following example. In complex notation $f(E) = 0$ except that $f(0) = 1$, $f(2) = i$ and $f(2i) = 1$.

Note added in proof (19 May 1976). A map $f\colon E \to E$ is a *collineation* COL if fx, fy, fz are colinear whenever x, y, z are colinear. Carter and Vogt jointly and Barnes independently have proved that every COL map with fE containing 4 points no 3 colinear is affine. Earlier it was proved [2] that every continuous COL map with fE not a subset of a line is affine, and this result was used to characterize the maps $f\colon E \to E$ for which there is an $\alpha > 0$ such that $\Delta(fx, fy, fz) \leqslant \alpha\Delta(x, y, z)$ for all $x, y, z \in E$.

Note added in proof (18 October 1977). The conjecture from [1], that any TC self-map of a real or complex finite dimensional Hilbert space has a fixture, has just been proved by Dang Dinh Ang and Le Hoan Hoa.

References

[1] D. E. Daykin and J. K. Dugdale, *Triangle contractive self maps of a Hilbert space*, Fund. Math. 83 (1974), pp. 187–195.

[2] — *Triangle contractive self maps*, Problem 5986, Amer. Math. Monthly 83 (1976), pp. 295-297.

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Fixtures for triangle contractive self maps

by

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Abstract. The Banach contraction principle is generalized in [2] to maps which contract three or more points, giving rise to self mappings of a Hilbert space which have either a fixed point, a fixed line, or both. In this paper the definitions of [2] are extended to a much wider class of mappings. For this larger class it is shown that most of the results of [2] remain true.

Let H be a Hilbert space. For $y, z \in H$, the line $L(y, z)$ passing through y and z is the collection of points $x = \alpha y + \beta z$ for all scalars α, β such that $\alpha + \beta = 1$. For arbitrary points $x, y, z \in H$, let $a = x - y$, $b = y - z$. Then the distance Π of x from $L(y, z)$ is

$$\Pi(x, L(y, z)) = \begin{cases} \|a\| & \text{if } \quad y = z, \\ \dfrac{1}{\|b\|}\sqrt{a^2 b^2 - (a, b)(b, a)} & \text{if } \quad y \neq z. \end{cases}$$

The area of triangle x, y, z written $\Delta(x, y, z)$ will be half the base $\|b\|$ times the height Π, and will be the same whichever side of the triangle is used as the base. (The above terminology appears in [2]. It has been reproduced here for the convenience of the reader.)

Let $f\colon H \to H$. f will be called *generalized triangle expansion bounded*, written GTEB, if there exists a positive constant h such that for every three points $x, y, z \in H$, either

(1) $\Delta(fx, fy, fz) \leqslant h\max\{\Delta(x, y, z), \Delta(fx, fy, z), \tfrac{1}{2}[\Delta(x, fy, z) + \Delta(fx, y, z)]\}$

or

(2) $$\|fx - fy\| \leqslant h\max\{\|x-y\|, \|x-fx\|, \|y-fy\|, \tfrac{1}{2}[\|x-fy\| + \|y-fx\|]\}.$$

and

$$\|fy - fz\| \leqslant h\max\{\|y-z\|, \|y-fy\|, \|z-fz\|, \tfrac{1}{2}[\|y-fz\| + \|z-fy\|]\},$$

and

$$\|fx - fz\| \leqslant h\max\{\|x-z\|, \|x-fx\|, \|z-fz\|, \tfrac{1}{2}[\|x-fz\| + \|z-fx\|]\}.$$

If f is GTEB with $0 < h < 1$, then f is called *generalized triangle contractive*, written f is GTC.

If f satisfies (2) for each triple of points, then f has a unique fixed point p, and iteration from any point leads to p. (See, e.g. [1, p. 21].) In [3] it is shown that (2) is one of the most general definitions for contractive type mappings.

Since the results of this paper extend those of [2] to GTEB or GTC mappings, the same numbering scheme will be used.

LEMMA 1. *Let f be* GTC, *p, q, r fixed points of f. Then they are collinear.*

If $p \neq q$, then f does not satisfy (2). From (1),

$$\Delta(p, q, r) = \Delta(fp, fq, fr) \leqslant h\Delta(p, q, r),$$

which implies $\Delta(p, q, r) = 0$; i.e., p, q, and r are collinear.

LEMMA 2. *Let f be* GTC, *L a line with $x, y \in L$ such that $fx, fy \in L$ and f does not satisfy* (2) *at x, y. Then L is a fixed line. Further, $\{f^n w\} \to L$ for every $w \in H$.*

Let $z \in L$. Then $\Delta(x, y, z) = 0$, and from (1), $\Delta(fx, fy, fz) = 0$, so that $fL(x, y) \subset L(fx, fy) = L(x, y)$ and L is a fixed line.

Since (1) must hold,

$$\tfrac{1}{2}\|fx-fy\| \Pi(f^{n+1}w, L)$$
$$= \Delta(fx, fy, f^{n+1}w)$$
$$\leqslant h\max\{\Delta(x, y, f^n w), \Delta(fx, fy, f^n w), \tfrac{1}{2}[\Delta(x, fy, f^n w)+\Delta(fx, y, f^n w)]\}.$$

By hypothesis, $L(x, y) = L(fx, fy) = L(x, fy) = L(fx, y)$, so that the above inequality becomes

$$\|fx-fy\| \Pi(f^{n+1}w, L) \leqslant hM\Pi(f^n w, L), \tag{3}$$

where $M = \max\{\|x, y\|, \|fx, fy\|, \tfrac{1}{2}[\|x-fy\|+\|y-fx\|]\}$.

If $M = \|fx-fy\|$, then, from (3), $\Pi(f^{n+1}w, L) \leqslant h\Pi(f^n w, L)$. If $M \neq \|fx-fy\|$, then, since f does not satisfy (2) at x, y, $\|fx-fy\| > hM$ so that, from (3), $\Pi(f^{n+1}w, L) < \Pi(f^n w, L)$. In either case $\{\Pi(f^n w, L)\}$ converges. Call the limit d.

Suppose $d > 0$. Then (3) implies $\|fx-fy\| \leqslant hM$. $M = \|fx-fy\|$ leads to a contradiction, since $h < 1$. If $M \neq \|fx-fy\|$, since f does not satisfy (2), we have $\|fx-fy\| \leqslant hM < \|fx-fy\|$, a contradiction.

The proof of Lemmas 3–8 are similar to their counterparts in [2], so they have been omitted. For completeness we list Theorem 2.

THEOREM 2. *Let f be* GTC. *If two different fixed lines of f meet at a point p, then p is a fixed point. If f has no fixed points, then it has at most one fixed line and if it has one such line L then $\{f^n w\} \to L$ for every $w \in H$. If f has exactly one fixed point p, then its fixed lines, if any, all pass through p. If f has two or more fixed points then they all lie on a fixed line L. Moreover, any other fixed line M will intersect L and $\{f^n w\} \to L$ for every $w \in H$.*

Theorem 3 of [2] cannot be extended to GTEB functions.

For an example, define f: $R^2 \to R^2$ by $f(x, y) = (x+1, 2y)$, $(x, y) \neq (0, 0)$, and $f(0, 0) = (0, 0) = q$, say. Then f is not continuous at q and satisfies (2) for $h = 2$. Let $x_n = y_n = 1/n$, $L = L(f(x_1, y_1), q)$. Let $z = (x, x) \in L$. Then $f(z) = (x+1, 2x) \notin L$ for any $x \neq 1$, and L is not a fixed line.

Lemma 9 cannot be extended to GTEB maps. Let $f(x, y) = (x+1, 2y)$, $p = (q, r)$, $r > \sqrt{3}$. Note that if $\|(x, y)-p\| < \delta$, then each point (q, y), with $|y-\frac{1}{3}r| < \frac{1}{3}(r^2-3)^{1/2}$ satisfies $\|f(q, y)-p\| \leqslant \|(q, y)-p\|$. Let $L = L(p, fp)$, $z = (x, y) \in L$. Then $f(z) = (x+1, 2y)$ and $2y \neq r(x+1-q)$, so that L is not a fixed line.

We can prove Lemma 9 for f a GTC map.

LEMMA 9. *Let f be* GTC, *p a point such that every neighborhood of p contains a point x and its image fx. Then either p is a fixed point or $L(p, fp)$ is a fixed line containing fH.*

Suppose $p \neq fp$. Let $L = L(p, fp)$. Suppose also that $z \in L$ but $fz \notin L$. Choose sequences of points $\{x_n\} \to p$ and $\{fx_n\} \to p$. Thus $\Delta(x_n, p, z) \to 0$ but $\Delta(fx_n, fp, fz) \to \Delta(p, fp, fz) > 0$, and $\|x_n - p\| \to 0$ but $\|fx_n - fp\| \to \|p-fp\| > 0$.

If f satisfies (2) for an infinite number of values of n we get $\|fx_n - fp\| \leqslant h\|p-fp\|$, a contradiction. Therefore f satisfies (1) for an infinite number of values of n. But

$$\Delta(fx_n, fp, fz) \leqslant h\max\{\Delta(x_n, p, z), \Delta(fx_n, f, z), \tfrac{1}{2}[\Delta(x_n, fp, z)+\Delta(fx_n, p, z)]\} \to 0,$$

a contradiction.

Lemma 10 remains unchanged, and the proof of Lemma 11 parallels that in [2]. We state Theorems 5 and 6. Their proofs parallel their counterparts in [2].

THEOREM 5. *If f is* GTC *and there is a sequence of points $\{x_n\}$ in a finite dimensional H with $\|x_n - fx_n\| \to 0$ then f has a fixture.*

THEOREM 6. *If H is finite dimensional, f is* GTC *and has a sequence of iterates which converges to a line then f has a fixture.*

Lemma 12 is not true for f a GTC. Let $0 < \varepsilon < \frac{1}{2}$, f: $R^3 \to R^3$ defined by $f(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$, where $\bar{x}$, $\bar{y}$, $\bar{z}$ is ε if the corresponding coordinate lies in $D = [0, \frac{1}{2}]$ and 0 otherwise. Then it can be shown that f satisfies (2) with $h = 2\varepsilon$.

Consider the sequence $\{w_n\} = \{(x_n, y_n, z_n)\}$ defined by $x_{3n} = \frac{1}{2}-1/n$, $y_{3n} = z_{3n} = \frac{1}{2}+1/n$, $x_{3n+1} = z_{3n+1} = y_{3n}$, $y_{3n+1} = x_{3n}$, $x_{3n+2} = y_{3n+2} = y_{3n}$, $z_{3n+2} = x_{3n}$. Then $w_n \to \frac{1}{2}(1, 1, 1)$, and $\|w_n - fw_n\| \to \frac{1}{2}(2+(1-2\varepsilon)^2)^{1/2} > 0$, so that f satisfies the hypotheses of Lemma 12. However, $f(w_{3n}) \to (\varepsilon, 0, 0)$, $f(w_{3n+1}) \to (0, \varepsilon, 0)$ and $f(w_{3n+2}) \to (0, 0, \varepsilon)$, and $\{fw_n\}$ has three distinct cluster points.

Lemma 13 does not hold for f a GTEB. Define f: $R \to R$ by $f(x) = \frac{1}{2}$, $0 \leqslant x \leqslant \frac{1}{2}$, $f(x) = 0$, $|x| > \frac{1}{2}$. Then f satisfies (2) with $h = 1$. Let $x_n = \frac{1}{2}+1/n$. Then $\|x_n - fx_n\| \to \frac{1}{2} > 0$. $\Delta(x_n, fx_n, f^2 x_n) = \Delta(\frac{1}{2}+1/n, 0, \frac{1}{2}) \to 0$, and $\|fx_n - f^2 x_n\| = \frac{1}{2} > 0$. f has no fixed line through $\frac{1}{2}$.

We simply state Theorems 7 and 8 since their proofs parallel those in [2].

THEOREM 7. *If H is finite dimensional, f is* GTC *and $\{x_n\}$ is a sequence of iterates such that* $\liminf \|x_n - x_{n+1}\|$ *is finite and $\Delta(x_n, x_{n+1}, x_{n+2}) \to 0$ then f has a fixture.*

THEOREM 8. *If H is finite dimensional, f has a bounded sequence of iterates and f satisfies* (1) *for all $x, y, z \in H$ where $0 < h < 1$, then f has a fixture.*

References

[1] L. B. Ciric, *Generalized contractions and fixed-point theorems*, Publ. L'Inst. Math. (Beograd) 12 (26) (1971), pp. 19–26.

[2] D. E. Daykin and J. K. Dugdale, *Triangle contractive self maps of a Hilbert space*, Fund. Math. 83 (1974), pp. 187–195.

[3] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. 226 (1977), pp. 257–290.

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Partitions of pairs of reals

by

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Abstract. We prove that there is essentially only one simple counterexample to the partition relation $2^{\aleph_0}\to(2^{\aleph_0})^2_{\aleph_0}$. The partition relation $2^{\aleph_0}\to(2^{\aleph_0})^2_{\aleph_1}$ is also considered, and some independence results concerning it are derived from some known independence results in set theory.

1. Introduction. Despite the title of this paper, we will primarily work with the set ${}^{\omega}2$ of all functions from the set $\omega=\{0,1,2,\ldots\}$ into the two element set $2=\{0,1\}$, rather than work with the real line R itself. The set ${}^{\omega}2$ can be endowed with a topology and a measure in a natural way be regarding it as a countable product of the two element set 2 where 2 is equipped with both the discrete topology and the probability measure that assigns both $\{0\}$ and $\{1\}$ measure one-half. By considering the binary expansion of a real number, it will be clear that all our results stated in terms of ${}^{\omega}2$ carry over to the real line R. We will also identify $[{}^{\omega}2]^2$ with $\{(x,y)\in{}^{\omega}2\times{}^{\omega}2\colon x<y\}$ where $<$ denotes the usual lexicographic ordering. This not only equips $[{}^{\omega}2]^2$ with a topology and a measure, but gives meaning to assertions such as "$A\times B\subseteq[X]^2$". For all relevant topological notions (e.g. analytic set, restricted property of Baire) we refer the reader to [4] or [5].

Our starting point is the following observation of Sierpiński. If we let $<$ be the usual ordering of R and $\lessdot$ be a well ordering of R of type $2^{\aleph_0}$ and define $f\colon [R]^2\to 2$ by declaring that $f(\{x,y\})=0$ iff the two orderings agree on $\{x,y\}$, then there is no uncountable set $X\subseteq R$ that is *homogeneous* for f (i.e. such that f is constant on $[X]^2$). Thus, using the arrow notation of Erdös-Rado [2], this example shows that $2^{\aleph_0}\not\to(\aleph_1)^2_2$. Since this counterexample makes heavy use of the axiom of choice, it is natural to ask if it can be replaced by a constructive counterexample. Silver observed that by combining a special case of a theorem of Mycielski [7] with a special case of a theorem of Galvin (unpulished) one obtains the following.

THEOREM 1.1 (Galvin, Mycielski, Silver). *Suppose $f\colon [{}^{\omega}2]^2\to 2$ and $f^{-1}(\{i\})$ has the property of Baire for all $i<2$. Then there exists a perfect set $P\subseteq{}^{\omega}2$ that is homogeneous for f.*

This theorem was first brought to our attention by Baumgartner, who rediscovered it independently of the work of Galvin, Mycielski and Silver. It has since been rediscovered by Burgess [1] and probably by others as well. Actually, the Galvin–