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# Undefinable ordinals and the rank hierarchy

by

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**Abstract.** This paper shows that certain definability properties concerning the ordinals $\alpha$ and $\beta$ are equivalent to the property of $\langle R\alpha, \varepsilon\rangle$ being a proper elementary substructure of $\langle R\beta, \varepsilon\rangle$.

**1. Introduction.** This note was motivated by [5]. Section 2 starts by answering a question from [5] and then it gives a number of conditions involving undefinable ordinals, each of which is equivalent to $R\alpha \prec R\beta$ (where $R\alpha \prec R\beta$ means $\langle R\alpha, \varepsilon\rangle$ is a proper elementary substructure of $\langle R\beta, \varepsilon\rangle$).

Most of our notation is standard but $\mathrm{Df}(x, y)$ is the set of those elements of $x$ which are definable in $\langle x, \varepsilon\rangle$ using a first order $\varepsilon$ formula with parameters from $y \cap x$. Also, $\mathrm{Df}(x) = \mathrm{Df}(x, \varphi)$ and $\bar{\bar{x}}$ is the cardinality of $x$.

It is well known that $V = L$ implies the existence of certain definable well orderings and we shall make use of this fact in the following form (see Theorem 4.11 of [4], for instance).

THEOREM 1. *Suppose that* $V = L$ *holds and that* $\beta \geqslant \omega$. *Then there is an* $\varepsilon$ *formula* $\varphi$ *with two free variables such that* $\{\langle x, y\rangle \mid \varphi^{R\beta+1}(x, y)\}$ *is a well ordering of* $R\beta+1$.

**2. Results.** The following notions were introduced in [5]. An ordinal $\alpha$ ($\in x$) is said to be *inconceivable* in $x$ if $\alpha \notin \mathrm{Df}(x, \alpha)$, *strongly inconceivable* in $x$ if $\beta \geqslant \alpha \to \beta \notin \mathrm{Df}(x, \alpha)$ and *weakly inconceivable* in $x$ if it is inconceivable, but not strongly inconceivable in $x$. Then Theorem 2.4 (i) of [5] gives

$$R\beta \models \mathrm{ZF} \to (\alpha \text{ is strongly inconceivable in } R\beta \to R\alpha \prec R\beta),$$

and Rucker asks if this result can be proved without assuming $R\beta \models \mathrm{ZF}$. More precisely, he asks "If $x$ is a model of $Z$ and there is an $\alpha \in x$ such that $\alpha$ is strongly inconceivable in $x$, then is $x$ a model of ZF?"

Theorem 2 shows that the answer to Rucker's question is no, in general, as there is an $\alpha$ which is strongly inconceivable in $R\omega_1$ and $R\omega_1$ is not a model of ZF. However, Theorem 3 shows that if $V = L$ holds, then we get a positive answer to Rucker's question when $x = R\beta$ and $\beta$ is a singular ordinal.

THEOREM 2. *If* $\beta$ *is a regular ordinal* $> \omega$, *then there is an* $\alpha < \beta$ *such that* $\alpha$ *is strongly inconceivable in* $R\beta$.

Proof. Suppose that $\beta$ is a regular ordinal $>\omega$ and consider the ordinals

$$\delta_0 = \sup(\mathrm{Df}(R\beta)\cap\beta),$$
$$\delta_{n+1} = \sup(\mathrm{Df}(R\beta,\delta_n)\cap\beta),$$
$$\alpha = \sup_{n\in\omega}\delta_n .$$

For every $n\in\omega$, $\bar{\delta}_n<\beta$ so that $\delta_{n+1}<\beta$ and then as $\mathrm{cf}(\beta)>\omega$ we get $\alpha<\beta$. It is also clear that $\alpha$ is strongly inconceivable in $R\beta$ as

$$\xi\in\mathrm{Df}(R\beta,\alpha)\to\xi\in\mathrm{Df}(R\beta,\delta_n)\quad\text{for some } n$$
$$\to\xi<\delta_{n+1}<\alpha. \blacksquare$$

THEOREM 3. *Suppose that* V = L *holds,* $\beta$ *is a singular ordinal and* $\alpha$ *is strongly inconceivable in* $R\beta$. *Then* $R\beta\models\mathrm{ZF}$.

Proof. Suppose that V = L holds, $\mathrm{cf}(\beta)<\beta$, $\alpha$ is strongly inconceivable in $R\beta$ and $R\beta$ not $\models$ ZF. Then $\langle R\beta,\varepsilon\rangle$ is not a model of the Replacement axiom and it easily follows that there is a set $y\in R\beta$ and a formula $\theta(x,\gamma)$ such that $\{\langle x,\gamma\rangle|\ \theta^{R\beta}(x,\gamma)\}$ is an injection from $y$ to $\beta$ which is cofinal in $\beta$. (We have assumed that there are no parameters in $\theta$, but it is straightforward to generalise our proof if this is not the case.)

Let $\eta$ be the least ordinal for which there is a $y\in R\eta$ satisfying the above condition and then let $y$ be the least such set in $R\eta$ (using the definable well ordering given by Theorem 1). Then $y\in\mathrm{Df}(R\beta)$.

$\theta$ induces a well ordering of $y$ and then we can define its cofinality which is, of course, $\mathrm{cf}(\beta)$. As $\mathrm{cf}(\beta)<\beta$ we can suppose that the cofinality is actually an ordinal and then we get $\mathrm{cf}(\beta)<\alpha$ as $\alpha$ is strongly inconceivable in $R\beta$ and $\mathrm{cf}(\beta)\in\mathrm{Df}(R\beta)$.

Using the definable well ordering given by Theorem 1 again, we can now let $f$ be the least injection from $\mathrm{cf}(\beta)$ to $y$ which is cofinal in $y$ with the induced ordering. Then combining $f$ with the function given by $\theta$ we can get a formula $\psi(\delta,\gamma)$ such that $\{\langle\delta,\gamma\rangle|\ \psi^{R\beta}(\delta,\gamma)\}$ is an injection from $\mathrm{cf}(\beta)$ to $\beta$ which is cofinal in $\beta$. As $\mathrm{cf}(\beta)<\alpha$, this contradicts $\alpha$ being strongly inconceivable in $R\beta$. $\blacksquare$

Next, we point out that it is not possible to prove Theorem 3 without using some assumption such as V = L. Following the notation of [1], let $\alpha_0$ be the least ordinal $\alpha$ such that $\exists\gamma>\alpha,\ \alpha\notin\mathrm{Df}(R\gamma)$ and let $\gamma_0$ be the least $\gamma$ such that $\exists\alpha<\gamma\alpha\notin\mathrm{Df}(R\gamma)$. Then Theorem 2.5 of [1] shows that $\alpha_0$ is strongly inconceivable in $R\gamma_0$ and Theorem 4.4 of that paper shows that it is relatively consistent to have $\gamma_0<\omega_1$. In this case, $\gamma_0$ is a singular ordinal and $R\gamma_0$ not $\models$ ZF, as required.

Theorem 4 is an improved version of Theorem 3.1 of [5] for the rank hierarchy, and we shall use it later on.

THEOREM 4. *Suppose that* V = L *holds and that* $\alpha$ *is weakly inconceivable in* $R\beta$. *Let* $\alpha^*$ *be the least ordinal* $>\alpha$ *satisfying* $\alpha^*\in\mathrm{Df}(R\beta,\alpha)$. *Then*

(i) $\alpha^*$ *is a regular, uncountable cardinal,*

(ii) *if* $\alpha$ *is a cardinal, then* $\alpha^*$ *is an inaccessible cardinal, and*



(iii) *if* $\alpha$ *is an inaccessible cardinal, then* $\alpha^*$ *is a Mahlo cardinal.*

Proof. The proof is exactly similar to Rucker's proof of Theorem 3.1 in [5], except that when "the least $x$ satisfying..." is taken, use the definable well ordering with respect to an appropriate $R\gamma$, as given by Theorem 1. $\blacksquare$

We end by noting five conditions involving definability, each of which is equivalent to $R\alpha\prec R\beta$. However, the last three of these are probably best thought of as relative consistency results. The first two conditions, which are due to Grewe ([2]) and Rucker ([5]), respectively, are

(i) $\beta\cap\mathrm{Df}(R\beta,R\alpha)=\alpha$, and

(ii) $R\beta\models\mathrm{ZF}$ and $\alpha$ is strongly inconceivable in $R\beta$.

If we are willing to assume that V = L holds, then another equivalent of $R\alpha\prec R\beta$ is

(iii) $\beta$ is a limit cardinal and $\alpha$ is strongly inconceivable in $R\beta$.

This can be seen as follows. It is clear that $R\alpha\prec R\beta$ implies that $\alpha$ and $\beta$ satisfy (iii). Now, suppose that V = L holds and that $\alpha$ and $\beta$ satisfy (iii). Then, if $\beta$ is regular cardinal it is also inaccessible and we have $R\beta\models\mathrm{ZF}$. If $\beta$ is a singular cardinal, then Theorem 3 gives $R\beta\models\mathrm{ZF}$ so that in either case $\alpha$ and $\beta$ satisfy (ii) and we have $R\alpha\prec R\beta$.

If we suppose that V = L holds and there are no inaccessible cardinals between $\alpha$ and $\beta$, then two further equivalents are

(iv) $\alpha\notin\mathrm{Df}(R\beta,R\alpha)$, and

(v) $\beta$ is a limit cardinal, $\alpha$ is a cardinal and $\alpha$ is inconceivable in $R\beta$.

We proved the equivalence of (iv), under these conditions, in [3], and it is clear that $R\alpha\prec R\beta$ implies that $\alpha$ and $\beta$ satisfy (v). Now suppose that V = L holds, there are no inaccessible cardinals between $\alpha$ and $\beta$ and that $\alpha$ and $\beta$ satisfy (v). Then Theorem 4 (ii) shows that $\alpha$ is strongly inconceivable in $R\beta$, so that $R\alpha\prec R\beta$ follows from (iii).

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