Local behavior and the Vietoris and Whitehead theorems in shape theory

by

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Abstract. THEOREM. *For* LC^n *paracompacta the shape groups and the homotopy groups are naturally isomorphic.* As an application we have the theorem: *For metric spaces a perfect map* $f\colon X\to Y$ *of* X *onto* Y *such that* $f^{-1}(y)$ *is* AC^n *(approximately* k*-connected,* $0\leqslant k\leqslant n$*), for all* $y\in Y$*, induces for each* $x\in X$ *an isomorphism of the* n*-th shape group of* (X,x) *with that of* $(Y,f(x))$. Finally we construct a movable continuum X^* (based on the Kahn space X, one of its ANR-sequences, and the map recently described by J. L. Taylor of X onto the Hilbert cube) which can be mapped onto a movable continuum of different shape by a Vietoris (or CE) map.

Introduction. In Section 2 we show that for a LC^n paracompactum X the nth shape group $\check{\pi}_n(X,x)$ and the nth homotopy group $\pi_n(X,x)$ are naturally isomorphic. As an application of this result we prove in Section 3 that for metric spaces X and Y a perfect map $f\colon X\to Y$ of X onto Y such that $f^{-1}(y)$ is AC^n (approximately k-connected, $0\leqslant k\leqslant n$), for all $y\in Y$, induces for each $x\in X$ an isomorphism of the nth shape group of (X,x) with that of $(Y,f(x))$. In Section 4 we construct a movable continuum X^* (based on the Kahn example X [7], one of its ANR-sequences, and the map recently described by J. L. Taylor [25] of X onto the Hilbert cube) which can be mapped onto a movable continuum of different shape by a Vietoris (or CE) map. By a Vietoris map $f\colon X\to Y$ we mean a map f for which the inverse image of each point in Y is of trivial shape.

Section 1. Shape and shape groups. Let $\boldsymbol{P}$ be the category of ANR's and homotopy classes of continuous maps between them. If X is a (topological) space, then Π_X is the functor from $\boldsymbol{P}$ to the category of sets and functions which assigns to an ANR P the set $\Pi_X(P)=[X,P]$ of all homotopy classes of maps of X into P and which assigns to any homotopy class $\theta\colon P\to Q$ between ANR's the induced function $\theta_\#\colon [X,P]\to[X,Q]$ which maps the homotopy class $f\colon X\to P$ into the composition $\theta f=\theta_\#(f)$ of the homotopy classes of f and θ. A natural transformation Ψ of the functor Π_X into the functor Π_Y assigns to each homotopy class $f\colon X\to P$ a homotopy class $\Psi(f)\colon Y\to P$ in such a way that for all homotopy classes $f\colon X\to P$, $g\colon X\to Q$,

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and $\theta\colon P\to Q$ such that $\theta f = g$ we have $\theta\Psi(f) = \Psi(g)$. If $f\colon X\to Y$ is a map, then there is a natural transformation $f^{\#}\colon \Pi_Y\to\Pi_X$ which assigns to the homotopy class $\theta\colon Y\to P$ the composition $\theta[f] = f^{\#}(\theta)$ of the homotopy class $[f]$ of f with θ.

A natural transformation $\Phi\colon \Pi_Y\to\Pi_X$ is also called a shape map from X to Y and $\mathrm{Mor}_{\mathrm{Sh}}(X, Y)$ is used to denote the collection of all such shape maps from X to Y (i.e., $\mathrm{Mor}_{\mathrm{Sh}}(X, Y) = \mathrm{Transf}\,(\Pi_Y, \Pi_X)$). Note that $\mathrm{Mor}_{\mathrm{Sh}}(X, Y)$ is a set (see [14]). If f is a map or homotopy class of a map, then $f^{\#}$ is called the shape map induced by f. $\mathcal{S}$ will be used to denote the shape category, i.e., the category whose objects are topological spaces and whose morphisms are shape maps. This is the category considered by Mardešić in [14].

THE COMPOSITION CONVENTION. The composition of two shape maps $\varphi\colon X\to Y$, $\psi\colon Y\to Z$ is denoted by $\psi\varphi$, although as natural transformations we have

$$\psi\varphi[h] = \varphi(\psi[h])$$

for any homotopy class $h\colon Z\to P$ into an ANR.

While we will usually work with natural transformations it will sometimes be conceptually more helpful to use shape maps.

A natural transformation $\Phi\colon \Pi_Y\to\Pi_X$ is characterized by the property that whenever $\alpha\colon Y\to P$, $\beta\colon Y\to Q$ and $\gamma\colon P\to Q$ are maps such that $[\gamma\alpha] = [\beta]$, then $[\gamma]\Phi[\alpha] = \Phi[\beta]$.

Instead of $\boldsymbol{P}$ one could use the category $\boldsymbol{Q}$ of all spaces which are dominated by ANR's and homotopy classes of maps between such spaces, because any natural transformation $\Phi\colon \Pi_Y\to\Pi_X$ (defined for $\boldsymbol{P}$) has a unique extension to a natural transformation $\Pi_Y\to\Pi_X$ (defined for $\boldsymbol{Q}$). By a well-known result of Milnor [20] a space belongs to $\boldsymbol{Q}$ if and only if it is dominated by a CW-complex. We use Δ to denote the index set for our inverse system. However, we surpress it unless needed.

DEFINITION 1.1. The space Y and the inverse system $\{Y_\beta, q_{\beta\beta'}\}$ in the shape category are said to *satisfy a continuity condition*, provided there is a family of shape maps $q_\beta\colon Y\to Y_\beta$ such that $q_{\beta\beta'}q_{\beta'} = q_\beta$ and such that for any ANR P the natural transformations q_β induce a bijection

$$\operatorname{dirlim}\{\Pi_{Y_\beta}(P), q_{\beta\beta'}\}\to\Pi_Y(P)\,.$$

In the notation of shape maps this means that (1) for any homotopy class $f\colon Y\to P$ there is an index β and a homotopy class $f_\beta\colon Y_\beta\to P$ such that

$$f^{\#} = f_\beta^{\#}q_\beta$$

and (2) if $f_{\beta'}\colon Y_{\beta'}\to P$ and $f_{\beta''}\colon Y_{\beta''}\to P$ are homotopy classes satisfying $f_{\beta'}^{\#}q_{\beta'} = f_{\beta''}^{\#}q_{\beta''}$, then there is a $\beta\geqslant\beta', \beta''$ such that

$$f_{\beta'}^{\#}q_{\beta'\beta} = f_{\beta''}^{\#}q_{\beta''\beta}\,.$$

The shape maps $q_\beta\colon Y\to Y_\beta$ are said to *induce* the continuity condition.

As the following examples show the inverse system in $\mathcal{S}$ is frequently derived from an inverse system in **TOP**. In these instances we shall write the inverse system in **TOP** and omit the obvious modification of the corresponding shape maps.

EXAMPLE 1.1. Y is the inverse limit in **TOP** with projection maps $q_\beta\colon Y\to Y_\beta$ of the inverse system $\{Y_\beta, q_{\beta\beta'}\}$ of compacta. Then Y and the inverse system $\{Y_\beta, q_{\beta\beta'}\}$ satisfy a continuity condition ([13, Theorems 3 and 4]).

EXAMPLE 1.2. Y is an arbitrary subset of a metrizable space Z, and $\{Y_\beta, q_{\beta\beta'}\}$ is any cofinal family of open neighborhoods of Y in Z with all maps the appropriate inclusions. Then Y and the inverse system $\{Y_\beta, q_{\beta\beta'}\}$ satisfy a continuity condition ([13, Corollary 4.8]).

EXAMPLE 1.3. Y is a paracompact space, Δ is cofinal family of open covers of Y, Y_β is the nerve of the corresponding open cover $\beta\in\Delta$, $q_{\beta\beta'}$ is the shape map induced by a projection $Y_{\beta'}\to Y_\beta$ of the nerve $Y_{\beta'}$ of a refinement β' of β, and q_β is the shape map induced by a barycentric (also called "canonical") map of Y into the nerve Y_β. Then Y and the inverse system $\{Y_\beta, q_{\beta\beta'}\}$ satisfy a continuity condition ([13, Corollary 2.8]).

The following theorem is a considerable generalization of a result of K. Morita in *On shapes of topological spaces*, Fund. Math. 86 (1975), p. 256.

THEOREM 1.1 (Continuity). *If the space Y and the inverse system $\{Y_\beta, q_{\beta\beta'}\}$ in $\mathcal{S}$ satisfy a continuity condition, then*

$$\mathrm{Mor}_{\mathrm{Sh}}(X, Y)\approx\operatorname{invlim}\{\mathrm{Mor}_{\mathrm{Sh}}(X, Y_\beta), q_{\beta\beta'\#}\}\,.$$

If the shape maps $q_\beta\colon Y\to Y_\beta$ induce the continuity condition, then the equivalence of the conclusion is given by the assignment of $\varphi\in\mathrm{Mor}_{\mathrm{Sh}}(X, Y)$ to the unique element of the inverse limit defined by the family of shape maps $q_\beta\varphi$.

Proof. We need to show that when a family of shape maps $q_\beta\colon Y\to Y_\beta$ induces a continuity condition between Y and the inverse system $\{Y_\beta, q_{\beta\beta'}\}$ (in $\mathcal{S}$), then the shape maps $q_\beta\colon Y\to Y_\beta$ represent Y as the inverse limit in $\mathcal{S}$ of the system $\{Y_\beta, q_{\beta\beta'}\}$. This means that if Z is any space and $\psi_\beta\colon Z\to Y_\beta$ are shape maps such that

$$q_{\beta\beta'}\psi_{\beta'} = \psi_\beta \quad\text{whenever}\quad \beta\leqslant\beta'\,,$$

then there is a unique shape map

$$\psi\colon Z\to Y$$

such that

$$\psi_\beta = q_\beta\psi \quad\text{for all } \beta\,.$$

This follows by an argument similar to the proof of Theorem 6 given in [14]. Once this is shown the assignment of $\varphi\in\mathrm{Mor}_{\mathrm{Sh}}(X, Y)$ to the family of shape maps $q_\beta\varphi\in\mathrm{Mor}_{\mathrm{Sh}}(X, Y_\beta)$ is easily seen to define the equivalence of the theorem.

We now introduce a group structure on the natural transformations from $\Pi_{(X,x)}$ to $\Pi_{(S^n,a)}$. We simplify the notation by deleting the base points but we only work in the pointed case.

DEFINITION 1.2 (Group structure). Let $\Phi, \Psi\colon \Pi_X\to\Pi_{S^n}$ be natural transformations from Π_X to Π_{S^n}. We define the addition of natural transformations

$$\Phi+\Psi = \Theta$$

by

$$\Theta[f] = \Phi[f]+\Psi[f]$$

where this latter addition is just addition in $\Pi_{S^n}(W)$, $W\in Q$, and $f\colon X\to W$.

We need only show that Θ is a natural transformation from Π_X to Π_{S^n}. Consider the homotopy commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f'} \\ W & \xrightarrow{\;h\;} & W' \end{array}$$

(i.e., we have $h_\#[f] = [f']$). From this and the fact that Φ and Ψ are natural transformations we get $h_\#\Theta[f] = \Theta[f']$ or $h_\#\Theta = \Theta h_\#$ since

$$\Theta h_\#[f] = \Phi h_\#[f]+\Psi h_\#[f] = h_\#\Phi[f]+h_\#\Psi[f]$$
$$= h_\#(\Phi[f]+\Psi[f]) = h_\#\Theta[f].$$

So we have $\Theta h_\# = h_\#\Theta$ and thus Θ is a natural transformation from Π_X to Π_{S^n}.

Let the nth *shape group* of X at x (denoted by $\check{\pi}_n(X, x)$) be the collection of natural transformations from Π_X to Π_{S^n} with addition defined as above. A natural transformation $\Phi\colon \Pi_Y\to\Pi_X$ (i.e., a shape map $\Phi\colon X\to Y$) induces a homomorphism $\Phi_\#\colon \check{\pi}_n(X, x)\to\check{\pi}_n(Y, y)$ given by $\Phi_\#(\Psi)$ = the composition $\Phi\Psi$ of natural transformations (composition convention). (A similar description is also given by J. Keesling in *Products in the shape category and some applications*, Symp. Math. Institute Nazionale di Alta Matematica Roma (Rome, 1973).

The correspondence from $\pi_n(X, x)$ to $\check{\pi}_n(X, x)$ given by $[\varphi]\to\varphi^\#$ for a map $\varphi\colon S^n\to X$ is a homomorphism. This is because for any map $f\colon X\to W$, $W\in Q$, we have

$$(\varphi+\psi)^\#[f] = [f\circ(\varphi+\psi)] = [f\varphi]+[f\psi] = \varphi^\#[f]+\psi^\#[f],$$

i.e., $(\varphi+\psi)^\# = \varphi^\#+\psi^\#$. If $f\colon X\to Y$ is a map, then we shall write $f_\#\colon \check{\pi}_n(X, x)\to\check{\pi}_n(Y, f(x))$ instead of $(f^\#)_\#$.

If $f\colon X\to Y$ is a map, then there is a commutative diagram

$$\begin{array}{ccc} \pi_n(X, x) & \longrightarrow & \check{\pi}_n(X, x) \\ {\scriptstyle f_\#}\downarrow & & \downarrow{\scriptstyle f_\#} \\ \pi_n(Y, f(x)) & \to & \check{\pi}_n(Y, f(x)) \end{array}$$

involving these homomorphisms.

If X is homotopy dominated by a CW-complex, then the natural homomorphism $\pi_n(X, x)\to\check{\pi}_n(X, x)$ is an isormorphism. The proof of this results from a standard argument along with the fact that if $X\xrightarrow{\alpha}K\xrightarrow{\beta}X$ satisfies $\beta\alpha\simeq \mathrm{id}_X$, then there is a $\beta'\colon K\to X$ such that $\beta'\alpha\simeq\mathrm{id}_X \operatorname{rel} x$.

If $X = S^n$, then the equivalence of Theorem 1.1 is an isomorphism of groups.

Now we relate this natural transformation approach to systems and continuity conditions.

Remark 1.2 (Systems). Let X and the system $\{X_\alpha, p_{\alpha\beta}\}$ of ANR's and maps satisfy a continuity condition. Corresponding to the natural transformation $\Phi\colon \Pi_X\to X_{S^n}$ is a family of maps, $\{\varphi_\alpha\}$ where $\varphi_\alpha\colon S^n\to X_\alpha$ and such that $p_{\alpha\beta}\varphi_\beta\simeq\varphi_\alpha$ for $\alpha<\beta$. Similarily for the natural transformations $\Psi\colon \Pi_X\to\Pi_{S^n}$ and $\Theta\colon \Pi_X\to\Pi_{S^n}$ we have $\psi_\alpha\colon S^n\to X_\alpha$ and $\theta_\alpha\colon S^n\to X_\alpha$ satisfying $p_{\alpha\beta}\psi_\beta\simeq\psi_\alpha$ and $p_{\alpha\beta}\theta_\beta\simeq\theta_\alpha$ for $\alpha<\beta$. By definition of the continuity condition we have $\Phi[p_\alpha] = [\varphi_\alpha]$, $\Psi[p_\alpha] = [\psi_\alpha]$ and $\Theta[p_\alpha] = [\theta_\alpha]$, and consequently, $[\theta_\alpha] = \Theta[p_\alpha] = \Phi[p_\alpha]+\Psi[p_\alpha] = [\varphi_\alpha]+[\psi_\alpha]$. According to this, addition in $\check{\pi}_n(X, x)$ is seen to be accomplished by performing the usual addition of homotopy classes in each coordinate $\pi_n(X_\alpha, x_\alpha) = \check{\pi}_n(X_\alpha, x_\alpha)$ and then passing to the limit.

Section 2. The effect of local behavior on the shape groups. In this section we show that for LC^n paracompacta the shape groups are isomorphic with the homotopy groups. This is analogous to the situation as regards singular and Čech homology on lc^n paracompacta (see [17]).

We use the following conventions for pointed spaces. We will be considering pointed spaces (X, x_0) (usually paracompacta). In choosing open coverings $\mathcal{U}$ of such a space we will always choose them so that exactly one element U_0 of $\mathcal{U}$ contains x_0. Therefore x_0 is carried to the vertex $U_0\in K(\mathcal{U})$, the nerve of $\mathcal{U}$, by any barycentric map $u\colon X\to|K(\mathcal{U})|$. If $\mathcal{V}$ refines $\mathcal{U}$, then $x_0\in V_0\subset U_0$ and so $\pi_{\mathcal{U}\mathcal{V}}(V_0) = U_0$ and hence any refining function automatically gives a pointed map $\pi_{\mathcal{U}\mathcal{V}}\colon K(\mathcal{V})\to K(\mathcal{U})$. The full realizations $t\colon |K(\mathcal{U})|\to X$ are defined by vertex assignment and so we can (and do) make the choice $t(U_0) = x_0$. Homotopies between contiguous base point preserving maps are taken to be linear homotopies which are then base point preserving homotopies. Homotopies between close maps of an n-dimensional complex into an LC^n space are obtained through a familiar skeletonwise procedure and can be taken to be base point preserving. The homotopy for $ut\simeq\pi$ follows from the "Contractible carrier lemma" [11, Lemma 6.4] and the proof of that lemma clearly shows that we may take the homotopy to be base point preserving.

LEMMA 2.1 ([11]). *Suppose that*

(1) $\mathcal{P}$, $\mathcal{R}$ *and* $\mathcal{U}$ *are open covers of a topological space* X,

(2) $\mathcal{P}$ *refines* $\mathcal{R}$, $\mathcal{R}$ *star-refines* $\mathcal{U}$, *and* $\mathcal{U}$ *is locally finite, and*

(3) *any partial realization in* $\mathcal{P}$ *of any* n*-complex has a full realization in* $\mathcal{R}$.

Then for any star-refinement $\mathcal{V}$ *of* $\mathcal{P}$ *there is a full realization* $t\colon |K^n(\mathcal{V})|\to X$ *of* $K^n(\mathcal{V})$ *in* $\mathcal{R}$ *such that for any projection* $\pi\colon K(\mathcal{V})\to K(\mathcal{U})$ *and barycentric maps* $u\colon X\to|K(\mathcal{U})|$ *the maps* ut *and* $\pi\,|\,|K^n(\mathcal{V})|$ *are homotopic. Moreover, the map* t *can be chosen so that for any simplex* $s\in K^n(\mathcal{V})$ *the image* $t(|s|)$ *is contained in* $\bigcup \mathrm{St}(\bigcup s, \mathcal{R})$.

LEMMA 2.2. *Let* X *be a* LC^n *paracompactum. Then there exist an open cover* $\mathcal{U}$ *and maps* $u\colon X\to|K(\mathcal{U})|$ *and* $t\colon |K^n(\mathcal{U})|\to X$ *such that for any map* $\varphi\colon |K|\to X$ *of a com-*

plex K of dimension $\leqslant n$ into X and any simplicial approximation λ of $u\varphi$ the relation $\varphi\simeq t\lambda$ holds. (Note that the image of λ lies in $|K^n(\mathscr{U})|$ and that $\lambda\simeq u\varphi$.)

Proof. Let $\mathscr{H}$ be any open cover of X with the property that any two $\mathscr{H}$-close maps of a complex of dimension $\leqslant n$ into X are homotopic (see [10]). Let $\mathscr{R}$ be a star-refinement of $\mathscr{H}$ and let $\mathscr{P}$ be an open cover refining $\mathscr{R}$ such that any partial realization of an n-complex in $\mathscr{P}$ has a full realization in $\mathscr{R}$. Let $\mathscr{U}$ be a locally finite open cover which star refines $\mathscr{P}$.

Now consider

$$|K|\xrightarrow{\varphi}X\xrightarrow{u}|K(\mathscr{U})|$$

and let $\lambda\colon K'\to K(\mathscr{U})$ be the simplicial approximation to u defined on some suitable subdivision K' of K ([26]). Then $\lambda(|K'|)\subset|K^n(\mathscr{U})|$ and so

$$|K'|\xrightarrow{\lambda}|K^n(\mathscr{U})|\xrightarrow{t}X$$

where t is chosen as in Lemma 2.1. If $p\in|K'|=|K|$, then $\lambda(p)\in|s_0|$ of $K(\mathscr{U})$ which contains $u\varphi(p)$ in its interior so $\varphi(p)\in\bigcap s_0$. Therefore, there is a face s of s_0 such that $\lambda(p)\in|\dot{s}|$. Now $t\lambda(p)\in t(|s|)$. Therefore, $\varphi(p)$ and $t\lambda(p)$ are in $\bigcup\operatorname{St}(\bigcup s,\mathscr{R})$ which is contained in some member of $\mathscr{H}$. So we have $\varphi\simeq t\lambda$.

COROLLARY 2.1. *Let X be a LC^n paracompactum and K be a complex of dimension $\leqslant n$. If $\varphi_0,\varphi_1\colon K\to X$ are maps such that $\varphi_0^{\#}=\varphi_1^{\#}$, then $\varphi_0\simeq\varphi_1$.*

Proof. Take $\mathscr{U}$, u, t, etc. as in Lemma 2.2. Then $\varphi^{\#}[u]=[u\varphi_i]$ for $i=0,1$. Therefore, $u\varphi_0\simeq u\varphi_1$. Since the approximation λ_i of $u\varphi_i$ is homotopic to $u\varphi_i$ for $i=0,1$, we have

(1) $$\lambda_0\simeq u\varphi_0\simeq u\varphi_1\simeq\lambda_1\,.$$

Hence, by Lemma 2.2,

(2) $$\varphi_0\simeq t\lambda_0\simeq t\lambda_1\simeq\varphi_1\,.$$

THEOREM 2.1. *Let X be a LC^n paracompactum and K be a complex of dimension $\leqslant n$. Then for any natural transformation $\Phi\colon\Pi_X\to\Pi_K$ there is a map $\varphi\colon K\to X$ such that $\varphi^{\#}=\Phi$ and the homotopy class of φ is uniquely determined by this condition.*

Proof. Let $\mathscr{H}$ be any open cover of X such that any two $\mathscr{H}$-close maps of a complex of dimension $\leqslant n$ into X are homotopic. Let $\mathscr{W}$ be an open cover of X which star-refines $\mathscr{H}$.

Let $\{\mathscr{U}_\alpha\}$ be a cofinal family of locally finite open covers of X. Let $\mathscr{R}_\alpha$ be star-refinement of $\mathscr{W}$ and $\mathscr{U}_\alpha$ and let $\mathscr{P}_\alpha$ be an open cover refining $\mathscr{R}_\alpha$ such that any partial realization of a complex of dimension $\leqslant n+1$ in $\mathscr{P}_\alpha$ has a full realization in $\mathscr{R}_\alpha$ (see [6, Sections 4.1 and 4.4]). Then, for each α, $\mathscr{P}_\alpha$, $\mathscr{R}_\alpha$, $\mathscr{U}_\alpha$ are open covers of X satisfying (2) and (3) of Lemma 2.1 with n replaced by $n+1$. If $\mathscr{V}_\alpha$ is a star-refinement of $\mathscr{P}_\alpha$, then by Lemma 2.1 there is a full realization $t_\alpha\colon|K^{n+1}(\mathscr{V}_\alpha)|\to X$ of

$K^{n+1}(\mathscr{V}_\alpha)$ in $\mathscr{R}_\alpha$ such that for any projection $\pi_\alpha\colon K(\mathscr{V}_\alpha)\to K(\mathscr{U}_\alpha)$ and barycentric map $u_\alpha\colon X\to|K(\mathscr{U}_\alpha)|$ we have

(1) $$u_\alpha t_\alpha\simeq\pi_\alpha\big||K^{n+1}(\mathscr{V}_\alpha)|\,,$$

and for any simplex $s\in K^{n+1}(\mathscr{V}_\alpha)$ the image $t_\alpha(|s|)$ is contained in $\bigcup\operatorname{St}(\bigcup s,\mathscr{R}_\alpha)$.

Consider the homotopy-commutative diagram

(2) $$\begin{array}{ccc} & X & \\ {\scriptstyle v_\beta}\swarrow & & \searrow{\scriptstyle v_\alpha} \\ |K(\mathscr{V}_\beta)| & \xrightarrow[\pi_{\alpha\beta}]{} & |K(\mathscr{V}_\alpha)|. \end{array}$$

Since Φ is a natural transformation we have the homotopy-commutative diagram

(3) $$\begin{array}{ccc} & |K| & \\ {\scriptstyle \psi_\beta}\swarrow & & \searrow{\scriptstyle \psi_\alpha} \\ |K(\mathscr{V}_\beta)| & \xrightarrow[\pi_{\alpha\beta}]{} & |K(\mathscr{V}_\alpha)|\,. \end{array}$$

where $\psi_\alpha=\Phi[v_\alpha]$. We use $\pi_{\alpha\beta}$ for $\pi_{\alpha\beta}|K^{n+1}(\mathscr{V}_\beta)$ in the following. Using simplicial approximation we may replace ψ_α, ψ_β by maps φ_α, φ_β, respectively, such that the following diagram is homotopy commutative.

(4) $$\begin{array}{ccc} & |K| & \\ {\scriptstyle \varphi_\beta}\swarrow & & \searrow{\scriptstyle \varphi_\alpha} \\ |K^{n+1}(\mathscr{V}_\beta)| & \xrightarrow[\pi_{\alpha\beta}]{} & |K^{n+1}(\mathscr{V}_\alpha)|\,. \end{array}$$

For any simplex s of $K^{n+1}(\mathscr{V}_\beta)$ we have

(5) $$t_\beta(|s|)\subset\bigcup\operatorname{St}(\bigcup s,\mathscr{R}_\beta)$$

and

(6) $$t_\alpha\pi_{\alpha\beta}(|s|)\subset\bigcup\operatorname{St}(\bigcup\pi_{\alpha\beta}(s),\mathscr{R}_\alpha)$$

Since $\mathscr{V}_\beta$ refines $\mathscr{V}_\alpha$,

(7) $$\bigcup s\subset\bigcup\pi_{\alpha\beta}(s)$$

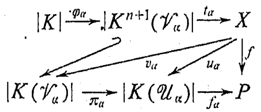
and, since $\mathscr{R}_\beta$ refines $\mathscr{R}_\alpha$,

(8) $$\bigcup\operatorname{St}(\bigcup s,\mathscr{R}_\beta)\subset\operatorname{St}(\bigcup\pi_{\alpha\beta}(s),\mathscr{R}_\alpha)\,.$$

Since $\pi_{\alpha\beta}(s)$ is a simplex of $K^{n+1}(\mathscr{V}_\alpha)$, $\pi_{\alpha\beta}(s)\subset\operatorname{St}(V,\mathscr{V}_\alpha)$ for some $V\in\mathscr{V}_\alpha$ and, therefore, $\bigcup\pi_{\alpha\beta}(s)\subset W$ for some $W\in\mathscr{W}$. Since $\mathscr{R}_\alpha$ refines $\mathscr{W}$, $\bigcup\operatorname{St}(\bigcup\pi_{\alpha\beta}(s),\mathscr{R}_\alpha)$ is contained in $\bigcup\operatorname{St}(W,\mathscr{W})$ and therefore in a member of $\mathscr{H}$. Thus, for any simplex s of $K^{n+1}(\mathscr{V}_\beta)$ the images $t_\beta(|s|)$ and $t_\alpha\pi_{\alpha\beta}(s)$ are contained in a member of $\mathscr{H}$ and hence, the maps $t_\beta\varphi_\beta$ and $t_\alpha\pi_{\alpha\beta}\varphi_\beta$ are $\mathscr{H}$-close. Since $\dim K \leqslant n$, $t_\alpha\pi_{\alpha\beta}\varphi_\beta \simeq t_\beta\varphi_\beta$, and, therefore, by (4)

$$(9) \qquad t_\alpha\varphi_\alpha \simeq t_\alpha\pi_{\alpha\beta}\varphi_\beta \simeq t_\beta\varphi_\beta .$$

So all the maps $t_\alpha\varphi_\alpha$ belong to a single homotopy class of $|K|\to X$. Claim: $[t_\alpha\varphi_\alpha]^{\#} = \Phi$. Consider the diagram



First we have

$$(11) \qquad \Phi[f] = [f_\alpha\pi_\alpha]\Phi[v_\alpha] = [f_\alpha\pi_\alpha\varphi_\alpha]$$

(where $\pi_\alpha = \pi_\alpha \big| |K^{n+1}(\mathscr{V}_\alpha)|: K^{n+1}(\mathscr{V}_\alpha)\to K^{n+1}(\mathscr{U}_\alpha)$) and

$$(12) \qquad [t_\alpha\varphi_\alpha]^{\#}[f] = [ft_\alpha\varphi_\alpha].$$

Then it follows from (1) that

$$(13) \qquad ft_\alpha\varphi_\alpha \simeq f_\alpha u_\alpha t_\alpha\varphi_\alpha \simeq f_\alpha\pi_\alpha\varphi_\alpha .$$

K. Kuperberg [12] has attained the following corollary in the compact metric case.

COROLLARY 2.2. *If X is an* LC^n *paracompactum then the homotopy groups $\pi_n(X, x)$ and the shape groups $\check{\pi}_n(X, x)$ are isomorphic under the natural homomorphism given in in Definition* 1.2.

Section 3. A Vietoris theorem for shape groups. The following definition is due to Borsuk [3] who also showed the property is a shape invariant.

DEFINITION 3.1. A pointed compactum (X, x) contained in an ANR (Q, x) is said to be *approximately k-connected*, if for every neighborhood U of X in Q there is a neighborhood V of X in Q such that each map $f\colon (S^k, a)\to(V, x)$ is null homotopic in (U, x). A compact X contained in an ANR Q is said to be *approximately k-connected*, if (X, x) is approximately k-connected for every $x\in X$. These definitions do not depend on the choice of containing ANR.

DEFINITION 3.2. A pointed compactum (X, x) is said to be AC^n if it is approximately k-connected for $0\leqslant k\leqslant n$.

DEFINITION 3.3. A map $f\colon X\to Y$ is said to be *strongly* LC^n, provided that every neighborhood U of an arbitrary point $y\in Y$ contains a neighborhood V of y such that for $k = 0, 1, 2, \ldots, n$ any map $S^k\to f^{-1}(V)$ extends to a map $B^{k+1}\to f^{-1}(U)$. (Such a map is called a strong local connection in dimension n in [10].)

LEMMA 3.1. *Let $f\colon X\to Y$ be strongly* LC^n *and map X onto a dense subset of a paracompactum Y, and let K be an $(n+1)$-complex with subcomplex L. Then a map $g\colon |L|\to X$ has an extension $h\colon |K|\to Y$.*

Proof. This result is a consequence of Lemma 1 of [10] after achieving a suitable subdivision of K.

THEOREM 3.1. *If $f\colon X\to Y$ is a strongly* LC^n *map of X onto a dense subset of a metric space Y, then Y is* LC^n *and for $k = 0, 1, 2, \ldots, n$ and for every $x\in X$ the induced homomorphism*

$$f_{\#}\colon \pi_k(X, x)\to\pi_k(Y, f(x))$$

is an isomorphism.

Proof. By Theorem 1 of [10] Y is LC^n and $f_{\#}\colon \pi_k(X, x)\to\pi_k(Y, f(x))$ is an isomorphism for $0\leqslant k<n$ and an epimorphism $k = n$. To verify that $f_{\#}$ is a monomorphism let $|K| = S^n\times I$ and $|L| = S^n\times\{0, 1\}\cup\{a\}\times I$ in Lemma 3.1 to conclude that: any two maps $g_1, g_2\colon (S^n, a)\to(X, x)$ such that $fg_1\simeq fg_2$ (relative to a) are already homotopic (relative to a).

DEFINITION 3.4. A map $f\colon X\to Y$ is said to be *perfect* if f is (1) closed, (2) surjective and (3) $f^{-1}(y)$ is compact for all $y\in Y$.

THEOREM 3.2. *For metric spaces and a perfect map $f\colon X\to Y$ of X onto Y such that $f^{-1}(y)$ is* AC^n *for all $y\in Y$ the induced homomorphism*

$$f_{\#}\colon \check{\pi}_n(X, x)\to\check{\pi}_n(Y, f(x))$$

is an isomorphism for each $x\in X$.

Proof. Consider X as a closed subset of an ANR P. (This is possible due to the Kuratowski–Wojdysławski embedding theorem). Let Q be the adjunction space $P\cup_f Y$ and let $F\colon P\to Q$ be the composite of the inclusion of P in $P+Y$ (free union) and the projection of $P+Y$ onto Q.

Denote by $\{Y_\alpha\}$ the family of all open neighborhoods of Y in Q and let $X_\alpha = F^{-1}(Y_\alpha)$. Then $\{X_\alpha\}$ comprises a cofinal family of open neighborhoods of X in P. Let $f_\alpha = F|X_\alpha\colon X_\alpha\to Y_\alpha$. So

$$(1) \qquad \begin{array}{ccc} (X_\alpha, x_\alpha) & \hookleftarrow & (X, x) \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f} \\ (Y_\alpha, y_\alpha) & \hookleftarrow & (Y, y) \end{array}$$

where $y = f(x)$, $x_\alpha = x$, $y_\alpha = y$. Further, we have the diagram

$$(2) \qquad \begin{array}{ccc} \pi_k(X_\alpha, x_\alpha) & \xrightarrow{\approx} & \check{\pi}_k(X_\alpha, x_\alpha) \\ {\scriptstyle f_{\alpha\#}}\downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle f_{\alpha\#}} \\ \pi_k(Y_\alpha, y_\alpha) & \xrightarrow[\approx]{} & \check{\pi}_k(Y_\alpha, y_\alpha) \end{array}$$

for all α and $k = 0, 1, \ldots, n$. Applying Corollary 2.2 we have that the top arrow is an isomorphism since X_α is an ANR; the bottom arrow is an isomorphism since Y_α is LC^n. The left arrow is an isomorphism by Theorem 3.1. Thus we have

$$f_{\alpha\#}\colon \check{\pi}_k(X_\alpha, x_\alpha)\to\check{\pi}_k(Y_\alpha, y_\alpha) \tag{3}$$

is an isomorphism for $k = 0, 1, \ldots, n$.

Finally, we have

$$\begin{array}{ccc} \check{\pi}_k(X_\alpha, x_\alpha) & \leftarrow & \check{\pi}_k(X, x) \\ f_{\alpha\#}\downarrow \approx & & \downarrow f_\# \\ \check{\pi}_k(Y_\alpha, y_\alpha) & \leftarrow & \check{\pi}_k(Y, y) \end{array} \tag{4}$$

for all α. Therefore, since $f_\#$ is a limit of isomorphisms

$$f_\#\colon \check{\pi}_k(X, x)\to\check{\pi}_k(Y, y) \tag{5}$$

is an isomorphism for $k = 0, 1, \ldots, n$.

Remark. Bogatyĭ [1] and [2] outlines such a result for metric compacta. K. Kuperberg [12] also obtained this result in the compact metric case.

Section 4. In [7] D. S. Kahn constructed an acyclic ∞-dimensional continuum X which could be essentially mapped onto spheres of arbitrarily high dimension. Recently, J. L. Taylor [25] has shown the existence of a Vietoris map (i.e., a proper map such that the inverse image of each point has the shape of a point) of X onto the Hilbert cube Q and thereby gave an example of a Vietoris map which was not a shape equivalence. In this section we show that a construction described in [24] can be used to obtain a movable continuum X^* which can be mapped onto a movable continuum by a Vietoris map which is not a shape equivalence. This shows that the Whitehead theorem fails in the ∞-dimensional case even for movable continua and maps. Thus we answer questions raised by several authors on the dimension restrictions which appear in various versions of the Whitehead theorem (see for example, [21], [15], [16], [8], and [12]).

Remark. A similar result has been obtained independently by J. Draper and J. Keesling in their paper *An example concerning the Whitehead theorem in shape theory*, Fund. Math. 92 (1976), pp. 255–259.

For each odd prime p, which we suppose fixed, Kahn constructs a continuum X as the inverse limit of a system

$$X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} \ldots$$

with the following properties:

(i) X_0 is a $(2p+1)$-dimensional, $(2p-1)$-connected, finite complex.

(ii) $X_{n+1} = \Sigma^{2p-2}X_n$, the $(2p-2)$-fold suspension of X_n, and $\alpha_{n+1} = \Sigma^{2p-2}\alpha$ for all $n\geqslant 0$.

(iii) There is a map $\alpha\colon X_0\to S^3$ such that the compositions $\alpha\alpha_1 \ldots \alpha_n\colon X_n\to S^3$ are essential for $n\geqslant 1$.

As shown in [5] X is not movable. Now we modify the construction of X so as to obtain movable continua X^* and $X^*\cup_f Q$ and a Vietoris map F between them which is not a shape equivalence. First let $x = (x_n)\in X = \operatorname{Invlim}\{X_n, \alpha_n\}$. Then define another ANR-system $\{X_a^*, \alpha_n^*\}$ where X_n^* is the one-point union of $X_0, X_1, \ldots, X_n$ with each X_i attached at x_i, $i = 0, 1, \ldots, n$, and $\alpha_{n+1}^*\colon X_{n+1}\to X_n^*$ is given by

$$\alpha_{n+1}^*|X_{n+1} = \alpha_{n+1} \quad\text{and}\quad \alpha_{n+1}^*|X_i = \mathrm{id}_{X_i}, \quad i\leqslant n.$$

Finally let $X^* = \operatorname{Invlim}\{X_n^*, \alpha_n^*\}$. Clearly $\{X_n^*, \alpha_n^*\}$ is a movable sequence since we can take $r^{nn+1}\colon X_n^*\to X_{n+1}^*$ to be the inclusion and have

$$\alpha_n^* r^{nn+1} = \mathrm{id}_{X_n}^*.$$

Hence X^* is a movable continuum.

Now we construct the map F of X^* onto $X^*\cup_f Q$ (X^* attached to Q by f) which we will show is also a movable continuum. Let $k\colon X^*\to X^*+Q$ be the inclusion of X^* into the free union of X^* and Q and let $p\colon X^*+Q\to X^*\cup_f Q$ be the projection of X^*+Q onto the adjunction space $X^*\cup_f Q$. Define $F\colon X^*\to X^*\cup_f Q$ to be the composite pk. Then clearly F is also a Vietoris map.

To show that $X^*\cup_f Q$ is movable we verify the condition of uniform movability in [11] which agrees with the other versions of movability on metric compacta. A compactum Y is *movable* provided, that for each map $h\colon Y\to P$ of Y into a polyhedron P, there exists a polyhedron R, maps $g\colon Y\to R$, $\varphi\colon R\to P$, and a natural transformation $\Phi\colon \Pi_Y\to\Pi_R$ such that $\varphi g\simeq h$ and $\Phi[h] = [\varphi]$. So consider any map h of $X^*\cup_f Q$ into a polyhedron P. Then $h|Q\simeq 0$ and by the Neighborhood Extension Property there is a neighborhood N of Q in $X^*\cup_f Q$ such that $h|N\simeq 0$. Then there is an integer m such that for $n\geqslant m$, $X_n\subset N$ and $h|X_n\simeq 0$. Let the polyhedron R be X_m^*, $g\colon X^*\cup_f Q\to X_m^*$ be given by $g|X_m^* = \mathrm{id}_{X_m}^m$ and elsewhere g maps everything to x, and $\varphi\colon X_m^*\to P$ be given by $\varphi = h|X_m^*$. Let $i\colon X_m^*\to X^*\cup_f Q$ be the inclusion and let $\Phi = i^\#$ be the natural transformation determined by i. Then $\varphi g = h$ and $\Phi[h] = i^\#[h] = [hi] = [h|X_m^*] = [\varphi]$.

We now show that the map $F\colon X^*\to X^*\cup_f Q$ is not a shape equivalence in spite of the fact that it is a Vietoris map between movable continua. The following diagram

$$\begin{array}{ccc} X^* & \hookleftarrow & X_n \\ F\downarrow & & \downarrow F|X_n = \mathrm{id}_{X_n} \\ X^*\cup_f Q & \hookleftarrow & X_n \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} [X^* \cup_f Q, S^3] & \to & [X_n, S^3] \\ {\scriptstyle F^\#}\downarrow & & \| \\ [X^*, S^3] & \to & [X_n, S^3]. \end{array}$$

If F were a shape equivalence, $F^\#$ would be a bijection. We shall show that $F^\#$ is not surjective. There is an essential map of X^* onto S^3 such that its restriction to each X_n is an essential map of X_n onto S^3 [10]. However, as we have seen any map of $X^* \cup_f Q$ into S^3 when restricted to X_n, for n sufficiently large, is an inessential map of X_n into S^3.

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