CHAPTER I.

Trigonometrical series and Fourier series.

1.1. Definitions. Trigonometrical series are series of the form

$$(1)\qquad \tfrac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx),$$

where the coefficients $a_0, a_1, \dots, b_1, b_2, \dots$ are independent of the real variable x. It is convenient to provide the constant term of trigonometrical series with the factor 1/2. Except when otherwise stated, we shall suppose, always, that the coefficients of the trigonometrical series considered are real. Since all the terms of (1) are of period 2π, it is sufficient to study trigonometrical series in any interval of length 2π, e. g. in $(0,2\pi)$ or $(-\pi,\pi)$.

Consider the power series

$$(2)\qquad \tfrac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k-ib_k)z^k$$

on the unit circle: $z=e^{ix}$. The series (1) is the real part of (2).

The series

$$(3)\qquad \sum_{k=1}^{\infty}(a_k\sin kx-b_k\cos kx),$$

(with vanishing constant term) which multiplied by i and added to (1) gives the power series (2), is called *conjugate* to (1).

1.12. Summation of certain trigonometrical series. The fact that trigonometrical series are the real parts of power series facilitates in many cases finding the sums of the former. For example, the series

$$(1)\qquad P_r(x)=\tfrac{1}{2}+\sum_{k=1}^{\infty}r^k\cos kx,\quad Q_r(x)=\sum_{k=1}^{\infty}r^k\sin kx,$$

where $0 \leqslant r < 1$, are the real and imaginary parts of the series $\frac{1}{2} + z + z^2 + \ldots$, where $z = re^{ix}$, and so we obtain without difficulty

$$(2) \quad P_r(x) = \tfrac{1}{2} \cdot \frac{1 - r^2}{1 - 2r\cos x + r^2}, \quad Q_r(x) = \frac{r \sin x}{1 - 2r \cos x + r^2}.$$

Similarly, from the formula $\log 1/(1-z) = z + z^2/2 + \ldots$, we obtain

$$(3) \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k} r^k = \tfrac{1}{2} \log \frac{1}{1 - 2r \cos x + r^2}, \qquad \sum_{k=1}^{\infty} \frac{\sin kx}{k} r^k = \operatorname{arctg} \frac{r \sin x}{1 - r \cos x},$$

where $0 \leqslant r < 1$, $\operatorname{arctg} 0 = 0$. Denoting by $p_n(x)$, $q_n(x)$ the n-th partial sums $(n = 0, 1, 2, \ldots)$ of the series (1) with $r = 1$, we obtain by the same argument

$$(4) \quad p_n(x) = \frac{\sin (n + \frac{1}{2}) x}{2 \sin \frac{1}{2} x}, \quad q_n(x) = \frac{\cos \frac{1}{2} x - \cos (n + \frac{1}{2}) x}{2 \sin \frac{1}{2} x}.$$

(A simple, although less natural, method of proving for example the first formula in (4) would be to multiply p_n by $2 \sin \frac{1}{2} x$ and to replace the products $\cos kx . 2 \sin \frac{1}{2} x$ by differences of sines; then all the terms, except the last, cancel). From (4) we deduce that $p_n(x)$ and $q_n(x)$ are uniformly bounded, indeed less than $1/\sin \frac{1}{2} \varepsilon$ in absolute value, in every interval $0 < \varepsilon \leqslant x \leqslant 2\pi - \varepsilon$.

1.13. The complex form of trigonometrical series. Applying Euler's formulae to $\cos kx$, $\sin kx$, we may write the n-th partial sum of 1.1(1) in the form

$$s_n(x) = \tfrac{1}{2} a_0 + \tfrac{1}{2} \sum_{k=1}^{n} [(a_k - ib_k) e^{ikx} + (a_k + ib_k) e^{-ikx}].$$

If we define a_k, b_k for any integral k by the conditions $a_{-k} = a_k$, $b_{-k} = -b_k$, (thus, in particular, $b_0 = 0$), we see that s_n is the n-th symmetric partial sum, i. e. the sum of $2n + 1$ terms with indices not exceeding n in absolute value, of the Laurent series

$$(1) \qquad \sum_{k=-\infty}^{+\infty} c_k e^{ikx} \qquad (2c_k = a_k - ib_k).$$

Here c_{-k} is conjugate to c_k. Conversely, any series (1) with this property can be written in the form 1.1(1). Whenever we speak of convergence or summability of series (1), we shall always mean the limit, ordinary or generalized, of the symmetric partial sums.

The series conjugate to (1) may be obtained from the latter, replacing in it c_k by $-ic_k \operatorname{sign} k$, where $\operatorname{sign} z = z/|z|$ if $z \neq 0$, and $\operatorname{sign} 0 = 0$.

1.2. Abel's transformation:

$$(1) \qquad \sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k (v_k - v_{k+1}) - U_{m-1} v_m + U_n v_n,$$

where $0 \leqslant m \leqslant n$, $U_k = u_0 + u_1 + \ldots + u_k$ if $k \geqslant 0$, $U_{-1} = 0$. This formula, which can be easily verified, corresponds to integration by parts in the theory of integration, and is a very useful tool in the general theory of series. We shall call a sequence $v_0, v_1, \ldots$ of *bounded variation* if the series $|v_0 - v_1| + |v_1 - v_2| + \ldots$ is convergent. Without aiming at complete generality, we mention the following consequences of (1) in the case $m = 0$.

1.21 a) *If a series $u_0(x) + u_1(x) + \ldots$ converges uniformly and $\{v_k\}$ is of bounded variation, the series $u_0(x)v_0 + u_1(x)v_1 + \ldots$ converges uniformly.*

b) *If $u_0(x) + u_1(x) + \ldots$ has its partial sums uniformly bounded, $\{v_k\}$ is of bounded variation and $v_k \to 0$, the series $u_0(x)v_0 + u_1(x)v_1 + \ldots$ converges uniformly.*

1.22. A corollary of Abel's formula. If $v_m, v_{m+1}, \ldots v_n$ are non-negative and non-increasing, the left-hand side of 1.2(1) does not exceed $2v_m \operatorname{Max} |U_k|$ $(m - 1 \leqslant k \leqslant n)$ in absolute value. In fact, it does not exceed $\operatorname{Max} |U_k|$ multiplied by $(v_m - v_{m+1}) + \ldots + (v_{n-1} - v_n) + v_m + v_n = 2v_m$.

1.23. Convergence of a class of trigonometrical series. The problems of convergence of 1.1(1) are, except in the trivial case when $|a_1| + |b_1| + |a_2| + |b_2| + \ldots < \infty$, always delicate. Some rather special but, none the less, important results follow from Theorem 1.21. Applying it to the series

$$(1) \qquad \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos kx, \qquad \sum_{k=1}^{\infty} a_k \sin kx,$$

and taking into account the last remark in § 1.12, we obtain:

If $\{a_k\}$ is of bounded variation and $a_k \to 0$, in particular if a_k monotonically decreases to 0, the series (1) converge uniformly in any interval $0 < \varepsilon \leqslant x \leqslant 2\pi - \varepsilon$.

As regards the neighbourhood of $x=0$, the behaviour of sine and cosine series may be quite different. In particular, the former always converge for $x=0$, whereas the convergence of the latter is equivalent to that of $\frac{1}{2}a_0+a_1+\ldots$ [1].

Transforming the argument x we may present the last theorem in other, equivalent, forms. We shall be contented with the following statement.

If $\{a_k\}$ *is of bounded variation and* $a_k\to 0$, *then the series* $\frac{1}{2}a_0-a_1\cos x+a_2\cos 2x-\ldots$, $a_1\sin x-a_2\sin 2x+\ldots$ *converge uniformly in* $(0,2\pi)$, *except in arbitrarily small neighbourhoods of* $x=\pi$.

For the proof it is sufficient to replace in (1) x by $x+\pi$.

1.3. Orthogonal systems of functions. Fourier series. A system of real functions $\varphi_0(x),\varphi_1(x),\ldots,\varphi_n(x),\ldots$ defined in an interval (a,b) is said to be *orthogonal* in this interval if

$$(1)\qquad \int_0^1 \varphi_m(x)\,\varphi_n(x)\,dx \begin{array}{ll} =0 & (m\neq n)\\ =\lambda_n>0 & (m=n)\end{array}\qquad m,n=0,1,\ldots$$

In particular, no φ_m vanishes identically. If $\lambda_0=\lambda_1=\ldots=1$, the system is said, in addition, to be *normal*. If $\{\varphi_n\}$ is orthogonal, $\{\varphi_n/\lambda_n^{1/2}\}$ is orthogonal and normal. The importance of orthogonal systems is based on the following fact. Suppose that a series $c_0\varphi_0(x)+c_1\varphi_1(x)+\ldots$, where $c_0,c_1,\ldots$ are constants, converges in (a,b) to a function $f(x)$. Multiplying both sides of the formula $f(x)=c_0\varphi_0(x)+\ldots+c_n\varphi_n(x)+\ldots$ by $\varphi_n(x)$ and integrating over the range (a,b), we find, in virtue of (1), that

$$(2)\qquad c_n=\frac{1}{\lambda_n}\int_a^b f\varphi_n\,dx\qquad (n=0,1,\ldots).$$

This argument is purely formal, but in some cases, for example if the series defining f converges uniformly, φ_n are continuous and (a,b) is finite, it is easily justified. It suggests the following very important problem. Suppose that we have a function $f(x)$ defined (a,b). Having formed the numbers c_n by means of (2), we write, quite formally,

$$(3)\qquad f(x)\sim c_0\varphi_0(x)+c_1\varphi_1(x)+\ldots$$

[1]) See also Chapter V.

and call the series on the right the *Fourier series* of $f(x)$, with respect to the system $\{\varphi_n\}$. The numbers c_n are called the *Fourier coefficients* of f. The sign $\sim$ in (3) only means that the numbers c_n are connected with f by the formula (2) and does not imply in the least that the series is convergent, still less that it converges to f. Now, what are the properties of this series? In what sense does it 'represent' f?

This book is devoted to the study of one, very special but extremely important, orthogonal system, viz. the trigonometrical system, and so we shall study the general theory only in so far as it bears relation on this system [1]).

If an orthogonal system is to be at all useful for the development of functions, it should be *complete,* that is, whatever function ψ is added to $\{\varphi_m\}$, the new system ceases to be orthogonal. In fact, otherwise there would exist a function, just the function ψ, not vanishing identically, whose Fourier series with respect to $\{\varphi_n\}$ would consist entirely of zeros.

1.31. The notion of orthogonality, and hence that of Fourier coefficients and Fourier series, may be extended to the case of complex φ_n. We need only modify conditions 1.3(1) slightly, by replacing the products $\varphi_m\varphi_n$ by $\varphi_m\overline{\varphi}_n$, or, what is the same thing, by $\overline{\varphi}_m\varphi_n$ [2]). Similarly in (2) we replace $f\varphi_n$ by $f\overline{\varphi}_n$.

1.32. Rademacher's system. The following very instructive orthogonal and normal system was first considered by Rademacher [3]): $\varphi_n(x)=\text{sign}\sin(2^{n+1}\pi x)$ $(0\leqslant x\leqslant 1)$. The function $\varphi_n(x)$ assumes alternately the values ± 1 in the interior of the intervals $(0,2^{-n-1})$, $(2^{-n-1},2.2^{-n-1})$, ... The proof of orthogonality is very simple and may be left to the reader. The system is not complete, since e. g. the function $\psi(x)\equiv 1$ may be added to it.

[1]) We refer the reader interested in wider problems to a book by Kaczmarz and Steinhaus which is to appear in this series.

[2]) We denote by $\overline{z}=x-iy$ the number conjugate to $z=x+iy$. However the bar will also be used to denote the conjugate series, functions etc, where the word 'conjugate' has a different meaning. No misunderstanding will occur if the reader takes into account the context.

[3]) Rademacher [1]. See also Kaczmarz and Steinhaus [1].

1.4. The trigonometrical system. The system of functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$, i. e. the trigonometrical system, is orthogonal in $(-\pi, \pi)$. In fact, let $I_{m,n} = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx$, and let $I'_{m,n}$, $I''_{m,n}$ denote the corresponding integrals with $\cos mx \sin nx$ and $\cos mx \cos nx$. Integrating the formula $2 \sin mx \sin nx = \cos (m-n)x - \cos (m+n)x$ and taking into account the periodicity of trigonometrical functions, we find that $I_{m,n} = 0$ whenever $m \neq n$. Similarly $I'_{m,n} = 0$, $I''_{m,n} = 0$, the former result being true even when $m = n$. The λ's are now $2\pi, \pi, \pi, \ldots$, and so, if for a given f we put

$$(1) \qquad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt,$$

the Fourier series of f may be written in the form 1.1(1). Changing the definition of the preceding paragraph slightly in the case of a_0, we shall call a_k, b_k the Fourier coefficients of f. We shall denote by $\mathfrak{S}[f]$ the Fourier series of f and by $\bar{\mathfrak{S}}[f]$ the conjugate series. It is obvious that, if μ_1, μ_2 are two constants, then $\mathfrak{S}[\mu_1 f_1 + \mu_2 f_2] = \mu_1 \mathfrak{S}[f_1] + \mu_2 \mathfrak{S}[f_2]$.

1.41. *If a series* 1.1(1) *converges uniformly to a function* $f(x)$, *the coefficients* a_k, b_k *are given by the formulae* 1.4(1). The proof is the same as that which led to the formula 1.3(2).

1.42. If the function f is even, that is if $f(-x) = f(x)$, the coefficients b_k vanish and the integral defining a_k may be replaced by twice the integral over the interval $(0, \pi)$. If f is odd, that is if $f(-x) = -f(x)$, then $a_k = 0$ and the second integral in (1) may be replaced by twice the integral over $(0, \pi)$.

1.43. The complex form of Fourier series. The system of complex functions e^{kix} $(k = 0, \pm 1, \pm 2, \ldots)$ is orthogonal in $(-\pi, \pi)$. Putting

$$(1) \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt \qquad (k = 0, \pm 1, \ldots),$$

we may write the Fourier series, with respect to this system, in the form 1.13(1). Let us suppose, as we always shall do, except when it is stated otherwise, that f is real, and put $2c_k = a_k - ib_k$. Then a_k, b_k are given by 1.4(1) and we see that this Fourier series is equivalent to the trigonometrical Fourier series. However the complex form is very convenient and we shall frequenly use it.

1.44. It is also convenient to suppose that the functions whose Fourier series we consider are defined not only in $(-\pi, \pi)$, but for all real x by the condition of periodicity: $f(x + 2\pi) = f(x)$, and, unless a statement to the contrary is made, we shall always assume this. Hence, we assume, in particular, that $f(-\pi) = f(\pi)$, a condition which we may always suppose satisfied [1]). Whenever we say that a series is the Fourier series of a continuous function f, we mean that f is continuous in $(-\infty, +\infty)$.

It is obvious that if a function $\psi(x)$ is of period 2π, the integrals of ψ, taken over arbitrary intervals of length 2π, are all equal. In particular, in 1.4(1) we may integrate over the interval $(0, 2\pi)$.

1.45. However, sometimes it is more convenient to consider the trigonometrical system not in $(0, 2\pi)$ but in another interval, e. g. in $(0,1)$. The system $\{e^{2\pi ikx}\}$ is orthogonal and normal in the latter interval, so that the complex Fourier coefficients assume now the form

$$c_k = \int_0^1 f(t) e^{-2\pi ikt} \, dt \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

1.46. Integration and Fourier series. The problems of the theory of Fourier series are closely connected with the notion of integration. In the preceding definitions we assumed tacitly that the products $f \cos kx$, $f \sin kx$ were integrable. Hence we may consider Fourier-Riemann, Fourier-Lebesgue, Fourier-Denjoy series, according to the way in which the integrals are defined [2]). Except when otherwise stated, integrals are always Lebesgue integrals. It is assumed that the reader knows the elements of the Lebesgue theory of integration. Proofs of results of a more special character will be given in the text [3]).

Every integrable function $f(x)$ $(0 \leqslant x \leqslant 2\pi)$ has its Fourier series. It is even sufficient for f to be defined almost everywhere in $(0, 2\pi)$, i. e. everywhere, except in a set of measure 0. Two

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[1]) See § 1.46.

[2]) For a general discussion see Lusin [1], [2].

[3]) The few passages in which the Denjoy integral is mentionned are not essential and may be omitted.

functions f_1 and f_2 which are equal almost everywhere have the same Fourier series and, following the usage of the Lebesgue theory, we call them equivalent: $f_1(x) \equiv f_2(x)$ and do not distinguish them from each other.

1.47. Fourier-Stieltjes series. Let $F(x)$ be a function of bounded variation, defined in $(0, 2\pi)$. Consider the series 1.1(1) with coefficients given by the formulae

$$(1) \qquad a_k = \frac{1}{\pi}\int_0^{2\pi} \cos kt\, dF(t), \quad b_k = \frac{1}{\pi}\int_0^{2\pi} \sin kt\, dF(t),$$

the integrals being Riemann-Stieltjes integrals. We shall write

$$(2) \qquad dF(x) \sim \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

and call the series on the right the *Fourier-Stieltjes series* of dF. If F is absolutely continuous and $F'(x) = f(x)$, then $\mathfrak{S}[dF] = \mathfrak{S}[f]$.

It is convenient to define $F(x)$ for all x by the condition $F(x+2\pi) - F(x) = F(2\pi) - F(0)$. We may then integrate in the formulae (1) over any interval of length 2π. A necessary and sufficient condition for F to be periodic is: $\pi a_0 = F(2\pi) - F(0) = 0$. It follows that the function $F(x) - a_0 x/2$ is periodic.

1.5. *The trigonometrical system is complete.* This result is a simple corollary of theorems which we encounter later, but the following elementary proof, due to Lebesgue, is interesting in itself. Suppose first that there is a continuous and periodic $f \not\equiv 0$, whose Fourier coefficients all vanish. It follows that

$$(1) \qquad \int_{-\pi}^{\pi} f(x) T_n(x)\, dx = 0$$

for every trigonometrical polynomial T_n [1]). We may suppose without loss of generality that there exists a point x_0 and two numbers ε, $\delta > 0$, such that $f(x) > \varepsilon$ for $x \in I = (x_0 - \delta, x_0 + \delta)$ [2]). It will be enough to show that there exists a sequence $\{T_n(x)\}$, such

[1]) Trigonometrical polynomials of order n are finite sums of the form $\frac{1}{2}\alpha_0 + (\alpha_1 \cos x + \beta_1 \sin x) + \dots + (\alpha_n \cos nx + \beta_n \sin nx)$.

[2]) $x \in A$ means: x belongs to a set A; $x \bar{\in} A$ means: x does not belong to A; $A \subset B$ means: A is a subset of B.

that (i) $T_n(x) \geqslant 1$ in I, (ii) $T_n(x)$ tends uniformly to $+\infty$ in every interval I' interior to I, (iii) $T_n(x)$ are uniformly bounded outside $I \pmod{2\pi}$. For the left-hand side of (1) is the sum of two integrals, extended respectively over I and the rest of $(-\pi, \pi)$. The first of them exceeds $|I'|$. Max $T_n(x)$ $(x \in I') \to \infty$ [1]). The second is bounded and so (1) is impossible for large n. We put $T_n = t^n$, where $t(x) = 1 + \cos(x - x_0) - \cos\delta$. In this case $t(x) \geqslant 1$ in I, $t(x) > 1$ in I', $|t(x)| \leqslant 1$ for $x \bar{\in} I \pmod{2\pi}$.

Suppose now f only integrable and let $F(x)$ be the integral of f over $(-\pi, x)$. Hence $F(-\pi) = 0$, and the condition $a_0 = 0$ involves $F(\pi) = 0$. Integrating 1.4(1) by parts we obtain $A_1 = B_1 = A_2 = B_2 = \dots = 0$, where $A_0, A_1, B_1, \dots$ are the Fourier coefficients of F. Hence, for a suitable constant c, the continuous function $F - c$ will have all its Fourier coefficients equal to 0, and so $F(x) = c$. Since $F(-\pi) = 0$, we obtain ultimately $F(x) = 0$, i. e. $f \equiv 0$. The reader will observe that the proof remains valid with more general definitions of an integral than that of Lebesgue.

1.51. Corollaries. (i) *If f_1 and f_2 have the same Fourier series then $f_1 \equiv f_2$.* (ii) *If, for f continuous, $\mathfrak{S}[f]$ converges uniformly, it converges to f.* Let $g(x)$ denote the sum of $\mathfrak{S}[f]$. Then the coefficients of $\mathfrak{S}[f]$ are the Fourier coefficients of g (see § 1.41), and so $f = g$.

1.6. Bessel's inequality. Parseval's relation. We may also be led to the notion of Fourier coefficients by the following considerations. Let $\{\varphi_n\}$ be a system of functions orthogonal and normal in an interval (a, b), and let f be a function such that f^2 is integrable in (a, b). We fix an integer $n \geqslant 0$, put $T = \gamma_0 \varphi_0 + \gamma_1 \varphi_1 + \dots + \gamma_n \varphi_n$ and then ask what values of the constants $\gamma_0, \gamma_1, \dots \gamma_n$ make the integral

$$(1) \quad \int_a^b (f-T)^2\, dx = \int_a^b (f^2 - 2fT + T^2)\, dx = \int_a^b f^2\, dx - 2\sum_{k=0}^{n} c_k \gamma_k + \sum_{k=0}^{n} \gamma_k^2$$

a minimum, $c_0, c_1, \dots$ being the Fourier coefficients of f. The last two sums can be written as $-\gamma_0(2c_0 - \gamma_0) - \dots - \gamma_n(2c_n - \gamma_n)$ and since the function $u(\alpha - u)$ assumes its maximum when $u = \alpha/2$, we see that *the left-hand side of* (1), *which is called the quadratic*

[1]) $|E|$ denotes the measure of a set E.

approximation to f *by* T, *is a minimum when* $\gamma_k = c_k$ $(k = 0, 1, \ldots n)$, *that is when* T *is the n-th partial sum of the Fourier series of* f [1]).

Putting $\gamma_k = c_k$ and taking into account that the integral on the left in (1), is non-negative, we obtain the very important relation

$$\sum_{k=0}^{n} c_k^2 \leqslant \int_a^b f^2\, dx, \tag{2}$$

which is called 'Bessel's inequality'. Since n in (2) is arbitrary, we have also:

$$\sum_{k=0}^{\infty} c_k^2 \leqslant \int_a^b f^2\, dx. \tag{3}$$

For some systems $\{\varphi_n\}$ the sign $\leqslant$ in (3) may be replaced by $=$ and the equation we then obtain is called 'Parseval's relation'.

Since the system $1/\sqrt{2\pi}, (\cos x)/\sqrt{\pi}, (\sin x)/\sqrt{\pi}, \ldots$ is orthogonal and normal, we obtain from (3), using the notation 1.4(1), that

$$\tfrac{1}{2} a_0^2 + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leqslant \frac{1}{\pi} \int_0^{2\pi} f^2 dx, \tag{4}$$

for any f with integrable square.

Corollary. If f^2 is integrable, then $a_k \to 0$, $b_k \to 0$.

1.61. The argument used in § 1.6 shows that, if $\mathfrak{S}[f]$ converges uniformly, in particular, if f is a trigonometrical polynomial, there is equality in (4).

1.7. Remarks on series and integrals. It will be convenient to collect here a few elementary theorems on series and integrals, which will often be used in the sequel. Let $f(x)$ and $g(x) > 0$ be two functions defined for $x > x_0$. We say that $f(x) = o(g(x))$ if $f(x)/g(x) \to 0$ as $x \to \infty$. If $f(x)/g(x)$ is bounded for all x sufficiently large, we write $f(x) = O(g(x))$. The same notation is used when x tends to a finite limit, or to $-\infty$, or even when x tends to its limit through a discrete sequence of values. In particular, an expresion is $o(1)$ or $O(1)$ if it tends to 0 or is bounded, as the case may be.

[1]) Toepler [1].

Two functions $f(x)$ and $g(x)$ will be called asymptotically equal in the neighbourhood of x_0 if $f(x)/g(x) \to 1$ as $x \to x_0$, and we write $f(x) \simeq g(x)$. If there exist two constants $A > 0$, $B > 0$, such that $A \leqslant f(x)/g(x) \leqslant B$ for x sufficiently near x_0, we shall say that f and g are of the same order in the neighbourhood of x_0 and write $f(x) \sim g(x)$. Similar definitions and notations will be used for sequences.

Examples: $x = O(x^2)$ as $x \to \infty$, $x^2 = o(x)$ as $x \to 0$, $\log r = O(|1-r|)$ as $r \to 1$, $n^{-1} = o(1)$ as $n \to \infty$, $n + \sqrt{n} \simeq \sqrt{n}$ as $n \to \infty$, $\exp n \sim \exp(n + \sin n)$ as $n \to \infty$ [1]).

1.71. *Let* $f(x)$ *and* $g(x) > 0$ *be two functions defined for* $a \leqslant x < b$ *and integrable in any interval* $(a, b - \varepsilon)$. *Let* $F(x)$ *and* $G(x)$ *be the integrals of* f, g *over* (a, x). *If* $f(x) = o(g(x))$ *and* $G(x) \to \infty$ *as* $x \to b$, *then* $F(x) = o(G(x))$. Suppose that $|f(x)|/g(x) < \varepsilon/2$ for $a < x_0 \leqslant x < b$. For such values of x we have the inequality

$$|F(x)| \leqslant \int_a^{x_0} |f|\, dt + \int_{x_0}^{x} |f|\, dt \leqslant \int_a^{x_0} |f|\, dt + \frac{\varepsilon}{2} G(x).$$

Since $G(x) \to \infty$, the last sum is less than $\varepsilon G(x)$ for $x \geqslant x_1$ $(x_0 \leqslant x_1 < b)$ and, since ε is arbitrary, the theorem follows.

1.72. In the above theorem the rôle played by a and b can, obviously, be reversed. If $a = 0$, $b = \infty$, it has an analogue for finite sums: *Let* f_n *and* $g_n > 0$ *be two sequences,* $F_n = f_0 + \ldots + f_n$, $G_n = g_0 + \ldots + g_n$. *If* $f_n = o(g_n)$, $G_n \to \infty$, *then* $F_n = o(G_n)$. The proof is essentially the same as for integrals.

1.73. The proof of the following result is still simpler. *If the series* $f_0 + f_1 + \ldots$, $g_0 + g_1 + \ldots$, $g_n > 0$, converge and if, $F_n = f_n + f_{n+1} + \ldots$, $G_n = g_n + g_{n+1} + \ldots$, then $f_n = o(g_n)$ implies $F_n = o(G_n)$.

1.74. *Let* $f(x)$ $(x \geqslant 0)$ *be a positive, finite, monotonic function. Let* $F(x)$ *be the integral of* f *over* $(0, x)$ *and* $F_n = f(0) + f(1) + \ldots + f(n)$. *Then* (i) *if* f *is decreasing,* $F(n) - F_n$ *tends to a finite limit* C, (ii) *if* f *increases, then* $F(n) \leqslant F_n \leqslant F(n) + f(n)$. In order to prove (i) we observe that, from geometrical considerations, we may write $f(k) \leqslant F(k) - F(k-1) \leqslant f(k-1)$ or, what is the same thing, $0 \leqslant F(k) - F(k-1) - f(k) \leqslant f(k-1) - f(k)$, $k = 1, 2, \ldots$. Since

[1]) $\exp x$ means e^x.

the series with terms $f(k-1)-f(k)$ converges, the same may be said of the series with terms $F(k)-F(k-1)-f(k)$ and partial sums $F(n)-F_n+f(0)$.

For example, the difference $1+1/2+\dots+1/n-\log n$ tends to a constant C, usually called Euler's constant.

To obtain (ii) we proceed similarly, summing the inequalities $f(k-1)\leqslant F(k)-F(k-1)\leqslant f(k)$ from $k=1$ to n.

1.741. *If either $f(x)$ decreases and $F(x)\to\infty$, or $f(x)$ increases and $f(x)/F(x)\to 0$, then $F_n\simeq F(n)$.*

1.742. If $f(x)>0$ is decreasing and integrable over $(0,\infty)$, $F(x)$ denotes the integral of f over (x,∞), and $F_n=f(n)+f(n+1)+\dots$, then $0\leqslant F_n-F(n)\leqslant f(n)$. In particular, if $f(x)/F(x)\to 0$, we have $F_n\simeq F(n)$.

In cases when $F(n)$ can be easily obtained, the above theorems give us approximate expressions for F_n.

$$\textit{Examples:}\ \sum_{k=1}^{n} k^{\alpha}\simeq\frac{n^{\alpha+1}}{\alpha+1},\quad \sum_{k=n}^{\infty} k^{-\beta}\simeq\frac{n^{-\beta+1}}{\beta-1}\quad(\alpha>-1,\ \beta>1).$$

1.8. Miscellaneous theorems and examples.

1. Show that $\sin x+{}^1/_2\sin 2x+{}^1/_3\sin 3x+\dots$ converges to $(\pi-x)/2$ in the interior, and to 0 at the ends, of $(0,2\pi)$.

2. Let (i) $f_1(x)$, $|x|\leqslant\pi$, be even, equal to 1 in $(0,h)$ and to 0 in (h,π), $0<h<\pi$, (ii) $f_2(x)$, $|x|\leqslant\pi$, be even, continuous, vanishing in $(2h,\pi)$, $0<h\leqslant\pi/2$, equal to 1 at $x=0$, and linear in $(0,2h)$, (iii) $f_3(x)=\operatorname{sign}x$, $|x|<\pi$, (iv) $\varphi(x)=(\pi-x)/2$, $0<x<2\pi$, (v) $F(x)=\pi[x/2\pi]$ [1]). Show that

$$f_1(x)\sim\frac{2h}{\pi}\left[\tfrac12+\sum_{n=1}^{\infty}\left(\frac{\sin nh}{nh}\right)\cos nx\right],\qquad f_2(x)\sim\frac{2h}{\pi}\left[\tfrac12+\sum_{n=1}^{\infty}\left(\frac{\sin nh}{nh}\right)^2\cos nx\right],$$

$$f_3(x)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1},\qquad \varphi(x)\sim\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$dF(x)\sim\tfrac12+\sum_{n=1}^{\infty}\cos nx,\qquad |\sin x|=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2 nx}{4n^2-1}.$$

3. Let $f(x)\not\equiv 0$ be even, $g(x)\not\equiv 0$ odd, both non-negative in $(0,\pi)$, and let $a_0,a_1,\dots$, $b_1,b_2,\dots$ be the Fourier coefficients of f and g respectively. Show that $|a_m|<a_0$, $|b_n|<nb_1$, $m=1,2,\dots$, $n=2,3,\dots$

[Prove, by induction, the inequality $|\sin nt|\leqslant n|\sin t|$. Carathéodory [1], Rogosinski [1]].

[1]) $[y]$ denotes the integral part of y.

4. Each of the systems $1,\cos x,\cos 2x,\dots$ and $\sin x,\sin 2x,\dots$ is orthogonal and complete in $(0,\pi)$.

5. Let $\{\varphi_n\}$ denote Rademacher's system. Put $\chi_0(t)=1$, $\chi_N(t)=\varphi_{n_1}(t)\varphi_{n_2}(t)\dots\varphi_{n_k}(t)$, if $N=2^{n_1}+2^{n_2}+\dots+2^{n_k}$. Show that the system $\{\chi_N\}$ is orthogonal normal and complete in $(0,1)$.

[If $\int_0^1 f(t)\prod_{k=0}^{n}(1+\varphi_k(x)\varphi_k(t))\,dt=0$ for every x and n, and if F is an integral of f then $F'(x)=0$ at almost every x. The system $\{\chi_N\}$ was first considered by Walsh [1]; see also Kaczmarz [1], Paley [1]].

6. Orthogonal and normal systems may be defined also in spaces of higher dimensions, the interval of integration being replaced by any measurable set. Show that if $\{\varphi_m(x)\}$ and $\{\psi_n(y)\}$ are orthogonal, normal and complete in the intervals $a\leqslant x\leqslant b$, $c\leqslant y\leqslant d$ respectively, then the doubly infinite system $\{\varphi_m(x)\psi_n(y)\}$ is orthogonal, normal and complete in the rectangle R with opposite corners at the points (a,c), (b,d).

[If $\iint_R f(x,y)\varphi_m(x)\psi_n(y)\,dx\,dy=0$ for all m,n, the functions $f_m(y)=\int_a^b f(x,y)\varphi_m(x)\,dx$ vanish for almost every y, and so $f(x,y)$ vanishes almost everywhere on almost every line $y=\text{const}$].