Hyperspaces of polyhedra are Hilbert cubes *

by

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Abstract. Let 2^X be the hyperspace of nonempty closed subsets of a metric continuum X, and let $C(X)$ be the space of nonempty subcontinua of X, both with the Hausdorff metric. The main results of this paper are that if P is a nondegenerate connected polyhedron, then 2^P is homeomorphic to the Hilbert cube Q, $C(P)\times Q$ is homeomorphic to Q, and if P contains no principal 1-cells, then $C(P)$ is homeomorphic to Q. Proofs of these theorems are based on theorems of Schori and West (*Hyperspaces of graphs are Hilbert cubes*, Pacific J. Math. 53 (1974), pp. 239–251).

§ 1. **Introduction.** Let 2^X be the hyperspace of nonempty closed subsets of a metric continuum X, and let $C(X)$ be the space of nonempty subcontinua of X, both with the Hausdorff metric. In [4], we announced the following results.

THEOREM 1.1. *$2^X\approx Q$, the Hilbert cube, if and only if X is a nondegenerate Peano space (locally connected metric continuum).*

THEOREM 1.2. *$C(X)\times Q\approx Q$ if and only if X is a Peano space, and $C(X)\approx Q$ if and only if X is a nondegenerate Peano space containing no free arcs.*

In this paper, we introduce some techniques and apply them to prove the above theorems for polyhedra X. In [5], we apply these techniques to prove the above stated general theorems.

We refer the reader to [3], [4], [7], [8], [9] and [10] for background material and previous results on hyperspace problems. In particular, the proofs of the above theorems are based on the recent results of Schori and West [10] that $2^\Gamma\approx Q$ for every nondegenerate compact connected graph Γ, and $C(L)\approx Q$ for every compact connected local dendron L with a dense set of branch points.

Certain relative versions of these theorems are also obtained. For $A\in 2^X$, let $2^X_A=\{B\in 2^X\colon A\subset B\}$, and for $A\in C(X)$, let $C_A(X)=\{B\in C(X)\colon A\subset B\}$.

THEOREM 1.3. *$2^X_A\approx Q$ if X is a Peano space and $A\neq X$. $C_A(X)\times Q\approx Q$ if X is a Peano space, and $C_A(X)\approx Q$ if X is a Peano space, $A\neq X$, and $X\setminus A$ contains no free arcs.*

In §§ 2, 3 and 5, we develop some of the necessary tools (an inverse sequence approximation lemma, and techniques for obtaining near-homeomorphisms between hyperspaces of graphs). These are applied in §§ 4, 6 and 7 to hyperspaces of polyhedra, and will be applied in [5] to complete the proofs of the general results.

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§ 2. Structure of the proof. A map $f\colon X_1\to X_2$ between copies of a compact metric space is a *near-homeomorphism* if it is the uniform limit of (onto) homeomorphisms.

We shall construct inverse sequences satisfying the hypotheses of the following lemma.

APPROXIMATION LEMMA 2.1. *Let Y be a compact metric space, and let*

$$Q_1 \xleftarrow{f_1} Q_2 \xleftarrow{f_2} \dots$$

be an inverse sequence of maps and copies of the Hilbert cube in Y such that

(i) $Q_i\to Y$ *(in 2^Y)*;

(ii) $\sum_{i=1}^{\infty} d(f_i, \mathrm{id})<\infty$;

(iii) $\{f_i\circ\dots\circ f_j\colon j\geqslant i\}$ *is an equi-uniformly continuous family for each i; and*

(iv) *each f_i is a near-homeomorphism. Then $Y\approx Q$.*

Thus, for instance, we apply the Approximation lemma (to be proved below) to the hyperspace 2^X of a nondegenerate compact, connected polyhedron X by constructing an inverse sequence

$$2^{\Gamma_1} \xleftarrow{f_1} 2^{\Gamma_2} \xleftarrow{f_2} \dots$$

where $\{\Gamma_i\}$ is a sequence of compact connected graphs in X converging to X (thus $2^{\Gamma_i}\approx Q$ and $2^{\Gamma_i}\to 2^X$), and the maps $\{f_i\}$ are near-homeomorphisms satisfying conditions (ii) and (iii) of the lemma.

Each map $f_i\colon 2^{\Gamma_{i+1}}\to 2^{\Gamma_i}$ is induced by a map $\varphi_i\colon \Gamma_{i+1}\to C(\Gamma_i)$, i.e.,

$$f_i(A)=\bigcup\{\varphi_i(a)\colon a\in A\}.$$

The particular type of map φ_i used for this purpose (a *C-monotone piecewise-linear map*) is discussed in § 3 and § 5, where it is shown that the induced maps f_i are near-homeomorphisms.

Proof of Lemma 2.1. If we denote $\mathrm{invlim}(Q_i, f_i)$ by Q_∞, then the fact that $Y\approx Q_\infty$ follows from [1], Theorem I. As an aid to the reader we outline this proof. Define $h\colon Q_\infty\to Y$ as follows. For $(q_i)\in Q_\infty$, the sequence (q_i) in Y is Cauchy by Dondition (ii) and hence converges to a point $q\in Y$. Let $h(q_i)=p$. Condition (ii) also implies that h is continuous. With an easy proof by contradiction, Condition (iii) implies that h is one-to-one and Condition (i) implies that h is onto. Thus, h is a homeomorphism and hence $Q_\infty\approx Y$.

Since each $Q_i\approx Q$ and each f_i is a near-homeomorphism, it follows by Morton Brown's theorem [2] that $Q_\infty\approx Q$ and hence $Y\approx Q$.

§ 3. Piecewise-linear induced maps on hyperspaces of graphs. Let Γ be a compact connected graph and for every compact connected subgraph S of Γ let ϱ_S be the minimum path-length metric. For $D=\mathrm{diam}(S, \varrho_S)$, let $e_S\colon C(S)\times I\to C(S)$ be the expansion homotopy defined by $e_S(A, t)=\{x\in S\colon \varrho_S(x, A)\leqslant tD\}$. Thus $e_S(A, 0)=A$ and $e_S(A, 1)=S$ for each $A\in C(S)$. In the following Γ_i will always denote a compact connected graph.

DEFINITION 3.1. A map $\varphi\colon \Gamma_1\to C(\Gamma_2)$ is *simplicial* if for each vertex $v\in\Gamma_1$, $\varphi(v)$ is a connected subgraph (possibly degenerate) of Γ_2, and for each edge e of Γ_1, either

(i) $\varphi|e$ is a linear map onto an edge of Γ_2, or

(ii) $\varphi(v)\subset\varphi(w)\subset \mathrm{St}\,\varphi(v)$, where $\dot e=\{v, w\}$, and $\varphi(tw+(1-t)v)=e_{\varphi(w)}(\varphi(v), t)$, for every $t\in I$.

DEFINITION 3.2. A map $\varphi\colon \Gamma_1\to C(\Gamma_2)$ is *piecewise-linear* if there exist triangulations of Γ_1 and Γ_2 with respect to which φ is simplicial.

Remark. If $\varphi\colon \Gamma_1\to C(\Gamma_2)$ is piecewise-linear, then there exist arbitrarily fine subdivisions of Γ_1 and Γ_2 with respect to which φ is simplicial.

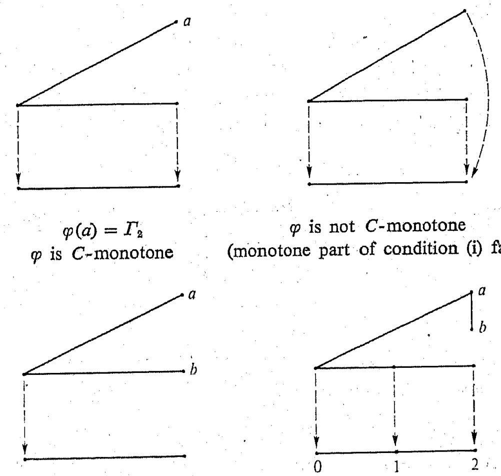
We are now ready to introduce C-monotone maps. Let $\tilde\Gamma_2\subset C(\Gamma_2)$ be the collection of degenerate subcontinua, and let $\Gamma_1^*=\varphi^{-1}(\tilde\Gamma_2)$.

DEFINITION 3.3. A piecewise-linear map $\varphi\colon \Gamma_1\to C(\Gamma_2)$ is *C-monotone* if

(i) $\varphi|\Gamma_1^*$ is a monotone map onto $\tilde\Gamma_2$, and

(ii) for each $x\in\Gamma_1$ there exists a subcontinuum C_x of Γ_1 such that $x\in C_x$, $C_x\cap\Gamma_1^*\neq\emptyset$, and $\varphi(y)\subset\varphi(x)$ for each $y\in C_x$.

C-monotone piecewise-linear maps $\Gamma_1\to C(\Gamma_2)$ may be regarded as generalizations of monotone piecewise-linear maps $\Gamma_1\to\Gamma_2$. The following examples may serve to clarify the above definitions.



$\varphi(a)=\Gamma_2$
φ is C-monotone

φ is not C-monotone
(monotone part of condition (i) fails)

$\varphi(a)=\varphi(b)=\Gamma_2$
φ is not C-monotone
(onto part of condition (i) fails)

$\varphi(a)=[0, 2]$, $\varphi(b)=[1, 2]$
φ is not C-monotone
(condition (ii) fails)

Fig. 1

Every map $\varphi\colon \Gamma_1 \to C(\Gamma_2)$ induces hyperspace maps $f\colon 2^{\Gamma_1} \to 2^{\Gamma_2}$ and $g\colon C(\Gamma_1) \to C(\Gamma_2)$. Furthermore, if $\varphi(p) = q \in \tilde{\Gamma}_2$, then φ induces relative hyperspace maps $f_{pq}\colon 2_p^{\Gamma_1} \to 2_q^{\Gamma_2}$ and $g_{pq}\colon C_p(\Gamma_1) \to C_q(\Gamma_2)$.

THEOREM 3.5. *Let $\varphi\colon \Gamma_1 \to C(\Gamma_2)$ be a C-monotone piecewise-linear map. Then all induced maps f, g, f_{pq}, g_{pq} stabilize to near-homeomorphisms (i.e., $f \times \mathrm{id}_Q\colon 2^{\Gamma_1} \times Q \to 2^{\Gamma_2} \times Q$, etc., are near-homeomorphisms).*

The proof of Theorem 3.5 that we give is relatively short but uses a good deal of recently established and powerful apparatus. Our original proof of Theorem 3.5, on which our announcement in [4] was based, was more elementary but much longer and used rather involved constructions of Q-factor decompositions.

A closed subset of the Hilbert cube has *trivial shape* if it is contractible in each neighborhood of itself, and a surjection between Hilbert cubes is *cell-like* if each point inverse has trivial shape. The next theorem is a powerful theorem originally proved by T. A. Chapman with a more direct and shorter proof supplied by A. Fathi in [6].

THEOREM (Chapman). *A cell-like map between Hilbert cubes is a near-homeomorphism.*

Proof of Theorem 3.5. By the previous theorem, it is sufficient to show that each point-inverse has trivial shape, but we in fact will show that each point inverse is contractible. For each $K \in C(\Gamma_2)$, let $K^1 = \{x \in \Gamma_1\colon \varphi(x) \subset K\}$. Then by condition (i) of Definition 3.3, $K^1 \cap \Gamma_1^* = \{x \in \Gamma_1\colon \varphi(x) \in K\} = (\varphi|\Gamma_1^*)^{-1}(K)$ is connected, and by Condition (ii) each component of K^1 meets $K^1 \cap \Gamma_1^*$. Thus K^1 is connected. Now consider $A \in 2^{\Gamma_2}$, and let $\mathrm{Comp}\,A$ be the set of components of A. It is clear that $\{K^1\colon K \in \mathrm{Comp}\,A\}$ is the set of components of $A^1 = \{x \in \Gamma_1\colon \varphi(x) \subset A\}$. For each $B \in 2^{\Gamma_1}$ such that $f(B) = A$ we have $B \subset A^1$ and $B \cap K^1 \neq \emptyset$ for each component K^1 of A^1. Thus there exists an "expansion homotopy"

$$E\colon f^{-1}(A) \times I \to f^{-1}(A)$$

such that $E_0(B) = B$ and $E_1(B) = A^1$ for each $B \in f^{-1}(A)$. (Specifically, we can set $E_t(B) = \{x \in A^1\colon \varrho_{K^1(x)}(x, B \cap K^1(x)) \leqslant tD\}$, where $\varrho_{K^1(x)}$ is the minimum path-length metric in the component $K^1(x)$ of A^1 containing x, and

$$D = \sup\{\mathrm{diam}\,K^1(x)\colon x \in A^1\}.)$$

Recall that B must meet each component of A^1. The same argument shows that the other induced maps are also cell-like.

§ 4. Hyperspaces of polyhedra. In this section, we state the Subdivision lemma, postponing its proof to Section 5, and use it to prove our main result for polyhedra. By a *geometric cell complex* K we mean a finite collection of convex cells intersecting only along common faces. For $i \geqslant 0$, the *i-skeleton* of K, K^i, is the collection of all i-dimensional faces of K. This should not be confused with the earlier use of the notation K^1, in the proof of Theorem 3.5.

SUBDIVISION LEMMA 4.1. *If K is a cell complex and $\varepsilon > 0$, then there exists a subdivision L of K and a C-monotone piecewise-linear map $\varphi\colon L^1 \to C(K^1)$ such that*

(i) $\mathrm{mesh}\,L < \varepsilon$;

(ii) $\varphi(x) \subset P^1$ *if $x \in P$, a cell of K; and*

(iii) $\mathrm{diam}\,\varphi(R^1) < \varepsilon$ *for each cell R of L.*

(Here we consider $\varphi(R^1) \subset (C(K), d^*)$ where d^* is the induced Hausdorff metric.)

THEOREM 4.2. *If K is a nondegenerate compact connected polyhedron, then $2^K \approx Q$, and $C(K) \times Q \approx Q$.*

Proof. As remarked earlier, we apply the Approximation lemma 2.1 by inductively constructing a sequence $\{K_i\}$ of subdivisions of K (with repeated applications of the Subdivision lemma 4.1), and a corresponding sequence $\{\varphi_i\colon \Gamma_{i+1} \to C(\Gamma_i)\}$ of C-monotone piecewise-linear maps, where Γ_i is the 1-skeleton of K_i. We use an arbitrary metric d on K, and the induced Hausdorff metric d^* on 2^K.

Suppose that subdivisions $K_1, \ldots, K_i$ and the corresponding C-monotone maps $\varphi_1, \ldots, \varphi_{i-1}$ have been constructed, with mesh $K_j < 2^{-j}$, for each j. Let $f_1, \ldots, f_{i-1}$ be the hyperspace maps induced by $\varphi_1, \ldots, \varphi_{i-1}$, respectively. For $1 \leqslant m < n$, define $f_m^n = f_m \circ \ldots \circ f_{n-1}\colon 2^{\Gamma_n} \to 2^{\Gamma_m}$. Choose $0 < \delta_i < 1/i$ such that for $A, B \in 2^{\Gamma_i}$ with $d^*(A, B) < \delta_i$, we have $d^*(f_j^i(A), f_j^i(B)) < 1/i$ for each j, $1 \leqslant j < i$. By 4.1, take a subdivision K_{i+1} of K_i with respect to $\varepsilon = \mathrm{minimum}\,\{2^{-(i+1)}, \frac{1}{2}\delta_i\}$, and this completes the inductive construction.

Obviously, this construction of the inverse sequence

$$2^{\Gamma_1} \xleftarrow{f_1} 2^{\Gamma_2} \xleftarrow{f_2} \ldots$$

satisfies Conditions (i) and (iv) of the Approximation lemma. For $x \in \Gamma_{i+1}$, we have $\varphi_i(x) \subset P^1$ where P is a face of K_i containing x, and since $\mathrm{mesh}\,K_i < 2^{-i}$ it follows that $d^*(f_i, \mathrm{id}) < 2^{-i}$. Thus, Condition (ii) is satisfied. To verify Condition (iii), let $\varepsilon > 0$ and $k \geqslant 1$ be given. Choose $j \geqslant k$ such that $1/j < \varepsilon$. Choose $\mu > 0$ such that for $x, y \in K$ with $d(x, y) < \mu$, there exist intersecting faces P_x and P_y of K_{j+1} containing x and y, respectively. Now consider points $x, y \in \Gamma_i$, $i \geqslant j+1$, with $d(x, y) < \mu$. With P_x and P_y as above, we have $f_{j+1}^i(\{x\}) \subset P_x^1$ and $f_{j+1}^i(\{y\}) \subset P_y^1$, and it follows from the construction of K_{j+1} and φ_j that $d^*(f_j^i(\{x\}), f_j^i(\{y\})) < \delta_j$. Thus for $A, B \in 2^{\Gamma_i}$, with $i \geqslant j+1$ and $d^*(A, B) < \mu$, we have $d^*(f_j^i(A), f_j^i(B)) < \delta_j$, and therefore $d^*(f_k^i(A), f_k^i(B)) < 1/j < \varepsilon$. This shows that for each k, the sequence of maps $\{f_k^i\colon i > k\}$ is equi-uniformly continuous. Thus $2^K \approx Q$.

To obtain the result $C(K) \times Q \approx Q$, we consider the same sequence of 1-skeletons $\{\Gamma_i\}$ and piecewise-linear maps $\{\varphi_i\}$ and form the inverse sequence

$$C(\Gamma_1) \times Q \xleftarrow{g_1 \times \mathrm{id}} C(\Gamma_2) \times Q \xleftarrow{g_2 \times \mathrm{id}} \ldots,$$

where the maps $\{g_i\}$ are those induced by $\{\varphi_i\}$. Each $C(\Gamma_i) \times Q \approx Q$ (by Lemma 4.1, [10]), and each map $g_i \times \mathrm{id}$ is a near-homeomorphism. Since $\mathrm{mesh}\,K_i \to 0$, $C(\Gamma_i) \to C(K)$

and therefore $C(\Gamma_i)\times Q\to C(K)\times Q$. Since Conditions (ii) and (iii) of the Approximation lemma are clearly satisfied, we conclude that $C(K)\times Q\approx Q$.

§ 5. Proof of the Subdivision lemma. Let K be a compact connected polyhedron. We shall view K and its subdivisions as geometric cell complexes. For $i\geqslant 0$, the i-skeleton of K, denoted K^i, is the collection of all i-dimensional faces of K. For each face P of K, let $\hat{P}$ be an arbitrarily chosen point in the interior of P, and consider P as a cone over its boundary $\dot{P}$ with cone point $\hat{P}$. We use the *cone coordinates* given by the map $C_P\colon \dot{P}\times I\to P$, where $C_P(b, t) = (1-t)b+t\hat{P}$.

As in § 3, on every subcontinuum S of the 1-skeleton K^1 we use the minimum path-length metric ϱ_S. For $D = \operatorname{diam}(S, \varrho_S)$, let $e_S\colon C(S)\times I\to C(S)$ be the expansion homotopy defined by $e_S(A, t) = \{x\in S\colon\ \varrho_S(x, A)\leqslant tD\}$. Thus $e_S(A, 0) = A$ and $e_S(A, 1) = S$ for each $A\in C(S)$.

LEMMA 5.1. *For every cell complex K there exists a unique map $\alpha\colon K\to C(K^1)$ such that*

(i) $\alpha(x) = \{x\}$ *for each* $x\in K^1$;

(ii) $\alpha(C_P(b, t)) = e_{P^1}(\alpha(b), t)$ *for each cell P with* $\dim P>1$;

(iii) $\alpha(P)\subset C(P^1)$ *for each cell P.*

Proof. The conditions define the map for K^2 and the extension of the map to the rest of K is by the obvious induction on the skeleton $\{K^i\}$.

For each $n\geqslant 1$ we construct the *n-th radial-transverse subdivision* $K(n)$ of K by inductively describing the nth subdivision $K^i(n)$ of the i-skeletons of K, $i\geqslant 0$. With $K^0(n) = K^0$, let $K^{i+1}(n)$ be the cell subdivision of K^{i+1} given by the convex cells $\{C_P(\sigma\times[m/n, m+1/n])\colon\ P\in K^{i+1}, \sigma\in K^i(n)\ \text{with}\ \sigma\subset\dot{P},\ 0\leqslant m<n\}$. Thus $K(1)$ is simply a barycentric subdivision of K and in constructing $K^1(n)$, each element of K^1 is subdivided into $2n$ subintervals.

Clearly the mesh $K(n)\to 0$ as $n\to\infty$ where we can use an arbitrary metric on K. The 1-skeleton $\Gamma(n)$ of the radial-transverse subdivision $K(n)$ is the union of two subcomplexes: $R(n)$ (the radial segments) and $T(n)$ (the transverse segments), where $R(n) = \{C_P(v, [m/n, m+1/n])\colon P$ is a cell of K, v is a vertex of $K(n)$ in $\dot{P}$, $0\leqslant m<n\}$ and $T(n) = \{C_P(\sigma, m/n)\colon P$ is a cell of K with $\dim P>1$; $\sigma\in\Gamma(n)$ with $\sigma\subset\dot{P}$, $1\leqslant m<n\}$. Thus $R(n)$ covers all the vertices of $K(n)$, and also the 1-skeleton K^1.

We now restate and prove the Subdivision lemma 4.1.

SUBDIVISION LEMMA. *If K is a cell complex and $\varepsilon>0$, then there exists a subdivision L of K and a C-monotone piecewise-linear map $\varphi\colon L^1\to C(K^1)$ such that*

(i) $\operatorname{mesh} L<\varepsilon$;

(ii) $\varphi(x)\subset P^1$ *if $x\in P$, a cell of K; and*

(iii) $\operatorname{diam}\varphi(R^1)<\varepsilon$ *for each cell R of L.*

(Here we consider $\varphi(R^1)\subset C(K)$.)

Proof. Let $\varepsilon>0$ and let ϱ be the minimum path length metric of K^1 and ϱ^* the induced Hausdorff metric on $C(K^1)$. By the uniform continuity of the map α from Lemma 5.1, and the fact that mesh $K(n)\to 0$ as $n\to\infty$, pick n sufficiently large such that if a, b belong to the same cell of $K(n)$, then $\varrho^*(\alpha(a), \alpha(b))<\frac{1}{3}\varepsilon$. Let $L = K(n)$ and define $\varphi\colon L^1\to C(K^1)$ as follows. We have $L^1 = \Gamma(n) = R(n)\cup T(n)$. Let $\varphi|R(n) = \alpha|R(n)$ and for $\tau\in T(n)$, where $\dot{\tau} = \{a, b\}$, let $\hat{\tau}$ be an interior point of τ and let P be the smallest cell of K containing τ. Since $\varphi(a) = \alpha(a)$ and $\varphi(b) = \alpha(b)$ are subcontinua of P^1 and ϱ is the minimum path length metric, there exists a subcontinuum M of P^1 such that $\varphi(a)\cup\varphi(b)\subset M$ and

$$\varrho^*(\varphi(a), M)\leqslant\varrho^*(\varphi(a), \varphi(b))\geqslant\varrho^*(\varphi(b), M).$$

Let $\varphi(\hat{\tau}) = M$ and for $c\in\dot{\tau}$ and $t\in I$, let $\varphi((1-t)c+t\hat{\tau}) = e_M(\varphi(c), t)$.

In the notation of the C-monotone Definition 3.3, we have that $(L^1)^* = \{x\in L^1|\varphi(x)$ is degenerate$\}$ is precisely the subset K^1 of L^1, thus $\varphi|_{(\alpha^1)_*}\colon (L^1)^*\to\tilde{K}^1$ is actually a homeomorphism. For a point $x = C_P(v, t)$ in an edge $C_P(v, [m/n, (m+1)/n])$ of $R(n)$, a subcontinuum C_x satisfying Condition (ii) of the C-monotone definition is given by $C_x = C_P(v, [0, t])$, and for a point $x = (1-t)c+t\hat{\tau}$ in an edge $\tau = C_P(\sigma, m/n)$ of $T(n)$, we may take $C_x = \{(1-s)c+s\hat{\tau}|\ 0\leqslant s\leqslant t\}\cup C_P(v, [0, m/n])$, where $v\in P^1$ is the vertex of σ such that $C_P(v, m/n) = c\in\dot{\tau}$.

It is easily seen that φ satisfies Conditions (i) and (ii) of the Subdivision lemma. Regarding Condition (iii), we first observe that for $x\in\tau$ and $c\in\dot{\tau}$ we have $\varrho^*(\varphi(x), \varphi(c))\leqslant\varrho^*(\varphi(a), \varphi(b))$. For a cell R of L and for $x, y\in R^1$, there exists $\tau = \langle c, d\rangle\in\Gamma(n)$ where c is a vertex of an edge of R containing x and d is a vertex of an edge containing y and thus by using the triangle inequality $\varrho^*(\varphi(x), \varphi(y))<\varepsilon$.

§ 6. $C(K)$ for polyhedra K with no principal 1-cells.

LEMMA 6.1 [4]. *Let $S = \operatorname{invlim}(X_n, f_n)$ and $T = \operatorname{invlim}(Y_n, g_n)$, where all the spaces are compact metric and for each n let $h_n\colon X_n\to Y_n$ be a map such that $g_n\circ h_{n+1} = h_n\circ f_n$. If for each n, both f_n and h_n are near-homeomorphisms, then the induced map $h = \lim h_n\colon S\to T$ is a near-homeomorphism.*

THEOREM 6.2. *If K is a nondegenerate compact connected polyhedron with no principal 1-cells, then $C(K)\approx Q$.*

Proof. We proceed essentially as before in constructing the sequence $\{K_i\}$ of radial-transverse subdivision, but add at the ith stage of the construction finite collections of stickers to Γ_i and to each of its predecessors $\Gamma_{i-1}, \ldots, \Gamma_1$. These stickers are obtained from Γ_{i+1}, and do not change the homology of the graphs $\Gamma_i, \ldots, \Gamma_1$. In this manner, we eventually add countably many stickers to each Γ_i, and obtain (upon forming the closures) a sequence $\{\Gamma_i^*\}$ of compact connected local dendra whose sets of branch points are dense. Thus each $C(\Gamma_i^*)\approx Q$ (by Theorem 5.7, [10]). We construct an inverse sequence

$$C(\Gamma_1^*)\xleftarrow{g_1^*}C(\Gamma_2^*)\xleftarrow{g_1^*}\ldots$$

to which the Approximation lemma applies, and thereby obtain the desired results.

Let $\{K_i\}$ be the sequence of radial-transverse subdivisions constructed in the proof of Theorem 4.2. We may assume that for each i, $K_{i+1} = K_i(n)$ for some

$n>1$; i.e., there are transverse segments in each subdivision. Let $\Gamma_{ii} = \Gamma_i$, and inductively define $\Gamma_{ij} = \mathrm{St}(\Gamma_{i,j-1}; \Gamma_j)$ for $j>i$. Note that $\Gamma_{ij} \subset \Gamma_{i+1,j}$. Let $\tau_{ij}\colon \Gamma_{i+1,j+1} \to \Gamma_{i+1,j} \cup \Gamma_{i,j+1}$ be the unique monotone retraction.

For each i we define a C-monotone piecewise-linear map $\gamma_i\colon \Gamma_{i+1} \to C(\Gamma_{i,i+1})$, similar to the map φ_i, as follows

(i) $\gamma_i(x) = \{x\}$ if $x \in \Gamma_{i,i+1}$;

(ii) $\gamma_i(a\hat{P} + (1-a)v) = e_{P \cap \Gamma_{i,i+1}}(\{v\}, a)$, for P a cell of K_i and v a vertex of $\Gamma_{i,i+1}$ such that $v \in \mathrm{int}P \setminus \Gamma_i$,

(iii) $\gamma_i(a\hat{P} + (1-a)v) = e_{P \cap \Gamma_{i,i+1}}(\gamma_i(v), a)$, for P a cell of K_i and v a vertex of Γ_{i+1} such that $v \in \dot{P} \setminus \Gamma_i$ (which situation occurs in case $\dim P \geqslant 3$);

(iv) $\gamma_i | T$ is defined as in the proof of the Subdivision lemma, for the subgraph T of transverse segments.

Set $\gamma_{i,i+1} = \gamma_i$, and inductively define $\gamma_{i,j+1}\colon \Gamma_{i+1,j+1} \to C(\Gamma_{i,j+1})$ as follows:

(i) $\gamma_{i,j+1}(x) = \{x\}$ if $x \in \Gamma_{i,j+1}$;

(ii) $\gamma_{i,j+1}(x) = (\gamma_{ij} \circ \tau_{ij})(x)$ if $x \in \Gamma_{i+1,j+1} \setminus \Gamma_{i,j+1}$. Then each $\gamma_{i,j+1}$, $i \leqslant j$, is a C-monotone piecewise-linear map.

For each $i \leqslant j$, let $\sigma_{ij}\colon \Gamma_{i,j+1} \to \Gamma_{ij}$ be the unique monotone retraction (thus σ_{ij} collapses all $(j+1)$-stage stickers). Regarded as a map into $C(\Gamma_{ij})$, σ_{ij} is C-monotone and piecewise-linear. Let $s_{ij}\colon C(\Gamma_{i,j+1}) \to C(\Gamma_{ij})$ and $g_{i,j+1}\colon C(\Gamma_{i+1,j+1}) \to C(\Gamma_{i,j+1})$ be the maps induced by σ_{ij} and $\gamma_{i,j+1}$, respectively.

We now consider the following commutative diagram of inverse sequences:

$$\begin{array}{ccccccc}
 & & C(\Gamma_{i+1,i+1}) & \xleftarrow{s_{i+1,i+1}} & C(\Gamma_{i+1,i+2}) & \xleftarrow{s_{i+1,i+2}} & \ldots \\
 & & \downarrow{\scriptstyle g_{i,i+1}} & & \downarrow{\scriptstyle g_{i,i+2}} & & \\
C(\Gamma_{ii}) & \xleftarrow{s_{ii}} & C(\Gamma_{i,i+1}) & \xleftarrow{s_{i,i+1}} & C(\Gamma_{i,i+2}) & \xleftarrow{s_{i,i+2}} & \ldots
\end{array}$$

From the construction of the $\{K_i\}$, as in the proof of Theorem 4.2, it follows that the inverse sequence $\{\Gamma_{ij}, \sigma_{ij}\}$ satisfies the hypothesis of the Approximation lemma and hence, for each i, the limit space $\Gamma_i^* = \mathrm{invlim}(\Gamma_{ij}, \sigma_{ij})$ is homeomorphic with the closure of $\bigcup_{k=i}^{\infty} \Gamma_{ik}$ and is a compact connected local dendron with a dense set of branch points. Clearly $C(\Gamma_i^*) \approx \mathrm{invlim}(C(\Gamma_{ij}), s_{ij})$. By Lemma 6.1, the map $g_i^*\colon C(\Gamma_{i+1}^*) \to C(\Gamma_i^*)$ induced by the maps $\{g_{ij}\}$ stabilizes to a near-homeomorphism, and (by Lemma 5.2, [10]) is therefore a near-homeomorphism. It is easily seen that the Approximation lemma applies to the inverse sequence

$$C(\Gamma_1^*) \xleftarrow{g_1^*} C(\Gamma_2^*) \xleftarrow{g_2^*} \ldots$$

yielding $C(K) \approx Q$.

The requirement that K have no principal 1-cells was used above to insure that each local dendron Γ_i^* has a dense set of branch points, and is obviously necessary for the result $C(K) \approx Q$, since otherwise $C(K)$ would at some point locally look like $C(I) \approx I^2$.

§ 7. The relative hyperspaces 2_p^K and $C_p(K)$.

THEOREM 7.1. *Let K be a nondegenerate compact connected polyhedron with $p \in K$. Then $2_p^K \approx Q$, $C_p(K) \times Q \approx Q$, and $C_p(K) \approx Q$ if K has no principal 1-cell.*

Proof. We may assume p is a vertex of K. The arguments are exactly the same as for Theorems 4.2 and 6.2, with all induced hyperspace maps f and g replaced by their restrictions f_{pp} and g_{pp}. We also use the results from [10] that $2_p^{\Gamma} \approx Q$, and $C_p(\Gamma) \times Q \approx Q$, for every nondegenerate compact connected graph Γ; and that $C_p(\Gamma^*) \approx Q$, for every compact connected local dendron Γ^* with a dense set of branch points.

References

[1] R. D. Anderson and G. Choquet, *A plane continuum no two of whose nondegenerate subcontinua are homeomorphic; An application of inverse limits*, Proc. Amer. Math. Soc. 10 (1959), pp. 347–353.

[2] M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. 11 (1960), pp. 478–483.

[3] D. W. Curtis, *The hyperspace of subcontinua of a Peano continuum*, General Topology and its Applications, Proc. of the Second Pittsburgh International Conference 1972, Lecture Notes in Mathematics, Vol. 378 (1974), pp. 108–118.

[4] — and R. M. Schori, *2^X and $C(X)$ are homeomorphic to the Hilbert cube*, Bull. Amer. Math. Soc. 80 (1974), pp. 927–931.

[5] — — *Hyperspaces of Peano continua are Hilbert cubes* (submitted for publication).

[6] A. Fathi, *A cell-like map between Hilbert cube manifolds is a near-homeomorphism, a short proof* (preprint).

[7] R. M. Schori, *Inverse limit and near-homeomorphism techniques for hyperspace problems*, General Topology and its Applications, Proc. of the Second Pittsburgh International Conference 1972, Lecture Notes in Mathematics, Vol. 378 (1974), pp. 421–428.

[8] — and J. E. West, *2^I is homeomorphic to the Hilbert cube*, Bull. Amer. Math. Soc. 78 (1972), pp. 402–406.

[9] — — *The hyperspace of the closed unit interval is a Hilbert cube*, Trans. Amer. Math. Soc. 213 (1975), pp. 217–235.

[10] — — *Hyperspaces of graphs are Hilbert cubes*, Pacific J. Math. 53 (1974), pp. 239–251.

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