Some observations on Uniform Reduction for properties invariant on the range of definable relations

by

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Abstract. Several Uniform Reduction Properties for definable operations and relations for modeltheoretic languages are studied. They are generalizations and variants of corresponding properties definied by Feferman and Gaifman. It is shown that the unary case is equivalent to Craig's theorem. They are applied to prove Feferman–Vaught-type preservation theorems. $L_{\omega\omega}$ is characterized in terms of a Löwenheim–Skolem theorem and a Uniform Reduction Property.

1. Introduction. In [3] Feferman studies an application of his many-sorted interpolation theorem, concerning PC_L-definable binary relations on structures. The main value of this lies in uniformizing proofs of preservation theorems such as Gaifman's theorem on definable operations or definability results such as Beth's theorem. In [3] n-ary relations ($n>2$) are not considered and it is argued that they can be subsumed in the binary case by considering two structures as one two-sorted structure. But this presupposes a uniform reduction theorem for this pairing operation. The aim of this note is to study the precise assumption hidden behind Fefermans claim. We shall introduce several Uniform Reduction Properties and determine their logical interdependence and their impact on Definability properties such as Feferman–Vaught–type theorems, Beth's theorem and the interpolation theorem. It will turn out that the corresponding Uniform Reduction Theorem for n-ary ($n>2$) relations does not follow from Feferman's Interpolation theorem without further assumptions. The main result is that Feferman's many-sorted Interpolation theorem is equivalent to this Uniform Reduction Theorem. In Section 4 we discuss some Feferman–Vaught–type theorems which can be derived from our Uniform Reduction Theorem but not from Feferman's. As a corollary to this we shall get another characterization of the classical predicate calculus, namely that it is the strongest logic having both the Karp property introduced by Barwise and this stronger Uniform Reduction Property. I am indebted to S. Feferman for valuable suggestions. We assume the reader is familiar with Feferman's [3]. All other notation follows Chang and Keisler [2].

2. Preliminaries and definitions. Unless otherwise stated unexplained notation follows [3]. In particular *model-theoretic languages, types* and (many-sorted) *structures, elementary* and *projective classes* and *compactness properties* $\delta = (F, I)$ are taken from [4]. We now proceed to define several definability properties. Throughout the remainder of the paper we assume that L is any regular, $L_{\omega\omega}$-closed language, and that δ is any compactness property for L.

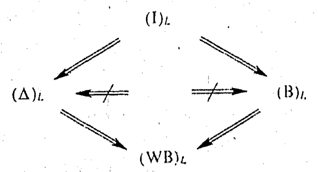
$(\mathrm{I})_L$: For each $\tau \in \mathrm{Typ}_L$ and $K_1, K_2 \in \mathrm{PC}_{L(\tau)}$, if $K_1 \cap K_2 = \emptyset$, then there exists $K_3 \in \mathrm{EC}_{L(\tau)}$ with $K_1 \leqslant K_3$ and $K_2 \leqslant \overline{K}_3$.

$(\Delta)_L$: For each $\tau \in \mathrm{Typ}_L$, if $K \in \mathrm{PC}_{L(\tau)}$ and $\overline{K} \in \mathrm{PC}_{L(\tau)}$ then $K \in \mathrm{EC}_{L(\tau)}$.

$(\mathrm{B})_L$: For each $\tau \in \mathrm{Typ}_L$, if $\mathfrak{A}$ is of type τ and there is at most one R such that $\langle \mathfrak{A}, R\rangle \in K$ and $K \in \mathrm{EC}_{L(\tau\cup R)}$ then $\{\langle \mathfrak{A}, \bar{a}\rangle$: there is an R with $\langle \mathfrak{A}, R\rangle \in K$ and $\bar{a} \in R\} \in \mathrm{EC}_{L(\tau\cup\bar{a})}$.

$(\mathrm{WB})_L$: As $(\mathrm{B})_L$ but "there is at most one R" replaced by "there is exactly one R".

PROPOSITION 1. *The mutual implications between these properties are given in the following diagram*:



The positive implications are well known. For the counterexamples one may consult Makowsky–Shelah [7], which contains a fairly complete survey.

3. Uniform Reduction Properties. If $\mathfrak{A}$ and $\mathfrak{B}$ are structures of typ τ_1 and τ_2 respectively then $[\mathfrak{A}, \mathfrak{B}]$ is a structure of type $[\tau_1, \tau_2]$. In first order logic the theory of $[\mathfrak{A}, \mathfrak{B}]$ depends only on the theory of $\mathfrak{A}$ and the theory of $\mathfrak{B}$. This is a consequence of the Feferman–Vaught theorem in [5]. In fact they prove this result with a reduction theorem which motivated the following generalization: (Uniform Reduction for Pairs)

$(\mathrm{URP})_L$: Let τ_1, τ_2, τ_3, be in Typ_L and $\tau_3 = [\tau_1, \tau_2]$. Then for every φ in $\mathrm{Stc}(\tau_3)$ there exists a pair of sequences of formulas $\psi_1^1, \ldots, \psi_{n_1}^1$ and $\psi_1^2, \ldots, \psi_{n_2}^2$ with ψ_j^i in $\mathrm{Stc}(\tau_i)$ and a Boolean function $B \in 2^{n_1+n_2}$ such that $[\mathfrak{A}, \mathfrak{B}] \models \varphi$ iff $B(a_1^1, \ldots, a_{n_1}^1, a_1^2 \ldots, a_{n_2}^2) = 1$ where

$$(*) \qquad a_j^i = \begin{cases} 1 & \text{if} \quad \mathfrak{A}_i \models \psi_j^i, \\ 0 & \text{else.} \end{cases}$$

EXAMPLE 1. $(\mathrm{URP})_L$ holds for $L = L_{\omega\omega}$ and $L = L_{\omega\omega}(Q_\alpha)$, $\aleph_\alpha$ regular. (For the latter one may consult, among others, A. Wojciechowska [9].)

EXAMPLE 2. $(\mathrm{URP})_L$ holds also for $L = L_{\infty\omega}$ (cf. Malitz [8]).

EXAMPLE 3. $(\mathrm{URP})_L$ does not hold for $L = L_{\omega_1\omega}$ or more generally for $L = L_{\kappa\lambda}$ and k not strongly inaccessible (cf. Malitz [8]).

Let $\tau_0, \ldots, \tau_n$ be disjoint types and $R \subset \mathrm{Str}(\tau_0) \times \ldots \times \mathrm{Str}(\tau_n)$. A sentence φ in $\mathrm{Stc}_L(\tau_n)$ is said to be *invariant on the range of* R if for all $\mathfrak{A}_0, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n, \mathfrak{A}_n'$

(i) $R(\mathfrak{A}_0, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n)$ & $R(\mathfrak{A}_0, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n')$ implies $\mathfrak{A}_n \models \varphi$ iff $\mathfrak{A}_n' \models \varphi$.

An n-tuple of sequences of formulas $\overline{\psi}_0, \ldots, \overline{\psi}_{n-1}$ with $\overline{\psi}_k = \psi_1^k, \ldots, \psi_{m_k}^k$ and ψ_i^k in $\mathrm{Stc}(\tau_k)$ together with a Boolean function $B \in 2^{m_1+\ldots+m_{n-1}}$ is called an *associate pair* for φ on the domain of R if for all $\mathfrak{A}_0, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n$ we have

(ii) $R(\mathfrak{A}_0, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n)$ implies $\mathfrak{A}_n \models \varphi$ iff $B(a_1^1, \ldots, a_{m_1}^1, a_1^2, \ldots, a_{m_{n-1}}^{n-1}) = 1$ where a_j^i is defined as in (*).

$(\mathrm{UR}(n))_L$: Let R be an $(n+1)$-ary PC_L^δ-relation on structures for L. If φ in $\mathrm{STc}_L(\tau_n)$ is invariant on the range of R, then there is an associate pair for φ on the domain of R. (Uniform Reduction on n-ary domains)

PROPOSITION 2. (i) $(\mathrm{UR}(n+1))_L \Rightarrow (\mathrm{UR}(n))_L$ *for all* n.

(ii) $(\mathrm{UR}(2))_L \Rightarrow (\mathrm{URP})_L$.

(iii) $(\mathrm{UR}(1))_L$ & $(\mathrm{URP})_L \Rightarrow (\mathrm{UR}(n))_L$ *for all* $n < \omega$.

Proof. We just have to observe that the relation $R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ iff $\mathfrak{C} = [\mathfrak{A}, \mathfrak{B}]$ is PC_L for all $L_{\omega\omega}$-closed regular languages. Uniform Reduction for unary domains has been introduced by Feferman [3]. There he derives it from $(\mathrm{I})_L$ and shows that $(\mathrm{B})_L$ is a consequence from $(\mathrm{UR}(1))_L$. In fact we have more:

THEOREM 3. $(\mathrm{I})_L$ *iff* $(\mathrm{UR}(1))_L$, *provided all* $\tau \in \mathrm{Typ}_L$ *are finite.*

Proof. We only have to prove $(\mathrm{UR}(1))_L \Rightarrow (\mathrm{I})_L$. Let K_1 and K_2 be two disjoint $\mathrm{PC}_{L(\tau_0)}$-classes. Hence there is a type τ_1 extending τ_0 such that K_1 and K_2 are both $\mathrm{EC}_{L(\tau_1)}$. Let $K_1 = \mathrm{Mod}_L(\psi_1)$ and $K_2 = \mathrm{Mod}_L(\psi_2)$. Now put $R \subset \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1)$ with $R(\mathfrak{A}, \mathfrak{B})$ iff $\mathfrak{A} =_{\tau_0} \mathfrak{B} \restriction \tau_0$ and $\mathfrak{B}$ is in K_1 or in K_2. Clearly R is PC_L using an additional predicate for the isomorphism and the finiteness of τ_0.

Claim. ψ_1 and ψ_2 are invariant in the range of R. For assume there are $\mathfrak{A}, \mathfrak{B}_1$ and $\mathfrak{B}_2$ with $R(\mathfrak{A}, \mathfrak{B}_1)$ and $R(\mathfrak{A}, \mathfrak{B}_2)$ and, say, $\mathfrak{B}_1 \models \psi_1$ but $\mathfrak{B}_2 \models \psi_1$. Then $\mathfrak{B}_2 \models \psi_2$ and $\mathfrak{A}$ is in $K_1 \cap K_2$, a contradiction. Let θ_1 and θ_2 be associates for ψ_1, ψ_2 respectively. Now assume $\mathfrak{A}$ is in K_1. So $\mathfrak{A}$ has an expansion $\mathfrak{A}^* \models \psi_1$. Therefore $R(\mathfrak{A}, \mathfrak{A}^*)$ and $\mathfrak{A} \models \theta_1 \wedge \neg\theta_2$. Similarly we get, if $\mathfrak{A} \in K_2$ then $\mathfrak{A} \models \theta_2 \wedge \neg\theta_1$. Therefore $\theta_1 \wedge \neg\theta_2$ is the desired interpolant. ∎

Remark. For those familiar with Feferman [4], it is clear that we only need that τ_0 is L-finite, or, to put it anotherway, that "f is a τ_0-isomorphism" is PC_L.

PROPOSITION 4. (i) $(\mathrm{UR}(1))_L \not\Rightarrow (\mathrm{URP})_L$, *hence* $(\mathrm{UR}(1))_L \not\Rightarrow (\mathrm{UR}(2))_L$.

(ii) $(\mathrm{URP})_L \not\Rightarrow (\mathrm{UR}(1))_L$, *in fact.*

(iii) $(\mathrm{URP})_1 \not\Rightarrow (\Delta)_1$ *and* $(\mathrm{URP})_L \not\Rightarrow (\mathrm{B})_L$.

Proof. For (i) take $L_{\omega_1\omega}$. Since $L_{\omega_1\omega}$ satisfies interpolation it satisfies $(\mathrm{UR}(1))_{L_{\omega_1\omega}}$. The rest is stated in Example 3. (ii) follows from (iii) by Theorem 3 and Proposition 1. The counterexample is $L_{\omega\omega}(Q_\alpha)$ (cf. Makowsky–Shelah [7]).

4. Feferman-Vaught-type theorems. Let F be an operation on structures, $F: \mathrm{Str}(\mathfrak{A}_0)\times\ldots\times\mathrm{Str}(\mathfrak{A}_{n-1})\to\mathrm{Str}(\mathfrak{A}_n)$. We are interested in the following type of preservation results inspired by the work of Feferman–Vaught [5].

$\mathrm{FV}(F)_L$ If F is an n-ary operation and for all structures $\mathfrak{A}_i, \mathfrak{A}_i'$ of type τ_i $(i<n)$ we have $\mathfrak{A}_i\equiv_L\mathfrak{A}_i'$ then $F(\mathfrak{A}_0,\ldots,\mathfrak{A}_{n-1})\equiv_L F(\mathfrak{A}_0',\ldots,\mathfrak{A}_{n-1}')$.

EXAMPLE 4. The pairing operation for structures: $P(\mathfrak{A},\mathfrak{B})=[\mathfrak{A},\mathfrak{B}]$. P is a PC_L-operation for any $L_{\omega\omega}$-closed logic L.

EXAMPLE 5. The ordered sum (ORDS) and ordered product (ORDP) for ordered structures. Both are PC_L operations.

More examples may be found in Feferman [3].

PROPOSITION 5. (i) $(\mathrm{URP})_L\Rightarrow\mathrm{FV}(P)_L$.

(ii) *If F is a PC_L^δ-operation then* $(\mathrm{UR}(2))_L\Rightarrow\mathrm{FV}(F)_L$.

This is an obvious generalization of the results in Feferman [3]. It extends Gaifman's theorem to n-ary operations for a seemingly small price $((\mathrm{URP})_L)$ and gives a new result when applied to $L_{\omega\omega}$ and the operation Q (cf. also Gaifman [6]).

5. A Lindström-type result. For the following definitions of abstract modeltheory we refer to Barwise [1]. A logic L has the Karp property if partially isomorphic structures are L-elementarily equivalent. A logic L has the $\mathrm{LS}(\omega)$-property if every $K\in\mathrm{EC}_L$ which has a model has a countable (or finite) model. The following theorem is due to Shelah (cf. Makowsky and Shelah [7]).

THEOREM 6. *If L satisfies $(\mathrm{WB})_L$ and $\mathrm{FV}(P)_L$ and has the $\mathrm{LS}(\omega)$-property, then $L=L_{\omega\omega}$.*

From this we shall deduce the following

THEOREM 7. *If L has the Karp property and satisfies* $(\mathrm{UR}(2))_L$ *then* $L=L_{\omega\omega}$.

Proof. By a theorem due to Barwise every logic L with the Karp property which satisfies $(\mathrm{I})_L$ has the $\mathrm{LS}(\omega)$-property (cf. [1]). So by Theorem 3 (or for infinitary types by Feferman's theorem stated before Theorem 3) L has the $\mathrm{LS}(\omega)$-property. By Propositions 5 and 2 L satisfies $\mathrm{FV}(P)_L$ and by Theorem 3 and Proposition 1 L satisfies $(\mathrm{WB})_L$. Now apply Theorem 6. ■

6. Discussion. The results in the previous sections teach us several lessons.

1. $(\mathrm{UR}(2))_L$ is a relatively rare property (Theorem 7).

2. One could argue that for many-sorted logics $(\mathrm{URP})_L$ is a natural thing to ask for. At least it does not contradict our (admittedly poor) intuition about logics.

3. But $(\mathrm{URP})_L$ together with $(\mathrm{I})_L$ implies $(\mathrm{UR}(2))_L$.

4. So $(\mathrm{I})_L$ is, in the abstract context, both too weak (since it is equivalent to $(\mathrm{UR}(1))_L$ but not to $(\mathrm{UR}(2))_L$) and too strong (since for most of the definability applications either $(\Delta)_L$ or $(\mathrm{B})_L$ are good enough and strictly weaker).

Interpolation properties shall be discussed more in detail in Makowsky–Shelah [7]. Uniform reduction properties for n-ary Relations could be defined in many ways: Doing it via Boolean functions seems the easiest which works for $L_{\omega\omega}$. It could be replaced by a formula for Boolean algebras in the abstract language, allowing even infinitely many free variables, but for finitary relations this seems going too far.

References

[1] K. J. Barwise, *Axioms for abstract model theory*, Ann. Math. Logic 7 (1973), pp. 221–266.
[2] C. C. Chang and J. Keisler, *Model Theory*, Amsterdam 1973.
[3] S. Feferman, *Two notes on abstract model theory I*, Fund. Math. 82 (1974), pp. 153–165.
[4] — *Two notes on abstract model theory II*, Fund. Math. 89 (1975), pp. 111–130.
[5] — and R. Vaught, *The first order properties of products of algebraic systems*, Fund. Math. 47 (1959), pp. 57–103.
[6] H. Gaifman, *Operations on relational structures, functors and classes*, I. Proceedings of the Tarski Symposion 1971, Proceedings of Symposia in Pure Mathematics 25 (1974), pp. 21–40.
[7] J. A. Makowsky and S. Shelah, *The theorems of Beth and Craig in abstract model theory*, I, to appear.
[8] J. Malitz, *Infinitary analogs of theorems from first order model theory*, J. Symbolic Logic 36 (1971), pp. 216–228.
[9] A. Wojciechowska, *Generalized products for Q_α-languages*, Bull. Acad. Polon. Sci. 17 (1969), pp. 337–339.

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Accepté par la Rédaction le 31. 1. 1976