Proof of Theorem 7.1. Let (X, x_0) be shape deformable into (A, x_0), i.e. there is a shape map $\underline{f}\colon (X, x_0)\to(A, x_0)$ such that

(1) $$\underline{i}\,\underline{f} = \underline{1}_{(X,x_0)}\,.$$

Then, there is an ANR-sequence $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0)$ and a representative $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x}_0)\to(\boldsymbol{A}, \boldsymbol{x}_0)$ of $\underline{f}$, such that

$$\boldsymbol{i}\boldsymbol{f}\simeq \boldsymbol{1}_{(\boldsymbol{X},\boldsymbol{x}_0)}\,,$$

i.e. the system $(\boldsymbol{X}, \boldsymbol{x}_0)$ is deformable into $(\boldsymbol{A}, \boldsymbol{x}_0)$. Hence, by Theorem 7.2,

$$\pi_n(\boldsymbol{A}, \boldsymbol{x}_0)\approx\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)\times\pi_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0) \quad \text{for} \quad n\geqslant 2\,.$$

Passing to inverse limits and applying 1.3, we obtain the condition (a) for shape groups. The condition (b) follows directly by (1) and 4.1. ■

Theorem 7.1 is an analogue of Proposition 5.2, p. 151 [2].

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.
[2] Sze-Tsen Hu, *Homotopy Theory*, 1959.
[3] S. Mardešić, *On the Whitehead theorem in shape theory I, II*, Fund. Math. 91 (1976), pp. 51–64, 93–103.
[4] — *Retracts in shape theory*, Glasnik Mat. 6 (26) (1971), pp. 153–163.
[5] M. Moszyńska, *On shape and fundamental deformation retracts I*, Fund. Math. 75 (1972), pp. 145–167.
[6] — *Uniformly movable compact spaces and their algebraic properties*, Fund. Math. 77 (1972), pp. 125–144.
[7] — *The Whitehead Theorem in the theory of shapes*, Fund. Math. 80 (1973), pp. 221–263.
[8] — *Various approaches to the fundamental groups*, Fund. Math. 78 (1973), pp. 107–118.
[9] T. J. Sanders, *Shape groups and products*, Pacific J. Math. 48 (2) (1973), pp. 485–495.
[10] H. Schubert, *Categories*, 1972.

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On the Lusternik-Schnirelmann category in the theory of shape

by

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Abstract. A modification of the notion of the Lusternik–Schnirelmann category gives a monotonous shape invariant $\underline{\varkappa}(X)$ defined for all compacta X. Some properties of $\underline{\varkappa}(X)$ are established. In particular it is shown that if X is a continuum, then $\underline{\varkappa}(X)\leqslant \mathrm{Fd}(X)+1$, where $\mathrm{Fd}(X)$ denotes the fundamental dimension of X.

1. Coefficients $\varkappa(X)$ and $\underline{\varkappa}_M(X)$. By the Lusternik-Schnirelmann (absolute) category of a compactum X one understands (compare [1], [5] and [7]) the number $\varkappa(X)$ defined as follows:

If there exist natural numbers n such that $X = X_1 \cup X_2 \cup \ldots \cup X_n$, where X_i are (for $i = 1, 2, \ldots, n$) compacta contractible in X, then $\varkappa(X)$ denotes the smallest of such numbers n.

If such natural numbers n do not exist, then $\varkappa(X) = \infty$.

Observe that

(1.1) *If compactum X homotopically dominates compactum Y, then $\varkappa(X)\geqslant\varkappa(Y)$.*

In fact, assume that there exist two maps

$$f\colon X\to Y \quad \text{and} \quad g\colon Y\to X$$

such that fg is homotopic to the identity map $i_Y\colon Y\to Y$. If $\varkappa(X)\leqslant n$, then there exist compacta $X_1, X_2, \ldots, X_n$ such that $X = X_1 \cup X_2 \cup \ldots \cup X_n$ and that for $i = 1, 2, \ldots, n$ there is a homotopy

$$\varphi_i\colon X_i\times\langle 0, 1\rangle\to X$$

satisfying the conditions

$$\varphi_i(x, 0) = x \quad \text{and} \quad \varphi_i(x, 1) = a_i\,,$$

where a_i is a fixed point of X.

Setting $Y_i = g^{-1}(X_i)$ for $i = 1, 2, \ldots, n$, one gets compacta $Y_1, \ldots, Y_n$ such that $Y = Y_1 \cup \ldots \cup Y_n$. It remains to show that Y_i is contractible in Y.

The relation $fg \simeq i_Y$ means that there exists a homotopy

$$\vartheta\colon Y\times\langle 0,1\rangle \to Y$$

such that $\vartheta(y,0) = y$ and $\vartheta(y,1) = fg(y)$ for every $y\in Y$. Setting

$$\psi_i(y,t) = \begin{cases} \vartheta(y,2t) & \text{for } y\in Y_i,\ 0\leqslant t\leqslant \frac{1}{2},\\ f\varphi_i[g(y),2t-1] & \text{for } y\in Y_i,\ \frac{1}{2}\leqslant t\leqslant 1,\end{cases}$$

one gets a homotopy

$$\psi_i\colon Y_i\times\langle 0,1\rangle\to Y$$

such that $\psi_i(y,0) = y$ and $\psi_i(y,1) = f(a_i)$ for every $y\in Y_i$. Hence Y_i is contractible in Y and the proof of (1.1) is finished.

It follows by (1.1) that $\varkappa(X)$ depends only on the homotopy type of X. One sees easily that if X is an ANR-set, then $\varkappa(X)$ is finite.

It is clear that $\varkappa(X)$ is not a shape-invariant (concerning notions belonging to the theory of shape, see [3]), because if X is a continuum which is not arcwise connected, then $\varkappa(X) = \infty$, though there exist continua with trivial shape, which are not arcwise connected.

However it is easy to modify the notion of the Lusternik-Schnirelmann category in order to obtain a shape invariant. In order to do this, consider a space $M\in$ AR containing the given compactum X. Denote by $\underset{\sim}{\varkappa}_M(X)$ the number defined as follows:

If $X = \emptyset$, then $\underset{\sim}{\varkappa}_M(X) = 0$.

If $X\neq\emptyset$ and if there exist natural numbers n such that

(1.2) For every neighborhood U of X in M there exist compacta $X_1, X_2, \ldots, X_n$ contractible in U and such that $X = X_1\cup X_2\cup\ldots\cup X_n$,

then $\underset{\sim}{\varkappa}_M(X)$ denotes the smallest of all such numbers n.

If $X\neq\emptyset$ and if no natural number n satisfies (1, 2), then $\underset{\sim}{\varkappa}_M(X) = \infty$.

2. Coefficient $\underset{\sim}{\varkappa}(X)$. Now let us prove that

(2.1) *If X is a compactum homeomorphic to another compactum Y and if $X\subset M\in$ AR, $Y\subset N\in$ AR, then $\underset{\sim}{\varkappa}_M(X) = \underset{\sim}{\varkappa}_N(Y)$.*

Proof. It suffices to show that if $\underset{\sim}{\varkappa}_M(X)\leqslant n$, where n is a natural number, then $\underset{\sim}{\varkappa}_N(Y)\leqslant n$.

Let $h\colon X\xrightarrow[\text{onto}]{} Y$ be a homeomorphism. Then there exists a map

$$g\colon M\to N$$

such that

$$g(x) = h(x) \quad \text{for every } x\in X.$$

Then for every neighborhood V of Y (in N) there is a neighborhood U of X (in M) such that

$$g(U)\subset V.$$

Since $\underset{\sim}{\varkappa}_M(X)\leqslant n$, there exist compacta $X_1, X_2, \ldots, X_n$ satisfying (1.2). Let

$$\varphi_i\colon X_i\times\langle 0,1\rangle\to U$$

be a homotopy contracting X_i to a point $a_i\in U$, that is such that $\varphi_i(x,0) = x$ and $\varphi_i(x,1) = a_i$ for every $x\in X_i$. Setting

$$Y_i = h(X_i)$$

and

$$\psi_i(y,t) = g\varphi_i(h^{-1}(y),t) \quad \text{for} \quad (y,t)\in Y_i\times\langle 0,1\rangle,$$

one gets compacta $Y_1, Y_2, \ldots, Y_n$ with $Y = Y_1\cup Y_2\cup\ldots\cup Y_n$ and homotopies

$$\psi_i\colon Y_i\times\langle 0,1\rangle\to g(U)\subset V$$

contracting Y_i in V to the point $b_i = g(a_i)$, because

$$\psi_i(y,0) = gh^{-1}(y) = hh^{-1}(y) = y$$

and

$$\psi_i(y,1) = g(a_i)\in V \quad \text{for every } y\in Y_i.$$

Hence $\underset{\sim}{\varkappa}_N(Y)\leqslant\underset{\sim}{\varkappa}_M(X)$ and the proof of (2.1) is finished.

It follows by (2.1) that the number $\underset{\sim}{\varkappa}_M(X)$ does not depend on the choice of the space $M\in$ AR containing X. Thus we can omit in the notation $\underset{\sim}{\varkappa}_M(X)$ the index M writing shortly $\underset{\sim}{\varkappa}(X)$ instead of $\underset{\sim}{\varkappa}_M(X)$. Moreover (2.1) implies that $\underset{\sim}{\varkappa}(X)$ is a topological invariant. Finally let us observe that

(2.2) $\underset{\sim}{\varkappa}(X)\leqslant\varkappa(X)$ *for every compactum X.*

Notice that

(2.3) *The condition $\underset{\sim}{\varkappa}(X) = 1$ characterizes among all compacta X the* FAR*-sets.*

This is a direct consequence of (2.1) and of the fact that a compactum X lying in the Hilbert cube Q is an FAR-set if and only if X is contractible in each of its neighborhoods (in Q). See [3], p. 262.

3. Case of ANR-spaces. Let us próve that

(3.1) *If $X\in$ ANR, then $\underset{\sim}{\varkappa}(X) = \varkappa(X)$.*

By (2.2) it suffices to show that $\underset{\sim}{\varkappa}(X)\geqslant\varkappa(X)$, i.e., to prove that if $X\in$ ANR and if $\underset{\sim}{\varkappa}(X)\leqslant n$ for a natural number n, then $\varkappa(X)\leqslant n$.

Assume that $X\subset M\in$ AR. Then there is a neighborhood U of X in M such that there exists a retraction

$$r\colon U\to X.$$

The inequality $\underset{\sim}{\varkappa}(X)\leqslant n$ implies that there exist compacta $X_1, X_2, \ldots, X_n$ satisfying (1.2). Thus for every $i = 1, 2, \ldots, n$ there are a point $a_i\in U$ and a homotopy

$$\varphi_i\colon X_i\times\langle 0,1\rangle\to U$$

such that $\varphi_i(x, 0) = x$ and $\varphi_i(x, 1) = a_i$ for every $x \in X_i$. Setting $\psi_i = r\varphi_i$, one gets a homotopy

$$\psi_i\colon X_i \times \langle 0, 1\rangle \to X$$

contracting X_i in X to the point $b_i = r(a_i)$. Hence $\underset{\sim}{\varkappa}(X) \leqslant n$ and the proof of (3.1) is finished.

4. $\underset{\sim}{\varkappa}(X)$ as a monotonous shape invariant. The main aim of this note is to prove the following

(4.1) THEOREM. *If X, Y are compacta with* $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$, *then* $\underset{\sim}{\varkappa}(X) \leqslant \underset{\sim}{\varkappa}(Y)$.

Proof. It suffices to show that if n is a natural number such that $\underset{\sim}{\varkappa}(Y) \leqslant n$, then $\underset{\sim}{\varkappa}(X) \leqslant n$.

We may assume that X and Y are subsets of the Hilbert cube Q. The hypothesis $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ implies that there exist two fundamental sequences

$$\underline{f} = \{f_k, X, Y\} \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k, Y, X\}$$

such that

(4.2) $$\underline{\hat{f}}\underline{f} = \{\hat{f}_k f_k, X, X\} \simeq \underline{i} = \{i, X, X\}.$$

Then for any neighborhood U of X in Q there exists an open neighborhood V of Y in Q such that

(4.3) $$\hat{f}_k(V) \subset U \quad \text{for almost all } k.$$

Moreover, there exists a neighborhood $\hat{U} \subset U$ of X in Q such that

(4.4) $$\hat{f}_k f_k/\hat{U} \simeq i/\hat{U} \quad \text{in } U \quad \text{for almost all } k.$$

The hypothesis $\underset{\sim}{\varkappa}(Y) \leqslant n$ implies that there exist compacta $Y_1, \ldots, Y_n$ such that $Y = Y_1 \cup \ldots \cup Y_n$ and that for every $i = 1, 2, \ldots, n$ there are a point $b_i \in V$ and a homotopy

$$\psi_i\colon Y_i \times \langle 0, 1\rangle \to V$$

such that $\psi_i(y, 0) = y$ and $\psi_i(y, 1) = b_i$ for every $y \in Y_i$.

Since V, as open in Q, is an ANR, there exists for every $i = 1, 2, \ldots, n$ a compact neighborhood $W_i \subset V$ of Y_i (in Q) and a homotopy

$$\bar{\psi}_i\colon W_i \times \langle 0, 1\rangle \to V$$

such that $\bar{\psi}_i(y, 0) = y$ and $\bar{\psi}_i(y, 1) = b_i$ for every $y \in W_i$. Then the set $W = \bigcup_{i=1}^{n} W_i$ is a neighborhood of Y in Q. Hence

(4.5) $$f_k(X) \subset W \quad \text{for almost all } k.$$

It follows by (4.3), (4.4) and (4.5) that there is an index k_0 such that

(4.6) $$\hat{f}_{k_0}(V) \subset U, \quad \hat{f}_{k_0} f_{k_0}/\hat{U} \simeq i/\hat{U} \text{ in } U, \quad f_{k_0}(X) \subset W.$$

It is clear that setting, for $i = 1, 2, \ldots, n$:

$$X_i = X \cap f_{k_0}^{-1}(W_i),$$

one gets a system of compacta $X_1, X_2, \ldots, X_n$ such that

$$X = X_1 \cup X_2 \cup \ldots \cup X_n$$

and that

$$f_{k_0}(X_i) \subset W_i \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Since the homotopy $\bar{\psi}_i$ contracts the set W_i in V to the point b_i and since $\hat{f}_{k_0}(V) \subset U$, we infer that setting $a_i = \hat{f}_{k_0}(b_i)$, one gets the relation

(4.7) $$\hat{f}_{k_0} f_{k_0}/X_i \simeq a_i \quad \text{in } U \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Moreover, it follows by (4.6) that

(4.8) $$\hat{f}_{k_0} f_{k_0}/X_i \simeq i/X_i \quad \text{in } U.$$

Relations (4.7) and (4.8) imply that $i/X_i \simeq a_i$ in U, hence the compactum X_i is contractible in U. Thus $\underset{\sim}{\varkappa}(X) \leqslant n$ and the proof of Theorem (4.1) is finished.

(4.9) COROLLARY. *$\underset{\sim}{\varkappa}(X)$ is a monotonous shape invariant.*

5. An addition theorem. Let us prove the following

(5.1) THEOREM. *If $X = Y \cup Z$, where Y and Z are disjoint compacta, then* $\underset{\sim}{\varkappa}(X) = \underset{\sim}{\varkappa}(Y) + \underset{\sim}{\varkappa}(Z)$.

Proof. We may assume that both sets Y and Z are not empty. Then each of them is a retract of X, hence $\mathrm{Sh}(Y) \leqslant \mathrm{Sh}(X)$ and $\mathrm{Sh}(Z) \leqslant \mathrm{Sh}(X)$. It follows by Theorem (4.1) that $\underset{\sim}{\varkappa}(Y) \leqslant \underset{\sim}{\varkappa}(X)$ and $\underset{\sim}{\varkappa}(Z) \leqslant \underset{\sim}{\varkappa}(X)$. We infer that if at least one of the numbers $\underset{\sim}{\varkappa}(Y)$, $\underset{\sim}{\varkappa}(Z)$ is infinite, then $\underset{\sim}{\varkappa}(X) = \underset{\sim}{\varkappa}(Y) + \underset{\sim}{\varkappa}(Z)$.

In order to prove that

(5.2) $$\underset{\sim}{\varkappa}(X) \leqslant \underset{\sim}{\varkappa}(Y) + \underset{\sim}{\varkappa}(Z),$$

we may assume that both numbers $\underset{\sim}{\varkappa}(Y) = m$ and $\underset{\sim}{\varkappa}(Z) = n$ are finite. Assume that $X \subset M \in \mathrm{AR}$ and let U be a neighborhood of X in M. Then U is also a neighborhood of Y and of Z and we infer that there exist compacta $Y_1, \ldots, Y_m, Z_1, \ldots, Z_n$ contractible in U and such that

$$Y = Y_1 \cup Y_2 \cup \ldots \cup Y_m, \qquad Z = Z_1 \cup Z_2 \cup \ldots \cup Z_n.$$

Then $X = Y_1 \cup \ldots \cup Y_m \cup Z_1 \cup \ldots \cup Z_n$ and consequently the inequality (5.2) holds true.

It follows that in the sequel we can assume that the number $k = \underset{\sim}{\varkappa}(X)$ is finite.

Consider a sequence $\{U_\nu\}$, $\nu = 1, 2, \ldots$ of open neighborhoods of X in M shrinking to X. If we cancel in this sequence a finite number of sets, then we may assume that

$$U_\nu = V_\nu \cup W_\nu \quad \text{for} \quad \nu = 1, 2, \ldots,$$

where V_ν is an open neighborhood of Y (in M) and W_ν is an open neighborhood of Z (in M) and that $V_\nu \cap W_\nu = \emptyset$. Clearly $\{V_\nu\}$ shrinks to Y and $\{W_\nu\}$ shrinks to Z.

Since $\underline{\varkappa}(X) = k$, there exists for every $\nu = 1, 2, \ldots$ a system of compacta $X_{\nu 1}, X_{\nu 2}, \ldots, X_{\nu k}$ such that $X_{\nu i}$ is contractible in U_ν for $i = 1, 2, \ldots, k$. It is clear that every set $X_{\nu i}$ is contractible in only one of the sets V_ν and W_ν. Thus we can assume that there exists a natural number m_ν such that $X_{\nu 1}, X_{\nu 2}, \ldots, X_{\nu m_\nu}$ are contractible in V_ν and $X_{\nu, m_\nu+1}, X_{\nu, m_\nu+2}, \ldots, X_{\nu, k}$ are contractible in W_ν. Setting $n_\nu = k - m_\nu$ and $Y_{\nu i} = X_{\nu i}$ for $i = 1, 2, \ldots, m_\nu$ and $Z_{\nu j} = X_{\nu, m_\nu + j}$ for $j = 1, 2, \ldots, n_\nu$ we get a system of compacta

$$Y_{\nu 1}, Y_{\nu 2}, \ldots, Y_{\nu m_\nu}, Z_{\nu 1}, Z_{\nu 2}, \ldots, Z_{\nu n_\nu}$$

such that $m_\nu + n_\nu = k$, $Y = Y_{\nu 1} \cup \ldots \cup Y_{\nu m_\nu}$, $Z = Z_{\nu 1} \cup \ldots \cup Z_{\nu n}$ and that $Y_{\nu i}$ is contractible in V_ν for $i = 1, 2, \ldots, m_\nu$ and $Z_{\nu j}$ is contractible in W_ν for $j = 1, 2, \ldots, n_\nu$.

It is clear that there exist natural numbers m and n such that $m + n = k$ and that for an increasing sequence of indices $\nu_1 < \nu_2 < \ldots$

$$m_{\nu_l} = m, \quad n_{\nu_l} = n \quad \text{for} \quad l = 1, 2, \ldots$$

If we recall that the sequence of neighborhoods $\{V_{\nu_l}\}$ shrinks to Y and the sequence of neighborhoods $\{W_{\nu_l}\}$ shrinks to Z, we infer that $\underline{\varkappa}(Y) \leqslant m$ and $\underline{\varkappa}(Z) \leqslant n$. Hence

$$\underline{\varkappa}(Y) + \underline{\varkappa}(Z) \leqslant k = \underline{\varkappa}(X)$$

and the proof of Theorem (5.1) is finished.

A simple consequence of Theorem (5.1) is that if X has an infinite number of components, then $\underline{\varkappa}(X) = \infty$. Thus one gets the following

(5.3) Corollary. *If $\{X_\mu\}$ is the class of all components of a compactum X, then $\underline{\varkappa}(X) = \sum_\mu \underline{\varkappa}(X_\mu)$.*

6. A relation between $\underline{\varkappa}(X)$ and $\mathrm{Fd}(X)$**.** We denote by $\mathrm{Fd}(X)$ the fundamental dimension of X. Let us prove the following

(6.1) Theorem. *If X is a continuum, then $\underline{\varkappa}(X) \leqslant \mathrm{Fd}(X) + 1$.*

Proof. We can assume that $\mathrm{Fd}(X) = n$ is finite. Then there exists an n-dimensional compactum Y such that $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. We may assume that Y lies in the Hilbert cube Q. If V is a neighborhood of Y in Q, then there is a homotopy

$$\varphi\colon\ Y \times \langle 0, 1\rangle \to V$$

such that $\varphi(y, 0) = y$ for every $y \in Y$ and that $P = \varphi(Y, 1)$ is a connected polyhedron of dimension $\leqslant n$. One knows [2] that there exists a system of contractible in itself polyhedra $P_1, P_2, \ldots, P_{n+1}$ such that

$$P = P_1 \cup P_2 \cup \ldots \cup P_{n+1}.$$

Let Y_i denote (for $i = 1, 2, \ldots, n+1$) the set consisting of all points $y \in Y$ such that $\varphi(y, 1) \in P_i$. It is clear that $Y_1, Y_2, \ldots, Y_{n+1}$ are compacta such that

$$Y = Y_1 \cup Y_2 \cup \ldots \cup Y_{n+1}$$

and that the homotopy

$$\varphi/(Y_i \times \langle 0, 1\rangle)\colon\ Y_i \times \langle 0, 1\rangle \to V$$

carries Y_i to the polyhedron $P_i \subset V$, Since P_i is contractible in itself, we conclude that Y_i is contractible in V. Hence $\underline{\varkappa}(X) \leqslant \underline{\varkappa}(Y) \leqslant n+1$ and the proof of Theorem (6.1) is finished.

7. Final remarks. As has been proved by L. Schnirelmann ([7], p. 134), if T_n denotes the Cartesian product of n circles, then $\varkappa(T_n) = n+1$. Since $T_n \in \mathrm{ANR}$, we infer by (3.1) that $\underline{\varkappa}(T_n) = n+1$. Since $\mathrm{Fd}(T_n) = n$, we infer that for every natural number n, $\underline{\varkappa}(T_n) = \mathrm{Fd}(T_n) + 1$. This shows that the inequality given in Theorem (6.1) can not be replaced by any more restrictive one.

Consider now in the Hilbert cube Q a sequence $\{X_n\}$, where X_n is homeomorphic to T_n, the diameters of X_n converge to zero and there exists a point c such that for $m \neq n$ the common part of X_m and of X_n consist of only one point c. It is clear that the set

$$X = \bigcup_{n=1}^{\infty} X_n$$

is an infinite-dimensional continuum such that X_n is a retract of X for every $n = 1, 2, \ldots$ Hence $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(X_n)$ and we infer by Theorem (4.1) that

$$\underline{\varkappa}(X) \geqslant \underline{\varkappa}(X_n) = n+1 \quad \text{for} \quad n = 1, 2, \ldots$$

Hence X is a continuum for which $\underline{\varkappa}(X) = \infty$.

Observe that

(7.1) *The homology groups and the fundamental group of a compactum X do not determine $\underline{\varkappa}(X)$.*

In fact, the well known Case-Chamberlin curve (see [4]) X is a compactum for which all homology groups and also the fundamental group are trivial, but X is non-movable (see [6], p. 653), hence X is not an FAR-set, and we infer by (2.3) that $\underline{\varkappa}(X) > 1$.

(7.2) Problem. *Do there exist two movable compacta X and Y such that:*

1^0 *All homology properties of X are the same as homology properties of Y.*

2^0 *There exists a one-to-one function $f\colon X \to Y$ such that for every point $x \in X$ and for every $n = 0, 1, \ldots$ the fundamental group $\underline{\pi}_n(X, x)$ is isomorphic to the fundamental group $\underline{\pi}_n(Y, f(x))$.*

3^0 $\underline{\varkappa}(X) \neq \underline{\varkappa}(Y)$?

Observe that a slight modification of the definition of $\underline{\varkappa}(X)$ leads to another topological invariant $\lambda(X)$, which however is not a shape-invariant.

Let X be a compactum lying in a space $M \in \mathrm{AR}$. A compactum $Y \subset X$ is said to be *weakly contractible* in X if it is contractible in every neighborhood of X in M. It is clear that the choice of the space $M \in \mathrm{AR}$ containing X is here immaterial.

Let $\lambda(X)$ denote the number defined as follows:

If there exists a natural number n such that X is the union of n weakly contractible in X compacta, then $\lambda(X)$ denotes the smallest of such numbers n.

If a such natural number n does not exist, then $\lambda(X) = \infty$.

It is clear that $\lambda(X)$ is a topological invariant of X and that

$$\underline{\varkappa}(X) \leqslant \lambda(X) \leqslant \varkappa(X).$$

It follows by (3.1) that if $X \in \mathrm{ANR}$, then $\lambda(X) = \underline{\varkappa}(X)$. However this last relation does not hold true if one omits the hypothesis $X \in \mathrm{ANR}$. In fact, if A denotes the well-known universal plane curve of Sierpiński, then $\lambda(A) = \infty$, because for every finite decomposition $A = A_1 \cup A_2 \cup \ldots \cup A_n$ of A into compacta, at least one of A_i contains a simple closed curve and consequently it is not weakly contractible in A. On the other hand, $\underline{\varkappa}(A) = 2$, because of Theorem (6.1). Observe that this example shows also that $\lambda(X)$ is not a shape invariant of X. In fact, there exist in the plane E^2 two dendrits D_1, D_2 such that $B = D_1 \cup D_2$ is a curve decomposing E^2 into an infinite number of regions. Hence $\mathrm{Sh}(A) = \mathrm{Sh}(B)$ and $\lambda(B) = 2 \neq \lambda(A)$.

References

[1] K. Borsuk, *Über den Lusternik–Schnirelmann Begriff der Kategorie*, Fund. Math. 26 (1936), pp. 123–136.

[2] — *Sur la décomposition des polyèdres n-dimensionnels en polyèdres contractiles en soi*, Comp. Math. 3 (1936), pp. 431–434.

[3] — *Theory of Shape*, Warszawa 1975, p. 379.

[4] J. H. Case and E. R. Chamberlin, *Characterization of tree-like continua*, Pacific J. Math. 10 (1960), pp. 73–84.

[5] L. Lusternik, *Topologische Grundlagen der allgemeinen Eigenwerttheorie*, Monatsh. Math. Phys. 37 (1930), pp. 125–130.

[6] S. Mardešić and J. Segal, *Movable compacta and* ANR-*systems*, Bull. Acad. Polon. Sci. 18 (1970), pp. 649–654.

[7] L. Schnirelmann, *Über eine neue kombinatorische Invariante*, Monatsh. Math. Phys. 37 (1930), pp. 131–134.

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Triangle contractive self maps of the plane

by

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Abstract. Let E be the real Euclidean plane. A map $f\colon E \to E$ is triangle contractive TC if $0<\alpha<1$ and for each $x, y, z \in E$ either

(i) $\|fx-fy\| \leqslant \alpha\|x-y\|$ and $\|fy-fz\| \leqslant \alpha\|y-z\|$ and $\|fz-fx\| \leqslant \alpha\|z-x\|$ or

(ii) $\Delta(fx, fy, fz) \leqslant \alpha\Delta(x, y, z)$ where $\Delta(x, y, z)$ is the area of triangle x, y, z. We prove that every TC map $f\colon E \to E$ has a fixed point $p = fp$ or a fixed line $L \supset fL$.

1. Introduction. Let E be the real Euclidean plane and $f\colon E \to E$. We call $p \in E$ a *fixed point* of f if $fp = p$. Also a line L of E is called a *fixed line* of f if $fL \subset L$. By a fixture of f we mean either a fixed point or a fixed line.

We say that f is *triangle contractive* TC if there is a coefficient α in $0<\alpha<1$ such that for each $x, y, z \in E$ either

(i) $\|fx-fy\| \leqslant \alpha\|x-y\|$ and $\|fy-fz\| \leqslant \alpha\|y-z\|$ and $\|fz-fx\| \leqslant \alpha\|z-x\|$ or

(ii) $\Delta(fx, fy, fz) \leqslant \alpha\Delta(x, y, z)$, where $\Delta(x, y, z)$ denotes the area of the triangle x, y, z. Such maps were discussed in [1] where it was conjectured that every TC self map of a Hilbert space has a fixture. The object of this note is to present

THEOREM 1. *Each triangle contractive self map of the real plane has a fixed point or a fixed line.*

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2. Proof of Theorem 1. Let $f\colon E \to E$ be TC with coefficient α. We will assume f is continuous, because otherwise fE is contained in a fixed line ([1], Theorem 3). Also we will assume that every circle C contains a point w with fw outside C, otherwise f has a fixed point by the Brouwer theorem. So for $n = 1, 2, \ldots$ let w_n be a point inside the circle C_n of radius n centred at the origin with fw_n outside C_n. If the sequence $\{w_n\}$ had an accumulation point q then f would be discontinuous at q. Hence $\{w_n\}$ is unbounded and we can choose a subsequence $\{x_n\}$ of $\{w_n\}$ such that $0<\|x_n\| \to \infty$.

Let us write $\angle x$ for the principal angle subtended at the origin by x. Then $\{\angle x_n\}$ has an accumulation point ψ. We take a subsequence $\{y_n\}$ of $\{x_n\}$ such that