Images of $B(k)$

by

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Abstract. There are a number of classical results concerning the existence of Borel isomorphisms of specified class between the space of irrational numbers and other separable (metric) absolute Borel sets. In this paper we consider analogous results for nonseparable (metric) absolute Borel sets. In particular, after proving a technical decomposition lemma for certain complete spaces, we characterize those complete spaces which are $(0, 1)$ homeomorphs of $B(k)$, and prove that, roughly, every (metric) absolute Borel set of multiplicative class $\alpha+1$ which is of weight k and not $\sigma\mathrm{LW} < k$ is, after the deletion of a "small" subset, a $(0, \alpha+1)$ image of $B(k)$.

1. Preliminaries. All spaces considered are metrizable. The main facts about absolute Borel sets can be found in [5], [6], and [7]. We summarize some of the basic facts here for convenience.

Let P denote one of the phrases "Borel", "Borel of additive class $1 < \alpha < \omega_1$", or "Borel of multiplicative class $1 \leqslant \alpha < \omega_1$". X is said to be *absolutely P* if X is P in every metric space in which it is embedded. This is equivalent to the statement that X is P in some complete space. X is completely metrizable if and only if X is absolutely of multiplicative class 1, i.e., if and only if X is an absolute G_δ-set.

A bijection f from X to Y is called a *generalized homeomorphism of class* (α, β) if whenever V is open in Y, $f^{-1}(V)$ is of additive class α in X and whenever U is open in X, $f(U)$ is of additive class β in Y. Such a map will be called an (α, β) *map* for short. In particular, a $(0, \beta)$ map is continuous.

A space X is said to be of *σ-local weight $< k$* (abbreviated $\sigma\mathrm{LW} < k$), where k is an infinite cardinal, if X is the union of countably many subsets of local weight $< k$. Stone has shown [6, 7] that among absolute Borel sets, weight and $\sigma\mathrm{LW} < k$ are preserved by generalized homeomorphisms.

For k an infinite cardinal, $B(k)$ denotes the product of countably many discrete spaces of cardinal k. In particular, $B(\aleph_0)$ is the space of irrational numbers. In the classical theory of separable Borel sets [5] as well as in the nonseparable case [6, 7], the spaces $B(k)$ play a prominent role. In particular, Stone [7] has shown that an absolute Borel set of weight k is either $\sigma\mathrm{LW} < k$ or else contains a copy of $B(k)$.

2. The main lemma. A space X of weight k is called *weight-homogeneous* if every nonempty open set has weight k. In [1], Comfort and Hager showed that if X is

a weight-homogeneous nonseparable metric space of weight k, then X must contain a closed discrete set of cardinal k. This result is used in the proof of the following decomposition lemma.

LEMMA 1. *Suppose (X, d) is a weight-homogeneous metric space of weight $k > \aleph_0$. Then there exists a sequence of families of well-ordered sets of cardinal k, denoted A_1, $A_2(a_1)$ (for each $a_1 \in A_1$), ..., $A_{n+1}(a_1, ..., a_n)$ (for each n-tuple $(a_1, ..., a_n)$ with $a_1 \in A_1$ and $a_i \in A_i(a_1, ..., a_{i-1})$ for $i \geq 2$), ..., and, for each such n-tuple $(a_1, ..., a_n)$, there exist subsets $F(a_1, ..., a_n)$ and $D(a_1, ..., a_n)$ of X such that:*

1) $X = \bigcup \{D(a_1): a_1 \in A_1\}$,

2) $D(a_1, ..., a_n) = F(a_1, ..., a_n) - \bigcup \{F(a_1, ..., a_{n-1}, a): a \in A_n(a_1, ..., a_{n-1}),$ $a < a_n\}$ *(where $A_1(a_0) = A_1$)*,

3) *for $n > 1$,* $D(a_1, ..., a_{n-1}) = \bigcup \{D(a_1, ..., a_{n-1}, a): a \in A_n(a_1, ..., a_{n-1})\}$,

4) *for each n, the sets $D(a_1, ..., a_n)$ are nonempty pairwise disjoint F_σ-sets and the family $\{D(a_1, ..., a_n): a_n \in A_n(a_1, ..., a_{n-1}), ..., a_2 \in A_2(a_1), a_1 \in A_1\}$ is σ-discrete,*

5) *for each n, $F(a_1, ..., a_n)$ is a closed set of diameter $\leq 1/n$, and, for each $a_n \in A_n(a_1, ..., a_{n-1})$, $\bigcup \{F(a_1, ..., a_{n-1}, a): a \in A_n(a_1, ..., a_{n-1}), a < a_n\}$ is closed in X,*

6) *if $a_1 \in A_1, a_2 \in A_2(a_1), ..., a_n \in A_n(a_1, ..., a_{n-1}), ...,$ then*

$$F(a_1) \supseteq D(a_1) \supseteq F(a_1, a_2) \supseteq D(a_1, a_2) \supseteq ... ,$$

7) *each $D(a_1, ..., a_n)$ has weight k and is weight-homogeneous.*

Proof. Let $\mathfrak{B} = \{B(\alpha, n): \alpha < k, n = 1, 2 ...\}$ be a σ-discrete open base of cardinality k for X, where, for each n, $\{B(\alpha, n): \alpha < k\}$ is discrete. We assume each $B(\alpha, n) \neq \emptyset$. Well-order $\mathfrak{B}$ by "last differences", i.e., $(\alpha, n) < (\beta, m)$ if either $n < m$ or else $n = m$ and $\alpha < \beta$. All sets of n-tuples occurring in the proof are to be ordered in this way.

Since X is weight-homogeneous, X contains a closed discrete set P of cardinality k. For each $x \in X$, pick a set $B(\alpha_x, n_x) \in \mathfrak{B}$ which contains x, has diameter < 1 and such that $\mathrm{cl}B(\alpha_x, n_x) \cap (P - \{x\}) = \emptyset$. Let $F(\alpha_x, n_x) = \mathrm{cl}B(\alpha_x, n_x)$ and $D(\alpha_x, n_x) = F(\alpha_x, n_x) - \bigcup \{F(\alpha_y, n_y): y \in X, (\alpha_y, n_y) < (\alpha_x, n_x)\}$. Clearly,

$$X = \bigcup \{D(\alpha_x, n_x): x \in X\}.$$

The family $A_1 = \{(\alpha_x, n_x): D(\alpha_x, n_x) \neq \emptyset\}$ has cardinality k since, on the one hand, $|\mathfrak{B}| = k$, and, on the other, the family $\{D(\alpha_x, n_x): x \in X\}$ covers X and each of its members meets P in at most one point.

The sets $D(a_1)$ and $F(a_1)$ (for $a_1 \in A_1$) have the required properties. Indeed, (1), (2), (3) and (6) are immediate. If $a_1, a_1' \in A_1$, then $F(a_1) = \mathrm{cl}B(a_1)$ and $F(a_1') = \mathrm{cl}B(a_1')$ for some $B(a_1), B(a_1') \in \mathfrak{B}$. It is easy to check that if $a_1 \neq a_1'$, then $B(a_1) \neq B(a_1')$. From this, and the fact that $\mathfrak{B}$ is σ-discrete, it follows that $\{F(a_1): a_1 \in A_1\}$ and $\{D(a_1): a_1 \in A_1\}$ are σ-discrete families, and then, from the definition of the ordering, that $\bigcup \{F(a): a < a_1\}$ is closed in X for each $a_1 \in A_1$.

The other items in (4) and (5) are now immediate. Finally, if U is open in X and $U \cap D(a_1) \neq \emptyset$, then

$$U \cap D(a_1) = U \cap \mathrm{cl}B(a_1) \cap (X - \bigcup \{F(a): a < a_1\})$$
$$\supseteq U \cap B(a_1) \cap (X - \bigcup \{F(a); a < a_1\}) \neq \emptyset,$$

since $X - \bigcup \{F(a): a < a_1\}$ is open. Therefore $U \cap D(a_1)$ has weight k and $D(a_1)$ is weight-homogeneous.

Assume now that $n \geq 1$ and that we have defined for $i = 1, ..., n$, the families of well ordered sets and the sets $F(a_1, ..., a_i)$ and $D(a_1, ..., a_i)$ as required. Since $D(a_1, ..., a_n)$ is weight-homogeneous of weight k, it contains a (relatively) closed discrete set $P(a_1, ..., a_n)$ of cardinality k. For each $x \in D(a_1, ..., a_n)$, pick an open set U_x in X which contains x and such that

$$U_x \cap D(a_1, ..., a_n) \cap (P(a_1, ..., a_n) - \{x\}) = \emptyset .$$

For each $i = 1, ..., n$, let $A_i = \bigcup \{F(a_1, ..., a_{i-1}, a): a < a_i, a \in A_i(a_1, ..., a_{i-1})\}$ and let F be the closed set $A_1 \cup ... \cup A_n$. Using (5) and (6), we check that $D(a_1, ..., a_n)$ is disjoint from F. Pick an open set O_x in X which contains x and misses F. Now choose a set $B(a_1, ..., a_n, (\alpha_x, n_x))$ from $\mathfrak{B}$ which contains x and such that $\mathrm{cl}B(a_1, ..., a_n, (\alpha_x, n_x)) \subseteq U_x \cap O_x$, and whose diameter is $< 1/(n+1)$. Let $F(a_1, ..., a_n, (\alpha_x, n_x)) = \mathrm{cl}[B(a_1, ..., a_n, (\alpha_x, n_x)) \cap D(a_1, ..., a_n)]$.

Since $D(a_1, ..., a_n) \subseteq F(a_1, ..., a_n)$, and because $F(a_1, ..., a_n, (\alpha_x, n_x)) \subseteq O_x$ (and therefore misses F), it follows from (2) that $F(a_1, ..., a_n, (\alpha_x, n_x)) \subseteq D(a_1, ..., a_n)$. Then

$$F(a_1, ..., a_n, (\alpha_x, n_x)) \subseteq U_x \cap D(a_1, ..., a_n),$$

so

$$F(a_1, ..., a_n, (\alpha_x, n_x)) \cap (P(a_1, ..., a_n) - \{x\}) = \emptyset .$$

Define

$$D(a_1, ..., a_n, (\alpha_x, n_x)) = F(a_1, ..., a_n, (\alpha_x, n_x)) -$$
$$- \bigcup \{F(a_1, ..., a_n, (\alpha_y, n_y)): (\alpha_y, n_y) < (\alpha_x, n_x), y \in D(a_1, ..., a_n)\} .$$

Clearly $D(a_1, ..., a_n) = \bigcup \{D(a_1, ..., a_n, (\alpha_x, n_x)): x \in D(a_1, ..., a_n)\}$. Let

$$A_{n+1}(a_1, ..., a_n) = \{(\alpha_x, n_x): x \in D(a_1, ..., a_n), D(a_1, ..., a_n, (\alpha_x, n_x)) \neq \emptyset\}.$$

As in the case $n = 1$, $|A_{n+1}(a_1, ..., a_n)| = k$.

It is easy to verify that (1), (2), (3) and (6) hold for these sets. As before, we claim that if $(a_1, ..., a_{n+1})$ and $(a_1', ..., a_{n+1}')$ from $A_{n+1}(a_1, ..., a_n)$ are distinct, then $B(a_1, ..., a_{n+1}) \neq B(a_1', ..., a_{n+1}')$. For suppose $a_j < a_j'$ and $a_i = a_i'$ for each $i < j$. Then, since $\mathrm{cl}B(a_1', ..., a_{n+1}') \subseteq O_x$ for some $x \in D(a_1', ..., a_n')$, we get that

$$\mathrm{cl}B(a_1', ..., a_{n+1}') \cap \bigcup \{F(a_1', ..., a_{j-1}', a): a < a_j'\} = \emptyset ;$$

so that, by (6),

$$\emptyset = \mathrm{cl}\,B(a_1', ..., a_{n+1}') \cap F(a_1', ..., a_{j-1}', a_j) \supseteq \mathrm{cl}\,B(a_1', ..., a_{n+1}') \cap F(a_1, ..., a_{n+1}).$$

But $F(a_1, ..., a_{n+1}) \subseteq \mathrm{cl}\,B(a_1, ..., a_{n+1})$, so if $B(a_1, ..., a_{n+1}) = B(a_1', ..., a_{n+1}')$, then $\emptyset = F(a_1, ..., a_{n+1}) = D(a_1, ..., a_{n+1})$, a contradiction. From this and the fact that $\mathfrak{B}$ is σ-discrete, it follows that $\{D(a_1, ..., a_{n+1}): a_{n+1} \in A_n(a_1, ..., a_n), ..., a_1 \in A_1\}$ and $\{F(a_1, ..., a_n, a): a < a_{n+1} \in A_{n+1}(a_1, ..., a_n)\}$ are σ-discrete, and, from the definition of the ordering, that the union of the latter family is closed. The other items in (4) and (5) now follow immediately.

Finally, to verify (7), suppose U is open in X and $U \cap D(a_1, ..., a_{n+1}) \neq \emptyset$. Letting O denote the open set $X - \bigcup \{F(a_1, ..., a_n, a): a < a_{n+1}\}$, we get $\emptyset \neq U \cap D(a_1, ..., a_{n+1}) = U \cap \mathrm{cl}\,[B(a_1, ..., a_{n+1}) \cap D(a_1, ..., a_n)] \cap O$, so $U \cap O \cap B(a_1, ..., a_{n+1}) \cap D(a_1, ..., a_n) \neq \emptyset$; since this is a nonempty open set in $D(a_1, ..., a_n)$ it has weight k by (7) of the induction hypothesis; hence $D(a_1, ..., a_{n+1})$ is weight-homogeneous of weight k.

3. Images of $B(k)$. The following theorem characterizes the complete metric spaces which are $(0, 1)$ images of $B(k)$.

THEOREM 2. *Suppose X is a complete metric space of infinite weight k. Then there exists a $(0, 1)$-map from $B(k)$ onto X if and only if X is weight-homogeneous.*

Proof. If $k = \aleph_0$, then every nonempty open set in X must be dense-in-itself and therefore uncountable. It then follows from a classical theorem [5] that X is a $(0, 1)$ homeomorph of $B(\aleph_0)$.

If $k > \aleph_0$, decompose X according to Lemma 1, making the index sets disjoint. Let $Y = \{(a_1, ..., a_n, ...): a_1 \in A_1, ..., a_n \in A_n(a_1, ..., a_{n-1}), ...\}$. Since each index set has cardinality k, Y, with the "first difference" metric, is homeomorphic to $B(k)$.

For each $y = (a_1, ..., a_n, ...) \in Y$, $\bigcap_{n=1}^{\infty} D(a_1, ..., a_n) \supseteq \bigcap_{n=2}^{\infty} F(a_1, ..., a_n)$ contains a single point $f(y)$. It follows from (1), (3), and (4) of Lemma 1 that f is a bijection from Y to X.

If $y = (a_1, ..., a_n, ...)$ and $y' = (a_1', ..., a_n', ...)$ are at distance $\leq 1/n < 1$ in Y, then $f(y)$ and $f(y')$ are both in $D(a_1, ..., a_{n-1})$ which has diameter $< 1/(n-1)$. Hence f is (uniformly) continuous. Let

$$S_1 = A_1, \quad S_2 = \{(a_1, a_2): a_2 \in A_2(a_1)\}, \quad ...$$
$$..., S_n = \{(a_1, ..., a_n): a_n \in A_n(a_1, ..., a_{n-1}), ..., a_1 \in A_1\}$$

and let $V(a_1, ..., a_n)$ be the (basic) open set in Y consisting of all points with initial coordinates $(a_1, ..., a_n)$. Then, if O is open in Y,

$$O = \bigcup \{V(a_1): a_1 \in S_1'\} \cup ... \cup \bigcup \{V(a_1, ..., a_n): (a_1, ..., a_n) \in S_n'\} \cup ...,$$

where $S_i' \subseteq S_i$. Clearly, $f(V(a_1, ..., a_n)) = D(a_1, ..., a_n)$, so

$$f(O) = \bigcup \{D(a_1): a_1 \in S_1'\} \cup ... \cup \bigcup \{D(a_1, ..., a_n): (a_1, ..., a_n) \in S_n'\} \cup ...$$

But since each family of sets $\{D(a_1, ..., a_n): (a_1, ..., a_n) \in S_n'\}$ is a σ-discrete family of F_σ-sets, it follows that $f(O)$ is an F_σ-set in X; so f is a $(0, 1)$-map.

On the other hand, if X is a complete metric space (or even any absolute Borel set) of weight k, and if $f: B(k) \to X$ is a $(0, \alpha)$-map, then X must be weight homogeneous. Indeed, if O is a nonempty open set in X, then the open set $f^{-1}(O)$ must have weight k. But the absolute Borel sets O and $f^{-1}(O)$ are generalized homeomorphic and therefore have the same weight.

We remark that Stone [6] showed that if X is a complete metric space of weight k, then there is a $(0, 1)$-map from a closed subset A of $B(k)$ onto X. The additional assumption of weight-homogeneity allows us to take $A = B(k)$.

COROLLARY 3. *Suppose X is a complete metric space of infinite weight k. Then there exists a $(0, 1)$-map from $B(k)$ onto X if and only if no nonempty open set in X is $\sigma\mathrm{LW} < k$.*

Proof. If no nonempty open set in X is $\sigma\mathrm{LW} < k$, then X is weight-homogeneous. On the other hand, if X is weight-homogeneous and O is a nonempty open set, then O, by Theorem 2, is generalized homeomorphic to $B(k)$ and therefore is not $\sigma\mathrm{LW} < k$.

Hausdorff showed [3] that every separable absolute Borel set of multiplicative class $\alpha + 1 \geq 2$ in which every point is a condensation point is a $(0, \alpha)$ homeomorph of $B(\aleph_0)$. Since a separable set is $\sigma\mathrm{LW} < \aleph_0$ if and only if it is countable, Corollary 3 generalizes Hausdorff's result, for complete spaces, to the nonseparable case. A further step toward a full generalization of Hausdorff's result is in Theorem 6. Corollary 3 also generalizes the classical result that every uncountable complete separable metric space is, after the deletion of a countable set, a $(0, 1)$ homeomorph of $B(\aleph_0)$.

The following corollaries all generalize classical results which can be found in [5].

COROLLARY 4. *Let X be a metric space of infinite weight k. Then there is a $(0, 1)$ map f from a subset A of $B(k)$ onto X.*

Proof. Let S be the metric space consisting of k copies of $[0, 1]$ identified at O, with the "shortest path" metric, and let P be the product of countably many copies of S. Then P is a completely metrizable, weight-homogeneous space of weight k, and therefore there is a $(0, 1)$-map $g: B(k) \to P$. Since P is universal for metric spaces of weight k [4], we may assume $X \subseteq P$, and put $A = g^{-1}(X)$ and $f = g \,| A$.

COROLLARY 5. *If X and Y are complete, weight-homogeneous metric spaces of weight k, then there is a $(1, 1)$-map $h: X \to Y$.*

Proof. Let $h = f \circ g^{-1}$, where g and f are $(0, 1)$-maps from $B(k)$ to X and Y respectively.

The following theorem moves in the direction of a full generalization of Hausdorff's result. It also generalizes to the nonseparable case and to higher Borel classes the classical result that if X is an uncountable complete separable metric space, then there exists a countable set $C \subseteq X$ and a $(0, 1)$-map of $B(\aleph_0)$ onto $X - C$.

THEOREM 6. *Let X be an absolute Borel set, of multiplicative class $\alpha+1 \geqslant 2$ which has weight k, and is not $\sigma LW < k$. Then there is a set $C \subseteq X$ of additive class α and $\sigma LW < k$ and a $(0, \alpha+1)$-map of $B(k)$ onto $X-C$.*

Proof. By a theorem of Hansell [2], there exists a $(0, \alpha)$-map $g: A \to X$ where A is a closed subset of $B(k)$. Since X is not $\sigma LW < k$, neither is A. Let B be the largest open subset of A which is $\sigma LW < k$ (B, for example, can be described as the set of all points of A which are not in a homeomorph of $B(k)$ in A [7]). Then $A-B$ is a completely metrizable space of weight k in which no nonempty (relatively) open set is $\sigma LW < k$. Hence, by Corollary 3, there is a $(0, 1)$-map $h: B(k) \to A-B$. Then $f = (g|(A-B) \circ h)$ is a $(0, \alpha+1)$-map of $B(k)$ onto $X-C$, where $C = g(B)$. Since B is open in A and is $\sigma LW < k$, C is of additive class α in X and has $\sigma LW < k$.

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