x+t) - \alpha|\,dt \to |f(x) - \alpha|$ does not hold. In virtue of the theorem just mentionned, any E_α is of mesure 0, and so the sum E of all E_α is of mesure 0. We will prove that $F_x(h) = o\,(h)$ for $x \,\overline{\epsilon}\, E$. Suppose that $\varepsilon > 0$ is given and let β be a rational number such that $|f(x) - \beta| < \varepsilon/2$. In the inequality

$$F_x(h) \leqslant \int_0^h |f(x+t) - \beta|\,dt + \int_0^h |\beta - f(x)|\,dt,$$

where, for simplicity, $h > 0$, the ratio of the first integral on the right to h tends to $|f(x) - \beta| < \varepsilon/2$. Hence, for small h, we have $F_x(h) < \varepsilon h/2 + \varepsilon h/2 = \varepsilon h$, and, ε being arbitrary, the result follows.

2.71. The Dini-Lipschitz test. *If f is continuous and its modulus of continuity satisfies the condition* $\omega(\delta)\log 1/\delta \to 0$, *as* $\delta \to 0$, *then* $\mathfrak{S}[f]$ *converges uniformly*. This follows from Lemma 2.701. Since $|\varphi(t) - \varphi(t+\eta)| \leqslant |f(x+t) - f(x+t+\eta)| + |f(x-t) - f(x-t-\eta)| < 2\omega(\eta)$,

[1] The upper limit of integration π may be replaced by any fixed $\sigma > 0$ (§ 2.201).

the first term in 2.701(1) is $\leqslant 2\omega(\eta)\log \pi/\eta \to 0$. Similarly, since $\varphi(t) \to 0$ uniformly in x, the remaining terms in 2.701(1) tend uniformly to 0 (§ 1.71).

The result holds in particular if $f \,\epsilon\, \mathrm{Lip}\,\alpha$ $(\alpha > 0)$.

In virtue of the theorems on localization, we conclude that *if f is continuous in an interval $I = (a, b)$ and if the modulus of continuity of f in this interval is* $o\,(\log 1/\delta)^{-1}$, $\mathfrak{S}[f]$ *converges uniformly in every interval* $(a+\varepsilon, b-\varepsilon)$. This test is known as the Dini-Lipschitz test and is primarily a condition for uniform convergence. We shall see in Chapter VIII that the condition $f(x_0 + h) - f(x_0) = o\,(\log 1/|h|)^{-1}$ does not ensure the convergence of $\mathfrak{S}[f]$ at x_0.

2.72. In the preceding section we proved that, if in an interval (a, b) the function f satisfies a Lipschitz condition of positive order, then $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ converge uniformly in every interval $(a+\varepsilon, b-\varepsilon)$. We will now prove a slightly more precise result, which completes that established in § 2.4.

If $f(x) \,\epsilon\, \mathrm{Lip}\,\alpha$, $\alpha > 0$, *in an interval* (a, b), *and if, moreover,* $|f(b+t) - f(b)| < At^\alpha$, $|f(a) - f(a-t)| < At^\alpha$, $0 < t \leqslant h$, *where A is a constant, then* $\mathfrak{S}[f]$ *and* $\bar{\mathfrak{S}}[f]$ *converge uniformly in* (a, b) [1]. There exists a constant $B > 0$, such that $|f(x+t) - f(x)| \leqslant Bt^\alpha$, if only $a \leqslant x \leqslant b$, $|t| \leqslant h$, and so, in the equation

$$s_n^*(x) - f(x) = \frac{1}{\pi}\left\{\int_{-\sigma}^{\sigma} + \left(\int_{-\pi}^{-\sigma} + \int_\sigma^\pi\right)\right\}[f(x+t) - f(x)]\,D_n^*(t)\,dt = P + Q,$$

where $0 < \sigma \leqslant h$, the integrand of P does not exceed $B\,|t|^{\alpha-1}$ in absolute value. Hence, taking σ small enough, we have $|P| < \varepsilon/2$, uniformly in (a, b). Since Q is the Fourier coefficient of the function $[f(x+t) - f(x)]\,g(t)$, where $g(t) = \frac{1}{2}\operatorname{ctg}\frac{1}{2}t$ for $\sigma \leqslant |t| \leqslant \pi$, $g(t) = 0$ for $|t| < \sigma$, we see, by Theorem 2.501, that $Q \to 0$ uniformly as $n \to \infty$, so that $|Q| < \varepsilon/2$, $|P + Q| < \varepsilon$, for $n > n_0$, $a \leqslant x \leqslant b$. In the same way we prove the uniform convergence of $\bar{s}_n^*(x) - \bar{f}(x)$.

Let $\omega(\delta)$ denote the modulus of continuity of f in the interval (a, b). If $\omega(\delta)/\delta$ is integrable in a neighbourhood of $\delta = 0$, and if $|f(b+t) - f(b)| < A\omega(t)$, $|f(a) - f(a-t)| < A\omega(t)$, $0 < t \leqslant h$, then $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ converge uniformly in (a, b). The proof re-

[1] Hobson, *Theory of functions*, 2, 535.

mains the same as above. The result holds, in particular, if $\omega(\delta) = O(\log 1/\delta)^{-1-\varepsilon}$, $\varepsilon > 0$. For $\varepsilon = 0$ the argument fails, and, as we shall see later, the theorem itself is false.

2.73. As we shall see in Chapter VIII, the partial sums of $\mathfrak{S}[f]$ may be unbounded almost everywhere. However

If at a point x, we have $\Phi_x(h) = o(h)$, then $s_n(x) = o(\log n)$, and, if $\Psi_x(h) = o(h)$, then $\bar{s}_n(x) = o(\log n)$ [1]. We know that $\Phi_x(h) = o(h)$, $\Psi_x(h) = o(h)$ almost everywhere. From 2.3(4) we see that the expression $\pi\,|\,s_n^*(x) - f(x)\,|$ does not exceed

$$n\int_0^{1/n} |\,\varphi(t)\,|\,dt + \int_{1/n}^{\pi} t^{-1}\,|\,\varphi(t)\,|\,dt = n\Phi(1/n) + [\Phi(t)\,t^{-1}]_{1/n}^{\pi} + \int_{1/n}^{\pi} \Phi(t)\,t^{-2}\,dt.$$

The first two terms on the right give $\Phi(\pi)/\pi = O(1) = o(\log n)$, the third, in virtue of the relation $\Phi(t) = o(t)$, is $o(\log n)$ (§ 1.71). We proceed similarly with $|\,\bar{s}_n^*(x)\,|$, taking into account that $|\,\bar{D}_n^*(t)\,| \leqslant n$ for $0 \leqslant t \leqslant 1/n$, and $|\,\bar{D}_n^*(t)\,| < 2/t$ if $1/n \leqslant t \leqslant \pi$.

If f is continuous in (a, b), then $s_n(x)/\log n$ and $\bar{s}_n(x)/\log n$ tend uniformly to 0 for $x \in (a+\delta, b-\delta)$ $(\delta > 0)$. The proof is still simpler since in the inequalities for $|\,\bar{s}_n^*\,|$ and $|\,s_n^* - f\,|$ no integration by parts is necessary.

2.74. Lebesgue's criterion has an analogue for conjugate series. Let $\Psi_x(h)$ be the integral of $|\,\psi_x(t)\,|$ over $(0, h)$ and let $\bar{f}(x, h)$ have the same meaning as in § 2.61. Then, *the conditions*

$$(1) \qquad \Psi_x(h) = o(h), \qquad \int_{\eta}^{\pi} \frac{|\,\psi(t) - \psi(t+\eta)\,|}{t}\,dt \to 0 \quad (h, \eta \to 0)$$

involve the relation $\bar{s}_n^*(x) - \bar{f}(x, \pi/n) \to 0$. In other words, under the above conditions, $\bar{\mathfrak{S}}[f]$ converges at a point x if and only if the integral 2.61(1) exists [2]. The conditions (1) will certainly be satisfied if f satisfies the Dini-Lipschitz condition in an interval containg x. The proof we leave to the reader.

If $f \in \mathrm{Lip}\,\alpha$, then $\bar{\mathfrak{S}}[f]$ converges uniformly. This follows from the fact that $\bar{s}_n^*(x) - \bar{f}(x, \pi/n)$ tends uniformly to 0 and that the integral $\bar{f}(x, \eta)$ converges uniformly.

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1) Hardy [2]; Young [3].

2) If $\pi/(n+1) < h \leqslant \pi/n$, then $|\,\bar{f}(x, h) - \bar{f}(x, \pi/n)\,| \leqslant (n+1)\,\Psi_x(h)/\pi^2 \to 0$.

2.8. de la Vallée Poussin's test. *If the function $\chi(t) = \chi_x(t) = \dfrac{1}{t}\displaystyle\int_0^t \varphi_x(u)\,du$ is of bounded variation in an interval to the right of $t = 0$, and if $\chi(t) \to 0$ as $t \to 0$, then $\mathfrak{S}[f]$ converges at x to the value $f(x)$* [1].

The convergence of $\mathfrak{S}[f]$ at x to $f(x)$ is the same thing as the convergence of $\mathfrak{S}[\varphi]$ at the point $t = 0$ to the value 0. Now $\varphi(t) = t\chi'(t) + \chi(t)$ and, since the derivative of a function of bounded variation is integrable, φ is the sum of two functions, the first of which satisfies Dini's condition at $t = 0$ and the second is of bounded variation.

2.81. Young's test [2]. *$\mathfrak{S}[f]$ converges at the point x to the value $f(x)$, provided that* (1) *$\varphi_x(t) \to 0$ as $t \to 0$,* (2) *the function $\theta(t) = t\varphi_x(t)$ is of bounded variation in an interval to the right of $t = 0$, and* (3) *the total variation $v(h)$ of θ over $(0, h)$ is $\leqslant Ah$ for small h, where A is a constant.*

Consider the decomposition of the integral 2.52(1) defining $s_n - f$ into three integrals P, Q, R, extended over the intervals $(0, k/n)$, $(k/n, \eta)$, (η, π), where k is large but fixed, and η is defined by the condition that θ is of bounded variation in $(0, \eta)$. We have $|\,P\,| \leqslant n\Phi_x(k/n) \to 0$. Similarly $R \to 0$. Q is the sine coefficient of a function $\xi(t) = \xi_n(t)$ of bounded variation, and the theorem will have been proved when we have shown that the total variation of ξ over $(0, \pi)$ is less than εn (ε arbitrary > 0), if only k is made large enough [3]. Since $\xi(k/n) = o(n)$, $\xi(\eta) = O(1)$, $\xi(t) = 0$ outside $(k/n, \eta)$, it is enough to prove the same thing for the variation of $\xi(t) = \theta(t)/t^2$ over the interior of $(k/n, \eta)$.

Let $(a, b) = (k/n, \eta)$, $\alpha(t) = t^{-2}$, $\beta(t) = \theta(t)$, $u(t) =$ the total variation of α over (a, t), $v(t) =$ the total variation of β over (a, t), and let $a = t_0 < t_1 < \ldots < t_m = b$ be any subdivision of (a, b). If we add the obvious inequalities $|\,\alpha(t_i)\,\beta(t_i) - \alpha(t_{i-1})\,\beta(t_{i-1})\,| \leqslant$ $\leqslant |\,\alpha(t_i)\,|\,|\,\beta(t_i) - \beta(t_{i-1})\,| + |\,\beta(t_{i-1})\,|\,|\,\alpha(t_i) - \alpha(t_{i-1})\,| \leqslant |\,\alpha(t_i)\,|\,[v(t_i) -$ $- v(t_{i-1})] + |\,\beta(t_{i-1})\,|\,[u(t_i) - u(t_{i-1})]$, $i = 1, 2, \ldots, m$, we find that the total variation of $\alpha\beta$ over (a, b) does not exceed

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1) de la Vallée Poussin [1].

2) Young [4]; Hardy and Littlewood [1].

3) The argument is similar to that used in § 2.6.

$$\int_a^b |\alpha(t)|\, dv(t) + \int_a^b |\beta(t)|\, du(t) = \int_{k/n}^{\eta} t^{-2}\, dv(t) + 2\int_{k/n}^{\eta} |\theta(t)|\, t^{-3}\, dt.$$

Since $|\theta(t)| = |\theta(t) - \theta(0)| \leqslant v(t) \leqslant At$, the last integral is less than $2An/k$. An integration by parts shows the preceding integral to be less than $[v(\eta)\eta^{-2} - v(k/n)(k/n)^{-2}] + 2An/k$. Altogether the two integrals yield less than $O(1) + 4An/k < \varepsilon n$, if k is large enough.

2.82. The following theorem, in which $\bar{f}(x, \eta)$ has the same meaning as in § 2.61, is an extension to the case of conjugate series of the results proved in §§ 2.8, 2.81.

The difference $\bar{s}_n(x) - \bar{f}(x, \pi/n)$ tends to 0 as $n \to \infty$, if one of the following two conditions is satisfied: [1].

(i) *the function $\chi(t) = \dfrac{1}{t}\int_0^t \psi_x(u)\, du$ is of bounded variation in an interval to the right of $t = 0$.*

(ii) *$\psi_x(t) \to 0$ with t, $t\psi(t)$ is of bounded variation in an interval to the right of $t = 0$, and the total variation of $t\psi(t)$ over $(0, h)$ is $O(h)$.*

To prove the first part of the theorem we observe that $\bar{\mathfrak{S}}[f]$ at the point x is the same thing as $\frac{1}{2}\bar{\mathfrak{S}}[\psi]$ at $t = 0$. Now $\psi(t) = \psi_1(t) + \psi_2(t)$, $\psi_1(t) = t\chi'(t)$, $\psi_2(t) = \chi(t)$ and so we have $2[\bar{s}_n(x; f) - \bar{f}(x, \pi/n)] = \bar{s}_n(0; \psi) - \bar{\psi}(0, \pi/n) = [\bar{s}_n(0; \psi_1) - \bar{\psi}_1(0; \pi/n)] + [\bar{s}_n(0; \psi_2) - \bar{\psi}_2(0, \pi/n)]$.

Since $\bar{\psi}_1(0, \pi/n) \to \bar{\psi}_1(0)$, $\bar{s}_n(0; \psi_1) \to \bar{\psi}_1(0)$ (§ 2.4), $\bar{s}_n(0; \psi_2) - \bar{\psi}_2(0, \pi/n) \to 0$ (§ 2.61), we obtain that $\bar{s}_n(x) - \bar{f}(x, \pi/n) \to 0$ and this gives the first part of the theorem.

The proof of the second part is much the same as that of Theorem 2.81.

2.83. The Hardy-Littlewood test. This test is interesting because it takes into account not only the behaviour of the function, but also that of the Fourier coefficients.

[1]) Y o u n g [5].

$\mathfrak{S}[f]$ converges at the point x to the value $f(x)$, if the following two conditions are satisfied (i) *$f(x+h) - f(x) = o\,(\log 1/|h|)^{-1}$,* (ii) *the coefficients of $\mathfrak{S}[f]$ are $O(n^{-\delta})$, $\delta > 0$* [1]).

Since instead of $\mathfrak{S}[f]$ we may consider $\mathfrak{S}[\varphi]$, let us assume that $x = 0$, $f(0) = 0$, $f(x) = f(-x)$, $|a_n| < n^{-\delta}$, $0 < \delta < 1$. It is also convenient to suppose that $a_0 = 0$ [2]). Let $r = \delta/2$. We have

$$s_n^*(0) = \frac{2}{\pi}\int_0^{\pi} f(t)\frac{\sin nt}{2\operatorname{tg}\frac{1}{2}t}\, dt = \int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-r}} + \int_{n^{-r}}^{\pi} = P + Q + R.$$

Since f is continuous at the point 0, $P \to 0$ as $n \to \infty$. If $\varepsilon(t) = \operatorname{Max}\{|f(u)|\log 1/u\}$ for $0 < u \leqslant t$, then

$$|Q| \leqslant \varepsilon(n^{-r})\int_{n^{-1}}^{n^{-r}} \frac{dt}{t\log 1/t} = \varepsilon(n^{-r})\log 1/r \to 0,$$

and it remains only to show that $R \to 0$. Using the theorem (which will be established in Chapter IV) that Fourier series may be integrated term by term after having been multiplied by an arbitrary function of bounded variation, we have

$$R = \sum_{k=1}^{\infty} a_k \frac{2}{\pi}\int_{n^{-r}}^{\pi} \frac{\sin nt \cos kt}{2\operatorname{tg}\frac{1}{2}t}\, dt.$$

Replacing the products $\cos kt \sin nt$ by differences of sines, and applying the second mean-value theorem to the factor $\frac{1}{2}\operatorname{ctg}\frac{1}{2}t$, we see that the coefficient of a_k, $k \neq n$, does not exceed $4n^r/\pi|k-n|$ in absolute value, and so

$$|R| \leqslant o(1) + \frac{4}{\pi}\sum_{k=1}^{\infty}{}' \frac{k^{-\delta} n^r}{|k-n|} = o(1) + \sum_{k=1}^{n-1} + \sum_{k=n+1}^{\infty} = o(1) + R_1 + R_2,$$

where $'$ denotes that the term $k = n$ is omitted. Now

$$\frac{\pi}{4}R_1 < \frac{n^r}{\frac{1}{2}n}\sum_{k=1}^{[\frac{1}{2}n]} k^{-\delta} + n^r(\tfrac{1}{2}n)^{-\delta}\sum_{k=[\frac{1}{2}n]+1}^{n-1}\frac{1}{n-k} = O(n^{-\frac{\delta}{2}}) + O(n^{-\frac{\delta}{2}}\log n) = o(1)$$

$$\frac{\pi}{4}R_2 < n^{r-\delta}\sum_{k=n+1}^{2n}\frac{1}{k-n} + n^r\sum_{k=2n+1}^{\infty}\frac{k^{-\delta}}{\frac{1}{2}k} = O(n^{-\frac{\delta}{2}}\log n) + O(n^{-\frac{\delta}{2}}) = o(1),$$

and this completes the proof. The same argument shows that

Under the conditions of the above theorem, $\bar{s}_n(x) - \bar{f}(x, \pi/n) \to 0$.

[1]) H a r d y and L i t t l e w o o d [2], [3].

[2]) We can secure this by adding $-\frac{1}{2}a_0(1 - \cos x)$ to $\mathfrak{S}[f]$.

2.84. Relations between tests [1]. We shall consider only tests for the convergence of Fourier series.

Dini's and Jordan's tests are not comparable [2]. Let $f(x)$, $g(x)$ be even and let $f(x) = 1/\log(x/2\pi)$, $g(x) = x^{\alpha} \sin 1/x$ $(0 < \alpha < 1)$ for $0 < x \leqslant \pi$. At the point 0, f satisfies Jordan's condition but not Dini's, and conversely g satisfies Dini's condition but not Jordan's.

de la Vallée Poussin's test includes both Dini's and Jordan's. Let $\Phi(t)$ be the integral of φ over $(0, t)$, and let $\chi(t) = \Phi(t)/t$. If φ is positive and non-decreasing, so is χ. If φ is of bounded variation, i. e. if $\varphi = \varphi_1 - \varphi_2$, where φ_1, φ_2 are positive and non-decreasing, then $\chi = \chi_1 - \chi_2$ is also of bounded variation. This proves the second part of the theorem. To prove the first, let $\mu(t)$ be the integral of $\varphi(u)/u$ over $(0, t)$. A simple calculation shows that

$$\frac{1}{t}\int_0^t \varphi(u)\, du = \mu(t) - \frac{1}{t}\int_0^t \mu(u)\, du,$$

and if μ is of bounded variation the same is true for the expression on the left.

de la Vallée Poussin's and Young's tests are not comparable. Let $g(x)$ be even, $g(x) = (-1)^n x^{\alpha}$ for $\pi/(n+1) < x \leqslant \pi/n$, $n = 1, 2, \dots$ The total variation of $xg(x)$ over $(0, \pi/n)$ is exactly of order $n^{-\alpha}$. It follows that, if $0 < \alpha < 1$, $x = 0$, g satisfies Dini's condition but not Young's. Thus Young's condition does not include Dini's, and, à fortiori, de la Vallée Poussin's.

Let $h(x)$ be even and equal to $(-1)^n \beta_n$ in the interval $(\pi 2^{-n-1}, \pi 2^{-n})$, $n = 0, 1, 2, \dots$, where $1 > \beta_0 > \beta_1 > \dots \to 0$. A simple calculation shows that the total variation of $H(x) = x^{-1}\int_0^x h(t)\, dt$ over $(\pi 2^{-n-1}, \pi 2^{-n})$ is equal to $[\beta_n/2 + \beta_{n+1}/2^2 - \beta_{n+2}/2^3 + \beta_{n+3}/2^4 - \dots] > \beta_n/2$, so that, if $\beta_1 + \beta_2 + \dots = \infty$, $h(x)$ does not satisfy de la Vallée Poussin's condition at the point $x = 0$. From the graph of the curve $y = \theta(x) = xh(x)$ we deduce that, if $\pi 2^{-n-1} \leqslant x < \pi 2^{-n}$, the total

[1]) Hardy [3]. See also Gergen [1], Pollard [1].

[2]) We say that f satisfies Jordan's condition at a point x_0, if $f(x)$ is of bounded variation in a neighbourhood of x_0 (§ 2.62).

variation of θ over $(0, x)$ is less than $o(x) + 2\pi[\beta_n 2^{-n-1} + \beta_{n+1} 2^{-n-2} + \dots] \leqslant o(x) + \beta_n \pi 2^{1-n} = o(x)$, and so Young's condition is satisfied.

We state without proof the following result: *de la Vallée Poussin's and Young's tests are both included in Lebesgue's test* [1]), which, consequently, turns out to be the most powerful, although not always the most convenient, of all the tests discussed in this section.

2.85. Poisson's formula. Let $g(x)$ be a function defined for $-\infty < x < \infty$, tending to 0 as $x \to \pm\infty$, and integrable in any finite interval. Suppose, moreover, that the series

$$(1) \qquad \sum_{k=-\infty}^{+\infty} g(k + x) = G(x),$$

whose symmetric partial sums we denote by $G_n(x)$, converges uniformly [2]) for $0 \leqslant x \leqslant 1$. The sum $G(x)$ is of period 1, and its Fourier coefficients c_ν with respect to the system $\{\exp 2\pi i\nu x\}$ are

$$(2) \qquad \lim_{N\to\infty}\int_0^1 G_N\, e^{-2\pi i\nu x}\, dx = \lim_{N\to\infty}\int_{-N}^{N+1} g e^{-2\pi i\nu x}\, dx = \int_{-\infty}^{+\infty} g(x)\, e^{-2\pi i\nu x}\, dx.$$

Hence, supposing that, at the point $x = 0$, G satisfies one of the conditions ensuring the convergence of $\mathfrak{S}[G]$ to the value $G(0)$, we obtain immediately the Poisson formula

$$(3) \qquad \sum_{k=-\infty}^{+\infty} g(k) = \sum_{\nu=-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)\, e^{-2\pi i\nu x}\, dx.$$

This formula is true if, for example, g is of bounded variation over $(-\infty, +\infty)$, $2g(x) = g(x+0) + g(x-0)$, and if the series (1) converges at a point. In fact, let v_k be the total variation of $g(x)$ over $(k, k+1)$. Since the oscillation of $g(x+k)$ in $(0, 1)$ does not exceed v_k, and $\dots + v_{-1} + v_0 + v_1 + \dots = V < \infty$, the series in (1) converges uniformly. $G(x)$ is of bounded variation since its total variation over $(0, 1)$ does not exceed V. Moreover, it is easy to see that $2G(x) = G(x+0) + G(x-0)$.

An additional remark on the Fourier coefficients of the function $G(x)$ in (1) will be useful later. It may happen that $G_n(x)$

[1]) Hardy [3]; Hobson, *Theory of functions*, 2, 533.

[2]) This condition might be relaxed.

itself does not tend to any limit, but that there exists a sequence of constants K_n such that the sequence $H_n(x) = G_n(x) - K_n$ does tend, uniformly, to a limit $H(x)$. Changing, if necessary, the values of K_n, we may suppose that the integral of H over $(0, 1)$ vanishes, so that, if now c_ν are the complex Fourier coefficients of H, we have $c_0 = 0$. Taking into account that the integral of $K_n \exp(-2\pi i\nu x)$ over $(0, 1)$ vanishes, and replacing in (2) G_N by H_N, we find the same formula as before for c_ν. In other words, since K_n may be taken as the mean-value of G_n over $(0, 1)$, we may write

$$(4)\qquad \lim_{n\to\infty}\left\{G_n(x) - \int_0^1 G_n(t)\,dt\right\} \sim \sum_{\nu=-\infty}^{+\infty}{}' e^{2\pi i\nu x}\int_{-\infty}^{+\infty} g(t)\,e^{-2\pi i\nu t}\,dt,$$

where $'$ denotes that the term $\nu = 0$ is omitted.

Example. Let $g(x) = x^{-\alpha}$ for $x > 0$, $g(x) = 0$ elsewhere, $0 < \alpha < 1$. Here $G_n(x) = x^{-\alpha} + (x+1)^{-\alpha} + \dots + (x+n)^{-\alpha}$, $K_n = (n+1)^{1-\alpha}/(1-\alpha)$. Therefore, since $(n+1)^{1-\alpha} - n^{1-\alpha} \to 0$, the numbers $c_\nu = \int_0^\infty x^{-\alpha} e^{2\pi i\nu x}\,dx$ are the Fourier coefficients of the function

$$\lim_{n\to\infty}\,[x^{-\alpha} + (x+1)^{-\alpha} + \dots + (x+n)^{-\alpha} - n^{1-\alpha}/(1-\alpha)]\quad (0 < x < 1).$$

2.9. Miscellaneous theorems and examples.

1. If $\omega_1(\delta; f) = o(\delta)$, then $f \equiv \text{const}$. Titchmarsh, *Theory of functions*, 372.

[Consider $\int_{x_1}^{x_2} [f(t+h) - f(t)]\,dt$].

2. Given an arbitrary sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$, there exists a continuous f such that $|a_n| + |b_n| \geqslant \varepsilon_n$ for infinitely many n. Lebesgue [1].

[If $n_1 < n_2 < \dots$ and $\varepsilon_{n_1} + \varepsilon_{n_2} + \dots < \infty$, put $f(x) = \varepsilon_{n_1}\cos n_1 x + \varepsilon_{n_2}\cos n_2 x + \dots$].

3. Let $f(x) = a\cos bx + a^2\cos b^2 x + \dots + a^n \cos b^n x + \dots$, $0 < a < 1$, $ab > 1$. Show that (i) $f \in \mathrm{Lip}\,\alpha$, where $\alpha = \log a^{-1}/\log b$, (ii) the Fourier coefficients of f are $O(n^{-\alpha})$, but not $o(n^{-\alpha})$ (iii) if $ab = 1$, then $\omega(\delta; f) = O(\delta\log 1/\delta)$. Hardy [4].

[Let $\nu = \nu(h)$ be the largest n such that $b^n h \leqslant 1$. In the formula

$$f(x+h) - f(x-h) = -\sum_{n=1}^{\infty} 2a^n \sin b^n h \sin b^n x = \sum_{\nu=1}^{\nu} + \sum_{n=\nu+1}^{\infty} = P + Q$$

the terms of P do not exceed $2a^n b^n h$, so that $P = O(h^\alpha)$. The terms of Q are $\leqslant 2a^n$, and so $Q = O(h^\alpha)$].

4. Using Theorem 2.622 and the equation $\sum_{\lambda}^{\mu} (a_n \sin nx - b_n \cos nx)/n = \frac{1}{\pi}\int_0^{2\pi} f(t) \sum_{\lambda}^{\mu} \frac{\sin n(x-t)}{n}\,dt$, prove Theorem 2.621 and the formula

$$\sum_{n=1}^{\infty} \frac{b_n}{n} = \frac{1}{\pi}\int_0^{2\pi} f(t)\,\frac{\pi - t}{2}\,dt.$$

5. The numbers $C_k = 1 + 2^{-2k} + 3^{-2k} + \dots + n^{-2k} + \dots$, $k = 1, 2, 3, \dots$, are all rational multiples of π^{2k}.

[Integrate the series $\sin x + \frac{1}{2}\sin 2x + \dots$ an odd number of times].

6. If $f(x)$ has k derivatives, the Fourier coefficients of f satisfy the relation $|c_n| \leqslant \omega(\pi/n; f^{(k)})/2n^k$, $n > 0$. If $f^{(k)}$ is of bounded variation, then $c_n = O(n^{-k-1})$.

7. If $f(x)$ vanishes in (a, b), the function $\bar{f}(x)$ defined by 2.4(2) has derivatives of any order for $a < x < b$.

8. Considering $\mathfrak{S}[\cos\alpha x]$, prove the formulae

$$\frac{\alpha\pi}{\sin\alpha\pi} = 1 + 2\alpha^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha^2 - k^2},\qquad \alpha\pi\operatorname{ctg}\alpha\pi = 1 + 2\alpha^2 \sum_{k=1}^{\infty} \frac{1}{\alpha^2 - k^2}\qquad (\alpha \neq 0, \pm 1, \dots).$$

9. If $\varphi_x(t)$ increases monotonically to $+\infty$ as $t \to +0$, $0 < t \leqslant t_0$, $\mathfrak{S}[f]$ diverges to $+\infty$ at the point x.

[Let $\varphi(t)/t = \chi(t)$. Then

$$\int_0^{\pi/n} \chi(t)\sin nt\,dt + o(1) \geqslant \pi s_n^*(x) \geqslant \int_0^{\pi/n} [\chi(t) - \chi(t + \tfrac{\pi}{n})]\sin nt\,dt + o(1) \geqslant$$

$$\geqslant \tfrac{1}{2}\int_0^{\pi/n} \chi(t)\sin nt\,dt + o(1)].$$

10. If (i) $\varphi_x(t) \to 0$ with t, (ii) $t\varphi'(t)$ is absolutely continuous except at $t = 0$, (iii) $t\varphi'(t) > -A$, $A > 0$, for small $t > 0$, then $\mathfrak{S}[f]$ converges at x. Tonelli [1]; Hardy and Littlewood [3].

[Apply Young's test].

11. $\bar{\mathfrak{S}}[f]$ is convergent at the point x, provided that (1) the integral 2.61(1) exists and (2) the total variation of $t\psi(t)$ over $(0, h)$ is $O(h)$. See Prasad [1].

[The proof is analogous to that of Theorem 2.82 (ii) except at one point: to estimate $P = \int_0^{k/n} \psi(t)\,\bar{D}_n^*(t)\,dt$ we cannot use the fact that $\psi(t) \to 0$, but integrating by parts and applying condition (1) we find that $P \to 0$].