On Fox's theory of overlays

by

Thomas T. Moore (Amherst. Mass.)

Abstract. Fox (Fund. Math. 74 (1972), pp. 47–71) obtains a classification of d-fold covering spaces equipped with overlay structures over an arbitrary metric space X. We consider the question: when does a covering space admit a unique overlay structure? We find that there is such a unique overlay structure when d is finite or when X is locally connected. We give a covering space with no overlay structure and one with infinitely many. In both examples, X is compact and connected.

In [3] R. H. Fox obtains a generalization of the fundamental theorem of covering spaces in which the hypothesis of local path connectedness is dropped. He defines a shape theoretic generalization of the fundamental group called the *fundamental trope* and a class of covering spaces which admit a certain kind of covering structure, called here an *overlay structure*. Assuming only that the base space is connected and metrizable, it is found that isomorphism classes of d-fold covering spaces equipped with overlay structures are in bijective correspondence with representations of the fundamental trope in the symmetric group of degree d. The purpose of this paper is to investigate the correspondence between isomorphism classes of covering spaces and isomorphism classes of covering spaces equipped with overlay structures, i.e., to discover to what extent Fox's theorem allows us to classify covering spaces. In § 2, we see that this correspondence is bijective when the base space is locally connected (Proposition 2.1) or when fibers are finite (Proposition 2.2). In § 3, examples are given showing that, in general, this correspondence need be neither injective or surjective. In particular, a connected covering space of a compact base space may have no overlay structures (Example 1) or more than one inequivalent overlay structure (Example 3).

Throughout, all spaces are assumed metrizable. By abuse of notation, we identify each covering space with its total space. We denote by p_E the covering projection corresponding to the covering space E. Unless otherwise stated, we assume the base of all covering spaces to be B. By [1], we can assume that B is a subset of a normed linear space and that it is closed in its convex hull $\bar{B}$.

1. General properties of overlays.

Definition 1.1. Let E be a covering space. An open cover $\{U_\alpha^i\}$ of E is called an *overlay structure* for E iff

(i) p_E induces a homeomorphism from each U_α^i to a well-defined open set U_α in B.

(ii) $p_E^{-1}(U_\alpha) = \bigcup_i U_\alpha^i$.

(iii) $i = j$ if U_β^k intersects both U_α^i and U_α^j.

We say overlay structures are *equivalent* iff they have a common refining overlay structure.

We call a covering space together with an equivalence class of overlay structures an *overlay*.

Fox in [3] defines an "overlay" to be a covering space which admits an overlay structure.

In view of Definition 1.1(iii), when $U_\alpha \cap U_\beta \neq \emptyset$, the relation $i = \mu_\beta^\alpha(j)$ iff $U_\alpha^i \cap U_\beta^j \neq \emptyset$ defines a bijection $\mu_\beta^\alpha\colon \{j\} \to \{i\}$. Clearly, $\{\mu_\beta^\alpha\}$ satisfies the condition $\mu_\beta^\alpha \mu_\gamma^\beta = \mu_\gamma^\alpha$ when $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. By standard fiber bundle arguments, $\{U_\alpha\}$ and $\{\mu_\beta^\alpha\}$ determine the covering space (up to isomorphism) together with its overlay structure $\{U_\alpha^i\}$.

Definition 1.2. By an *extension* of the covering space E, we mean a covering space $\tilde{E}$ of some open neighborhood of B in $\bar{B}$ where $E = p_{\tilde{E}}^{-1}(B)$ and $p_{\tilde{E}}(x) = p_E(x)$ for $x \in E$. We say $\tilde{E}_1$ and $\tilde{E}_2$ are *equivalent* extensions iff the identity map on E can be extended to an isomorphism from $\tilde{E}_1$ to $\tilde{E}_2$ over some open neighborhood of B in $\bar{B}$.

If $\{U_\alpha^i\}$ is an overlay structure for B and $\tilde{U}_\alpha$ consists of those elements of $\bar{B}$ closer to U_α than to $B - U_\alpha$, then the nerve of $\{\tilde{U}_\alpha\}$ is naturally equivalent to the nerve of $\{U_\alpha\}$, since $\tilde{U}_\alpha \cap \tilde{U}_\beta = \widetilde{U_\alpha \cap U_\beta}$. The open cover $\{\tilde{U}_\alpha\}$ of $\tilde{B} = \bigcup_\alpha \tilde{U}_\alpha$ together with $\{\mu_\beta^\alpha\}$ determines (up to isomorphism) a covering space $\tilde{E}$ of $\tilde{B}$ and an overlay structure $\{\tilde{U}_\alpha^i\}$ for $\tilde{E}$.

We identify E naturally as a subset of $\tilde{E}$ and call $\tilde{E}$ the extension of E *induced* by $\{U_\alpha^i\}$.

Definition 1.3. A metric d (possibly taking some infinite values) for a covering space E is said to be an *overlay metric* iff, for each $w \in B$ and $\{w^i\} = p_E^{-1}(w)$, there is an $\varepsilon > 0$ such that p_E induces an isometry from the ε-ball W^i about w onto the ε-ball W about w and $\{W^i\}$ partitions $p_E^{-1}(W)$.

If, for each $w \in B$, one ε and $\{W^i\}$ suffices for two overlay metrics, we call them *equivalent*.

Let the covering space $\tilde{E}$ of $\tilde{B}$ be an extension of E. Since $\tilde{E}$ is locally convex, $\tilde{B}$ has an overlay metric defined by letting $d(x, y)$ be the infimum of the set of lengths of paths in $\tilde{B}$ which can be lifted to paths from x to y in $\tilde{E}$. We call its restriction to E the overlay metric *induced* by $\tilde{E}$.

Lemma 1.4. *Let d be a metric for the covering space E.*

If d is overlay metric, then E has an open cover $\{V_\beta^j\}$ such that

(1) p_E induces a homeomorphism from each V_β^j to a well-defined open set V_β in B,

(2) $2\operatorname{diam} V_\beta^j < d(V_\beta^j, V_\beta^k)$ when $j \neq k$.

Furthermore, such an open cover $\{V_\beta^j\}$ is an overlay structure. (We call the maximal such overlay structure the overlay structure induced by d.)

Proof. For $w \in B$ and $\{w^i\} \in p_E^{-1}(w)$, let ε be as given by 1.3 with $0 < \delta < \frac{1}{6}\varepsilon$. Thus $d(w^i, w^j) < \varepsilon$ implies $i = j$. Let V_w be the δ-ball about w and V_w^i be the δ-ball about w^i. Thus $\operatorname{diam} V_w^i \leqslant 2\delta$ and $d(V_w^j, V_w^k) + 2\delta \geqslant \varepsilon$ when $j \neq k$. Therefore $2\operatorname{diam} V_w^i < d(V_w^j, V_w^k)$.

Let $\{V_\beta^j\}$ be as in Lemma 1.4. Suppose $x \in V_\alpha^i \cap V_\beta^k$ and $y \in V_\alpha^j \cap V_\beta^k$ with $i \neq j$. Let z be the unique element of $p_E^{-1}(p_E(y)) \cap V_\alpha^i$. z is contained in some V_β^l with $k \neq l$.

$$2d(x, z) < d(z, y) \text{ and } 2d(x, y) < d(x, z) \text{ imply } 2d(x, y) + d(x, z) < d(z, y),$$

which is impossible.

Theorem 1.5. *For a given covering space, equivalence classes of overlay structures, of extensions and of overlay metrics are in natural bijective correspondence.*

Proof. Let the overlay structure $\{V_\beta^j\}$ for E be induced by a metric which is induced by an extension which is in turn induced by an overlay structure $\{U_\alpha^i\}$.

Choose an open, convex cover $\{W_\gamma\}$ of the base space of the extension. Each W_γ has a unique even covering by sets W_α^k. For $W_\alpha \cap W_\beta \neq \emptyset$, the contractibility of $W_\alpha \cup W_\beta$ implies 1.1(iii) and thus $\{W_\alpha^i\}$ is an overlay structure for the extension. We can assume that $\{W_\alpha\}$ is fine enough so that each $B \cap W_\gamma$ lies in some U_α and also in some V_β. Since the overlay structure $\{E \cap W_\alpha^i\}$ for E refines both $\{U_\alpha^i\}$ and $\{V_\beta^j\}$, they are equivalent. Equivalent overlay structures induce equivalent extensions, since their common refining overlay structure induces an extension which can be naturally identified with an open subset in either extension. The fact that equivalent extensions induce equivalent metrics follows from the local convexity of $\bar{B}$. Equivalent metrics induce overlay structures whose intersection is an overlay structure and which thus are equivalent. If the metrics d_1 and d_2 induce overlay structures with the common refinement $\{W_\alpha^i\}$, then, for each $w \in B$, the choice of ε small enough to satisfy 1.3 for both d_1 and d_2 and also so that the ε-ball about w lies in some W_α suffices to show that d_1 and d_2 are equivalent. Let F be an extension of E. Let $\{W_\alpha^i\}$ be an overlay structure for F where each W_α is convex. Then $\{U_\alpha^i = E \cap W_\alpha^i\}$ is an overlay structure for E.

Let the covering space $\tilde{E}$ of $\tilde{B}$ be the extension of E induced by $\{U_\alpha^i\}$. $\{V_\alpha^i = W_\alpha^i \cap p_F^{-1}(\tilde{U}_\alpha)\}$ is an overlay structure for an open neighborhood of E in F which is naturally isomorphic to an open neighborhood of E in $\tilde{E}$.

Therefore, every extension F of E is equivalent to an extension $\tilde{E}$ of E induced by some overlay structure $\{U_\alpha^i\}$ for E.

In view of Theorem 1.5, the distinction between an overlay specified by an overlay structure, overlay metric, or extension is purely formal. For convenience, we define overlay transformation in terms of extensions.

DEFINITION 1.6. If E and E' are overlays, then an *overlay transformation* $T: E \to E'$ is a covering transformation which can be extended to a covering transformation between extensions of E and E'. (By an extension of an overlay, we mean an extension compatible with its overlay structure.)

An *overlay isomorphism* is an invertible overlay transformation.

Clearly, the problem of finding inequivalent overlay structures for a covering space is equivalent to finding a covering space isomorphism of overlays which is not an overlay isomorphism. The following is an immediate corollary of the fundamental theorem of covering spaces, Theorem 1.5, and the terminology and results of Fox [3] and Hyman [5].

THEOREM 1.7. *If B is connected, then isomorphism classes of n-fold overlays are in natural bijective correspondence with representations of the fundamental trope of B in the symmetric group of degree n.*

2. Uniqueness of overlay structure (1).

PROPOSITION 2.1. *Let E be a covering space. If B is locally connected, then E admits (up to equivalence) exactly one overlay structure.*

Proof. Let $\{U_\alpha\}$ be an open cover of B where each U_α is evenly covered by sets U_α^i. Let $\{V_\beta\}$ be a refinement of $\{U_\alpha\}$ where each V_β is connected and such that $V_\beta \cap V_\gamma \neq \emptyset$ implies $V_\beta \cup V_\gamma \subset U_\alpha$ for some α. (A barycentric refinement, whose existence is well known [2], is sufficient.) Each V_β is evenly covered by sets V_β^j. Since $V_\beta^j \cap V_\beta^k \neq \emptyset$ implies $V_\beta^j \cup V_\beta^k \subset U_\alpha^i$ for some α and i, $\{V_\beta^j\}$ is an overlay structure for E. Since $\{V_\beta\}$ determines $\{V_\beta^j\}$ and can be chosen arbitrarily fine, $\{V_\beta^j\}$ is the only overlay structure for E (up to equivalence).

PROPOSITION 2.2. *If the covering space E has finite fibers, then it admits (up to equivalence) exactly one overlay structure.*

Proof. Let d be a fixed metric for E. The finiteness of fibers allows us to find an open cover $\{V_\beta^i\}$ of E satisfying 1.4(1) and 1.4(2) and thus $\{V_\beta^i\}$ is an overlay structure. Again, $\{V_\beta\}$ determines $\{V_\beta^i\}$ and can be chosen arbitrarily fine. Therefore $\{V_\beta^i\}$ is the only overlay structure for E (up to equivalence).

3. Examples.

EXAMPLE 1. Let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, $Q = \{1, 2, \dots\}$, $C_0 = I\times 0 \cup A\times[0, \frac{1}{2}]$, $C_1 = I\times 1 \cup A\times[\frac{1}{2}, 1]$, and $B = C_0 \cup C_1$. Let E be $C_0\times Q\times 0 \cup C_1\times Q\times 1$ with the identification given by $(1/m, \frac{1}{2}, m, i) \sim (1/m, \frac{1}{2}, m+1, 1-i)$ for all $m \in Q$, $i \in \{0, 1\}$, $(y, \frac{1}{2}, 1, 0) \sim (y, \frac{1}{2}, 1, 1)$ for all $y \in A\setminus\{1\}$, and $(y, \frac{1}{2}, m, 0) \sim (y, \frac{1}{2}, m, 1)$

(1) It has been pointed out to me that versions of Propositions 2.1 and 2.2 in which only existence of overlay structure is considered have been proven by Fox in [4]. Also in [4], Example 1 is given.

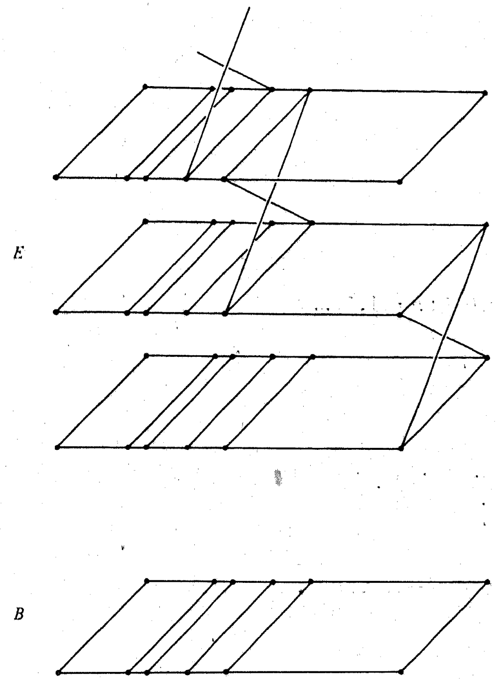

Fig. 1

for all $y \in A\setminus\{1/(m-1), 1/m\}$. Let $p_E: E \to B$ be given by $p_E[x, y, m, i] = (x, y)$. E is the covering space pictured in Figure 1.

E and B are path connected and B is compact. Suppose $p_E: E \to B$ has an overlay structure. Let $\bar{B} = I\times I$. By Theorem 1.7, there is an open neighborhood B^* of B in $\bar{B}$ and a covering space $p_{E^*}: E^* \to B^*$ such that p_{E^*} extends p_E. Since $\{0\}\times I$ is compact in $\bar{B}$, there is some $A_t = [0, t)\times I \subset B^*$. Let $p_t^*: p_{E^*}^{-1}(A) \to A_t$ be the restriction of p_{E^*}. Since A_t is contractible, $p_t^*: p_{E^*}^{-1}(A_t) \to A_t$ is trivial. Hence $p_t: p_E^{-1}(B \cap A_t) \to B \cap A_t$, defined to be the restriction of p_E (or p_{E^*}) is trivial, which is clearly not the case. Therefore, $p: E \to B$ has no overlay structure.

EXAMPLE 2. If A is a subset of the metric space (X, d) with $f: A \to X$ any map, let (δ) be the set of all pseudometrics [2] for X such that $\delta(x, y) \leqslant d(x, y)$ for all $x, y \in X$ and $\delta(x, f(x)) = 0$ for all $x \in A$. $d_f(x, y) = \sup\delta(x, y)$ defines the largest such pseudometric. The identification of x and y when $d_f(x, y) = 0$ defines a metric space (X_f, d_f).

Consider $X = \{(x, y) \in \mathbf{R}^2;\ 0 \leqslant y \leqslant 1$ and $x = 1/n$ for some $n = 1, 2, \dots$ or $y = 1$ and $-1 \leqslant x \leqslant 0\}$ and $\tilde{X} = X\times Z \subset \mathbf{R}^3$. Let $f: \{(1/n, 0);\ n = 1, 2, \dots\} \to X$ be the constant map given by $f(1/n, 0) = (-1, 1)$. X_f is as shown in Figure 2.

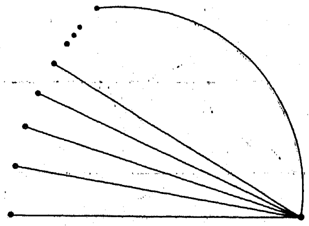

Fig. 2

Let F: $\{(1/n, 0, j);\ n = 1, 2, \dots \text{ and } j \in Z\} \to X$ be a map given by $F(1/n, 0, j) = (-1, 1, \beta_{nj})$ where, for given n, $j \to \beta_{nj}$ defines a permutation of the integers and, for given j, $\beta_{nj} = j+1$ holds for all but a finite number of n. $\tilde{X}_F$ is a covering space of X_f with projection map induced by the natural projection of $\tilde{X}$ onto X.

Up to covering space isomorphism, this covering space is independent of the choice of β_{nj}. However, the metric for $\tilde{X}_F$ is in fact an overlay metric, which depends on the β_{nj}. Indeed, $\beta_{nj} = j+1$ and $\beta_{nj} = j+1+\delta_{nj}-\delta_{n+1,j}$ yield non-isomorphic overlays.

EXAMPLE 3. The covering space ϱ: $\boldsymbol{R}^2 \to \boldsymbol{R}/Z \times \boldsymbol{R}$ has a natural overlay structure given by the Euclidean metric on $\boldsymbol{R}^2$. Let $\tilde{X} = \{(x, y) \in \boldsymbol{R}^2;\ y = (2/\pi)\operatorname{Arctg}(x+j)$ for some $j \in Z\} \cup \boldsymbol{R} \times \{-1, 1, 2\}$ and $X = \varrho(\tilde{X})$.

Let f: $\{(0, 2)\} \to X$ be given by $f(0, 2) = (0, 0)$. Let F: $Z \times \{2\} \to \tilde{X}$ be injective with image $Z \times \{0\}$. Using the terminology of Example 2, $\tilde{X}_F$ is a covering space of X_f, which is (up to covering space isomorphism) independent of the choice of F. Again however, the metric on $\tilde{X}_F$ is an overlay metric with $\tilde{X}_F$ and $\tilde{X}_{F'}$ isomorphic as overlays iff F and F' differ by a translation by some $(j, 0)$ with $j \in Z$ (cf. Fig. 3).

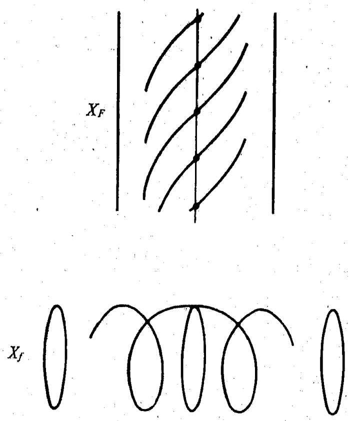

Fig. 3

References

[1] K. Borsuk, *Theory of Retracts*, Warszawa 1967.
[2] J. Dugundji, *Topology*, Boston 1966, pp. 168, 186, 198.
[3] R. H. Fox, *On shape*, Fund. Math. 74 (1972), pp. 47–71.
[4] — *Shape theory and covering spaces*, Lecture Notes in Mathematics 375 (1974), pp. 77–90.
[5] D. M. Hyman, *A remark on Fox's paper on shape*, Fund. Math. 75 (1972), pp. 205–208.

UNIVERSITY OF MASSACHUSETTS AT AMHERST

Accepté par la Rédaction le 31. 1. 1976