Concerning the shape groups of compact metric spaces

by

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Abstract. The paper concerns with the shape groups of the product and of the one-point union of two metric compact spaces. Also some relationships between the shape groups of a space (X, x_0) and its shape retract, and between the shape groups of a space (A, x_0) and a space shape deformable to (A, x_0) are established. All the results are analogues of classical theorems of homotopy theory.

The purpose of this paper is to establish some statements concerning the shape groups of metric compacta (see [1], [8], [9]). These statements are analogues of the well known theorems of homotopy theory.

The paper is devided into two parts. The first one, "Shape groups of the product" (§§ 1, 2), concerns with the multiplicativity of the functor of nth shape groups (Corollary 2.2). The second one, "One-point union, shape retracts and shape deformability" (§§ 3-7), concerns with the shape groups of the one-point union (Theorem 5.1) and with the relationships between the shape groups of (X, x_0) and (A, x_0) in two cases: for (A, x_0) being a shape retract of (X, x_0) (Theorem 6.1), and for (X, x_0) shape deformable into (A, x_0) (Theorem 7.1).

Each part is based upon some statements of category theory (§ 1 and §§ 3, 4). All these statements are given here in details, though some of them may be found in the literature, in the form more or less adequate to our purpose.

For any category $\mathcal{K}$, the category pro-$\mathcal{K}$ is understood here as the quotient category $\mathcal{K}^*|_{\cong}$, with $\mathcal{K}^*$ being the category of inverse sequences in $\mathcal{K}$ and with $\cong$ being the similarity relation (see e.g. [6]) (¹); (thus this is a subcategory of a pro-category in the usual sense). Morphisms of pro-$\mathcal{K}$ are denoted by $[f], [g], \ldots$ Morphisms of the shape category by $\underline{f}, \underline{g}, \ldots$

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Shape groups of the product

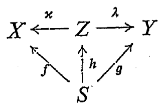
1. Products in pro-categories. Product in an arbitrary category $\mathcal{K}$ will be understood as usualy, i.e. an object Z is a *product* of X and Y whenever there exist two morphisms (*product morphisms*)

$$\varkappa\colon Z \to X \quad \text{and} \quad \lambda\colon Z \to Y$$

(¹) In [6] the symbol $\hat{\mathfrak{K}}^*$ was used for pro-$\mathfrak{K}$.

satisfying the following condition:

(i) For every pair of morphisms in $\mathscr{K}$, $f\colon S\to X$ and $g\colon S\to Y$, there is a unique morphism $h\colon S\to Z$ such that the diagram

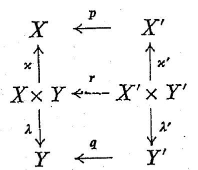


is commutative.

Then Z is determined uniquely up to isomorphism and denoted by $X\times Y$. It is easy to see that $\varkappa$ and λ are also determined uniquely up to isomorphism.

Condition (i) implies

(ii) For any pair of morphisms, $p\colon X'\to X$ and $q\colon Y'\to Y$, there is a unique morphism $r\colon X'\times Y'\to X\times Y$ such that the diagram



is commutative.

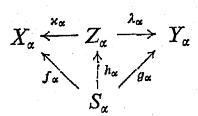
Then r is determined uniquely up to isomorphism and denoted by $p\times q$.

Let us prove the following

1.1. Proposition. *Let $X=(X_\alpha, p_\alpha^{\alpha'})$ and $Y=(Y_\alpha, q_\alpha^{\alpha'})$ be two objects of* pro-$\mathscr{K}$. *If $Z_\alpha = X_\alpha\times Y_\alpha$ and $r_\alpha^{\alpha'} = p_\alpha^{\alpha'}\times q_\alpha^{\alpha'}$ for $\alpha'\geqslant\alpha$, then $Z=(Z_\alpha, r_\alpha^{\alpha'})$ is a product of X and Y in* pro-$\mathscr{K}$, *i.e. $Z = X\times Y$. Moreover,*

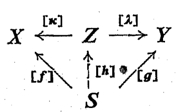
(a) *If $\varkappa_\alpha\colon Z_\alpha\to X_\alpha$ and $\lambda_\alpha\colon Z_\alpha\to Y_\alpha$ are product morphisms for every α, then $\boldsymbol{\kappa}=(1,\varkappa_\alpha)\colon Z\to X$ and $\boldsymbol{\lambda}=(1,\lambda_\alpha)\colon Z\to Y$ are special morphisms* (2) *in $\mathscr{K}^*$, which represent product morphisms $[\boldsymbol{\kappa}]$ and $[\boldsymbol{\lambda}]$ in* pro-$\mathscr{K}$.

(b) *Let $f\colon S\to X$ and $g\colon S\to Y$ be two special morphisms in $\mathscr{K}^*$, $f=(1,f_\alpha)$ and $g=(1,g_\alpha)$. Let h_α be the unique morphisms in $\mathscr{K}$, for which the diagram*



commutes.

Then $h=(1,h_\alpha)\colon S\to Z$ is a special morphism which represents the unique morphism $[h]$ in pro-$\mathscr{K}$ *such that the diagram*



is commutative.

(2) For the notion of special morphism (called also an ordinary morphism), see [3] I and [7].

Proof. Let $\varkappa_\alpha\colon Z_\alpha\to X_\alpha$ and $\lambda_\alpha\colon Z_\alpha\to Y_\alpha$ be product morphisms. Then, by (ii), $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$ defined as

$$\boldsymbol{\kappa}=(1,\varkappa_\alpha),\quad \boldsymbol{\lambda}=(1,\lambda_\alpha)$$

are special morphisms in $\mathscr{K}^*$, i.e.

(1) $$\varkappa_\alpha r_\alpha^{\alpha'} = p_\alpha^{\alpha'}\varkappa_{\alpha'} \quad\text{and}\quad \lambda_\alpha r_\alpha^{\alpha'} = q_\alpha^{\alpha'}\lambda_{\alpha'} \quad\text{for}\quad \alpha'\geqslant\alpha.$$

Take an object $S=(S_\alpha, s_\alpha^{\alpha'})$ and two special morphisms of $\mathscr{K}^*$, $f=(1,f_\alpha)\colon S\to X$ and $g=(1,g_\alpha)\colon S\to Y$. Since $Z_\alpha = X_\alpha\times Y_\alpha$, there is a unique $h_\alpha\colon S_\alpha\to Z_\alpha$ such that

(2) $$\varkappa_\alpha h_\alpha = f_\alpha \quad\text{and}\quad \lambda_\alpha h_\alpha = g_\alpha.$$

Since (2) and (1) imply

$$\varkappa_\alpha(h_\alpha s_\alpha^{\alpha'}) = f_\alpha s_\alpha^{\alpha'} = p_\alpha^{\alpha'} f_{\alpha'} = p_\alpha^{\alpha'}\varkappa_{\alpha'} h_{\alpha'} = \varkappa_\alpha(r_\alpha^{\alpha'} h_{\alpha'})$$

and

$$\lambda_\alpha(h_\alpha s_\alpha^{\alpha'}) = g_\alpha s_\alpha^{\alpha'} = q_\alpha^{\alpha'} g_{\alpha'} = q_\alpha^{\alpha'}\lambda_{\alpha'} h_{\alpha'} = \lambda_\alpha(r_\alpha^{\alpha'} h_{\alpha'}),$$

one gets

(3) $$h_\alpha s_\alpha^{\alpha'} = r_\alpha^{\alpha'} h_{\alpha'},$$

whence h is a special morphism in $\mathscr{K}^*$. The condition (2) implies

(4) $$\boldsymbol{\kappa}h\cong f \quad\text{and}\quad \boldsymbol{\lambda}h\cong g.$$

Let us prove the uniqueness of $[h]$. Take an $h'=(\xi, h'_\alpha)\colon S\to Z$ and suppose that

(4)′ $$\boldsymbol{\kappa}h'\cong f \quad\text{and}\quad \boldsymbol{\lambda}h'\cong g.$$

Then, for each α there is an $\alpha'\geqslant\alpha, \xi(\alpha)$ such that

(5) $$\varkappa_\alpha h'_\alpha s_{\xi(\alpha)}^{\alpha'} = f_\alpha s_\alpha^{\alpha'} \quad\text{and}\quad \lambda_\alpha h'_\alpha s_{\xi(\alpha)}^{\alpha'} = g_\alpha s_\alpha^{\alpha'};$$

i.e. for $\hat f_\alpha = f_\alpha s_\alpha^{\alpha'}$ and $\hat g_\alpha = g_\alpha s_\alpha^{\alpha'}$ there are two morphisms

$$h_\alpha s_\alpha^{\alpha'}\colon S_{\alpha'}\to Z_\alpha \quad\text{and}\quad h'_\alpha s_{\xi(\alpha)}^{\alpha'}\colon S_{\alpha'}\to Z_\alpha$$

such that

$$\varkappa_\alpha(h_\alpha s_\alpha^{\alpha'}) = \hat f_\alpha \quad\text{and}\quad \lambda_\alpha(h_\alpha s_\alpha^{\alpha'}) = \hat g_\alpha$$

and (by (5))

$$\varkappa_\alpha(h'_\alpha s_{\xi(\alpha)}^{\alpha'}) = \hat f_\alpha \quad\text{and}\quad \lambda_\alpha(h'_\alpha s_{\xi(\alpha)}^{\alpha'}) = \hat g_\alpha.$$

Thus $h_\alpha s_\alpha^{\alpha'} = h'_\alpha s_{\xi(\alpha)}^{\alpha'}$, i.e. $h'\cong h$. Hence $[h]$ is a unique morphism in pro-$\mathscr{K}$ such that

$$[\boldsymbol{\kappa}][h] = [f] \quad\text{and}\quad [\boldsymbol{\lambda}][h] = [g].$$

This completes the proof of (b).

For any pair of morphisms $f\colon S\to X$ and $g\colon S\to Y$ there exist an object S', isomorphism $i\colon S\to S'$ and special morphisms $f'\colon S'\to X$ and $g'\colon S'\to Y$ such that $f=f'i$ and $g=g'i$. Thus, to prove (a), it suffices to show that for every object

$S = (S_\alpha, s_\alpha^{\alpha'})$ and every pair of special morphisms, $f = (1, f_\alpha)\colon S \to X$ and $g = (1, g_\alpha)\colon S \to Y$, there is a unique (up to $\cong$) $h\colon S \to Z$ satisfying (4). Since for every α there is a unique $h_\alpha\colon S_\alpha \to Z_\alpha$ such that

$$\varkappa_\alpha h_\alpha = f_\alpha \quad \text{and} \quad \lambda_\alpha h_\alpha = g_\alpha ,$$

the existence and uniqueness of $[h]$ follows immediately by (b). Thus the proof is complete. ■

Consider now a covariant functor $\pi\colon \mathscr{K} \to \mathscr{L}$. The functor π is said to be *multiplicative* whenever

(iii) $\pi(X \times Y) = \pi(X) \times \pi(Y)$ for every pair of objects X, Y in $\mathscr{K}$

and

(iv) if $\varkappa$ and λ are product morphisms for $X \times Y$ then $\pi(\varkappa)$ and $\pi(\lambda)$ are product morphisms for $\pi(X) \times \pi(Y)$.

As known, any covariant functor $\pi\colon \mathscr{K} \to \mathscr{L}$ generates a covariant functor pro-π: pro-$\mathscr{K} \to$ pro-$\mathscr{L}$ [3]:

for any object $X = (X_\alpha, p_\alpha^{\alpha'})$ of pro-$\mathscr{K}$

$$\text{pro-}\pi(X) = (\pi(X_\alpha), \pi(p_\alpha^{\alpha'})) ,$$

if a morphism $[f]$ in pro-$\mathscr{K}$ is represented by $f = (\varphi, f_\alpha)\colon X \to Y$, then pro-$\pi\,[f]$ is represented by $(\varphi, \pi(f_\alpha))\colon \text{pro-}\pi(X) \to \text{pro-}\pi(Y)$.

Let us prove

1.2. Proposition. *If a functor $\pi\colon \mathscr{K} \to \mathscr{L}$ is multiplicative, then the functor pro-π: pro-$\mathscr{K} \to$ pro-$\mathscr{L}$ is also multiplicative.*

Proof. Consider a multiplicative functor $\pi\colon \mathscr{K} \to \mathscr{L}$. Take two objects $X = (X_\alpha, p_\alpha^{\alpha'})$ and $Y = (Y_\alpha, q_\alpha^{\alpha'})$ in pro-$\mathscr{K}$, and let

$$\varkappa_\alpha\colon X_\alpha \times Y_\alpha \to X_\alpha \quad \text{and} \quad \lambda_\alpha\colon X_\alpha \times Y_\alpha \to Y_\alpha$$

be product morphisms in $\mathscr{K}$. Then, by Proposition 1.1, the product of X and Y in pro-$\mathscr{K}$ is of the form

$$X \times Y = (X_\alpha \times Y_\alpha, p_\alpha^{\alpha'} \times q_\alpha^{\alpha'}) ,$$

and the product morphisms are represented by

$$\kappa = (1, \varkappa_\alpha) \quad \text{and} \quad \boldsymbol{\lambda} = (1, \lambda_\alpha) .$$

Let $r_\alpha^{\alpha'} = p_\alpha^{\alpha'} \times q_\alpha^{\alpha'}$. By (iii), for every α

(1) $$\pi(X_\alpha \times Y_\alpha) = \pi(X_\alpha) \times \pi(Y_\alpha)$$

and

(2) $\pi(\varkappa_\alpha)$ and $\pi(\lambda_\alpha)$ are product morphisms in $\mathscr{L}$.

[3] In the sequel we shall use the same symbol π for pro-π.

Thus, by 1.1,

(3) $$\text{pro-}\pi(X \times Y) = \text{pro-}\pi(X) \times \text{pro-}\pi(Y)$$

and, by 1.1 (a),

(4) pro-$\pi[\varkappa]$ and pro-$\pi[\lambda]$ are product morphisms in pro-$\mathscr{L}$.

Hence, by (iv), pro-π is multiplicative. ■

Assume now $\mathscr{K}$ to be a category with inverse limits and products. Let us prove the following

1.3. Proposition. *The functor $\underline{\lim}$: pro-$\mathscr{K} \to \mathscr{K}$ is multiplicative.*

Proof. Let $X = \underline{\lim}\, \boldsymbol{X}$ and $Y = \underline{\lim}\, \boldsymbol{Y}$ and let $\boldsymbol{p}\colon X \to \boldsymbol{X}$ and $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$ be projections. Take product morphisms

$$[\kappa]\colon \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}, \quad [\lambda]\colon \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Y} \quad \text{in pro-}\mathscr{K}$$

and

$$\varkappa\colon X \times Y \to X, \quad \lambda\colon X \times Y \to Y \quad \text{in } \mathscr{K} .$$

Let $\boldsymbol{r} = \boldsymbol{p} \times \boldsymbol{q}\colon X \times Y \to \boldsymbol{X} \times \boldsymbol{Y}$; then, by (ii), the diagram

(1) $$\begin{array}{ccccc} X & \xleftarrow{\varkappa} & X \times Y & \xrightarrow{\lambda} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\ \boldsymbol{X} & \xleftarrow[\kappa]{} & \boldsymbol{X} \times \boldsymbol{Y} & \xrightarrow[\boldsymbol{\lambda}]{} & \boldsymbol{Y} \end{array}$$ is commutative up to $\cong$.

In order to prove $X \times Y$ to be the inverse limit of $\boldsymbol{X} \times \boldsymbol{Y}$, it suffices to show that for every morphism $\boldsymbol{h}\colon S \to \boldsymbol{X} \times \boldsymbol{Y}$ in $\mathscr{K}^*$ there is a unique $h\colon S \to X \times Y$ in $\mathscr{K}$ such that $\boldsymbol{r}h \cong \boldsymbol{h}$.

Take $\boldsymbol{h}\colon S \to \boldsymbol{X} \times \boldsymbol{Y}$ and let

(2) $$\boldsymbol{f} = \kappa \boldsymbol{h}\colon S \to \boldsymbol{X} \quad \text{and} \quad \boldsymbol{g} = \boldsymbol{\lambda} \boldsymbol{h}\colon S \to \boldsymbol{Y} .$$

Since $X = \underline{\lim}\, \boldsymbol{X}$ and $Y = \underline{\lim}\, \boldsymbol{Y}$, there is a unique pair of morphisms $f\colon S \to X$, $g\colon S \to Y$, such that

(3) $$\boldsymbol{p}f = \boldsymbol{f} \quad \text{and} \quad \boldsymbol{q}g = \boldsymbol{g} .$$

In turn, there is a unique $h\colon S \to X \times Y$ satisfying

(4) $$\varkappa h = f \quad \text{and} \quad \lambda h = g .$$

By (1), (4) and (3), one gets

(5) $$\kappa(\boldsymbol{r}h) \cong (\boldsymbol{p}\varkappa)h = \boldsymbol{p}f = \boldsymbol{f} \quad \text{and} \quad \boldsymbol{\lambda}(\boldsymbol{r}h) \cong (\boldsymbol{q}\lambda)h = \boldsymbol{q}g = \boldsymbol{g} .$$

Conditions (2) and (5) imply $\boldsymbol{r}h \cong \boldsymbol{h}$, which proves $\underline{\lim}(\boldsymbol{X} \times \boldsymbol{Y})$ to be a product of X and Y.

By (1), it follows that

$$\varkappa = \underline{\lim}\, \kappa \quad \text{and} \quad \lambda = \underline{\lim}\, \boldsymbol{\lambda};$$

thus $\underline{\lim}\, \kappa$ and $\underline{\lim}\, \boldsymbol{\lambda}$ are product morphisms for $X \times Y$. This completes the proof. ■

2. Shape groups of the product. As known, the functor of nth homotopy group, π_n: $\mathcal{H}\to\mathcal{G}$, is multiplicative (see [2], Th. 2.1, p. 144). Thus Proposition 1.2 implies

2.1. COROLLARY. *The functor of n-th homotopy pro-groups,* π_n: pro-$\mathcal{H}\to$pro-$\mathcal{G}$ *is multiplicative.* ■

In turn, Proposition 1.3 implies

2.2. COROLLARY. *The functor of n-th shape group* $\hat{\pi}_n$: $\mathcal{S}\to\mathcal{G}$ *is multiplicative.* ■

Corollary 2.2 was obtained by T. J. Sanders in [9] (Theorem 4.1). However, Sanders was not interested in the multiplicativity of homotopy pro-groups (Corollary 2.1), which is needed here in § 5.

One-point union, shape retracts and shape deformability

3. Monomorphisms, epimorphisms, kernels and cokernels in pro-$\mathcal{G}$. We shall use the following notation. For any morphism f: $X\to Y$ of an arbitrary category $\mathcal{K}$, kernel and cokernel of f will be denoted by

$$\mathbf{Ker} f = (\mathrm{Ker} f, \ker f), \qquad \mathbf{Coker} f = (\mathrm{Coker} f, \mathrm{coker} f),$$

$\mathrm{Ker} f$ and $\mathrm{Coker} f$ being objects of $\mathcal{K}$, and $\ker f$, $\mathrm{coker} f$ being morphisms,

$$\ker f\colon \mathrm{Ker} f\to X, \qquad \mathrm{coker} f\colon Y\to\mathrm{Coker} f.$$

Then

$$\mathbf{Im} f = (\mathrm{Im} f, \mathrm{im} f), \quad \text{where} \quad \mathrm{Im} f = \mathrm{Ker}\,\mathrm{coker} f$$

and

$$\mathrm{im} f = \ker\mathrm{coker} f\colon \mathrm{Im} f\to Y, \quad \text{whence} \quad \mathbf{Im} f = \mathbf{Ker}\,\mathrm{coker} f.$$

Notice that

3.1. *In arbitrary category with kernels and cokernels, if the sequence*

$$X\xrightarrow{f} Y\xrightarrow{g} Z$$

is exact, then $g(\mathrm{im} f) = 0$.

Proof. By the assumption $\mathbf{Im} f = \mathbf{Ker} g$, thus $g(\mathrm{im} f) = g(\ker g) = 0$. ■

It is easy to prove

3.2. *Let* $\mathcal{K}$ *be a category with kernels and cokernels. Then for every morphism* f: $X\to Y$ *in* $\mathcal{K}$ *there is a unique morphism* f^0: $X\to\mathrm{Im} f$ *such that* $f = (\mathrm{im} f)f^0$, *i.e. the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{f^0} & \uparrow{\scriptstyle \mathrm{im} f} \\ & & \mathrm{Im} f \end{array}$$

is commutative. ■

This diagram will be referred to as the *natural diagram* for f.

Let us recall that a category $\mathcal{K}$ is said to be *balanced* if the class of isomorphisms coincides with the class of bimorphisms (see [10]).

3.3. *If* $\mathcal{K}$ *is balanced and* f^0 *is an epimorphism, then*

(a) f *is a monomorphism* $\Leftrightarrow$ f^0 *is an isomorphism* $\Rightarrow$ $\mathrm{Im} f\approx X$,

(b) f *is an epimorphism* $\Leftrightarrow$ $\mathrm{im} f$ *is an isomorphism* $\Rightarrow$ $\mathrm{Im} f\approx Y$,

(c) $\ker f$ *is an isomorphism* $\Leftrightarrow$ $\mathrm{Im} f = 0$ (4).

Proof. Let $i = \mathrm{im} f$. Since $f = if^0$ and i is a monomorphism, hence f is a monomorphism if and only if f^0 is a monomorphism. Since f^0 is simultaneously an epimorphism and $\mathcal{K}$ is balanced, we get (a).

Similarly we prove (b).

Let $\mathrm{Im} f = 0$. Then $f^0 = 0$ and thus $f = if^0 = 0$. Then $\ker f$ is an isomorphism. Conversely, if $\ker f$ is an isomorphism, then $f = 0$, whence $if^0 = 0$. Since i is a monomorphism, it follows that $f^0 = 0$. But f^0 was assumed to be an epimorphism, thus $\mathrm{Im} f = 0$, which proves (c). ■

We are interested in the category pro-$\mathcal{G}$. Consider a special morphism $\boldsymbol{f} = (1, f_\alpha)$: $\boldsymbol{X} = (X_\alpha, p_\alpha^{\alpha'})\to(Y_\alpha, q_\alpha^{\alpha'}) = \boldsymbol{Y}$ of $\mathcal{G}^*$. We have

3.4. $\mathrm{Ker}[\boldsymbol{f}] = (\mathrm{Ker} f_\alpha, p_\alpha^{\alpha'}|\ \mathrm{Ker} f_{\alpha'})$ *and* $\ker[\boldsymbol{f}] = [\boldsymbol{j}]$, $\boldsymbol{j} = (1, \ker f_\alpha)$.

Proof. Notice that

(1) $$p_\alpha^{\alpha'}(\mathrm{Ker} f_{\alpha'})\subset\mathrm{Ker} f_\alpha;$$

indeed, $f_{\alpha'}(x) = 0$ implies $f_\alpha p_\alpha^{\alpha'}(x) = q_\alpha^{\alpha'} f_{\alpha'}(x) = 0$, i.e. $p_\alpha^{\alpha'}(x)\in\mathrm{Ker} f_\alpha$. By (1), we can define maps $n_\alpha^{\alpha'}$: $\mathrm{Ker} f_{\alpha'}\to\mathrm{Ker} f_\alpha$ by the formula

(2) $$n_\alpha^{\alpha'} = p_\alpha^{\alpha'}|\ \mathrm{Ker} f_{\alpha'}.$$

Thus $\boldsymbol{N} = (\mathrm{Ker} f_\alpha, n_\alpha^{\alpha'})$ is an object of pro-$\mathcal{G}$. Since the diagram

$$\begin{array}{ccc} \mathrm{Ker} f_\alpha & \xleftarrow{n_\alpha^{\alpha'}} & \mathrm{Ker} f_{\alpha'} \\ {\scriptstyle \ker f_\alpha}\downarrow & & \downarrow{\scriptstyle \ker f_{\alpha'}} \\ X_\alpha & \xrightarrow[p_\alpha^{\alpha'}]{} & X_{\alpha'} \end{array}$$

obviously commutes,

$\boldsymbol{j} = (1, \ker f_\alpha)$ is a morphism of $\mathcal{G}^*$, $\boldsymbol{j}$: $\boldsymbol{N}\to\boldsymbol{X}$. It is easy to check that $\mathbf{Ker}[\boldsymbol{f}] = (\boldsymbol{N}, [\boldsymbol{j}])$. ■

Let us recall that in the category of groups, $\mathcal{G}$, for any homomorphism f: $X\to Y$ with $f(X)$ being a normal subgroup of Y,

$\mathrm{Im} f = f(X)$, $\mathrm{im} f$: $f(X)\to Y$ is the inclusion, $\mathrm{Coker} f = Y|f(X)$ and $\mathrm{coker} f$: $Y\to Y|f(X)$ is the natural projection.

Consider a commutative diagram in $\mathcal{G}$,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

(4) The condition (c) holds without the assumption on $\mathcal{K}$ to be balanced.

The homomorphism q induces a homomorphism $\hat{q}$: $\operatorname{Coker} f \to \operatorname{Coker} f'$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\operatorname{coker} f} & \operatorname{Coker} f \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \hat{q}} \\ Y' & \xrightarrow[\operatorname{coker} f']{} & \operatorname{Coker} f' \end{array} \qquad \text{is commutative.}$$

Consider a special morphism $\boldsymbol{f} = (1, f_\alpha)$: $\boldsymbol{X} = (X_\alpha, p_\alpha^{\alpha'}) \to (Y_\alpha, q_\alpha^{\alpha'}) = \boldsymbol{Y}$ in $\mathcal{G}^*$. The morphism $\boldsymbol{f}$ will be called *normal* if $f_\alpha(X_\alpha)$ is a normal subgroup of Y_α for all α. In particular, if all Y_α are abelian then $\boldsymbol{f}$ is obviously normal.

Let us prove

3.5. *If* $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ *is normal, then* $\operatorname{Coker}[\boldsymbol{f}] = (\operatorname{Coker} f_\alpha, \hat{q}_\alpha^{\alpha'})$ *and* $\operatorname{coker}[\boldsymbol{f}] = [\boldsymbol{p}]$, *where* $\boldsymbol{p} = (1, \operatorname{coker} f_\alpha)$.

Proof. Let $\boldsymbol{M} = (\operatorname{Coker} f_\alpha, \hat{q}_\alpha^{\alpha'})$. Take $\boldsymbol{p} = (1, p_\alpha)$, where p_α: $Y_\alpha \to \operatorname{Coker} f_\alpha$ is the natural projection. Let us show

(1) $$\operatorname{Coker}[\boldsymbol{f}] = (\boldsymbol{M}, [\boldsymbol{p}]).$$

By 3.1 and 3.2, I [7], $[\boldsymbol{p}]$ is an epimorphism and $[\boldsymbol{p}][\boldsymbol{f}] = \boldsymbol{0}$. Take $\boldsymbol{Z} = (Z_\alpha, r_\alpha^{\alpha'})$ and $\boldsymbol{g} = (\psi, g_\alpha)$: $\boldsymbol{Y} \to \boldsymbol{Z}$ satisfying the condition

(2) $$[\boldsymbol{g}][\boldsymbol{f}] = \boldsymbol{0}.$$

By (2), for every α there is an $\alpha^* = \eta(\alpha) \geqslant \psi(\alpha)$ such that

(3) $$g_\alpha q_{\psi(\alpha)}^{\alpha^*} f_{\alpha^*}(X_{\alpha^*}) = 0.$$

Notice that

(4) $$p_{\alpha^*}(y_1) = p_{\alpha^*}(y_2) \Rightarrow g_\alpha q_{\psi(\alpha)}^{\alpha^*}(y_1) = g_\alpha q_{\psi(\alpha)}^{\alpha^*}(y_2) \quad \text{for} \quad y_1, y_2 \in Y_{\alpha^*}.$$

Indeed, by (3),

$$p_{\alpha^*}(y_1) = p_{\alpha^*}(y_2) \Rightarrow p_{\alpha^*}(y_1 - y_2) = 0 \Rightarrow y_1 - y_2 \in f_{\alpha^*}(X_{\alpha^*})$$
$$\Rightarrow g_\alpha q_{\psi(\alpha)}^{\alpha^*}(y_1) - g_\alpha q_{\psi(\alpha)}^{\alpha^*}(y_2) = g_\alpha q_{\psi(\alpha)}^{\alpha^*}(y_1 - y_2) \in g_\alpha q_{\psi(\alpha)}^{\alpha^*} f_{\alpha^*}(X_{\alpha^*}) = 0.$$

The condition (4) enables us to define a homomorphism g'_α: $\operatorname{Coker} f_{\eta(\alpha)} \to Z_\alpha$ by the formula

(5) $$g'_\alpha[y] \underset{\mathrm{Df}}{=} g_\alpha q_{\psi(\alpha)}^{\eta(\alpha)}(y).$$

The diagram

$$\begin{array}{ccc} \operatorname{Coker} f_{\eta(\alpha)} & \xleftarrow{\hat{q}_{\eta(\alpha)}^{\eta(\alpha')}} & \operatorname{Coker} f_{\eta(\alpha')} \\ {\scriptstyle g'_\alpha}\downarrow & & \downarrow{\scriptstyle g'_{\alpha'}} \\ Z_\alpha & \xleftarrow[r_\alpha^{\alpha'}]{} & Z_{\alpha'} \end{array} \qquad \text{is commutative;}$$

indeed, by (5), since $g_\alpha q_{\psi(\alpha)}^{\psi(\alpha'')} = r_\alpha^{\alpha''} g_{\alpha''}$ for $\alpha'' \geqslant \alpha$, we get

$$g'_\alpha \hat{q}_{\eta(\alpha)}^{\eta(\alpha')}[y] = g'_\alpha p_{\eta(\alpha)} q_{\eta(\alpha)}^{\eta(\alpha')}(y) = g_\alpha q_{\psi(\alpha)}^{\eta(\alpha)} q_{\eta(\alpha)}^{\eta(\alpha')}(y) = g_\alpha q_{\psi(\alpha)}^{\eta(\alpha')}(y)$$
$$= g_\alpha q_{\psi(\alpha)}^{\psi(\alpha')} q_{\psi(\alpha')}^{\eta(\alpha')}(y) = r_\alpha^{\alpha'}\big(g_{\alpha'} q_{\psi(\alpha')}^{\eta(\alpha')}(y)\big) = r_\alpha^{\alpha'} g'_{\alpha'}[y].$$

Thus, setting $\boldsymbol{g}' = (\eta, g'_\alpha)$, we obtain a morphism $\boldsymbol{g}'$: $\boldsymbol{M} \to \boldsymbol{Z}$. The condition (5) implies

(6) $$\boldsymbol{g}'\boldsymbol{p} \cong \boldsymbol{g}.$$

Thus (1) is satisfied. ■

Statements 3.4 and 3.5 imply

3.6. *If* $\boldsymbol{f} = (1, f_\alpha)$: $\boldsymbol{X} \to \boldsymbol{Y}$ *is normal, then* $\operatorname{Im}[\boldsymbol{f}] = \big(f_\alpha(X_\alpha), q_\alpha^{\alpha'} | f_{\alpha'}(X_{\alpha'})\big)$ *and* $\operatorname{im}[\boldsymbol{f}] = [\boldsymbol{v}]$, *where* $\boldsymbol{v} = (1, \operatorname{im} f_\alpha)$. ■

Take now a normal morphism $\boldsymbol{f} = (1, f_\alpha)$: $\boldsymbol{X} \to \boldsymbol{Y}$, and consider the collection of homomorphisms $(f_\alpha^0$: $X_\alpha \to \operatorname{Im} f_\alpha)_{\alpha \in N}$ defined by the formula

$$f_\alpha^0(x) \underset{\mathrm{Df}}{=} f_\alpha(x) \quad \text{for every } x \in X_\alpha.$$

It is easy to show that $(1, f_\alpha^0)$ represents a morphism of pro-$\mathcal{G}$. Let $\boldsymbol{f}^0 = (1, f_\alpha^0)$. Notice that

3.7. $\boldsymbol{f} = (\operatorname{im} \boldsymbol{f}) \boldsymbol{f}^0$ *and* $[\boldsymbol{f}^0]$ *is an epimorphism in* pro-$\mathcal{G}$.

Proof. The equality $\boldsymbol{f} = (\operatorname{im} \boldsymbol{f}) \boldsymbol{f}^0$ follows directly by $f_\alpha = (\operatorname{im} f_\alpha) f_\alpha^0$. Each of f_α^0 is obviously an epimorphism in $\mathcal{G}$, whence $[\boldsymbol{f}^0]$ is an epimorphism in pro-$\mathcal{G}$ (see [7] I, 3.1). ■

As known, the category pro-$\mathcal{G}$ is balanced (see [3] II, Th. 8). Thus the statements 3.3 and 3.7 imply

3.8. *For any normal morphism* $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ *in* $\mathcal{G}^*$

(a) $[\boldsymbol{f}]$ *is a monomorphism* $\Leftrightarrow$ $[\boldsymbol{f}^0]$ *is an isomorphism* $\Rightarrow$ $\operatorname{Im}[\boldsymbol{f}] \approx \boldsymbol{X}$,

(b) $[\boldsymbol{f}]$ *is an epimorphism* $\Leftrightarrow$ $\operatorname{im}[\boldsymbol{f}]$ *is an isomorphism* $\Rightarrow$ $\operatorname{Im}[\boldsymbol{f}] \approx \boldsymbol{Y}$,

(c) $\ker[\boldsymbol{f}]$ *is an isomorphism* $\Leftrightarrow$ $\operatorname{Im}[\boldsymbol{f}] = \boldsymbol{0}$. ■

By [7] I, 1.5, 1.6 and 2.1, it follows that

3.9. *In the category* pro-$\mathcal{G}$, $[\boldsymbol{f}]$ *is a monomorphism* $\Leftrightarrow$ $\operatorname{Ker}[\boldsymbol{f}] = \boldsymbol{0}$. ■

Notice that

3.10. *For any* $\boldsymbol{f} = (1, f_\alpha)$: $(X_\alpha, p_\alpha^{\alpha'}) = \boldsymbol{X} \to \boldsymbol{Y} = (Y_\alpha, q_\alpha^{\alpha'})$ *in* $\mathcal{G}^*$

$$\operatorname{Ker}[\boldsymbol{f}] = \boldsymbol{0} \Leftrightarrow \bigwedge_{\alpha} \bigvee_{\alpha' \geqslant \alpha} p_\alpha^{\alpha'}(\operatorname{Ker} f_{\alpha'}) = 0\ (^5).$$

Proof. By 3.4, $\operatorname{Ker}[\boldsymbol{f}] = (\operatorname{Ker} f_\alpha, p_\alpha^{\alpha'} | \operatorname{Ker} f_{\alpha'})$. We have

$$\operatorname{Ker}[\boldsymbol{f}] = \boldsymbol{0} \Leftrightarrow [1_{\operatorname{Ker}[\boldsymbol{f}]}] = [0] \Leftrightarrow \bigwedge_{\alpha} \bigvee_{\alpha' \geqslant \alpha} p_\alpha^{\alpha'} | \operatorname{Ker} f_{\alpha'} = 0$$
$$\Leftrightarrow \bigwedge_{\alpha} \bigvee_{\alpha' \geqslant \alpha} p_\alpha^{\alpha'}(\operatorname{Ker} f_{\alpha'}) = 0. \ ■$$

(5) See also [3] II, Theorem 6.

Consider the subcategory pro-$\mathcal{G}_{\mathrm{Nor}}$ of pro-$\mathcal{G}$ with all the morphisms being normal. Let us prove

3.11. *For every exact diagram*

$$A \xrightarrow{[i]} B \xrightarrow{[\zeta]} C \xrightarrow{[\partial]} A' \xrightarrow{[i']} B' \quad \textit{in } \text{pro-}\mathcal{G}_{\mathrm{Nor}},$$

if $[i]$ *is an epimorphism then* $\mathrm{Ker}[i'] \approx C$.

Proof. Since $[i]$ is an epimorphism, 3.8(b) implies

(1) $\mathrm{im}[i]$ is an isomorphism.

By the exactness, $\ker[\zeta] = \mathrm{im}[i]$, whence, by (1) and 3.8(c) it follows

$$\mathrm{Im}[\zeta] = \mathbf{0}\,. \tag{2}$$

By the exactness, $\mathrm{Ker}[\partial] = \mathbf{0}$, thus, by 3.9, $[\partial]$ is a monomorphism. Hence, by 3.8(a), $\mathrm{Im}[\partial] \approx C$. Since $\mathrm{Ker}[i'] = \mathrm{Im}[\partial]$, we get $\mathrm{Ker}[i'] \approx C$. ■

4. Some special pairs of morphisms in pro-$\mathcal{G}$. We are interested in the properties of such a pair of morphisms $[f]$, $[g]$ in pro-$\mathcal{G}$, for which $[f][g]$ is an isomorphism. Notice that

4.1. *In arbitrary category* $\mathcal{K}$, *if* fg *is an isomorphism then* f *is an epimorphism and* g *is a monomorphism.* ■

Let $\mathcal{G}_{Ab}$ be the category of Abelian groups. For any two morphisms $\varphi\colon X \to Z$ and $\psi\colon Y \to Z$ in $\mathcal{G}_{Ab}$, let us define

$$\varphi \oplus \psi\colon X \times Y \to Z$$

by the formula

$$(\varphi \oplus \psi)(x, y) \underset{\mathrm{Def}}{=} \varphi(x) + \psi(y) \quad \text{for every } (x, y) \in X \times Y\,.$$

Since Z is Abelian, $\varphi \oplus \psi$ is a homomorphism. Obviously

4.2. *For every* $\zeta\colon Z \to Z'$

$$\zeta(\varphi \oplus \psi) = \zeta\varphi \oplus \zeta\psi\,. \ ■$$

In turn, by 1.1, for any two special morphisms $\boldsymbol{\varphi} = (1, \varphi_\alpha)\colon \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{\psi} = (1, \psi_\alpha)\colon \boldsymbol{Y} \to \boldsymbol{Z}$ in $\mathcal{G}^*_{Ab}$, we can define

$$\boldsymbol{\varphi} \oplus \boldsymbol{\psi}\colon \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Z}$$

by the formula

$$\boldsymbol{\varphi} \oplus \boldsymbol{\psi} \underset{\mathrm{Df}}{=} (1, \varphi_\alpha \oplus \psi_\alpha)\,.$$

It is easily seen that

$$(\boldsymbol{\varphi} \simeq \boldsymbol{\varphi}' \wedge \boldsymbol{\psi} \simeq \boldsymbol{\psi}') \Rightarrow (\boldsymbol{\varphi} \oplus \boldsymbol{\psi} \simeq \boldsymbol{\varphi}' \oplus \boldsymbol{\psi}');$$

this enables us to define $[\boldsymbol{\varphi}] \oplus [\boldsymbol{\psi}] \in \mathrm{Mor}\,\text{pro-}\mathcal{G}_{Ab}$ by the formula

$$[\boldsymbol{\varphi}] \oplus [\boldsymbol{\psi}] \underset{\mathrm{Df}}{=} [\boldsymbol{\varphi} \oplus \boldsymbol{\psi}]\,.$$

By 4.2 follows easily

4.3. *For any morphism* $[\boldsymbol{\zeta}]\colon \boldsymbol{Z} \to \boldsymbol{Z}'$ *in* pro-$\mathcal{G}_{Ab}$

$$[\boldsymbol{\zeta}]([\boldsymbol{\varphi}] \oplus [\boldsymbol{\psi}]) = [\boldsymbol{\zeta}][\boldsymbol{\varphi}] \oplus [\boldsymbol{\zeta}][\boldsymbol{\psi}]\,. \ ■$$

As known, for a pair of homomorphisms $f\colon X \to Y$ and $g\colon Y \to X$ in $\mathcal{G}_{Ab}$, if $fg = 1_Y$, then $X \approx \mathrm{Ker} f \times \mathrm{Im}\, g$. We are going to establish a similar assertion for pro-$\mathcal{G}_{Ab}$.

4.4. Proposition. *Take two special morphisms*

$$\boldsymbol{f} = (1, f_\alpha)\colon (X_\alpha, p_\alpha^{\alpha'}) = \boldsymbol{X} \to \boldsymbol{Y} = (Y_\alpha, q_\alpha^{\alpha'}) \quad \textit{and} \quad \boldsymbol{g} = (1, g_\alpha)\colon (\tilde{Y}_\alpha, \tilde{q}_\alpha^{\alpha'}) = \tilde{\boldsymbol{Y}} \to \boldsymbol{X}$$

in $\mathcal{G}^*_{Ab}$. *If the composition* $[\boldsymbol{f}][\boldsymbol{g}]$ *is an isomorphism in* pro-$\mathcal{G}_{Ab}$ *then*

$$\boldsymbol{X} \approx \mathrm{Ker}[\boldsymbol{f}] \times \mathrm{Im}[\boldsymbol{g}]\,,$$

moreover $\ker[\boldsymbol{f}] \oplus \mathrm{im}[\boldsymbol{g}]\colon \mathrm{Ker}[\boldsymbol{f}] \times \mathrm{Im}[\boldsymbol{g}] \to \boldsymbol{X}$ *is an isomorphism.*

Proof. By the assumption, there is a special morphism $\boldsymbol{e} = (1, e_\alpha)\colon \tilde{\boldsymbol{Y}} \to \boldsymbol{Y}$ in $\mathcal{G}^*_{Ab}$, such that $[\boldsymbol{f}][\boldsymbol{g}] = [\boldsymbol{e}]$ and $[\boldsymbol{e}]$ is an isomorphism in pro-$\mathcal{G}_{Ab}$. Let $\boldsymbol{d} = (\delta, d_\alpha)\colon \boldsymbol{Y} \to \tilde{\boldsymbol{Y}}$ represents the inverse of $[\boldsymbol{e}]$. Then, for every α there is a $\varphi(\alpha) \geqslant \delta(\alpha)$ such that

$$e_\alpha d_\alpha q_{\delta(\alpha)}^{\varphi(\alpha)} = q_\alpha^{\varphi(\alpha)}\,. \tag{1}$$

Since $[\boldsymbol{f}][\boldsymbol{g}] = [\boldsymbol{e}]$, for every α there is a $\psi(\alpha) \geqslant \varphi(\alpha)$ such that

$$f_\alpha g_\alpha \tilde{q}_\alpha^{\psi(\alpha)} = e_\alpha \tilde{q}_\alpha^{\psi(\alpha)}\,. \tag{2}$$

Let

$$b_\alpha(x) = g_\alpha d_\alpha q_{\delta(\alpha)}^{\delta\psi(\alpha)} f_{\delta\psi(\alpha)}(x) \quad \text{and} \quad a_\alpha(x) = p_\alpha^{\delta\psi(\alpha)}(x) - b_\alpha(x) \quad \text{for} \quad x \in X_{\delta\psi(\alpha)}\,. \tag{3}$$

Notice that

$$a_\alpha(x) \in \mathrm{Ker} f_\alpha; \tag{4}$$

indeed, by (3), (2) and (1),

$$\begin{aligned} f_\alpha b_\alpha &= f_\alpha g_\alpha d_\alpha q_{\delta(\alpha)}^{\delta\psi(\alpha)} f_{\delta\psi(\alpha)} = f_\alpha g_\alpha \tilde{q}_\alpha^{\psi(\alpha)} d_{\psi(\alpha)} f_{\delta\psi(\alpha)} \\ &= e_\alpha \tilde{q}_\alpha^{\psi(\alpha)} d_{\psi(\alpha)} f_{\delta\psi(\alpha)} = e_\alpha d_\alpha q_{\delta(\alpha)}^{\delta\psi(\alpha)} f_{\delta\psi(\alpha)} \\ &= q_\alpha^{\varphi(\alpha)} q_{\varphi(\alpha)}^{\delta\psi(\alpha)} f_{\delta\psi(\alpha)} = q_\alpha^{\delta\psi(\alpha)} f_{\delta\psi(\alpha)} = f_\alpha p_\alpha^{\delta\psi(\alpha)}; \end{aligned}$$

thus

$$f_\alpha a_\alpha(x) = f_\alpha p_\alpha^{\delta\psi(\alpha)}(x) - f_\alpha b_\alpha(x) = 0\,,$$

which proves (4). Obviously

$$b_\alpha(x) \in \mathrm{Im}\, g_\alpha\,. \tag{5}$$

Take $\alpha' \geqslant \alpha$ and consider two diagrams:

$$\begin{array}{ccc} X_{\delta\psi(\alpha)} & \xleftarrow{p_{\delta\psi(\alpha)}^{\delta\psi(\alpha')}} & X_{\delta\psi(\alpha')} \\ {\scriptstyle b_\alpha}\downarrow & & \downarrow{\scriptstyle b_{\alpha'}} \\ \mathrm{Im}\, g_\alpha & \xleftarrow[p_\alpha^{\alpha'}|\mathrm{Im}\, g_{\alpha'}]{} & \mathrm{Im}\, g_{\alpha'} \end{array} \qquad \begin{array}{ccc} X_{\delta\psi(\alpha)} & \xleftarrow{p_{\delta\psi(\alpha)}^{\delta\psi(\alpha')}} & X_{\delta\psi(\alpha')} \\ {\scriptstyle a_\alpha}\downarrow & & \downarrow{\scriptstyle a_{\alpha'}} \\ \mathrm{Ker} f_\alpha & \xleftarrow[p_\alpha^{\alpha'}|\mathrm{Ker} f_{\alpha'}]{} & \mathrm{Ker} f_{\alpha'} \end{array}$$

We have

$$p_\alpha^{\alpha'} b_{\alpha'}(x) = p_\alpha^{\alpha'} g_{\alpha'} d_{\alpha'} q_{\delta(\alpha')}^{\delta\psi(\alpha')} f_{\delta\psi(\alpha')}(x) = g_\alpha \tilde{q}_\alpha^{\alpha'} d_{\alpha'} q_{\delta(\alpha')}^{\delta\psi(\alpha')} f_{\delta\psi(\alpha')}(x)$$
$$= g_\alpha d_\alpha q_{\delta(\alpha)}^{\delta\psi(\alpha)} q_{\delta\psi(\alpha)}^{\delta\psi(\alpha')} f_{\delta\psi(\alpha')}(x) = (g_\alpha d_\alpha q_{\delta(\alpha)}^{\delta\psi(\alpha)} f_{\delta\psi(\alpha)}) p_{\delta\psi(\alpha)}^{\delta\psi(\alpha')}(x) = b_\alpha p_{\delta\psi(\alpha)}^{\delta\psi(\alpha')}(x),$$

i.e.

(6) the first diagram commutes.

Thus

$$p_\alpha^{\alpha'} a_{\alpha'}(x) = p_\alpha^{\alpha'} p_{\alpha'}^{\delta\psi(\alpha')}(x) - p_\alpha^{\alpha'} b_{\alpha'}(x) = p_\alpha^{\delta\psi(\alpha)} p_{\delta\psi(\alpha)}^{\delta\psi(\alpha')}(x) - b_\alpha p_{\delta\psi(\alpha)}^{\delta\psi(\alpha')}(x) = a_\alpha p_{\delta\psi(\alpha)}^{\delta\psi(\alpha')}(x),$$

i.e.

(7) the second diagram commutes.

By (4)-(7), setting

$$(8) \qquad \boldsymbol{h} = (\delta\psi, h_\alpha), \quad h_\alpha(x) = (a_\alpha(x), b_\alpha(x)) \quad \text{for} \quad x \in X_{\delta\psi(\alpha)},$$

we get a morphism $\boldsymbol{h}: X \to \mathrm{Ker}[f] \times \mathrm{Im}[g]$ in $\mathcal{G}^*_{Ab}$. Consider the morphism $\boldsymbol{i}: \mathrm{Ker}[f] \times \mathrm{Im}[g] \to X$ defined by

$$(9) \qquad \boldsymbol{i} = \ker \boldsymbol{f} \oplus \mathrm{im}\, \boldsymbol{g}.$$

It suffices to show that $[\boldsymbol{h}]$ is an inverse of $[\boldsymbol{i}]$ in pro-$\mathcal{G}_{Ab}$. We have

$$\boldsymbol{ih} = (\delta\psi, i_\alpha h_\alpha),$$

$$i_\alpha h_\alpha(x) = (\ker f_\alpha \oplus \mathrm{im}\, g_\alpha)(a_\alpha(x), b_\alpha(x)) = (\ker f_\alpha) a_\alpha(x) + (\mathrm{im}\, g_\alpha) b_\alpha(x)$$
$$= a_\alpha(x) + b_\alpha(x) = p_\alpha^{\delta\psi(\alpha)}(x) \quad \text{for} \quad x \in X_{\delta\psi(\alpha)},$$

thus

$$(10) \qquad \boldsymbol{ih} \simeq \boldsymbol{1}_X.$$

Let us take $\boldsymbol{hi} = (\delta\psi, h_\alpha i_{\delta\psi(\alpha)})$ and prove

$$(11) \qquad \boldsymbol{hi} \simeq \boldsymbol{1}_{\mathrm{Ker}[f] \times \mathrm{Im}[g]}.$$

There is an $\alpha' \geqslant \delta\psi(\alpha)$ satisfying the following two conditions

$$(12) \qquad f_{\delta(\alpha)} g_{\delta(\alpha)} \tilde{q}_{\delta(\alpha)}^{\alpha'} = e_{\delta(\alpha)} \tilde{q}_{\delta(\alpha)}^{\alpha'}$$

and

$$(13) \qquad d_\alpha e_{\delta(\alpha)} \tilde{q}_{\delta(\alpha)}^{\alpha'} = \tilde{q}_\alpha^{\alpha'}.$$

Take $(x_1, x_2) \in \mathrm{Ker} f_{\alpha'} \times \mathrm{Im}\, g_{\alpha'}$ and show that

$$(14) \qquad h_\alpha i_{\delta\psi(\alpha)}(p_{\delta\psi(\alpha)}^{\alpha'}(x_1), p_{\delta\psi(\alpha)}^{\alpha'}(x_2)) = (p_\alpha^{\alpha'}(x_1), p_\alpha^{\alpha'}(x_2)).$$

We have

$$(15) \qquad h_\alpha i_{\delta\psi(\alpha)}(p_{\delta\psi(\alpha)}^{\alpha'}(x_1), p_{\delta\psi(\alpha)}^{\alpha'}(x_2)) = h_\alpha p_{\delta\psi(\alpha)}^{\alpha'}(x_1 + x_2)$$
$$= (a_\alpha p_{\delta\psi(\alpha)}^{\alpha'}(x_1 + x_2), b_\alpha p_{\delta\psi(\alpha)}^{\alpha'}(x_1 + x_2)).$$

Since $f_{\alpha'}(x_1) = 0$ and there is a $y_2 \in \tilde{Y}_{\alpha'}$ such that $x_2 = g_{\alpha'}(y_2)$, applying (12) and (13) we obtain

$$(16) \qquad b_\alpha p_{\delta\psi(\alpha)}^{\alpha'}(x_1 + x_2) = g_\alpha d_\alpha q_{\delta(\alpha)}^{\delta\psi(\alpha)} f_{\delta\psi(\alpha)} p_{\delta\psi(\alpha)}^{\alpha'}(x_1 + x_2)$$
$$= g_\alpha d_\alpha q_{\delta(\alpha)}^{\delta\psi(\alpha)} q_{\delta\psi(\alpha)}^{\alpha'} f_{\alpha'}(x_1 + x_2) = g_\alpha d_\alpha q_{\delta(\alpha)}^{\alpha'} f_{\alpha'}(x_2)$$
$$= g_\alpha d_\alpha q_{\delta(\alpha)}^{\alpha'} f_{\alpha'} g_{\alpha'}(y_2) = g_\alpha d_\alpha f_{\delta(\alpha)} g_{\delta(\alpha)} \tilde{q}_{\delta(\alpha)}^{\alpha'}(y_2)$$
$$= g_\alpha d_\alpha e_{\delta(\alpha)} \tilde{q}_{\delta(\alpha)}^{\alpha'}(y_2) = g_\alpha \tilde{q}_\alpha^{\alpha'}(y_2) = p_\alpha^{\alpha'} g_{\alpha'}(y_2) = p_\alpha^{\alpha'}(x_2).$$

Thus

$$(17) \qquad a_\alpha p_{\delta\psi(\alpha)}^{\alpha'}(x_1 + x_2) = p_\alpha^{\alpha'}(x_1 + x_2) - b_\alpha p_{\delta\psi(\alpha)}^{\alpha'}(x_1 + x_2)$$
$$= p_\alpha^{\alpha'}(x_1 + x_2) - p_\alpha^{\alpha'}(x_2) = p_\alpha^{\alpha'}(x_1).$$

The conditions (15)-(17) imply (14), which results in (11).

By (10) and (11), $[\boldsymbol{h}]$ is an inverse of $[\boldsymbol{i}]$, whence $[\boldsymbol{i}]$ is an isomorphism. ∎

5. Shape groups of the one-point union. Consider two metric compacta (X, x_0) and (Y, y_0) and their one-point union $(X \vee Y, (x_0, y_0))$. The space $X \vee Y$ may be considered as a subspace of $X \times Y$ (see [2], p. 145). We are going to prove the following theorem concerning the shape groups

5.1. THEOREM.

$$\check{\pi}_n(X \vee Y, (x_0, y_0)) \approx \check{\pi}_n(X, x_0) \times \check{\pi}_n(Y, y_0) \times \check{\pi}_{n+1}(X \times Y, X \vee Y, (x_0, y_0))$$
$$\textit{for} \quad n \geqslant 2.$$

Let us start by proving the corresponding statement for homotopy pro-groups. Take two inverse sequences of compacta,

$$(\boldsymbol{X}, \boldsymbol{x}_0) = ((X_\alpha, x_\alpha), p_\alpha^{\alpha'}) \quad \text{and} \quad (\boldsymbol{Y}, \boldsymbol{y}_0) = ((Y_\alpha, y_\alpha), q_\alpha^{\alpha'})$$

and let

$$\boldsymbol{X} \vee \boldsymbol{Y} = (X_\alpha \vee Y_\alpha, (p_\alpha^{\alpha'} \times q_\alpha^{\alpha'}) | X_{\alpha'} \vee Y_{\alpha'}).$$

5.2. THEOREM.

$$\pi_n(\boldsymbol{X} \vee \boldsymbol{Y}, (\boldsymbol{x}_0, \boldsymbol{y}_0)) \approx \pi_n(\boldsymbol{X}, \boldsymbol{x}_0) \times \pi_n(\boldsymbol{Y}, \boldsymbol{y}_0) \times \pi_{n+1}(\boldsymbol{X} \times \boldsymbol{Y}, \boldsymbol{X} \vee \boldsymbol{Y}, (\boldsymbol{x}_0, \boldsymbol{y}_0))$$
$$\textit{for} \quad n \geqslant 2.$$

Proof of Theorem 5.2. Take $\boldsymbol{Z} = \boldsymbol{X} \times \boldsymbol{Y} = (Z_\alpha, r_\alpha^{\alpha'})$, $\boldsymbol{z}_0 = (z_\alpha) = (x_\alpha, y_\alpha)$, and $\boldsymbol{U} = \boldsymbol{X} \vee \boldsymbol{Y} = (U_\alpha, r_\alpha^{\alpha'} | U_{\alpha'})$. Let $\boldsymbol{i}: \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{j}: \boldsymbol{Y} \to \boldsymbol{Z}$, $\boldsymbol{k}: \boldsymbol{U} \to \boldsymbol{Z}$, $\boldsymbol{i}': \boldsymbol{X} \to \boldsymbol{U}$ and $\boldsymbol{j}': \boldsymbol{Y} \to \boldsymbol{U}$ be the inclusions. Evidently

$$(1) \qquad \boldsymbol{ki}' = \boldsymbol{i} \quad \text{and} \quad \boldsymbol{kj}' = \boldsymbol{j}.$$

Let $\varkappa_\alpha: Z_\alpha \to X_\alpha$ and $\lambda_\alpha: Z_\alpha \to Y_\alpha$ be product morphisms for every α. Then, by 1.1, $\boldsymbol{\varkappa} = (1, \varkappa_\alpha): \boldsymbol{Z} \to \boldsymbol{X}$ and $\boldsymbol{\lambda} = (1, \lambda_\alpha): \boldsymbol{Z} \to \boldsymbol{Y}$ are product morphisms.

Take $n \geqslant 2$. Let $\boldsymbol{\varkappa}_n = \pi_n(\boldsymbol{\varkappa})$, $\boldsymbol{\lambda}_n = \pi_n(\boldsymbol{\lambda})$, $\boldsymbol{i}_n = \pi_n(\boldsymbol{i})$ and so on. Since the functor pro-π_n is multiplicative (see 2.1), $[\boldsymbol{\varkappa}_n]$ and $[\boldsymbol{\lambda}_n]$ are product morphisms in pro-$\mathcal{G}$. Consider the morphism

$$\boldsymbol{l}_n: \pi_n(\boldsymbol{Z}, \boldsymbol{z}_0) \to \pi_n(\boldsymbol{U}, \boldsymbol{z}_0)$$

defined by the formula

(2) $$l_n = i'_n\kappa_n \oplus j'_n\lambda_n .$$

It is easy to check that (see [2] p. 145)

(3) $$i_n\kappa_n \oplus j_n\lambda_n = \mathbf{1}_{\pi_n(Z,z_0)} .$$

By (1)-(3) and 4.3, we get

(4) $$k_n l_n = \mathbf{1}_{\pi_n(Z,z_0)} .$$

Thus, by 4.1, k_n is an epimorphism and l_n is a monomorphism in $\mathcal{G}^*$. By 4.4 and (4), we infer that

(5) $$\pi_n(U, z_0) \approx \mathrm{Ker}[k_n] \times \mathrm{Im}[l_n] .$$

Since $n \geqslant 2$, all the groups $\pi_n(Z_\alpha, z_\alpha)$ and $\pi_n(U_\alpha, z_\alpha)$ are abelian, and thus k_n and l_n are normal. Since l_n is a monomorphism, 3.8.(a) implies

(6) $$\mathrm{Im}[l_n] \approx \pi_n(Z, z_0) ,$$

whence, by 2.1,

(7) $$\mathrm{Im}[l_n] \approx \pi_n(X, x_0) \times \pi_n(Y, y_0) .$$

Consider the exact sequence of homotopy pro-groups for (Z, U, z_0), (see [7] II, 1.2),

$$\mathcal{D}_{n+1}\colon \pi_{n+1}(U, z_0) \xrightarrow{k_{n+1}} \pi_{n+1}(Z, z_0) \xrightarrow{\xi_{n+1}} \pi_{n+1}(Z, U, z_0) \xrightarrow{\partial_{n+1}} \pi_n(U, z_0) \xrightarrow{k_n} \pi_n(Z, z_0) \to \ldots$$

Notice that ξ_{n+1} is normal, because $\pi_{n+1}(Z_\alpha, U_\alpha, z_\alpha)$ are abelian. Since $[k_{n+1}]$ is an epimorphism, 3.11 implies

(8) $$\mathrm{Ker}[k_n] \approx \pi_{n+1}(Z, U, z_0) .$$

By (5), (7) and (8) we get the desired formula

$$\pi_n(X \vee Y, z_0) \approx \pi_n(X, x_0) \times \pi_n(Y, y_0) \times \pi_{n+1}(X \times Y, X \vee Y, z_0) \quad \text{for} \quad n \geqslant 2. \ \blacksquare$$

Proof of Theorem 5.1. Take two pointed metric ompacta (X, x_0) and (Y, y_0) and let $(X, x_0) = \underline{\lim}(X, x_0)$ and $(Y, y_0) = \underline{\lim}(Y, y_0)$, where (X, x_0), (Y, y_0) are ARN-sequences. One can easily prove

(1) $$(X \vee Y, (x_0, y_0)) = \underline{\lim}(X \vee Y, (x_0, y_0)) .$$

Take $n \geqslant 2$. By 5.2

(2) $$\pi_n(X \vee Y, (x_0, y_0)) \approx \pi_n(X, x_0) \times \pi_n(Y, y_0) \times \pi_{n+1}(X \times Y, X \vee Y, (x_0, y_0)) .$$

Passing to inverse limits and applying (1) and 1.3, we obtain the required formula for shape groups. ■

6. Shape groups of a shape retract. We are going now to establish the following theorem concerning shape groups.

6.1. THEOREM. *Let (X, A, x_0) be a pointed pair of metric continua. If (A, x_0) is a shape retract of (X, x_0), then*

(a) $$\check{\pi}_n(X, x_0) \approx \check{\pi}_n(A, x_0) \times \check{\pi}_n(X, A, x_0) \quad \textit{for} \quad n \geqslant 2$$

and

(b) *the homomorphism $\underline{i}_n\colon \check{\pi}_n(A, x_0) \to \check{\pi}_n(X, x_0)$ induced by the inclusion map $i\colon A \to X$ is a monomorphism in $\mathcal{G}$ for $n \geqslant 1$.*

Let us start by proving the similar statement for homotopy pro-groups.

6.2. THEOREM. *Let (X, A, x_0) be an inverse sequence of pointed pairs of metric continua. If (A, x_0) is a retract of (X, x_0), then*

(a) $$\pi_n(X, x_0) \approx \pi_n(A, x_0) \times \pi_n(X, A, x_0) \quad \textit{for} \quad n \geqslant 2$$

and

(b) *the morphism $[i_n]\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ is a monomorphism in* pro-$\mathcal{G}$ *for $n \geqslant 1$.*

Proof of 6.2. Let $r = (\varrho, r_\alpha)\colon (X, x_0) \to (A, x_0)$ be a retraction, i.e.

(1) $$ri \approx \mathbf{1}_{(A,x_0)} .$$

By 3.8 of [6], we may assume that $\varrho(\alpha) \geqslant \alpha$ for every α. Both r and i can be replaced by special maps, r' and i' as follows: let

$$(X, x_0) = ((X_\alpha, x_\alpha), p_\alpha^{\alpha'}), \quad (A, x_0) = ((A_\alpha, x_\alpha), \tilde{p}_\alpha^{\alpha'}) \quad \text{and} \quad p_\alpha^{\alpha'} i_{\alpha'} = i_\alpha \tilde{p}_\alpha^{\alpha'} \quad \text{for} \quad \alpha' \geqslant \alpha;$$

let

$$(X', x'_0) = ((X_{\varrho(\alpha)}, x_{\varrho(\alpha)}), p_{\varrho(\alpha)}^{\varrho(\alpha')}), \quad (A', x'_0) = ((A_{\varrho(\alpha)}, x_{\varrho(\alpha)}), \tilde{p}_{\varrho(\alpha)}^{\varrho(\alpha')}) ,$$

$$r' = (1, r_\alpha)\colon (X', x'_0) \to (A, x_0), \quad i' = (1, i_{\varrho(\alpha)})\colon (A', x'_0) \to (X', x'_0)$$

and

$$e = (1, \tilde{p}_\alpha^{\varrho(\alpha)})\colon (A', x'_0) \to (A, x_0) .$$

Since e is homotopic to $\mathbf{1}_{(A,x_0)}$, hence

(2) e is a homotopy equivalence.

By (1) we get

(3) $$r'i' \simeq e .$$

Take $n \geqslant 1$ and let $[i'_n]$, $[r'_n]$ and $[e_n]$ be the induced morphisms of nth homotopy pro-groups. Obviously, by (2) and (3), $[e_n]$ is an isomorphism in pro-$\mathcal{G}$ and

(4) $$[r'_n][i'_n] = [e_n];$$

thus, by 4.1,

(5) $[r'_n]$ is an epimorphism in pro-$\mathcal{G}$

and

(6) $[i'_n]$ is a monomorphism in pro-$\mathcal{G}$.

Let $n \geqslant 2$. By (4) and 4.4 it follows that

(7) $\ker[r'_n] \oplus \operatorname{im}[i'_n]\colon \operatorname{Ker}[r'_n] \times \operatorname{Im}[i'_n] \to \pi_n(X', x'_0)$ is an isomorphism.

Consider the natural diagram for $[i'_n]$,

$$\begin{array}{ccc} \pi_n(A', x'_0) & \xrightarrow{[i'_n]} & \pi_n(X', x'_0) \\ & {\scriptstyle [i'_n]^0} \searrow & \uparrow {\scriptstyle \operatorname{im}[i'_n]} \\ & & \operatorname{Im}[i'_n] \end{array}$$

By (6) and 3.8(a) it follows that $[i'_n]^0$ is an isomorphism.

Consider now the exact sequence of homotopy pro-groups for (X', A', x'_0) (see [7] II, 1.2).

$$\mathcal{D}_n\colon \pi_n(A', x'_0) \xrightarrow{[i'_n]} \pi_n(X', x'_0) \xrightarrow{[\xi_n]} \pi_n(X', A', x'_0) \xrightarrow{[\partial_n]} \pi_{n-1}(A', x'_0).$$

Notice that all the morphisms i_n, ξ_n and ∂_n are normal. Indeed, for i_n and ∂_n the corresponding objects belong to pro-$\mathcal{G}_0$; for ξ_n we have

$$\xi_\alpha(\pi_n(X_\alpha, x_\alpha)) = \operatorname{Ker}\partial_{n\alpha},$$

thus $\xi_\alpha(\pi_n(X_\alpha, x_\alpha))$ is a normal subgroup of $\pi_n(X_\alpha, A_\alpha, x_\alpha)$ (even for $n = 2$, i.e. for the non-abelian case). Thus for these morphisms all the results of § 3 may be applied.

Let

(8) $$f_n = \xi_n \ker r'_n\colon \operatorname{Ker}[r'_n] \to \pi_n(X', A', x'_0).$$

Let us prove $[f_n]$ to be an isomorphism in pro-$\mathcal{G}$. Since pro-$\mathcal{G}$ is balanced it suffices to show that $[f_n]$ is a bimorphism in pro-$\mathcal{G}$. We have

$$f_n = (1, f_\alpha), \quad f_\alpha = \xi_{\varrho(\alpha)} \ker r'_{\alpha n}\colon \operatorname{Ker} r'_{\alpha n} \to \pi_n(X_{\varrho(\alpha)}, A_{\varrho(\alpha)}, x_{\varrho(\alpha)}).$$

First, let us prove

(9) $$\bigwedge_{\alpha} \bigvee_{\alpha' \geqslant \alpha} (p^{\varrho(\alpha')}_{\varrho(\alpha)})_n(\operatorname{Ker} f_{\alpha'}) = 0.$$

By (1), we have

(10) $$\bigwedge_{\alpha} \bigvee_{\alpha' \geqslant \alpha} r_\alpha i_{\varrho(\alpha)} \tilde{p}^{\alpha'}_{\varrho(\alpha)} \simeq p^{\alpha'}_{\alpha}.$$

Take an α. By (10) there is an $\alpha' \geqslant \varrho(\alpha)$ satisfying

(10′) $$r_{\varrho(\alpha)} i_{\varrho^2(\alpha)} \tilde{p}^{\alpha'}_{\varrho^2(\alpha)} \simeq \tilde{p}^{\alpha'}_{\varrho(\alpha)}.$$

By the assumption, $\varrho(\alpha') \geqslant \alpha'$. Then

(10″) $$r_{\varrho(\alpha)} i_{\varrho^2(\alpha)} \tilde{p}^{\varrho(\alpha')}_{\varrho^2(\alpha)} \simeq \tilde{p}^{\varrho(\alpha')}_{\varrho(\alpha)}.$$

Let $[\varphi] \in \operatorname{Ker} f_{\alpha'}$; then $f_{\alpha'}[\varphi] = \xi_{\varrho(\alpha')}[\varphi] = 0$, whence $[\varphi] \in \operatorname{Ker}\xi_{\varrho(\alpha')}$. Thus, by the exactness of homotopy sequence for $(X_{\varrho(\alpha')}, A_{\varrho(\alpha')}, x_{\varrho(\alpha')})$, we get $[\varphi] \in \operatorname{Im} i_{\varrho(\alpha')n}$, i.e. there exists $[\varphi'] \in \pi_n(A_{\varrho(\alpha')}, x_{\varrho(\alpha')})$ such that

(11) $$[i_{\varrho(\alpha')}\varphi'] = [\varphi].$$

Since $[\varphi] \in \operatorname{Ker} f_{\alpha'} \subset \operatorname{Ker} r'_{\alpha' n}$, we have

(12) $$[r_{\alpha'}\varphi] = 0.$$

By (10″), (11) and (12) we get

$$\begin{aligned}[] [\tilde{p}^{\varrho(\alpha')}_{\varrho(\alpha)}\varphi'] &= [r_{\varrho(\alpha)} i_{\varrho^2(\alpha)} \tilde{p}^{\varrho(\alpha')}_{\varrho^2(\alpha)}\varphi'] = [r_{\varrho(\alpha)} p^{\varrho(\alpha')}_{\varrho^2(\alpha)} i_{\varrho(\alpha')}\varphi'] = [r_{\varrho(\alpha)} p^{\varrho(\alpha')}_{\varrho^2(\alpha)}\varphi] \\ &= [\tilde{p}^{\alpha'}_{\varrho(\alpha)} r_{\alpha'}\varphi] = 0; \end{aligned}$$

thus

$$[p^{\varrho(\alpha')}_{\varrho(\alpha)}\varphi] = [p^{\varrho(\alpha')}_{\varrho(\alpha)} i_{\varrho(\alpha')}\varphi'] = [i_{\varrho(\alpha)} \tilde{p}^{\varrho(\alpha')}_{\varrho(\alpha)}\varphi'] = [i_{\varrho(\alpha)}](0) = 0,$$

i.e. $[\varphi] \in \operatorname{Ker}(p^{\varrho(\alpha')}_{\varrho(\alpha)})_n$, which proves (9). By 3.9 and 3.10, the condition (9) implies

(13) $[f_n]$ is a monomorphism in pro-$\mathcal{G}$.

Now, let us prove

(14) $[f_n]$ is an epimorphism in pro-$\mathcal{G}$.

Indeed, by the exactness of $\mathcal{D}_n$, together with 3.9, since (by (6)) $[i'_{n-1}]$ is a monomorphism, it follows that

$$\operatorname{Ker}[i'_{n-1}] = \mathbf{0} = \operatorname{Im}[\partial_n];$$

thus, by 3.8(c), we get

(15) $\ker[\partial_n]\colon \operatorname{Ker}[\partial_n] \to \pi_n(X', A', x'_0)$ is an isomorphism.

By (15) together with the exactness of $\mathcal{D}_n$,

(16) $\operatorname{im}[\xi_n]\colon \operatorname{Im}[\xi_n] \to \pi_n(X', A', x'_0)$ is an isomorphism,

thus, by 3.8(b), $[\xi_n]$ is an epimorphism. By 3.1, we have

(17) $$[\xi_n](\operatorname{im}[i_n]) = [\mathbf{0}].$$

By (17), 4.3 and (8), we get

(18) $$[f_n] = [\xi_n]\ker[r'_n] \oplus [\xi_n]\operatorname{im}[i'_n] = [\xi_n](\ker[r'_n] \oplus \operatorname{im}[i_n]),$$

thus, by (7), $[f_n]$ is an epimorphism, i.e. (14) is satisfied. By (13) and (14), $[f_n]$ is an isomorphism.

The condition (7) implies

(19) $$\pi_n(X', x'_0) \approx \operatorname{Ker}[r'_n] \times \operatorname{Im}[i'_n].$$

Since both $[i_n]^0\colon \pi_n(A', x'_0) \to \operatorname{Im}[i'_n]$ and $[f_n]\colon \operatorname{Ker}[r'_n] \to \pi_n(X', A', x'_0)$ were proved to be isomorphisms, we obtain

$$\pi_n(X', x_0') \approx \pi_n(X', A', x_0') \times \pi_n(A', x_0'),$$

whence obviously

$$(20) \qquad \pi_n(X, x_0) \approx \pi_n(A, x_0) \times \pi_n(X, A, x_0).$$

By (6) and (20) the proof of 6.2 is complete. ■

Proof of Theorem 6.1. Let (X, A, x_0) be an inverse sequence of pointed pairs of connected ANR's associated with (X, A, x_0). Let $\underline{\underline{i}}$ be the shape map generated by the inclusion $i\colon (A, x_0) \to (X, x_0)$. Since (A, x_0) is a shape retract of (X, x_0), the sequence (A, x_0) is a retract of (X, x_0) (see [4], [5]). Thus, by Theorem 6.2,

$$\pi_n(X, x_0) \approx \pi_n(A, x_0) \times \pi_n(X, A, x_0) \quad \text{for} \quad n \geqslant 2$$

and $[i_n]\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ is a monomorphism in pro-$\mathcal{G}$ for $n \geqslant 1$. Thus, passing to inverse limits and applying 1.3, one gets

$$\text{(a)} \qquad \check{\pi}_n(X, x_0) \approx \check{\pi}_n(A, x_0) \times \check{\pi}_n(X, A, x_0) \quad \text{for} \quad n \geqslant 2,$$

and, by 2.3 of [6],

(b) $\underline{\underline{i}}_n\colon \check{\pi}_n(A, x_0) \to \check{\pi}_n(X, x_0)$ is a monomorphism in $\mathcal{G}$ for $n \geqslant 1$. ■

Theorem 6.1 is an analogue of Proposition 5.1 [2], p. 150.

6.3. Corollary. *If (A, x_0) is a shape retract of (X, x_0) then the shape group $\check{\pi}_2(X, A, x_0)$ is abelian.* ■

7. Shape groups and shape deformability. Consider a pointed compact connected pair (X, A, x_0) and let $\underline{\underline{i}}$ be the shape map generated by the inclusion $i\colon (A, x_0) \to (X, x_0)$.

A shape map $\underline{\underline{f}}\colon (X, x_0) \to (A, x_0)$ is said to be a *shape deformation* provided $\underline{\underline{i}}\,\underline{\underline{f}} = \underline{\underline{1}}_{(X, x_0)}$.

If a shape deformation $\underline{\underline{f}}\colon (X, x_0) \to (A, x_0)$ does exist, then (X, x_0) is said to be *shape deformable* into (A, x_0).

We are going to prove the following theorem concerning shape groups:

7.1 Theorem. *If (X, x_0) is shape deformable into (A, x_0), then*

$$\text{(a)} \qquad \check{\pi}_n(A, x_0) \approx \check{\pi}_n(X, x_0) \times \check{\pi}_{n+1}(X, A, x_0) \quad \textit{for} \quad n \geqslant 2$$

and

(b) *the homomorphism $\underline{\underline{i}}_n\colon \check{\pi}_n(A, x_0) \to \check{\pi}_n(X, x_0)$ is an epimorphism for $n \geqslant 1$.*

Let us start by proving the corresponding statement for pro-groups.

Consider an inverse sequence $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0)$ of pointed pairs of metric continua and let $\boldsymbol{i}\colon (\boldsymbol{A}, \boldsymbol{x}_0) \to (\boldsymbol{X}, \boldsymbol{x}_0)$ be the inclusion.

A map $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{A}, \boldsymbol{x}_0)$ is said to be a *deformation* provided $\boldsymbol{if} \simeq \boldsymbol{1}_{(\boldsymbol{X}, \boldsymbol{x}_0)}$. The sequence $(\boldsymbol{X}, \boldsymbol{x}_0)$ is said to be *deformable* into $(\boldsymbol{A}, \boldsymbol{x}_0)$ whenever there exists a deformation $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{A}, \boldsymbol{x}_0)$.

7.2. Theorem. *If $(\boldsymbol{X}, \boldsymbol{x}_0)$ is deformable into $(\boldsymbol{A}, \boldsymbol{x}_0)$ then*

$$\text{(a)} \qquad \pi_n(\boldsymbol{A}, \boldsymbol{x}_0) \approx \pi_n(\boldsymbol{X}, \boldsymbol{x}_0) \times \pi_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0) \quad \textit{for} \quad n \geqslant 2$$

and

(b) *$[\boldsymbol{i}_n]\colon \pi_n(\boldsymbol{A}, \boldsymbol{x}_0) \to \pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$ is an epimorphism in* pro-$\mathcal{G}$ *for $n \geqslant 1$.*

Proof of 7.2. By the assumption, there is a map $\boldsymbol{f} = (\varphi, f_\alpha)\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{A}, \boldsymbol{x}_0)$ such that $\varphi(\alpha) \geqslant \alpha$ for every α (see [6] 3.8), and

$$(1) \qquad \boldsymbol{if} \simeq \boldsymbol{1}_{(\boldsymbol{X}, \boldsymbol{x}_0)}.$$

Let $(\boldsymbol{X}, \boldsymbol{x}_0) = (X_\alpha, p_\alpha^{\alpha'})$ and let $(\boldsymbol{X}', \boldsymbol{x}_0') = (X_{\varphi(\alpha)}, p_{\varphi(\alpha)}^{\varphi(\alpha')})$. Take

$$\boldsymbol{f}' = (1, f_\alpha)\colon (\boldsymbol{X}', \boldsymbol{x}_0') \to (\boldsymbol{A}, \boldsymbol{x}_0)$$

and the homotopy equivalence

$$\boldsymbol{e} = (1, p_\alpha^{\varphi(\alpha)})\colon (\boldsymbol{X}', \boldsymbol{x}_0') \to (\boldsymbol{X}, \boldsymbol{x}_0).$$

The condition (1) implies

$$(2) \qquad \boldsymbol{if}' \simeq \boldsymbol{e}.$$

and thus

$$(3) \qquad [\boldsymbol{i}_n][\boldsymbol{f}_n'] = [\boldsymbol{e}_n] \quad \text{for} \quad n \geqslant 1.$$

By (3) and 4.1 we get

(4) $[\boldsymbol{f}_n']$ is a monomorphism in pro-$\mathcal{G}$ for $n \geqslant 1$

and

(5) $[\boldsymbol{i}_n]$ is an epimorphism in pro-$\mathcal{G}$ for $n \geqslant 1$.

Take $n \geqslant 2$. By 4.4, the condition (3) implies

$$(6) \qquad \pi_n(\boldsymbol{A}, \boldsymbol{x}_0) \approx \operatorname{Ker}[\boldsymbol{i}_n] \times \operatorname{Im}[\boldsymbol{f}_n'].$$

The statement 3.8(a) together with (4) imply

$$(7) \qquad \operatorname{Im}[\boldsymbol{f}_n] \approx \pi_n(\boldsymbol{X}, \boldsymbol{x}_0).$$

Let us consider the exact sequence of homotopy pro-groups for $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0)$,

$$\mathcal{D}_{n+1}\colon \pi_{n+1}(\boldsymbol{A}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{i}_{n+1}]} \pi_{n+1}(\boldsymbol{X}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{\zeta}_{n+1}]} \pi_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{\partial}_{n+1}]} \pi_n(\boldsymbol{A}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{i}_n]} \pi_n(\boldsymbol{X}, \boldsymbol{x}_0).$$

By (5), $[\boldsymbol{i}_{n+1}]$ is an epimorphism, thus, by 3.11, we get

$$(8) \qquad \operatorname{Ker}[\boldsymbol{i}_n] \approx \pi_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0).$$

By (6), (7) and (8) it follows that

$$\pi_n(\boldsymbol{A}, \boldsymbol{x}_0) \approx \pi_n(\boldsymbol{X}, \boldsymbol{x}_0) \times \pi_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0) \quad \text{for} \quad n \geqslant 2.$$

This, together with (5), completes the proof of 7.2. ■

Proof of Theorem 7.1. Let (X, x_0) be shape deformable into (A, x_0), i.e. there is a shape map $\underline{f}$: $(X, x_0)\to(A, x_0)$ such that

(1) $$\underline{i}\,\underline{f} = \underline{1}_{(X,x_0)}\,.$$

Then, there is an ANR-sequence (X, A, x_0) and a representative f: $(X, x_0)\to(A, x_0)$ of $\underline{f}$, such that

$$if\simeq \mathbf{1}_{(X,x_0)}\,,$$

i.e. the system (X, x_0) is deformable into (A, x_0). Hence, by Theorem 7.2,

$$\pi_n(A, x_0)\approx\pi_n(X, x_0)\times\pi_{n+1}(X, A, x_0)\quad \text{for}\quad n\geqslant 2\,.$$

Passing to inverse limits and applying 1.3, we obtain the condition (a) for shape groups. The condition (b) follows directly by (1) and 4.1. ■

Theorem 7.1 is an analogue of Proposition 5.2, p. 151 [2].

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On the Lusternik-Schnirelmann category in the theory of shape

by

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Abstract. A modification of the notion of the Lusternik–Schnirelmann category gives a monotonous shape invariant $\underline{\varkappa}(X)$ defined for all compacta X. Some properties of $\underline{\varkappa}(X)$ are established. In particular it is shown that if X is a continuum, then $\underline{\varkappa}(X)\leqslant \mathrm{Fd}(X)+1$, where $\mathrm{Fd}(X)$ denotes the fundamental dimension of X.

1. Coefficients $\varkappa(X)$ and $\underline{\varkappa}_M(X)$. By the Lusternik-Schnirelmann (absolute) category of a compactum X one understands (compare [1], [5] and [7]) the number $\varkappa(X)$ defined as follows:

If there exist natural numbers n such that $X = X_1\cup X_2\cup\ldots\cup X_n$, where X_i are (for $i = 1, 2, \ldots, n$) compacta contractible in X, then $\varkappa(X)$ denotes the smallest of such numbers n.

If such natural numbers n do not exist, then $\varkappa(X) = \infty$.

Observe that

(1.1) *If compactum X homotopically dominates compactum Y, then $\varkappa(X)\geqslant\varkappa(Y)$.*

In fact, assume that there exist two maps

$$f\colon X\to Y \quad\text{and}\quad g\colon Y\to X$$

such that fg is homotopic to the identity map $i_Y\colon Y\to Y$. If $\varkappa(X)\leqslant n$, then there exist compacta $X_1, X_2, \ldots, X_n$ such that $X = X_1\cup X_2\cup\ldots\cup X_n$ and that for $i = 1, 2, \ldots, n$ there is a homotopy

$$\varphi_i\colon X_i\times\langle 0, 1\rangle\to X$$

satisfying the conditions

$$\varphi_i(x, 0) = x \quad\text{and}\quad \varphi_i(x, 1) = a_i\,,$$

where a_i is a fixed point of X.

Setting $Y_i = g^{-1}(X_i)$ for $i = 1, 2, \ldots, n$, one gets compacta $Y_1, \ldots, Y_n$ such that $Y = Y_1\cup\ldots\cup Y_n$. It remains to show that Y_i is contractible in Y.