Observe that a slight modification of the definition of $\underline{\varkappa}(X)$ leads to another topological invariant $\lambda(X)$, which however is not a shape-invariant.

Let X be a compactum lying in a space $M \in \mathrm{AR}$. A compactum $Y \subset X$ is said to be *weakly contractible* in X if it is contractible in every neighborhood of X in M. It is clear that the choice of the space $M \in \mathrm{AR}$ containing X is here immaterial.

Let $\lambda(X)$ denote the number defined as follows:

If there exists a natural number n such that X is the union of n weakly contractible in X compacta, then $\lambda(X)$ denotes the smallest of such numbers n.

If a such natural number n does not exist, then $\lambda(X) = \infty$.

It is clear that $\lambda(X)$ is a topological invariant of X and that

$$\underline{\varkappa}(X) \leqslant \lambda(X) \leqslant \varkappa(X).$$

It follows by (3.1) that if $X \in \mathrm{ANR}$, then $\lambda(X) = \underline{\varkappa}(X)$. However this last relation does not hold true if one omits the hypothesis $X \in \mathrm{ANR}$. In fact, if A denotes the well-known universal plane curve of Sierpiński, then $\lambda(A) = \infty$, because for every finite decomposition $A = A_1 \cup A_2 \cup \ldots \cup A_n$ of A into compacta, at least one of A_i contains a simple closed curve and consequently it is not weakly contractible in A. On the other hand, $\underline{\varkappa}(A) = 2$, because of Theorem (6.1). Observe that this example shows also that $\lambda(X)$ is not a shape invariant of X. In fact, there exist in the plane E^2 two dendrits D_1, D_2 such that $B = D_1 \cup D_2$ is a curve decomposing E^2 into an infinite number of regions. Hence $\mathrm{Sh}(A) = \mathrm{Sh}(B)$ and $\lambda(B) = 2 \neq \lambda(A)$.

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Triangle contractive self maps of the plane

by

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Abstract. Let E be the real Euclidean plane. A map $f\colon E \to E$ is triangle contractive TC if $0<\alpha<1$ and for each $x, y, z \in E$ either

(i) $\|fx-fy\| \leqslant \alpha\|x-y\|$ and $\|fy-fz\| \leqslant \alpha\|y-z\|$ and $\|fz-fx\| \leqslant \alpha\|z-x\|$ or

(ii) $\Delta(fx, fy, fz) \leqslant \alpha\Delta(x, y, z)$ where $\Delta(x, y, z)$ is the area of triangle x, y, z. We prove that every TC map $f\colon E \to E$ has a fixed point $p = fp$ or a fixed line $L \supset fL$.

1. Introduction. Let E be the real Euclidean plane and $f\colon E \to E$. We call $p \in E$ a *fixed point* of f if $fp = p$. Also a line L of E is called a *fixed line* of f if $fL \subset L$. By a fixture of f we mean either a fixed point or a fixed line.

We say that f is *triangle contractive* TC if there is a coefficient α in $0<\alpha<1$ such that for each $x, y, z \in E$ either

(i) $\|fx-fy\| \leqslant \alpha\|x-y\|$ and $\|fy-fz\| \leqslant \alpha\|y-z\|$ and $\|fz-fx\| \leqslant \alpha\|z-x\|$ or

(ii) $\Delta(fx, fy, fz) \leqslant \alpha\Delta(x, y, z)$, where $\Delta(x, y, z)$ denotes the area of the triangle x, y, z. Such maps were discussed in [1] where it was conjectured that every TC self map of a Hilbert space has a fixture. The object of this note is to present

THEOREM 1. *Each triangle contractive self map of the real plane has a fixed point or a fixed line.*

The author would like to thank his friend J. K. Dugdale for the benefit of many helpful discussions.

2. Proof of Theorem 1. Let $f\colon E \to E$ be TC with coefficient α. We will assume f is continuous, because otherwise fE is contained in a fixed line ([1], Theorem 3). Also we will assume that every circle C contains a point w with fw outside C, otherwise f has a fixed point by the Brouwer theorem. So for $n = 1, 2, \ldots$ let w_n be a point inside the circle C_n of radius n centred at the origin with fw_n outside C_n. If the sequence $\{w_n\}$ had an accumulation point q then f would be discontinuous at q. Hence $\{w_n\}$ is unbounded and we can choose a subsequence $\{x_n\}$ of $\{w_n\}$ such that $0<\|x_n\| \to \infty$.

Let us write $\angle x$ for the principal angle subtended at the origin by x. Then $\{\angle x_n\}$ has an accumulation point ψ. We take a subsequence $\{y_n\}$ of $\{x_n\}$ such that

$\angle y_n \to \psi$. Similarly $\{\angle fy_n\}$ has an accumulation point ω and there is a subsequence $\{z_n\}$ of $\{y_n\}$ with $\angle z_n \to \omega$. Thus

(1) $$\|z_n\| \to \infty ,$$

(2) $$\angle z_n \to \psi ,$$

(3) $$\|z_n\| < \|fz_n\| ,$$

(4) $$\angle fz_n \to \omega .$$

We shall let Ψ, Ω denote the lines of angle ψ, ω respectively which pass through the origin and go to infinity in both directions. Our first result is that lines parallel to Ψ contract under f into lines parallel to Ω.

LEMMA 1. *Suppose $\{z_n\}$ is a sequence of points satisfying* (1)-(4). *Then if L is a line parallel to Ψ its image fL is contained in a line cfL parallel to Ω. Further if the distance between two lines L and M parallel to Ψ is d, and the distance between cfL and cfM is e, then $e \leqslant \alpha d$.*

Proof. Let L and M be lines parallel to Ψ distance $d>0$ apart. If $x \in L$ and $y \in M$ for large n we have

$$\|fx - fz_n\| \sim \|fz_n\| > \|z_n\| \sim \|x - z_n\| .$$

Since f is TC we must therefore have $\Delta(fz, fy, fz_n) \leqslant \alpha\Delta(x, y, z_n)$. Let L' and M' be the lines parallel to Ω through fx and fy respectively, and let e be the distance between them. If $e = 0$ then $e \leqslant \alpha d$, while if $e>0$ for large n we have

$$\tfrac{1}{2}e\|z_n\| < \tfrac{1}{2}e\|fz_n\| \sim \tfrac{1}{2}e\|fx - fz_n\| \sim \Delta(fx, fy, fz_n)$$
$$\leqslant \alpha\Delta(x, y, z_n) \sim \tfrac{1}{2}\alpha d\|x - z_n\| \sim \tfrac{1}{2}\alpha d\|z_n\|$$

so again $e \leqslant \alpha d$. Since f is continuous $e \to 0$ as $d \to 0$ and the lemma follows.

Consider now the case when $\Psi = \Omega$ and let χ be a line perpendicular to Ψ. We define a map $g\colon \chi \to \chi$ by letting $g(x)$ be the perpendicular projection of $f(x)$ onto χ for each $x \in \chi$. This map is well defined and contractive because by Lemma 1 we know that f contracts lines parallel to Ψ. Hence by the Banach contraction mapping theorem g has a fixed point p, and the line parallel to Ψ through p is a fixed line of f. We state this formally as

LEMMA 2. *If there is a sequence $\{z_n\}$ of points satisfying* (1)-(4) *with $\Psi = \Omega$ then f has a fixed line.*

Up to this point our argument would apply to any finite dimensional Hilbert space, but we next consider the case $\Psi \neq \Omega$ and will use the fact that non-parallel lines of E intersect. We choose any line L parallel to Ψ and translate our axes till the origin 0 is at the intersection of L and cfL. Thus Ψ becomes L and Ω becomes cfL and $f(0) \in \Omega$. Then we rotate the axes until they bisect the angles between Ψ and Ω.

LEMMA 3. *Suppose $\{z_n\}$ is a sequence of points satisfying* (1)-(4) *with $f(0) \in \Omega \neq \Psi$, and axes of coordinates bisecting the angles between Ψ and Ω. Then there is a connected curve K passing through* 0 *such that $x - fx$ is parallel to Ω for each $x \in K$. Moreover K does not enter the two quadrants containing Ψ and K goes to infinity in both directions.*

Proof. Let L be any line parallel to Ψ. By Lemma 1 we know that cfL is parallel to Ω and so meets L at a point x with $fx \in cfL$. We simply let K be the locus of x as L varies. To show that K is connected we show that x is a continuous function of the distance of L from Ψ. So let L and M be two lines parallel to Ψ with points x and y respectively on K. If the distance between L and M is d, then by Lemma 1 the distance between cfL and cfM is $\leqslant \alpha d$, and hence the distance between x and y is $\leqslant (1+\alpha)d$. In other words x varies continuously with L. Suppose a point x of K lay in a quadrant containing Ψ. Let d, e be the distances from x to Ψ, Ω respectively. Then $d \leqslant e$ contradicting Lemma 1 for Ψ and the line through x parallel to Ψ. This ends the proof of the lemma.

To complete the proof of Theorem 1 we will assume further that f has no fixture and obtain contradictions. We have at least one sequence $\{z_n\}$ satisfying (1)-(4), and we choose a $\{z_n\}$ which makes the acute angle between Ψ and Ω as small as possible. In view of Lemma 2 this angle is not zero.

Consider the curve K of Lemma 3. Suppose $x_1, x_2 \in K$ with the parallel non-zero vectors $x_1 - fx_1$ and $x_2 - fx_2$ pointing in opposite directions. Then since f is continuous, as x moves along K from x_1 to x_2, there must be an x with $x - fx$ of zero magnitude. In other words f would have a fixed point, which we have disallowed. Thus the vectors $(fx) - x$ with $x \in K$ all point the same way, and we will work in the quadrant containing K where they point away from 0. Let P be a line perpendicular to Ω and meeting this quadrant. We proceed to define a point v on P.

Let k, l, m be points at which P meets K, Ω, Ψ respectively. If k lies between l and m then k is our choice for v. So we now deal with the alternative case when l lies between k and m. Suppose we had a point $r \in P$ between m and k where the vector $(fr) - r$ was parallel to Ω but pointing towards the origin side of P. Then we would obtain a contradiction of Lemma 1 for lines L, M parallel to Ψ passing through r, k respectively. The distance between cfL and cfM would be $\|r - k\|$. Hence there is no such point r. We recall that $fm \in \Omega$. Because f is continuous it now follows that as x moves along P from k to m it must come to a point at which $(fx) - x$ points *directly* away from 0. This point is our choice for v.

For $n = 1, 2, \ldots$ let P have distance n from 0 and v_n be the corresponding point v of P. If we chose v between l and m infinitely often then $\{v_n\}$ has an infinite subsequence lying in half the acute angle between Ψ and Ω. Clearly we could choose a subsequence $\{z_n'\}$ of this subsequence to satisfy (1)-(4) and giving Ψ' and Ω' with a sharper acute angle than Ψ and Ω, which is impossible. Hence we chose v between l and m only a finite number of times. All the other v_n have $(fv_n) - v_n$ pointing directly away from 0. Hence we get a subsequence $\{z_n'\}$ of $\{v_n\}$ satisfying (1)-(4) with $\Psi' = \Omega'$, a final contradiction.

Remark. It is natural to consider maps which contract either the area or the perimeter of each triangle. That these do not always have a fixture is seen from the following example. In complex notation $f(E) = 0$ except that $f(0) = 1$, $f(2) = i$ and $f(2i) = 1$.

Note added in proof (19 May 1976). A map $f\colon E \to E$ is a *collineation* COL if fx, fy, fz are colinear whenever x, y, z are colinear. Carter and Vogt jointly and Barnes independently have proved that every COL map with fE containing 4 points no 3 colinear is affine. Earlier it was proved [2] that every continuous COL map with fE not a subset of a line is affine, and this result was used to characterize the maps $f\colon E \to E$ for which there is an $\alpha > 0$ such that $\Delta(fx, fy, fz) \leqslant \alpha \Delta(x, y, z)$ for all $x, y, z \in E$.

Note added in proof (18 October 1977). The conjecture from [1], that any TC self-map of a real or complex finite dimensional Hilbert space has a fixture, has just been proved by Dang Dinh Ang and Le Hoan Hoa.

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Fixtures for triangle contractive self maps

by

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Abstract. The Banach contraction principle is generalized in [2] to maps which contract three or more points, giving rise to self mappings of a Hilbert space which have either a fixed point, a fixed line, or both. In this paper the definitions of [2] are extended to a much wider class of mappings. For this larger class it is shown that most of the results of [2] remain true.

Let H be a Hilbert space. For $y, z \in H$, the line $L(y, z)$ passing through y and z is the collection of points $x = \alpha y + \beta z$ for all scalars α, β such that $\alpha + \beta = 1$. For arbitrary points $x, y, z \in H$, let $a = x - y$, $b = y - z$. Then the distance Π of x from $L(y, z)$ is

$$\Pi(x, L(y, z)) = \begin{cases} \|a\| & \text{if} \quad y = z, \\ \dfrac{1}{\|b\|}\sqrt{a^2 b^2 - (a, b)(b, a)} & \text{if} \quad y \neq z. \end{cases}$$

The area of triangle x, y, z written $\Delta(x, y, z)$ will be half the base $\|b\|$ times the height Π, and will be the same whichever side of the triangle is used as the base. (The above terminology appears in [2]. It has been reproduced here for the convenience of the reader.)

Let $f\colon H \to H$. f will be called *generalized triangle expansion bounded*, written GTEB, if there exists a positive constant h such that for every three points $x, y, z \in H$, either

(1) $\quad \Delta(fx, fy, fz) \leqslant h\max\{\Delta(x, y, z), \Delta(fx, fy, z), \tfrac{1}{2}[\Delta(x, fy, z) + \Delta(fx, y, z)]\}$

or

(2) $$\|fx - fy\| \leqslant h\max\{\|x-y\|, \|x-fx\|, \|y-fy\|, \tfrac{1}{2}[\|x-fy\| + \|y-fx\|]\}.$$

and

$$\|fy - fz\| \leqslant h\max\{\|y-z\|, \|y-fy\|, \|z-fz\|, \tfrac{1}{2}[\|y-fz\| + \|z-fy\|]\},$$

and

$$\|fx - fz\| \leqslant h\max\{\|x-z\|, \|x-fx\|, \|z-fz\|, \tfrac{1}{2}[\|x-fz\| + \|z-fx\|]\}.$$

If f is GTEB with $0 < h < 1$, then f is called *generalized triangle contractive*, written f is GTC.