Dendroids and their endpoints

by

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Abstract. In this paper we introduce some non-negative real valued functions (which are of the 1st class of Baire) defined on dendroids. These functions can be used to characterize several properties of dendroids. We apply some of them to prove that in every dendroid $X$ the minimal arcwise connected set spanning the set of all endpoints of $X$ at which $X$ is semi-locally connected is a dense subset of $X$. Moreover, the set of endpoints at which $X$ is semi-locally connected is a $G_δ$-set in $X$ and the remaining endpoints form a subset of the first category in $X$.

1. Introduction. All spaces under considerations are assumed to be metric. By a continuum we mean a compact connected space. As usual by a dendroid is understood an arcwise connected hereditarily unicoherent continuum (see [2]). A continuum $X$ is said to be semi-locally connected at a point $x ∈ X$ provided that for every neighbourhood $U$ of $x$ there is a neighbourhood $V$ of $x$ contained in $U$ whose complement consists of a finite number of components (see [8, p. 19]). By a neighbourhood we always mean an open set. We say that $x$ is an endpoint of $X$ if there is no arc in $X$ containing $x$ in its interior. We follow [7] in writing $X^*$ for the set of endpoints of $X$. Denote by $X^*_c$ the set of points of $X^*$ at which $X$ is semi-locally connected.

In this paper we study the class of dendroids. We introduce a collection of non-negative real valued functions $α_n$ and $α_n^*$ defined on dendroids. The functions $α_n$ are of the 1st class of Baire (see [5]). In Section 3 these functions are used to characterize the following topological properties of dendroids: smoothness, semi-local connectedness, local connectedness and uniform arcwise connectedness. In the last two sections we apply these functions to prove some geometrical properties of dendroids. From now on by $X$ is denoted an arbitrary dendroid (not a one-point set).

B. J. Fugate raised the question if the set $X^*_c$ is not empty. Theorem 4.1 provides an affirmative answer to this question. We show also that $X^*_c$ is a $G_δ$ subset of $X$ (see 3.5). The set $X^*_c$ can be a second category subset of $X$. In fact one easily checks that the dendroid $X$ constructed in [7, p. 314] has this property, because the set $X^*_c$ is dense in $X$. However the set $X^*_c \setminus X_0^*$ is always a first category subset of $X$ (Theorem 5.1). Nevertheless, the example [7, p. 311] shows that $X^*_c \setminus X_0^*$ can be dense in $X$. The set $X_0^*$ is always "large" in the sense that the minimal arcwise connected set spanning it is dense in $X$ (Theorem 4.1). But the complement of the spanning set need not be of the first category in $X$ (Example 4.7).
We sincerely thank Professor D. Bellamy for his valuable remarks during the preparation of this paper.

2. On the functions $\alpha$. An arc with endpoints $a$ and $b$ will be denoted by $ab$. The arc $ab$ will be, in some cases, regarded as an ordered arc with $a$ as the first and $b$ as the last point. An order will be denoted by the symbol "<". Let $ab$ be an arc in a space $Y$ and let $U_1, U_2, ...$ be a sequence of subsets of $Y$. We say that the ordered arc $ab \in Y$ has type $(U_1, U_2, ...)$, and write $ab \in (U_1, U_2, ...)$, if there exists a sequence of points $a_1, a_2, ...$ satisfying the conditions:

\[ a_n \in ab \cap U_n \quad \text{for each } n \geq 1, \]

\[ a < a_1 < a_2 < ... < b. \]

(\*)

(\**)

It is not assumed in the definition above that $U_1, U_2, ...$ is an infinite sequence nor that $U_i \neq U_j$ for $i \neq j$.

In the case where $U_1, U_2, ...$ is a finite sequence consisting of $n$ terms we sometimes write $(U_1, U_2, ...)_n$, instead of $(U_1, U_2, ...)$, Similarly, if the sequence is infinite we sometimes write $(U_1, U_2, ...)_\infty$ instead of $(U_1, U_2, ...)$.

2.1. LEMMA. Let $U_1, U_2, ..., U_n$ be a finite sequence of open sets in a hereditarily unicoherent continuum $Y$ and let $ab$ be an arc in $Y$ of type $(U_1, U_2, ..., U_n)$. Then there are neighbourhoods $U$ of $a$ and $V$ of $b$ such that any arc in $Y$ joining $U$ and $V$ has type $(U_1, U_2, ..., U_n)$.

Proof. It is easy to check that the lemma holds for each arc of type $(U_1, U_2, ..., U_n)$, where $U$ is open in $Y$. Assume we have proved the lemma for all arcs of type $(U_1, ..., U_{n-1})$, where $n \geq 2$. From the definition it follows that there exists an arc $a_{n-1} \in ab \cap (U_n \setminus \{a, b\})$ such that the arc $aa_{n-1} \subseteq ab$ has type $(U_1, ..., U_{n-1})$ and the arc $a_{n-1}b$ has type $(U_n)$. From the assumption it follows that there exist open sets $U' \ni a$ and $V' \ni b$ such that each arc joining $U'$ and $G$ has type $(U_1, ..., U_{n-1})$ and each arc joining $G$ and $V'$ is of type $(U_n)$. Let $U \ni a$ and $V \ni b$. Such open sets do exist that each arc joining them has type $(U_1, ..., U_n)$. Clearly, we may assume that $U \subseteq U'$ and $V \subseteq V'$. Now it is easily seen that each arc $U$ joining $U$ and $V$ is of type $(U_1, ..., U_n)$. This completes the proof.

2.2. LEMMA. Let $U_1, U_2, ...$ be an infinite sequence of open sets in a space $Y$. If $ab \subseteq Y$ is an arc of type $(U_1, U_2, ...)$, then lim $d(U_n, U_{n+1}) = 0$ (for two sets $A, B \subseteq Y$ we define $d(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$).

Proof. Let $a_n \in U_n \cap ab, n = 1, 2, ...$, satisfy the relations:

\[ a < a_1 < a_2 < ... < b. \]

Observe that $\{a_n\}$ is a convergent sequence. Hence the conclusion follows.

2.3. LEMMA. Let $U_1, U_2, ...$ be an infinite sequence of open sets in a space $Y$, such that $U_n \cap U_{n+1} = \emptyset$ for each $n \geq 1$. If $ab$ is an arc in $Y$ of type $(U_1, U_2, ..., U_n)$ for each $n \geq 1$, then $ab$ is of type $(U_1, U_2, ...)_\infty$.

Proof. Define a sequence $a_1', a_2', ...$ in the following way (recursively):

\[ a_1' = \inf \{x \in ab : x \in U_1\}, \]

\[ a_{n+1}' = \inf \{x \in a_n' b : x \in U_{n+1}\} \]

(the infimum is taken with respect to the order on $ab$). Notice that the construction is possible. Observe that $a_n' \in U_n \cap a_{n+1}'$ for each $n \geq 1$. The points $a_n'$ lie on the arc $ab$ in the order:

\[ a < a_1' < a_2' < ... < b. \]

and $a_n' < a_{n+1}' \subseteq ab$ contains some point $a_n \in U_n$ in its interior for each $n \geq 1$. The points $a_1', a_2', ...$ satisfy conditions (\*) and (\**), which completes the proof.

Recall that by $X$ we always denote an arbitrary dendroid. For any two points $x, y \in X$ by the arc $xy$ is meant the unique arc in $X$ joining $x$ and $y$ provided $x \neq y$, and the one-point set $\{x\}$ in the case $x = y$.

Fix a point $a \in X$ and a natural number $n \geq 1$. For an arbitrary point $x \in X$ consider all the points $y \in ax$ such that for every neighbourhood $U$ of $x$ and every neighbourhood $V$ of $y$ there is an arc $a \in (U, V, U, V, ...)_n$. Observe that the points $y$ constitute a subarc of $ax$ with one endpoint at $x$. Denote this arc by $A_x(x, a)$ and define

\[ \alpha_x(x) = \inf \{\varepsilon > 0 : A_x(x, a) \subseteq K(x, \varepsilon)\}, \]

where $K(x, \varepsilon)$ denotes the $\varepsilon$-ball around $x$. Let us note that $A_x(x, a) \supseteq A_n(x, a)$ for $n \leq m$. It follows that the intersection $A_x(x, a) = \bigcap_{n=1}^\infty A_x(x, a)$ is again an arc.

Let us define analogously

\[ \alpha_n(x) = \inf \{\varepsilon > 0 : A_n(x, a) \subseteq K(x, \varepsilon)\}. \]

2.4. PROPOSITION. For each $a, x \in X$ and for natural numbers $n \leq m$ we have $\alpha_n(x) \geq \alpha_m(x)$, and $\alpha_n(x) = \inf \{\alpha_n(x)\}$.

In the definitions below let us agree to denote by $n$ a natural number or $\infty$.

The above formulas define a nonnegative real-valued functions

\[ \tau_n : X \rightarrow R, \]

Define also a function

\[ \tau : X \rightarrow R \]

obtained from the preceding ones as follows

\[ \tau_n(x) = \sup \{\tau_n(x) : a \in X\}, \]

2.5. PROPOSITION. For each $x \in X$ and for natural numbers $n \leq m$ we have $\tau_n(x) \geq \tau_n(x)$ and $\tau_n(x) \leq \inf \{\tau_n(x)\}$.

We shall show later that the last inequality can be replaced by an equality.
2.6. Lemma. Let $n$ be a natural number and let $x_1, x_2, \ldots$ be a sequence in $X$ converging to a point $x$. If $a_1, a_2, \ldots$ is another sequence such that $$a_{nm}(x_n) \rightarrow r$$ for some $r \in R$, then there exists a point $a \in X$ such that $a_{nm}(x_n) \geq r$.

Proof. For each $k \geq 1$ there is a point $x_k \in A_k(x_k, a_k)$ such that $a_k = a_{nm}(x_k)$. We may assume that the sequence $\{x_k\}$ is convergent. Denote its limit by $a$ and note that $a = a_{nm}(x)$ = $r$. Take a neighbourhood $U$ of $a$ and a neighbourhood $V$ of $y$. To complete the proof it suffices to construct an arc $\sigma$ such that

$$\sigma \in (V, U, V, \ldots)_{a_{nm}(x)}$$

We may assume that $a \neq x$. Then for some index $k$ there exist a neighbourhood $U_0$ of $x_k$ and a neighbourhood $V_0$ of $x_k$ such that

$$U_0 \subset U, \quad V_0 \subset V,$$

and (2) we have $a \in (V, U)$. This fact together with (5) and (8) imply that $a_{nm}(x)$ satisfies (1). Hence in the sequel we assume that (10) is fulfilled.

Conditions (10), (9), (7), (6) and (3) imply that the set $U_0 \cup (a \cup p)$ separates $X$ between $ab \cup p$ and $\mathfrak{x}_k$. It follows that there exists a neighbourhood $V_1 \subset U_0$ of $x_k$ such that every subsequence joining $ab \cup p$ with $V_1$ meets $U_0$. Since $x_k \in A_k(x_k, a_k)$, there is an arc $a_{n_k} q$ and a sequence $e_1, e_2, \ldots$ of $n \geq 2$ points such that

$$a_{n_k} q \in (V_1, U, V_1, \ldots)_{a_{nm}(x)}$$

We shall show that $q$ satisfies condition (1). Denote by $d$ the last point on $a_{n_k} q$ belonging to $a_{n_k} q$. First assume $a < d < c$. Then

$$a_{n_k} q = ab \cup bd \cup dq.$$

By the assumption, (8), (11) and (12) we have $a < d < q$. Since $ab \in (V)$ and $bd \in (U)$, conditions (11), (12) and (2) imply (1). Next assume $c < d \leq p$. Let $e$ be the first point on $ad$ belonging to $dq$. Then $aq = ae \cup eq$. We claim that $e < v < q$. Otherwise $v \in ed$ or $d$. But $d < c < d \leq p$. However this is impossible by (8) and (11). Now, $e < v < c$ and $v \in V_1$, hence $e_{n_k} q \in (U_0)$, by the construction of $V_1$. Finally, since $e \in (V)$ and $ae \leq c$, condition (1) follows from (12). This completes the proof of 2.6.

2.7. Lemma. Let $x_1, x_2, \ldots$ be a sequence in $X$ converging to a point $x \in X$. Let $n_1, n_2, \ldots$ be either a strictly increasing sequence of natural numbers or a constant sequence $n_0 = a$. If $a_1, a_2, \ldots$ is another sequence in $X$ such that

$$a_{nm}(x_k) \rightarrow r$$

for some $r \in R$, then there exists a point $a \in X$ such that $a_{nm}(x_k) \geq r$.

Proof. The proof is like the proof of the preceding lemma, but simpler. For each $k \geq 1$ there is a point $x_k \in A_k(x_k, a_k)$ such that $a_k = a_{nm}(x_k)$. We may assume that $\{x_k\}$ converges to some point $a \in X$. Note that $a_{nm}(x) = r$. Take a neighbourhood $U$ of $x$, a neighbourhood $V$ of $a$, and a natural number $m$. To complete the proof it suffices to show that there is an arc $a \in (V, U, V, \ldots)_{a_k}$. Take $a$ so large that $a > 2m$, $x \in U$ and $x_k \in V$. There is an arc $a_{n_k} q \in (V, U, \ldots, V, U)_{a_k}$. It is easy to see that for $x = a_k$ or $z = b$ the arc $a$ has the required properties. This completes the proof.

2.8. Corollary. For each $x \in X$ and for each $n \in \{1, 2, \ldots, \omega\}$ there exists a point $a \in X$ such that

$$a_{nm}(x) = a_{nm}(x).$$

Proof. Let $a_1, a_2, \ldots$ be a sequence such that

$$a_{nm}(x) = a_{nm}(x).$$
By 2.6 or 2.1 there is a point \( a \in X \) such that \( \alpha_n(x) \geq \alpha_n(x) \). Hence the conclusion follows from the definition of \( \alpha_n(x) \).

2.9. Corollary. For each \( x \in X \) we have

\[
\alpha_n(x) = \inf_{k \to \infty} \alpha_k(x).
\]

Proof. By 2.5 we have \( \alpha_n(x) \leq \inf_{k \to \infty} \alpha_k(x) \). Again by 2.5 the sequence \( \alpha_n(x), \alpha_n(x), \ldots \) converges to \( r \). By 2.8 for each \( k \geq 1 \) there is a point \( a_k \in X \) such that \( \alpha_k(x) = \alpha_{nk}(x) \). Hence

\[
\lim_{k \to \infty} \alpha_{nk}(x) = r.
\]

Using 2.7 we get a point \( a \in X \) such that \( \alpha_n(x) \geq r \). It follows that

\[
\alpha_n(x) \geq \alpha_{nk}(x) \geq r = \inf_{k \to \infty} \alpha_k(x),
\]

which completes the proof.

2.10. Theorem. For each \( n \in \{1, 2, \ldots, \omega \} \) the function \( \alpha_n : X \to R \) is upper semi-continuous, i.e. for each \( r \in R \) the set \( \alpha_n^{-1}(r, \infty) \) is closed in \( X \).

Proof. Let \( x_1, x_2, \ldots \) be a convergent sequence of points from \( \alpha_n^{-1}(r, \infty) \) and let \( x \) denote its limit. We have to prove that \( \alpha_n(x) \geq r \). For each \( k \geq 1 \) by 2.8 there is a point \( a_k \) such that \( \alpha_k(x) = \alpha_{nk}(x) \). Without loss of generality we may assume that the sequence \( \{\alpha_n(x_k)\} \) converges. Denote its limit by \( s \). Clearly, \( s \geq r \). By 2.6 or 2.7 there is a point \( x \in X \) such that \( \alpha_n(x) \geq s \). Hence the theorem is proved because \( \alpha_n(x) \geq \alpha_n(x) \).

Remark. Observe that if \( \alpha_n(x) \) vanishes on \( X \) for some \( a \in X \), then so does \( \alpha_n(x) \).

2.11. Corollary. In every closed subset \( X \) the function \( \alpha_n \) attains its greatest lower bound, for \( n \in \{1, 2, \ldots, \omega \} \).

Recall that a function \( f : Y \to Z \) is of the 1st class of Baire if for every open subset \( U \) of \( Z \) the inverse image \( f^{-1}(U) \) is an \( F_{(\omega)} \)-set in \( Y \) (comp. [4, p. 373]).

In the next theorem we show that each function \( \alpha_n \), \( n \in \{1, 2, \ldots, \omega \} \), is of the 1st class of Baire.

2.12. Theorem. For each \( n \in \{1, 2, \ldots, \omega \} \) and for every open subset \( U \) of \( R \) the inverse image \( \alpha_n^{-1}(U) \) is an \( F_{(\omega)} \)-set in \( X \). Consequently, the set of points of \( X \) at which \( \alpha_n \) is not continuous is an \( F_{(\omega)} \)-set in the first category of \( X \).

Proof. The set \( U \) can be represented as a union of open intervals, \( U = \bigcup_{k=1}^{\infty} (r_k, s_k) \), where \( r_k < s_k \). Then

\[
\alpha_n^{-1}(U) = \bigcup_{k=1}^{\infty} \alpha_n^{-1}(r_k, s_k).
\]

To complete the proof of the first part of 2.12 it suffices to show that the inverse image of an open interval \( (r, s) \), \( r < s \), in an \( F_{(\omega)} \)-set. But

\[
\alpha_n^{-1}(r, s) = X \setminus \bigcup_{k=1}^{\infty} \alpha_n^{-1}((r_k, s_k)),
\]

Now,

\[
\alpha_n^{-1}(r, \infty) = X \setminus \bigcup_{k=1}^{\infty} \alpha_n^{-1}((r_k + 1/k, \infty))
\]

and \( \alpha_n^{-1}(s, \infty) \) are \( G_{(\omega)} \)-sets by 2.10. Hence \( \alpha_n^{-1}((r, s)) \) is an \( F_{(\omega)} \)-set, as required.

Let \( M \) denote the set of points at which \( \alpha_n \) is coniguous. Let \( U_1, U_2, \ldots \) be a base for open sets in \( X \). It is easily seen that

\[
M = \bigcup_{k=1}^{\infty} \alpha_n^{-1}(U_k) \setminus \operatorname{Int} \alpha_n^{-1}(U_k).
\]

By the first part of 2.12 we obtain the conclusion.

3. Some properties of dendroids characterized by means of the functions \( \alpha_n \). In this section we express some topological properties of dendroids using the functions \( \alpha_n \) and \( \alpha_n(x) \). Among them are: smoothness, local connectedness, semi-local connectedness, and uniform arcwise connectedness.

Recall that a dendroid \( X \) is said to be smooth with respect to a point \( a \in X \) if for each \( x \in X \) and for every sequence \( \{x_n\} \), \( x_n \in X \), converging to \( x \), the sequence of arcs \( \{ax_n\} \) converges to \( ax \) in the Hausdorff metric dist \((\cdot, \cdot)\) (see [6, p. 47] for the definition of dist \((\cdot, \cdot)\), and [3] for the definition of smoothness). The dendroid is smooth if it smooths with respect to some point.

The following theorem characterizes smooth dendroids.

3.1. Theorem. A dendroid \( X \) is smooth with respect to \( a \in X \) if and only if the function \( \alpha_n(x) \) vanishes on \( X \).

Proof. Assume \( X \) is smooth with respect to \( a \in X \) and suppose \( \alpha_n(x) > 0 \) for some \( x \in X \). Take a point \( x \in A_n(x, 0) \setminus \{x\} \). Hence \( x \notin ax \). By the definition of \( \alpha_n(x) \), it is easy to construct a sequence of arcs \( ax_n, ax_n, \ldots \) such that \( \{x_n\} \) converges to \( x \) and \( d(x_n, ax_n) < 1/n \) for each \( n \). Hence the sequence \( \{ax_n\} \) converges to some continuum \( C \), then \( x \in C \). By our assumption \( \alpha_n(x) \) converges to \( ax \), hence \( x \notin ax \), a contradiction.

Next, assume \( \alpha_n(x) \) vanishes on \( X \) and suppose \( X \) is not smooth with respect to \( a \). Hence there is a point \( x \in X \) and a sequence \( \{x_n\} \) converging to \( x \) such that the sequence of arcs \( \{ax_n\} \) does not converge to \( ax \). Clearly, we may assume that \( \{ax_n\} \) converges to some continuum \( D \in X \). Since \( a, x \in D \), we have \( ax = bx \). Hence there is a point \( y \in D \setminus ax \). Let \( x \) be a point such that \( px \cap ax = \{x\} \). Clearly, \( x \neq y \) and \( ay = ax \cup ay \). To complete the proof it suffices to show that

\[
(1) \quad x \in A_n(y, 0).
\]

(in fact, since \( x \neq y \) this will imply \( \alpha_n(x) > q(x, y) > 0 \), contrary to our assumption).
So, let $U$ be a neighbourhood of $y$, and let $V$ be a neighbourhood of $x$. Since $ay \in (V)$, by 2.1 there is a neighbourhood $U_1 \subset U$ of $y$ such that for each $p \in U_1$, the arc $ap \in (V)$. Observe that $yx = yu \cup xu$, hence $yx \in (V)$. By the same argument as above, there is a neighbourhood $U_1 \subset U_1$ of $y$, and a neighbourhood $G$ of $x$, such that every arc joining $U_1$ and $G$ has type $(V)$. Since $y \in D_1$, $y \in U_1$ and $D$ is the topological limit of $ax_1$'s (see [6, § 43, II]), $U_1$ intersects all $ax_1$'s but a finite number of them. Since $[x_1] = x$ and $G$ is a neighbourhoud of $x$, by the above remark there is an index $m$ such that $x_m \in G$ and $ax_m \cap U_1 \neq \emptyset$. Let $p \in ax_m \cap U_2$. By the above constructions we have:

$$ap \in (V), \quad p \in U \quad \text{and} \quad px_m \in (V).$$

It follows that $ax_m \in (V', U, V)$. Hence for an arbitrary neighbourhood $U$ of $y$ and for an arbitrary neighbourhood $V$ of $x$, there is an arc $aq \in (V', U, V)$. This proves (i), and completes the proof.

3.2. COROLLARY. A dendroid $X$ is smooth if and only if there exists a point $a \in X$ such that the function $a_{ax}$ vanishes on $X$.

The following theorem characterizes the points at which $X$ is semi-locally connected.

3.3. THEOREM. A dendroid $X$ is semi-locally connected at $x \in X$ if and only if $a_{ax}(x) = 0$.

Proof. Assume $X$ is semi-locally connected at $x$ and suppose $a_{ax}(x) > 0$ for some $a \in X$. Take a point $z \in A(x, a \cap X)$. Let $G$ be a neighbourhood of $x$ such that $a \in G = \emptyset$. There is a neighbourhood $U \subset G$ of $x$ such that $X \setminus U$ has finitely many components. Let $C_U$ denote the component of $X \setminus U$ which contains $a$. Put $V = \text{Int} \, C_U$. Clearly, $a \in V$. By our supposition there is an arc $a \in (V', U, V')$ with $z \in V$. This is impossible because $a \subset C_U \subset X \setminus U$. Now, assume $a_{ax}(x) = 0$ and suppose $X$ is not semi-locally connected at $x$. Hence there is a neighbourhood $G$ of $x$ such that for every neighbourhood $U \subset G$ of $x$ the set $X \setminus U$ has infinitely many components. It is easy seen that there exists a point $a \notin G$ such that $a \notin \text{Int} \, C_U$, where $U$ is any neighbourhood of $x$ contained in $G$ and $C_U$ is the component of $X \setminus U$ containing $a$. Let $V$ be a neighbourhood of $a$ and let $U$ be as above. Take $z \in (V \cap C_U)$. Clearly, $a \in (V', U, V)$. It follows that $a_{ax}(x) \geq a_{ax}(x) \geq (a, x) > 0$, a contradiction.

By a dendrite we mean a locally connected dendroid. Theorem 3.3 implies the following.

3.4. THEOREM. A dendroid $X$ is a dendrite if and only if the function $a_{ax}$ vanishes on $X$.

Proof. Assume $X$ is a dendrite. Pick a point $x \in X$. To prove that $a_{ax}(x) = 0$ it suffices by 3.3 to show that $X$ is semi-locally connected at $x$. But this follows from [8, (13.21)], p. 20.

Now, assume $a_{ax}$ vanishes on $X$. Let $x \in X$ and let $U$ be a neighbourhood of $x$. To complete the proof it remains to show that there is a connected set $D \subset U$ such that $x \notin \text{Int} \, D$. By 3.3 the dendroid $X$ is semi-locally connected at each of its points. Since $X \setminus U$ is compact, it follows that there is a finite number of open sets $V_1, \ldots, V_m$ such that $x \notin \bigcup_j V_j$ and $X \setminus \bigcup_j V_j$ consists of a finite number of components, for each $j = 1, \ldots, m$. Let $D_j$ denote the component of $X \setminus V_j$ containing $x$. Clearly, $x \in \text{Int} \, D_j$ and $D = \bigcap_j D_j \subset U$. It follows that $x \in \text{Int} \, D$. Since $X$ is hereditarily unicoherent, $D$ is a continuum, which completes the proof.

We say that continuum $Y$ is colocally connected at a point $y \in Y$ provided that for every neighbourhood $U$ of $y$ in $Y$ there is a neighbourhood $V \subset U$ of $y$ such that $Y \setminus V$ is connected.

3.5. THEOREM. The set $X' = \{X \cap a_{ax}(x) \mid a \in X\}$ is the subset of $X$ consisting of all points at which $X$ is colocally connected. Moreover, it is a $G_\delta$-subset of $X$. For each $a \in X$ we have $X' \cup \{a\} = [X \cap a_{ax}(x)] \cup \{a\}$.

Proof. Let $x \in X'$. Pick an arbitrary neighbourhood $G$ of $x$. To prove that $X$ is colocally connected at $x$ it suffices to show that there is a neighbourhood $V \subset G$ of $x$ such that $X \setminus G$ is contained in some component of $X \setminus V$. Suppose it is not true. Pick a point $a \notin G$. There is a decreasing sequence $V_1 \supset V_2 \supset \ldots$ of neighbourhoods of $x$ with diameters converging to 0 such that $V_n \subset G$ and no component of $X \setminus V_n$ contains $X \setminus G$, for each $n \geq 1$. Define by $C_n$ the component of $X \setminus V_n$ which contains $a$. By the supposition there is a component $D_a \subset X \setminus G$ disjoint from $C_n$. Let $ab_a$ be an arc irreducible between $a$ and $D_a$. Hence $ab, \emptyset \neq B_0 \neq B_0 \notin G$. Let $b \in C_0$. We can assume that $[b, c_0] \subset D_a$. This contradicts our assumption.

One can similarly prove the first assertion of the theorem.

Now, assume $X$ is colocally connected at $x$. Clearly, $x \in X'$ and $X$ is semi-locally connected at $x$. By the definition $x \in X'$. Let $U \subset X$ be an open set such that $x \in \overline{U}$ and $X \setminus U$ is connected. Define by $\mathfrak{U}_U$ the union of the sets from $\mathfrak{U}_a$. Clearly,

$$X' = \bigcap_n \mathfrak{U}_U,$$

which follows from the first part of 3.5. This ends the proof.

We finish this section showing some relationship between the function $a_{ax}$ and the notion of uniform arcwise connectivity introduced in [2]. Recall that a dendroid $X$ is called uniformly arcwise connected, briefly: u.a.c., provided that for each $x > 0$
there is a natural number \(n\) such that for every arc \(ab \subset X, a \neq b\), there exist \(n+1\) points \(a_1, a_2, \ldots, a_{n+1} (a_i \in ab)\) such that

(i) \[ a = a_1 < a_2 < \ldots < a_{n+1} = b, \]

(ii) \[ \text{diam}_a(a_j) < \varepsilon \text{ for each } j = 1, \ldots, n. \]

3.6. Theorem. A dendroid \(X\) is uniformly arcwise connected if and only if the function \(\varphi\) vanishes on \(X\).

Proof. The necessity is obvious. Now we prove the sufficiency part of the theorem.

Suppose, to the contrary, that \(X\) is not u.a.c. Hence there is an \(e > 0\) and a sequence of arcs \(L_1, L_2, \ldots\) in \(X\) such that for each \(n\) there exist \(n+1\) points \(x_n(1), x_n(2), \ldots, x_n(n+1)\) on \(L_n\) having the properties:

(i) \[ L_n \supseteq x_n(1)x_n(n+1), \]

(ii) \[ x_n(1) < x_n(2) < \ldots < x_n(n+1) \text{ (with respect to an order on } L_n), \]

(iii) \[ \varphi(x_n(j), x_n(j+1)) \geq \varepsilon \text{ for each } j = 1, \ldots, n. \]

Let \(N\) denote the set of natural numbers (0 is not considered as a natural number).

Since \(X\) is a compact metric space there exist a sequence of points \(x(1), x(2), \ldots, x(j) \in X_n\) and a sequence of strictly increasing functions \(p_1, p_2, \ldots, p_j : N \to N\) such that

(4) \[ p_{j+1}(N) = p_j(N) \text{ for } j \in N, \]

(5) \[ \lim_{n \to \infty} x_{p_j(n)}(j) = x(j) \text{ for } j \in N. \]

Clearly, \(x_{p_j(n)}(j) = x(j)\) for \(i \in j\). Observe that the sequence \(p_1(1), p_2(2), \ldots\) is strictly increasing, and for \(j \geq 1\) we have

\[ p_j(j) \equiv p_j(N), \]

which follows from (4). Now it is a consequence of (5) that

(6) \[ \lim_{n \to \infty} x_{p_j(n)}(j) = x(j) \]

for each \(i \in N\) (it can happen that some of the symbols \(x_{p_j(n)}(j)\) are not defined, but this can only happen for a finite number of \(j\)'s). Without loss of generality we may assume that \(L_{p_1(1)}, L_{p_2(2)}, \ldots\) is our original sequence \(L_1, L_2, \ldots\). Hence condition (6) changes to

(7) \[ \lim_{n \to \infty} x(n) = x(i) \text{ for each } i \in N. \]

Observe also that conditions (1)-(3) are still valid. Conditions (3) and (7) imply

(8) \[ \varphi(x(i+1), x(i)) \geq \varepsilon \text{ for each } i. \]

It follows from the compactness of \(X\) that there exist a strictly increasing function \(q : N \to N\) and two points \(a, x \in X\) such that

\[ \lim_{n \to \infty} q(n) = a \text{ and } \lim_{n \to \infty} q(n+1) = x. \]

Condition (8) implies that

\[ \varphi(a, x) \geq \varepsilon. \]

Now we show that \(\varphi(x, a) > 0\).

Let \(U\) be a neighbourhood of \(x\) and let \(V\) be a neighbourhood of \(a\). Take an arbitrary natural number \(k\). There exists an index \(m\) such that

\[ x(q(m)) \in V \text{ and } x(q(m)+1) \in U \]

for each \(n \geq m\). Using (7) one can find an index \(r\) such that the symbols below make sense and satisfy the conditions:

\[ x(q(m+j)) \in V \text{ for } 1 \leq j \leq k \]

and

\[ x(q(m+j)+1) \in U \text{ for } 1 \leq j \leq k. \]

Let us note that the following inequalities hold true (with respect to the order on \(x(q(m+j))\))

\[ x(q(m)) \leq x(q(m+1)) \leq \ldots \leq x(q(m+k)) \leq x(q(m+k)+1) \leq x(q(m+k+1)). \]

Let \(ab\) be an arc irreducible between \(a\) and \(x(q(m))\). It is easily seen that either \(ax(q(m))\) or \(ax(q(m+1))\) is an arc of type \(V, U, U, U, \ldots\).

This proves that \(\varphi(x, a) \geq \varphi(a, x) \geq \varepsilon\), and the proof is completed.

In his paper [4] W. Kuperberg proved that a dendroid \(X\) can be represented as a continuous image of the Cantor fan if and only if \(X\) is u.a.c. Combining this result with 3.6 we get the following.

3.7. Corollary. A dendroid \(X\) is a continuous image of the Cantor fan if and only if the function \(\varphi\) vanishes on \(X\).

4. Necessity of the set \(X^*_a\). The aim of this section is to prove the following.

4.1. Theorem. The minimal arcwise connected set spanning the set \(X^*_a\) of all endpoints of a dendroid \(X\) at which \(X\) is semi-locally connected is a dense subset of \(X\).

Clearly this implies that \(X^*_a \neq \emptyset\), and answers the mentioned question of B. J. Fugate.

This theorem will follow from several lemmas which we are now going to state and prove. In the lemmas we use the following fixed notations.
Fix a point \(a\) in the dendroid \(X\). For any open subset \(U\) of \(X\) which does not contain \(a\) denote by \(C_U\) the component of \(X\setminus U\) containing \(a\), and let
\[
D_U = X \setminus (U \cup C_U).
\]

Denote
\[
P(U) = \{x \in X:\ \text{there is a neighbourhood } V \ni x \text{ such that } C_V \subseteq \text{Int } C_U\}.
\]

4.2. LEMMA. Let \(U\) be an open set not containing \(a\) and let \(x \in P(U) \cup \overline{U} \cup C_U\). Then for every neighbourhood \(V\) of \(x\) there exists an arc \(y \in (U, V, U)\).

Proof. Take a neighbourhood \(G\) of \(x\) such that \(C_G \subseteq \text{Int } G\) and \(G \cap (U \cup C_U) = \emptyset\). Since \(C_U\) is a component of the compact space \(X\setminus \overline{U}\), hence separates \(X\) between \(C_U\) and \(G\). There are disjoint closed subsets \(A\) and \(B\) in \(X\) such that \(X\setminus U = A \cup B\), \(C_U \subseteq A\) and \(\overline{B} \subseteq B\).

Since \(x \notin P(U)\), we have \(D_U \cap C_U \neq \emptyset\). There is a point \(y \in D_U \cap (A \cup U)\) because \(A \cup U\) is open in \(X\). One easily checks that \(y\) has the required properties.

4.3. LEMMA. Let \(U\) be an open set in the dendroid \(X\) such that \(a \notin U\) and \(x \in P(U)\) for some \(x \in X\). If \(x \neq a\), then \(y \in P(U)\).

Proof. Let \(V\) be a neighbourhood of \(x\) such that \(C_V \subseteq \text{Int } C_U\). Clearly, \(y \neq C_U\).

Let \(W\) be a neighbourhood of \(y\) contained in \(X \setminus C_U\). Then \(C_W \subseteq C_U\) and therefore \(C_U \subseteq \text{Int } C_W\), which completes the proof.

4.4. LEMMA. Let \(U\) and \(V\) be disjoint open subsets of \(X\) and let \(G_1, \ldots, G_n\) be arbitrary open subsets of \(X\) such that (1)
\[
P(U) \cap \{x \in X: x \in \text{Int } (V, U, V)\} = \emptyset,
\]
(2) there is a nonempty open set \(V_1 \subseteq V\) such that every arc joining \(a\) and \(V_1\) is of the type \((V, V, U)\) and \((G_1, \ldots, G_n)\).

Then there is an open nonempty set \(V_2 \subseteq V_1\) such that every arc joining \(a\) and \(V_2\) is of the type \((G_1, \ldots, G_n, U)\).

Proof. Let \(x_1 \in V_1\). It follows that \(x_1 \notin P(U)\) because \(x_1 \in \text{Int } (V, U, V)\). Also \(x_1 \notin U \cup C_U\). From 4.2 we infer that there is a point \(b \in U\) such that \(ab \in (V_1, U, V)\). Clearly, \(ab \in (G_1, \ldots, G_n, U)\) and \(b \notin P(U)\). It follows that \(b \notin P(V_1)\). Also it is clear that \(b \notin D_U \cup \overline{C_U}\). By 2.1 there is a neighbourhood \(W\) of \(x_1\) such that every arc joining \(a\) and \(W\) is of the type \((G_1, \ldots, G_n, U)\). By 4.2 there is a point \(x_2 \in V_1\) such that \(x_2 = x_1\). Again by 2.1 there is a neighbourhood \(V_2 \subseteq V_1\) of \(x_2\) such that every arc joining \(a\) and \(V_2\) is of the type \((V_1, V_2, V)\). Let \(e\) be an arbitrary point of \(V_2\). We shall show that \(ae \in (G_1, \ldots, G_n, U, V)\).

There exist points \(p \in \text{Int } V_1\) and \(q \in \text{Int } V\) such that \(a \prec p \prec q \prec c\). Clearly, \(ap \in (G_1, \ldots, G_n)\) and \(pq \in (U)\) and \(qc \in (V)\). This completes the proof.

4.5. LEMMA. Let \(x_0, x_0 \in X\) be such that \(x_0 \in A_1(x_0, a)\) (comp. the definition of \(x_0\)). Let \(U, V\) be neighbourhoods of respectively \(x_0\) and \(x_0\) such that \(U \cap V = \emptyset\). Then \(P(U) \cap \{x \in X: ax \in (V, U, V)\} \neq \emptyset\).

Proof. Suppose \(P(U) \cap \{x \in X: ax \in (V, U, V)\} = \emptyset\). By the assumption there is a point \(x \in P(V)\) such that \(ax \in (V, U, V)\). From 2.1 we infer that there is a neighbourhood \(V_1 \subseteq V\) of \(x\) such that every arc joining \(a\) and \(V_1\) is of the type \((V, U, V)\). By 4.4 there is a nonempty open set \(V_2 \subseteq V_1\) such that every arc joining \(a\) and \(V_2\) is of the type \((V, U, V)\). In particular, any such arc is of the type \((V, U, V)\). Again using 4.4 we can construct a nonempty open set \(V_3 \subseteq V_2\) such that every arc joining \(a\) and \(V_3\) is of the type \((V, U, V)\).

Repeating the argument we can construct a sequence of nonempty open sets \(V_4, V_5, \ldots\) such that \(V_{n+1} \subseteq V_n\) for each \(n \geq 1\) and every arc joining \(a\) and \(V_n\) is of the type \((V, U, V)\). The intersection \(\bigcap V_n\) is a nonempty set and every arc joining \(a\) with that intersection is of the type \((V, U, V)\) by Lemma 2.3. Hence 2.2 implies \(U \cap V = \emptyset\), contrary to our assumption.

4.6. LEMMA. For every nonempty open set \(G \subseteq X\) there is a point \(x^* \in X^*\) such that \(ax^* \in (G)\).

Proof. Let \(E_0 = \{x \in X \setminus \{a\}: ax \in (G)\}\). We may assume \(E_0 \neq \emptyset\), for otherwise \(X\) reduces to a one-point set according to a Borsuk's lemma [1] that every arc in a dendroid is a subarc of a maximal arc. By the last assertion of 3.5 we may assume that \(ax_0(x) > 0\) for each \(x \in E_0\). Choose \(x_0 \in E_0\) such that \(ax_0(x_0) > \sup \{ax_0(x): x \in E_0\}\).

Since \(A_1(x_0, a)\) is compact (an arc), by the definition of \(ax_0(x)\) there is a point \(x_0 \in A_1(x_0, a)\) such that \(ax_0(x_0) = \sup \{ax_0(x): x \in E_0\}\).

There exist small enough neighbourhoods \(U_0, V_0\) of respectively \(x_0\) and \(x_0\) with the properties
\[
a \notin U_0, \quad d(U_0, V_0) > \sup \{ax_0(x): x \in E_0\},
\]
and
\[
(1) \ ax \in (U_0) \Rightarrow ax \in (G) \text{ for each } x \in X \text{ (see Lemma 2.1)}.
\]

Set
\[
E_1 = P(U_0) \cap \{x \in X: ax \in (V_0, U_0, V_0)\} \cap X^*.
\]

By the quoted Borsuk's lemma each arc \(ay\) in \(X\) is a subarc of an arc \(ax\) with \(x \in X^*\), hence Lemmas 4.3 and 4.5 imply \(E_1 \neq \emptyset\). By 3.5 we may assume \(ax(x) > 0\) for each \(x \in E_1\). Choose \(x_1 \in E_1\) such that \(ax_1(x_1) > \sup \{ax_0(x): x \in E_1\}\).

As above there is a point \(x_1 \in A_1(x_1, a)\) such that \(ax_0(x_1) = \sup \{ax_0(x): x \in E_1\}\).

There exist small enough neighbourhoods \(U_1, V_1\) of respectively \(x_1\) and \(x_1\) such that
\[
a \notin U_1, \quad d(U_1, V_1) > \sup \{ax_1(x): x \in E_1\},
\]
and
\[
(2) \ ax \in (U_1) \Rightarrow ax \in (G) \text{ for each } x \in X \text{ (see Lemma 2.1)}.
\]
and
\[ d \in (U_x) \Rightarrow ax \in (V_a, U_a, V_a) \text{ for each } x \in X \text{ (by 2.1 and the definition of } E). \]

Repeating the above arguments one can construct the sets \( E_n, V_n, U_n \) and the points \( x_n, \hat{x}_n \in A(x_n, d) \) such that \( x_n(x_n) = d(\hat{x}_n, x_n) \) and

1. \( x_0 \in E_0 = P(U_{x-1}) \cap \{ x \in X : ax \in (V_{x-1}, U_{x-1}, V_{x-1}) \} \cap X^* \)
2. \( a \notin U_{x-1}, \quad U_{x-1} \) is a neighbourhood of \( x_{x-1} \) and \( V_{x-1} \) is a neighbourhood of \( \hat{x}_{x-1} \)
3. \[ d(U_x, V_x) \leq \sup \{ d(x, x) : x \in E_x \} \]
4. \[ C_{\hat{x}_n} = \text{Int} C_{x_n} \]
5. \( ax \in (U_x) \Rightarrow ax \in (V_{x-1}, U_{x-1}, V_{x-1}) \) for each \( x \in X \),
6. \[ a \in C \text{ and } C \text{ is an open and arcwise connected subset of } X. \]

By (3), (5) and (7) we infer that \( C \neq X \). By the Borsuk lemma and (8) there is a point \( x^* \in X^* \) such that \( ax^* \notin C \). Hence condition (8) implies

\[ x^* \notin C. \]

We claim that

1. \( ax^* \in (U^*) \) for each \( n \geq 0 \).

In fact, otherwise \( x^* \notin ax^* = C_{x^*} = C \), which contradicts (9). From (6) and (10) we get

\[ ax^* \in (V_{x-1}, U_{x-1}, V_{x-1}) \] for each \( n \geq 1 \).

We claim that

1. \( x^* \notin P(U_{x-1}) \) for each \( n \geq 1 \).

In fact, \( x^* \notin C_{x^*} \) by (9), hence there is a neighbourhood \( W \) of \( x^* \) disjoint with \( C_{x^*} \). Since \( C_{x^*} \subseteq C_W \), hence (12) follows from (5).

By (2), (11) and (12) we obtain \( x^* \in E_x \) for each \( n \geq 1 \), and by (4)

\[ \alpha_d(x^*) \leq d(U_x, V_x) \] for each \( n \geq 1 \).

For each \( n \geq 0 \) let \( \alpha_n \) be the component of \( ax^* \cap U_n \) containing \( a \). By (3) and (10) we have \( \alpha_n \neq x^* \).

Clearly, \( \alpha_n \cap ax^* \cap U_n \) and \( \alpha_n \cap C_{x^*} \). By (4) we have

\[ U_n \cap V_n = \emptyset \text{. Also } U_{n+1} \cap C_{x^*} = \emptyset \text{ by (5). These facts imply that there is a point } \alpha_n \in U_n \cap ax^* \text{ such that } a \leq \alpha_n \leq x^*. \]

\[ \begin{align*}
14. & \quad ax_n \cap U_{n+1} = \emptyset, \\
15. & \quad \alpha_n \cap V_n = \emptyset.
\end{align*} \]

It is evident by (14) that

\[ a \leq \alpha_n \leq \cdots \leq x^*. \]

We shall show that

\[ \alpha_n \cap V_n = \emptyset \text{ for each } n \geq 0. \]

Suppose \( \alpha_n \cap V_n \neq \emptyset \). By (6) there are points \( p, q, r \) such that

\[ p \in V_n, \quad q \in U_n, \quad r \in V_n, \quad a \leq p < q < r. \]

Since \( ax_n = aU_n \cup U_{n+1} \), by the assertion we have \( r \notin ax_n \). Also \( r \notin ax_n \) by (15), because \( ax_n = ax_n \cup ax_n \). However this implies that \( q \notin U_n \cup C_{x^*} \), a contradiction.

Conditions (16) and (17) imply that \( ax^* \in (U_n, V_n, U_{n+1}, V_{n+1}) \). Lemma 2.2 implies that \( d(U_n, V_n) \leq 0 \). Hence by (13) we get \( \alpha_d(x^*) = 0 \).

Applying 3.5 this gives \( x^* \in X^* \) because \( x^* \in X^* \). Moreover, by (10) and (1) we have \( ax^* \notin (G) \), which completes the proof of the lemma.

**Proof of Theorem 4.1.** By the preceding lemma \( X^*_a \neq \emptyset \). Take a point \( a \in X^*_a \). The minimal arcwise connected set spanning \( X^*_a \) in \( X \) is the union of all arcs joining \( a \) with the other points from \( X^*_a \). Again using the lemma one easily sees that the theorem holds.

Theorem 3.5 says that \( X^*_a \) is a \( G \)-set in \( X \). One might suppose that the minimal arcwise connected set spanning \( X^*_a \) is also a \( G \)-set. We give an example which shows that this is not true. Moreover, we construct a planar dendroid \( D \) for which the complement of the minimal arcwise connected set spanning \( X^*_a \) is a second (Baire) category subset of \( X \).

**4.7. Example (comp. 3).** Let \( C \) denote the ternary Cantor set on the interval \( I = [0, 1] \). Let \( \tau_k \), \( \tau_k \times \tau_s \), \( s = 1, 2, \ldots \), be the open intervals in \( I \) contiguous to \( C \). In the plane \( R \times R \) consider the set

\[ X = C \times I \cup \bigcup_{n=1}^{\infty} I_n, \]

where \( I_n \) is the straight segment in \( R \times R \) joining \((r_n, 0)\) and \((r_n + s_n, 1)\).
Clearly, $X$ is a (contractible) dendroid and $X^a = \{ (r, s) \in X : 0 < r < s < 1 \}$. The set $G = \{ (x, y) \in X : \frac{1}{2} < y < 1 \}$ is open in $X$ and the intersection of $G$ with the minimal arcwise connected set spanning $X^a$ (in $X$) is an $F_a$-set in $X$ with empty interior. Observe also that $X \setminus X^a$ is not an $F_a$-set in $X$.

5. On the set $X \setminus X^a$. In this section we prove the following.

5.1. Theorem. For a dendroid $X$ the set $X \setminus X^a$ is of the first category in $X$. First we have to prove two lemmas.

5.2. Lemma. For each $r > 0$ and for almost every $x \in X$, the set $\overline{a_x^a((r, \infty))}$ is nowhere dense in $X$.

Proof. Suppose, to the contrary, that the interior of $\overline{a_x^a((r, \infty))}$ is not empty. Then there are two open nonempty sets $U$ and $V$ such that

(1) $U = \overline{a_x^a((r, \infty))}$,
(2) $U \cap V = \emptyset$,
(3) $x \in U \cap a_x^a((r, \infty)) \implies A_x(x, V) \neq \emptyset$.

This follows from the fact that for some $s > 0$ and for some $x \in \text{Int}(a_x^a((r, \infty)))$, we have $K(x, s) = \text{Int}(a_x^a((r, \infty)))$ and for every $y \in K(x, s) \cap a_x^a((r, \infty))$, we have $A_x(y, s) \neq \emptyset$. We construct a sequence of open sets $U_1, U_2, \ldots$ such that for each $n \geq 1$ we have

(4) $\emptyset \neq U_n = U_{n+1}$,
(5) every arc joining $x$ with $U_n$ is of the type $(U_{n-1}, V, U_{n+1})$,

where $U_0 = U$. The set $U_n$ is constructed as follows. Let $x \in U_0 \cap a_x^a((r, \infty))$ (see (1)). By (3) there is a point $y \in U_0$ such that $y \in U_0 \cap a_x^a((r, \infty))$. From 2.1 we get a neighbourhood $U_1$ of $y$ satisfying (4) and (5). Similarly we construct $U_{n+1}$ having defined the set $U_n$.

By (4), for $n = 1, 2, \ldots$ there is a point $x \in U_n$. Again by (4), we have $x \in U_n$ for $n = 1, 2, \ldots$. By (5), we have $A_x(x, U_n) \neq \emptyset$. which implies that $A_x(x, U_n) \neq \emptyset$. Since $x \in U_n \cap a_x^a((r, \infty))$, we conclude that $A_x(x, U_n) \neq \emptyset$.

5.3. Corollary. For each $a \in X$ the set $a_x^a((0, \infty))$ is of the first category in $X$.

5.4. Lemma. For each $r > 0$ and for each point $a$ of a dendroid $X$ we have

$X^a \cap a_x^a((r, \infty)) \cap \text{Int}(X^a \cap a_x^a((r, \infty))) = a_x^a((0, \infty))$.

Proof. Let $x$ be an arbitrary endpoint with $a_x^a((r, \infty))$ belonging to the interior of $X^a \cap a_x^a((r, \infty))$. Let $G$ be a neighbourhood of $x$ contained in $X^a \cap a_x^a((r, \infty))$ such that $\text{diam}(G) < r$. Assume there is a point $y \in a_x^a((r, \infty))$ such that for every neighbourhood $M'$ of $y$ and for every neighbourhood $N' \subseteq G$ of $x$ there exists an arc joining $a$ and $N'$ of the type $(N', M', N')$. Then using 2.1 we can easily check that $a_x^a((r, \infty))$ is nowhere dense in $X$.

Therefore, for each $r > 0$ and for every $x \in X$, the set $a_x^a((r, \infty))$ is nowhere dense in $X$.

We claim that for some index $n$ the component $F_n$ of the set $X \setminus \bigcup_{n=1}^\infty M_n$ which contains $x$ is a subset of $G$. Otherwise $F_N \neq \emptyset$ for each $n \geq 1$. And since $F_{n+1} \subseteq F_n$ the intersection $\bigcap_{n=1}^\infty F_n$ would be a continuum joining $x$ and $X \setminus G$ having just the point $x$ in common with the arc $a_x$. Hence $x$ would be an interior point of some arc, contrary to the choice of $x$. Let

$M = \bigcup_{i=1}^n M_i$ and $N = \bigcap_{i=1}^n N_i$.

By the construction the component $F_n$ of the set $X \setminus M$ containing $x$ is a subset of $G$. Since $N$ is a neighbourhood of $x$ there is a point $y \in a_x^a((r, \infty))$ and a neighbourhood $V$ of $y$ such that

(2) $xy \subseteq G$,
(3) $V \subseteq N \cap M_x$.

We shall show that there is a neighbourhood $U$ of $x$ satisfying the conditions:

(4) $U \subseteq N \cap M_x$.

(5) for each $z \in V$ and for each $b \in az \cap U$ the arc $bz$ is contained in $G$.

Suppose there is no such $U$. Then for each $n > 1$ there are two points $s_n, b_n$ and a neighbourhood $U_n$ of $x$ such that $U_n \subseteq N \cap M_x$, $\text{diam}(U_n) < 1/n$, $z_n \in V$, $b_n \in az_n \cap U_n$ and $b_n \in \bar{G}$. For some index $k$, the arc $b_kz_k$ intersects $M_k$; for otherwise there would be a continuum joining $x$ and $X \setminus G$ outside $M$. This continuum would be a subset of $F_k$, contrary to the construction of $F_k$. There is an index $j < k$ such that $b_jz_j \cap M_j \neq \emptyset$. Since $z_j \in V_n \subseteq N \subseteq M_j$, we have $a_n \in \bar{G}$. This contradicts 1. Hence there is an $U$ with the required properties.

Since $x \in X^a \cap a_x^a((r, \infty))$, $xy \subseteq G$ by (2), and $\text{diam}(G) < r$, there is an arc $a_x^a((U, V))$ such that $x \in V$. Let $V' \subseteq V$ be a neighbourhood of $x$ such that every arc joining $a$ and $V'$ is of type $(U, V)$ (see 2.1). We have $V' \subseteq N \subseteq G$, hence there is a point $x' \in V' \cap a_x^a((r, \infty))$. By the construction $a_x^a((U, V'))$. Let $b \in az \cap U$. By (5) we have $bz \subseteq G$. Since $a_x^a((r, \infty))$ is a subset of $G$, by definition of $a_x^a((r, \infty))$
there is a point $c \in U$ such that $ac \in (V', U)$. By the construction of $V'$ we have $ae \in (U, V', U)$. By (3) and (4) we see that $ae \in (N, M, N')$ and $c \in N'$. This contradicts (1), and proves the lemma.

Proof of 5.1. Let $a \in X \setminus X'$. Let $B_n = X' \cap a^{-1}(1/n, \infty)$ for $n = 1, 2, \ldots$. The union $\bigcup_{n=1}^\infty B_n$ is equal to $X' \setminus X$ by 3.5. Note that for each $n \geq 1$ we have

$$B_n \subset \overline{B_n \cap \text{Int} B_n} \cup (B_n \cap \text{Int} B_n).$$

The set $B_n \setminus \text{Int} B_n$ is nowhere dense. By the above lemma $B_n \cap \text{Int} B_n = a_n^{-1}(0, \infty)$. Hence by 5.3 and the above remark $B_n$ is of the first category in $X$, which proves the theorem.

References


Accepté par la Réduction le 9. 2. 1978