Since $\bigcup_{k>k_2} h_k(P_k) \cup \bigcup_{k\geq k_2} \bigcup_{l=1}^{4^k} I_{kl} \cup [x\in Q\colon x_2 = 0]$ is a locally connected continuum, there is a mapping ψ of the interval $\varphi_{k_2}(I_{k_21})$ onto it such that the image by ψ of the point $p = \varphi_{k_2}(I_{k_21} \cap R_{k_2})$ is equal to the point $\varphi_{k_2}^{-1}(p) = I_{k_21} \cap R_{k_2}$.

Let us define $g_\varepsilon\colon Y \to X$ by the formula:

$$g_\varepsilon(y) = \begin{cases} \varphi_k^{-1}(y) & \text{if } k<k_2 \text{ and } y \in \varphi_k \circ h_k(P_k) \cup \bigcup_{l=1}^{4^k} \varphi_k(I_{kl}) \text{ or if} \\ & \qquad k = k_2 \text{ and } y \in \varphi_{k_2} \circ h_{k_2}(P_{k_2}), \\ \varphi_{k_2}^{-1} \circ \pi_2(y) & \text{if } y \in \bigcup_{k>k_2} [\varphi_k \circ h_k(P_k) \cup \bigcup_{l=1}^{4^k} \varphi_k(I_{kl})] \cup \bigcup_{l=2}^{4^{k_2}} \varphi_{k_2}(I_{k_2l}), \\ \psi(y) & \text{if } y \in \varphi_{k_2}(I_{k_21}). \end{cases}$$

It follows from the definitions of π_2 and ψ that g_ε is a map of Y onto X. Since each φ_k is a homeomorphism, we infer that for every $x\in X$ either $g_\varepsilon^{-1}(x)$ is a point or $g_\varepsilon^{-1}(x)$ is a subset of $\bigcup_{k\geq k_2} \hat{Q}'_k \cup (0)$, whence $\operatorname{diam}[g_\varepsilon^{-1}(x)] < \varepsilon$. Thus g_ε is the desired ε-mapping. This concludes the proof of (3.3), and therefore the following theorem is proved:

THEOREM. *There exist two quasi-homeomorphic locally connected continua X and Y such that $X \in \alpha$ and $Y \notin \alpha$.*

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A wildly embedded 1-dimensional compact set in S^3 each of whose components is tame

by

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Abstract. A compact set X in the 3-Sphere S^3 is said to be *definable by cubes with handles*, if $X = \bigcap_{i=0}^{\infty} H_i$ where each H_i is a compact polyhedral 3-manifold in S^3 whose components are cubes with handles (i.e. regular neighborhoods of connected finite polyhedral graphs in S^3), and $H_{i+1} \subseteq \operatorname{Int} H_i$. If X is a curve (i.e. 1-dimensional) and if these cubes with handles can be chosen thin in the sense that for each $\varepsilon>0$ there is an index i_0 such that for $i\geq i_0$ the retraction of H_i to the corresponding graph is an ε-retraction, X is called *definable by thin cubes with handles*. Each of these two properties of X is equivalent to some further geometrically reasonable tameness conditions of the embedding $X \subseteq S^3$. In the following paper examples of curves in S^3 are constructed with components which are definable by thin cubes with handles such that these curves themselves are not definable by cubes with handles or are definable by cubes with handles but not by thin cubes with handles.

1. Introduction. For topological embeddings of compact sets in manifolds several conditions were introduced in order to distinguish tame embeddings from wild ones. Here we are concerned with topological embeddings of curves X (i.e. compact sets each of whose components is a 1-dimensional continuum) in the euclidian 3-space or — what is almost the same — in the 3-sphere S^3. In this case the following conditions among others have proved to be useful (we prefer embeddings in S^3 for technical reasons).

(A) *X is definable by cubes with handles* if there is a sequence $H_1, H_2, \ldots$ of compact polyhedral manifolds in S^3 each of whose components is a cube with handles such that $H_{i+1} \subseteq \operatorname{Int} H_i$ $(i = 1, 2, \ldots)$ and $X = \bigcap_{i=1}^{\infty} H_i$. (A *cube with handles* is a connected 3-manifold which is the union of a finite number of closed 3-cells $Z_1, \ldots, Z_n$ such that $Z_i \cap Z_j$ is empty or a disk on $\operatorname{Bd} Z_i \cap \operatorname{Bd} Z_j$, and no three of the cells have a common point.)

(B) *X is definable by thin cubes with handles* if for each $\varepsilon>0$ the cubes with handles in (A) can be replaced by ε-thin cubes with handles. (An ε-thin cube with handles is a cube with handles for which the cells $Z_1, \ldots, Z_n$ in the definition above can be chosen with diameters smaller than ε.)

(C) *X has an unknotted complement in* S^3 if each simple closed polygon in $S^3\setminus X$ can be unknotted by a homotopy in $S^3\setminus X$ (for precise definitions see below).

An example in [4] shows that even in the case where X is connected (i.e. where the manifolds H_i can be chosen to be cubes with handles) (B) is more restrictive than (A). (In that paper a curve is called tangled if it has not the property (E) formulated below which is equivalent to (B).) Another example can be found in Section 7 below. Having in mind that a cube with handles in S^3 is a regular neighborhood of a finite polyhedral graph in S^3, it is easy to prove that (A) implies (C).

It is the aim of this paper to construct an example of a curve X with a knotted complement in S^3 each of whose components is definable by thin handlebodies. Therefore each component of X has all properties (A), (B), and (C) while X itself has none of them.

To illustrate the meaning of this example we mention some conditions which are equivalent to (A) or to (B).

(D) *X is definable by free 3-manifolds* if the cubes with handles in (A) are replaced by compact 3-manifolds with free fundamental groups.

The equivalence of (A) and (D) (for 1-dimensional compacta in E^3 or S^3 follows from [7], Theorem 10 using the fact that two boundary components of a polyhedral 3-manifold containing X can be connected by a tunnel in this manifold which does not meet X. (A tunnel in a 3-manifold M is a closed 3-cell T in M such that $T\cap \mathrm{Bd}M = \mathrm{Bd}T\cap \mathrm{Bd}M$ is the union of two separate disks.) We have mentioned condition (D) here, since it shows that the promised example answers the question of H. Row ([7], p. 226) whether a compact subset of a 3-manifold each of whose components is definable by free 3-manifolds is itself definable by free 3-manifolds.

(E) *X has the strong arc pushing property* if for each polygonal (or tame) arc A in S^3 and for each $\varepsilon>0$ there is an ε-homeomorphism f of S^3 onto itself such that $f(A)\cap X=\emptyset$.

(F) *X can be mapped by a homeomorphism of S^3 onto itself into Menger's universal curve* M_3^1. (M_3^1 is defined e.g. in [3] as a subset of E^3, but by the stereographic projection we may assume that M_3^1 is a subset of S^3.)

(G) *The embedding dimension of X is* 1, i.e. for each point x of X there are arbitrarily small tame 3-cells Z in S^3 such that $x\in \mathrm{Int}Z$ and $\mathrm{Bd}Z\cap X$ is 0-dimensional.

The implication (B)⇒(E) is obvious, (E)⇒(F) was proved in [3], (F)⇒(G) is obvious, and (G)⇒(B) is proved in [7], Theorem 13. So we see that (B), (E), (F), and (G) are equivalent. Investigations concerning the embedding dimension can be found in [9] and [5].

With a single exception in Section 6 where a more general set — our curve X — is considered, we shall work in the category of polyhedra and piecewise linear maps. Therefore all manifolds (no dimension exceeding 3 will occur) are assumed to carry a definite piecewise linear structure.

The following notational conventions are made: By $[t_1, t_2]$ we denote the closed

interval between t_1 and t_2, and I means the interval $[0, 1]$. For a subset A of a topological space, $\mathrm{Cl}A$ is the closure of A in this space which is always defined by the context. Manifolds are not assumed to be connected. The boundary and the interior of a manifold M will be denoted by $\mathrm{Bd}M$ or $\mathrm{Int}M$ respectively. A *spanning submanifold* N of a manifold M is a compact submanifold such that $N\cap \mathrm{Bd}M = \mathrm{Bd}N$. A *tunnel* in a 3-manifold M is a 3-cell T in M such that $T\cap \mathrm{Bd}M = D'\cup D''$ is the union of two separate disks which are called the *entrances* of T. We say that the 3-manifold $M' = \mathrm{Cl}(M\setminus T)$ is obtained from M by *boring out the tunnel* T. A tunnel T in a 3-cell Z is called *unknotted* in Z if there is a spanning disk D in $\mathrm{Cl}(Z\setminus T)$ such that $\mathrm{Bd}D$ is the union of two arcs A, A' with common end points where $A = D\cap \mathrm{Bd}Z$ and $A' = D\cap T$ connect the two antrances of T. Two disjoint tunnels T_1, T_2 in Z are called *unlinked* if there is a spanning disk D in Z which does not intersect $T_1\cup T_2$ and which separates in Z T_1 from T_2.

If V is a solid torus, a *meridian disk* of V is a spanning disk in V whose boundary does not bound a disk on $\mathrm{Bd}V$. A *meridian* of V is the boundary of a meridian disk of V. All meridians of V are isotopic on $\mathrm{Bd}V$. If V is embedded in S^3 we define a *longitude* of V to be a simple closed polygon on $\mathrm{Bd}V$ which is homologous to zero in $S^3\setminus \mathrm{Int}V$ but not in V. All longitudes of V are isotopic on $\mathrm{Bd}V$. A solid torus in S^3 is called *unknotted*, if its longitudes are unknotted in S^3. This is equivalent to the fact that $S^3\setminus \mathrm{Int}V$ is a solid torus too. Two simple closed curves K, L in a space G are called *homotopic* in G if there is a mapping h: $K\times I\to G$ such that $h(\ ,0)$ is the identity of K, and $h(\ ,1)$ is a homeomorphism of K onto L. We say that a simple closed polygon K in a subset G of S^3 or of a 3-cell *E can be unknotted* in G, if it is homotopic in G to a simple closed polygon which is unknotted in S^3 or in E. The set G itself is called *unknotted* if each simple closed polygon in G can be unknotted in G.

2. The manifold N. Here we define a compact 3-manifold N in S^3 which will be the starting-point of our construction. Let Δ be a triangle, and let $C = \Delta\times I$ be the cylinder over Δ. We assume that C is a subpolyhedron of S^3. The manifold N will be a subpolyhedron of C. By C' we denote the 3-cell $\Delta\times[0, \frac{1}{2}]$ and by R a solid torus in $C'' = \Delta\times[\frac{1}{2}, 1]$ such that $R\cap \mathrm{Bd}C''$ is a disk in $\mathrm{Int}(\Delta\times\{\frac{1}{2}\})$. Of course $C'\cup R$ is a solid torus too. The set N will be obtained from $C'\cup R$ by boring out a tunnel T with one entrance in $\mathrm{Int}\Delta$ ($\Delta = \Delta\times\{0\}$) and the other in the annulus $(\Delta\times\{\frac{1}{2}\})\setminus(R\cap C')$. We assume that R is unknotted and that T runs around in $C'\cup R$ as indicated in Figure 1. More precisely, T is chosen in such a way that there is a 3-cell C^* in R such that the following holds:

(1) $C^*\cap \mathrm{Bd}R = \mathrm{Bd}C^*\cap \mathrm{Bd}R = D$ is a disk which contains the disk $C'\cap R$ in its interior. Therefore, $\mathrm{Bd}C^*\setminus \mathrm{Int}D = D'$ is a disk which subdivides R in C^* and a solid torus R'.

(2) $C^*\cap T$ is the union of two separate tunnels T', T'' in C^* each of which has one entrance on $\mathrm{Int}(C'\cap R)$ and the other on $\mathrm{Int}D'$.

(3) T' and T'' are equally knotted in the following sense: There is a knotted

tunnel T^* in C^* with entrances in $C' \cap R$ and D' such that T' and T'' are unknotted and unlinked in T^*.

(4) $T \cap C'$ is the union of two separate tunnels which are unknotted and unlinked in C'.

(5) $T \cap R'$ is a tunnel in R' which is unknotted but linked with the hole of R' in the following sense: There is a meridian disk G in R' such that $G \cap D'$ is a spanning arc in D' which separates the two disks of $T \cap D'$. Moreover, if we split R' along G we get a 3-cell in which $R' \cap T$ is an unknotted tunnel.

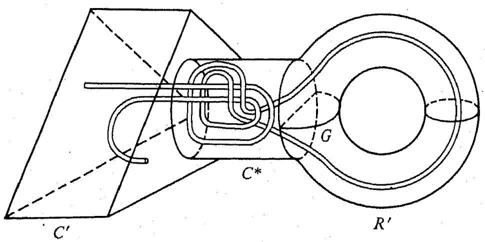

Fig. 1

The tunnel T is unknotted in the following sense: There is an unknotted tunnel in the 3-cell C with one entrance on $\Delta \times \{1\}$ and the other in the entrance $T \cap \Delta$ of T which does not intersect $N = \mathrm{Cl}((C' \cup R) \setminus T)$, i.e. there is an unknotted tunnel in C which does not meet N and which connects the two components of $\mathrm{Bd}\, C \setminus N$. The following remark is an immediate consequence of this fact.

Remark 1. If K is a polygonal are in S^3 such that $K \cap \mathrm{Bd}\, C \subseteq \mathrm{Bd}\, C \setminus N$, then there is a piecewise linear homeomorphism f of S^3 onto itself which is the identity outside C such that $f(K) \cap N = \emptyset$.

Now we consider a solid torus V_0 and a cube with two handles H_0 which is embedded in $\mathrm{Int}\, V_0$ as in Figure 2.

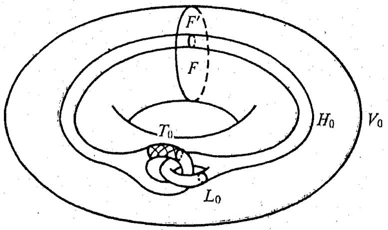

Fig. 2

Remark 2. Each simple closed polygon on $\mathrm{Bd} H_0$ which bounds a disk in $V_0 \setminus \mathrm{Int}\, H_0$ bounds disk on $\mathrm{Bd} H_0$. (This can be proved by standard techniques.)

If we look at Figure 3, we get the following lemma.

Lemma 1. *In S^3 there is a knotted solid torus V such that $H = S^3 \setminus \mathrm{Int}\, N$ is contained in $\mathrm{Int}\, V$ and the pairs (V, H) and (V_0, H_0) are homeomorphic.* (In Figure 3c P denotes a 3-cell with a knotted tunnel in $\mathrm{Int}\, N$, and V is the closure of the complement of P in S^3.)

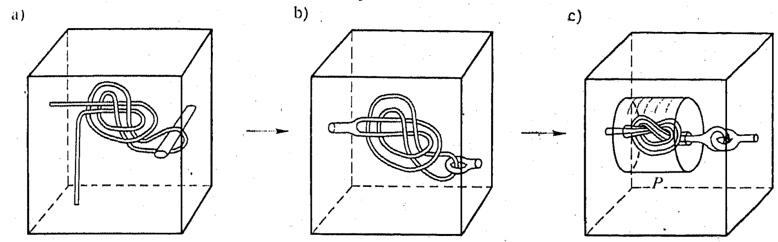

Fig. 3

Remark 3. In V_0 we may choose a meridian disk F such that the following holds:

(1) $F' = F \cap H_0$ is a disk which subdivides H_0 in two solid tori V_1, V_2 ($V_1 \cap V_2 = F'$).

(2) If $f\colon V \to V_0$ is a suitably chosen homeomorphism such that $f(S^3 \setminus \mathrm{Int}\, N) = H_0$, then the image of the tunnel T is contained in $V_1 \setminus F'$, and the images of the entrances of T are meridian disks in V_1.

(3) There is a longitude of R which is mapped by f onto a meridian L_0 of V_2 which does not intersect F'.

(4) On $\mathrm{Bd}\, N$ there is a meridian of R which is mapped by f onto a longitude of V_2 which does not intersect F'.

To prove this remark it is sufficient to remember the proof of Lemma 1, i.e. to look at Figure 3.

3. Unknotted simple closed polygons in $S^3 \setminus N$. In this section we shall prove the following lemma:

Lemma 2. (A) *If K is an unknotted simple closed polygon in $S^3 \setminus N$, then there is a spanning disk D in $S^3 \setminus \mathrm{Int}\, N$ which does not intersect K and whose boundary is either the boundary of an entrance of T or a longitude of R.*

(B) *If moreover there is a disk Q in $S^3 \setminus K$ whose boundary lies in $S^3 \setminus N$ and is homotopic in $S^3 \setminus \mathrm{Int}\, N$ to a meridian of R, then the disk D in* (A) *can be chosen such that it is the boundary of an entrance of T.*

Corollary. *Let N', N'' be obtained as images of N under autohomeomorphisms f', f'' of S^3. If $N' \cap N'' = \emptyset$ and the tori $f'(R)$, $f''(R)$ are linked in the sense that a longitude of $f'(R)$ is homotopic in $S^3 \setminus \mathrm{Int}\, N''$ to a meridian of $f''(R)$ and a longitude of $f''(R)$ is homotopic in $S^3 \setminus \mathrm{Int}\, N'$ to a meridian of $f'(R)$, then for each* *unknotted simple closed polygon* K *in* $S^3\setminus(N'\cup N'')$ *there is a spanning disk* D *in* $S^3\setminus\mathrm{Int}N'$ *which does not intersect* K *and whose boundary is the boundary of an entrance of* $f'(T)$ *in* N' *or there is a spanning disk* D *in* $S^3\setminus\mathrm{Int}N''$ *which does not intersect* K *and whose boundary is the boundary of an entrance of* $f''(T)$ *in* N''.

Proof of Lemma 2. We use the notations of Section 2. By [8], p. 164 each unknotted simple closed polygon in S^3 which is contained in a knotted solid torus bounds a disk in this torus. Therefore, since V is knotted in S^3, K is the boundary of a disk E in $\mathrm{Int}\,V$. By a similar argument the disk Q in (B) can be chosen in $\mathrm{Int}\,V$. So we get in $\mathrm{Int}\,V_0$ a simple closed polygon $K_0=f(K)$, a disk $E_0=f(E)$ with boundary K_0, and a disk $Q_0=f(Q)$ whose boundary lies in $\mathrm{Int}H_0$ and is homotopic in H_0 to a longitude of V_2 on $\mathrm{Bd}H_0$. (f: $(V, S^3\setminus\mathrm{Int}N)\to(V_0, H_0)$ is a homeomorphism.)

Each disk D_0 in $\mathrm{Int}\,V_0$ whose boundary lies in $\mathrm{Int}H_0$ can be replaced by another disk D_0' in $\mathrm{Int}\,V_0$ which has the following properties:

(1) $\mathrm{Bd}\,D_0'=\mathrm{Bd}\,D_0$.

(2) $D_0'\cap F\subseteq\mathrm{Int}F'$.

(3) D_0' and $\mathrm{Bd}H_0$ have general position in the following sense: Each component of $D_0'\cap\mathrm{Bd}H_0$ is a simple closed curve C, and D_0' intersects $\mathrm{Bd}H_0$ along C transversely.

(4) No component of $D_0'\cap\mathrm{Bd}H_0$ bounds a disk on $\mathrm{Bd}H_0$.

From these properties and Remark 2 we get:

(*) If D_1 is a subdisk of D_0' such that $D_1\cap\mathrm{Bd}H_0=\mathrm{Bd}\,D_1$ (if $D_0'\cap\mathrm{Bd}H_0\neq\emptyset$, such a subdisk must exist!), then D_1 lies in H_0 and $\mathrm{Bd}\,D_1$ is a meridian of one of the solid tori V_1 or V_2, or $\mathrm{Bd}\,D_1$ is isotopic on $\mathrm{Bd}H_0$ to $\mathrm{Bd}F'$.

In the proof of the lemma we may assume that the disks E_0 and Q_0 have the properties of D_0'. To prove (A) it is sufficient to find a meridian disk in V_1 or in V_2 which does not intersect K_0 and whose boundary lies on $\mathrm{Bd}H_0$. We note that one of the following cases must occur: $E_0\subseteq\mathrm{Int}H_0$, E_0 contains a meridian disk of V_1 or of V_2, or E_0 contains a spanning disk E_1 in H_0 whose boundary is isotopic in $\mathrm{Bd}H_0$ to $\mathrm{Bd}F'$. In the first and in the second case we are ready. In the third case H_0 is subdivided by E_1 in two solid tori V_1', V_2' where V_i' contains a meridian of V_i which lies on $\mathrm{Bd}H_0$ $(i=1,2)$. Since $K_0\cap E_1=\emptyset$, one of the solid tori V_i' does not intersect K_0, and we are ready too.

To prove (B) we have to show that there is a meridian disk in V_1 which does not intersect K_0 and whose boundary lies on $\mathrm{Bd}H_0$. Since $\mathrm{Bd}\,Q_0$ is homotopic in H_0 to a longitude of V_2, by (*) one of the following cases must occur: Q_0 contains a meridian disk of V_1 whose boundary lies on $\mathrm{Bd}H_0$, or Q_0 contains a spanning disk Q_1 in H_0 whose boundary is isotopic on $\mathrm{Bd}H_0$ to $\mathrm{Bd}F'$. In the first case we are ready. In the second case Q_1 subdivides H_0 in two solid tori V_1' and V_2' where V_i' contains a meridian of V_i which lies on $\mathrm{Bd}H_0$ $(i=1,2)$. Since K_0 does not intersect Q_0, we have $K_0\subseteq\mathrm{Int}\,V_1'$ or $K_0\subseteq\mathrm{Int}\,V_2'$. In the second case we are ready. In the first case we find a meridian disk in V_1' which does not intersect K_0 and whose boundary lies on $\mathrm{Bd}H_0$ by the following simple fact: if a simple closed polygon K_0 lies in a solid torus V_1' in S^3 and if there is a disk Q_0 in $S^3\setminus K_0$ whose boundary lies in $S^3\setminus\mathrm{Int}\,V_1'$ and is homotopic there to a meridian of V_1', then there is a meridian disk in V_1' which does not intersect K_0.

4. Unknotting simple closed polygons in the complement of a manifold with a tunnel. In this section we shall prove the following lemma.

Lemma 3. *Let* Q *be a compact* 3*-manifold in* S^3, *and let* T *be tunnel in* Q *with entrances* D' *and* D''. *We consider the manifold* $Q'=\mathrm{Cl}(Q\setminus T)$ *and a simple closed polygon* K *in* $G=S^3\setminus Q$. *This polygon* K *can be unknotted in* G, *provided it can be unknotted in* $G'=S^3\setminus Q'$ *by a homotopy* h: $K\times I\to G'$ *for which the following holds*: *There is a disk* D *in* S^3 *such that* $D\cap(T\cap Q')=\mathrm{Bd}\,D=\mathrm{Bd}\,D'$, *and* D *does not intersect the simple closed polygon* $L=h(K\times\{1\})$.

Proof. The annulus $T\cap Q'$ will be denoted by A. Since $E=S^3\setminus A$ is homeomorphic to an open solid torus, the total space $\tilde{E}$ of the universal covering π: $\tilde{E}\to E$ of E is an open 3-cell which is equipped via π with a piecewise linear structure. For a subset Z of S^3 we shall denote $\pi^{-1}(Z\setminus A)$ by $\tilde{Z}$. We consider the open 3-cell $U=S^3\setminus T$. Each component of $\tilde{U}$ is a 3-cell which is mapped by π homeomorphically onto U. Let $\tilde{U}_0$ be one of these components. The complement of $\mathrm{Cl}\,\tilde{U}_0$ in $\tilde{E}$ is the union of two open 3-cell $\tilde{U}'$, $\tilde{U}''$, and $\tilde{D}_0'=\mathrm{Cl}\,\tilde{U}'\cap\mathrm{Cl}\,\tilde{U}_0$, $\tilde{D}_0''=\mathrm{Cl}\,\tilde{U}''\cap\mathrm{Cl}\,\tilde{U}_0$ are open disks which are mapped by π homeomorphically onto $\mathrm{Int}\,D'$ or $\mathrm{Int}\,D''$ respectively. The intersection $\tilde{K}_0=\tilde{U}_0\cap\tilde{K}$ is a component of $\tilde{K}$ which is contained in $\tilde{U}_0\cap\tilde{G}=\tilde{U}_0\setminus\tilde{Q}$. The homotopy h: $K\times I\to G'$ can be lifted to a homotopy $\tilde{h}$: $\tilde{K}_0\times I\to\tilde{G}'$, and $\tilde{h}(\tilde{K}_0\times\{1\})=\tilde{L}_0$ is a simple closed polygon in $\tilde{G}'$ which is a component of $\tilde{L}$. Since L is contained in the open 3-cell $E\setminus D$, $\tilde{L}_0$ is unknotted in $\tilde{E}$. We have $\tilde{D}_0'\cup\tilde{D}_0''\subseteq G'$ and it is easy to find an autohomeomorphism f of $\tilde{E}$ onto itself which is the identity outside a neighborhood of $\mathrm{Cl}\,\tilde{U}'\cup\mathrm{Cl}\,\tilde{U}''$ and which throws $h(\tilde{K}_0\times I)$ into $\tilde{U}_0\cap\tilde{G}'=\tilde{U}_0\cap\tilde{G}$ without moving any point of K_0. The homotopy $\tilde{k}=f\tilde{h}$ is an unknotting of $\tilde{K}_0$ in $\tilde{U}_0\cap\tilde{G}$, and $k=\pi\tilde{k}$ unknots K in $G=\pi(\tilde{U}_0\cap\tilde{G})$.

5. The main construction. Let M be a compact 3-manifold in S^3 and let $\mathfrak{T}$ be a triangulation of $\mathrm{Bd}M$. By $\Delta_1, \ldots, \Delta_s$ we denote the triangles of $\mathfrak{T}$. As proved in [6] there is a piecewise linear collar of $\mathrm{Bd}M$ in M, i.e. a piecewise linear embedding $\varkappa$: $\mathrm{Bd}M\times I\to M$ such that $\varkappa(\mathrm{Bd}M\times I)$ is a closed neighborhood of $\mathrm{Bd}M$ in M and $\varkappa(x,0)=x$ for all $x\in\mathrm{Bd}M$. For each index i $(1\leqslant i\leqslant 2^s)$ and each mapping k: $\{1,\ldots,s\}\to\{0,\ldots,2^s\}$ we shall define a compact 3-manifold M_{ik} in $\varkappa(\mathrm{Bd}M\times I)$. In the simplest case M_{ik} will be the layer $\varkappa(\mathrm{Bd}M\times[(i-1)\cdot 2^{-s}, (i-\frac{1}{2})\cdot 2^{-s}])$, and in the general case M_{ik} will be obtained from this layer by adding some thin handles and boring out some tunnels in such a way that for fixed k the manifolds $M_{1k}, \ldots, M_{2^sk}$ are still mutually disjoint but linked in a special way.

We use the notations of Section 2. Over the triangle Δ with which we started the construction of the manifold N we consider the cylinder $\Delta\times[1,2^s+1]$ and choose piecewise linear embeddings φ_i of $\Delta\times I$ in $\Delta\times[i,2^s+1]$ $(1\leqslant i\leqslant 2^s)$ such that:

(1) If $(x,t)\in\Delta\times[0,\frac{1}{2}]$, then $\varphi_i(x,t)=(x,i+t)$.

(2) The sets $N_i=\varphi_i(N)$ are mutually disjoint.

(3) For $1\leqslant i\leqslant 2^s$ we have $\varphi_i(R')\subseteq\Delta\times[2^s+\frac{1}{2},2^s+1]$.

(4) Each two of the solid tori $\varphi_i(R')$ are linked in the sense that for $i_1\neq i_2$ a longitude of $\varphi_{i_1}(R')$ is homotopic in $S^3\setminus\varphi_{i_2}(\mathrm{Int}R')$ to a meridian of $\varphi_{i_2}(R')$.

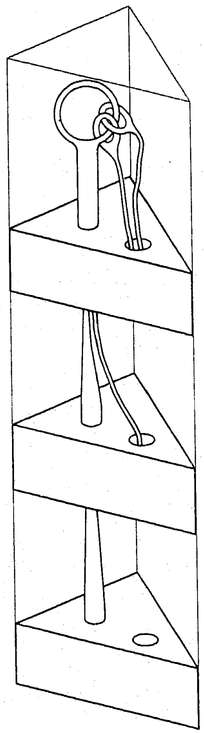

Fig. 4

For $i<2^s$ the images of the 3-cell C^* under φ_i must be thin worms which run through the tunnels $\varphi_{i+1}(T),\dots,\varphi_{2^s}(T)$. These tunnels will be denoted by T_i. Figure 4 illustrates the situation. Let J be a subset of $\{1,\dots,2^s\}$ and $0\leqslant k\leqslant 2^s$.

In our construction the following polyhedra will be used

$$W_{Jk}=\bigcup_{\substack{1\leqslant i\leqslant k\\ i\in J}}\Delta\times[i,i+\tfrac{1}{2}]\cup\bigcup_{\substack{k<i\leqslant 2^s\\ i\in J}}N_i\,.$$

As an immediate consequence of the corollary of Lemma 2 we get the following fact.

Remark 4. If ψ is a piecewise linear embedding of $\Delta\times[1,2^s+1]$ in S^3 and K is an unknotted simple closed polygon in $S^3\setminus\psi(W_{Jk})$, then for each pair $i_1,i_2>k$ of indices in J there is a spanning disk in $S^3\setminus\mathrm{Int}(\psi(N_{i_1}))$ whose boundary is the boundary of an entrance of the tunnel $\psi(T_{i_1})$ or there is a spanning disk in $S^3\setminus\mathrm{Int}(\psi(N_{i_1}))$ whose boundary is the boundary of an entrance of the tunnel $\psi(T_{i_2})$.

The following remark is a simple consequence of Remark 1 in Section 2.

Remark 5. Let ψ be as in Remark 4, and let K be a polygonal arc in S^3. We assume that for a fixed index i $(k<i\leqslant 2^s,\ i\in J)$ K does not intersect

$$\psi(N_i\cap\mathrm{Bd}(\Delta\times[1,2^s+1]))\,.$$

Then there is a piecewise linear homeomorphism h of S^3 onto itself which is the identity outside $\psi(\Delta\times[1,2^s+1])$ such that $h(K)\cap\psi(N_i)=\emptyset$.

For each index j $(1\leqslant j\leqslant s)$ we consider a linear map λ_j of our standard triangle Δ onto Δ_j and the piecewise linear homeomorphism

$$\psi_j\colon\ \Delta\times[1,2^s+1]\to\varkappa(\Delta_j\times I)$$

which is defined by

$$\psi_j(x,t)=\varkappa(\lambda_j(x),(t-1)\cdot 2^{-s})\,.$$

If a mapping $k\colon\{1,\dots,s\}\to\{0,\dots,2^s\}$ is given, then for $1\leqslant i\leqslant 2^s$

$$M_{ik}=\bigcup_{\substack{j\\ k(j)\geqslant i}}\psi_j(\Delta\times[i,i+\tfrac{1}{2}])\ \bigcup_{\substack{j\\ k(j)<i}}\psi_j(N_i)$$

is a compact 3-manifold in $\varkappa(\mathrm{Bd}M\times I)$ which coincides with

$$\varkappa(\mathrm{Bd}M\times[(i-1)\cdot 2^{-s},(i-\tfrac{1}{2})\cdot 2^{-s}])$$

provided $k(j)\geqslant i$ for all j. Two of the manifolds M_{ik} corresponding to different indices i but to the same mapping k are disjoint. For a subset J of $\{1,\dots,2^s\}$ we define

$$M_{Jk}=\bigcup_{i\in J}M_{ik}\,.$$

Obviously for each triangle Δ_j of $\mathfrak{T}$ we have

$$M_{Jk}\cap\varkappa(\Delta_j\times I)=\psi_j(W_{Jk(j)})\,.$$

If $k\equiv 2^s$, then $M_{Jk}=\bigcup_{i\in J}(\mathrm{Bd}M\times[(i-1)\cdot 2^{-s},(i-\frac{1}{2})\cdot 2^{-s}])$. If $J=\{1,\dots,2^s\}$ and $k(j)\equiv 0$, we denote M_{Jk} by M'.

In this way for a given compact 3-manifold M with boundary collar $\varkappa$ and a triangulation $\mathfrak{T}$ of BdM whose triangles are indexed by the numbers $1, \ldots, s$ to each subset J of $\{1, \ldots, 2^s\}$ and each mapping $k\colon \{1, \ldots, s\} \to \{0, \ldots, 2^s\}$ there corresponds a compact 3-manifold M_{Jk} in $\varkappa(\mathrm{Bd}M \times I)$. (Besides on $M, \varkappa, \Delta_1, \ldots, \Delta_s, J,$ and k the manifold depends on the linear maps $\lambda_j\colon \Delta \to \Delta_j$, but this dependence is inessential.)

Given a manifold M_{Jk} ($J \neq \emptyset$, i_0 the maximal number in J, $k \not\equiv 2^s$, j_0 the minimal index in $\{1, \ldots, s\}$ for which $k(j_0) < 2^s$) and an unknotted simple closed polygon L in $S^3 \setminus M_{Jk}$ we define a manifold $M_{J'k'}$ by the following standard modification of M_{Jk}.

First case: $k(j_0) \geqslant i_0$. Then

$$J' = J, \quad k'(j) = \begin{cases} k(j) & \text{if} \quad j \neq j_0, \\ 2^s & \text{if} \quad j = j_0. \end{cases}$$

Second case: $k(j_0) < i_0$, but besides i_0 there is no other number in J which exceeds $k(j_0)$. Then

$$J' = J \setminus \{i_0\}, \quad k'(j) = \begin{cases} k(j) & \text{if} \quad j \neq j_0; \\ 2^s & \text{if} \quad j = j_0. \end{cases}$$

Third case: There are indices i_1, i_2 in J such that $k(j_0) < i_1 < i_2$. Then choose i_1, i_2 minimal and consider $\psi_{j_0}(N_{i_1})$ and $\psi_{j_0}(N_{i_2})$. By Remark 4 for one of the indices i_1, i_2 (we call it i) there is a spanning disk D in $S^3 \setminus \mathrm{Int}(\psi_{j_0}(N_i))$ which does not intersect L and whose boundary is the boundary of an entrance of the tunnel $\psi_{j_0}(T_i)$ in $\psi_{j_0}(N_i)$. We define

$$J' = J, \quad k'(j) = \begin{cases} k(j) & \text{if} \quad j \neq j_0, \\ i_1 & \text{if} \quad j = j_0 \end{cases}$$

if $i = i_1$ and

$$J' = J \setminus \{i_1\}, \quad k'(j) = \begin{cases} k(j) & \text{if} \quad j \neq j_0, \\ i_2 & \text{if} \quad j = j_0 \end{cases}$$

if $i = i_2$.

Remark 6. It is a simple combinatorial fact that a repeated application of this standard modification (possibly with different curves L in each step) to the manifold M' (i.e. $J = \{1, \ldots, 2^s\}$, $k \equiv 0$) leads to a manifold $M_{J^*k^*}$ where $J^* \neq \emptyset$ and $k^* \equiv 2^s$.

Remark 7. Let K be a simple closed polygon in $S^3 \setminus M$ and let L be an unknotted simple closed polygon in $S^3 \setminus M_{Jk}$ ($J \neq \emptyset$, $k \not\equiv 2^s$) such that K is homotopic in $S^3 \setminus M_{Jk}$ to L. Then, if $M_{J'k'}$ is obtained from M_{Jk} and L by the standard modification, K can be unknotted in $S^3 \setminus M_{J'k'}$.

This remark is trivial if the standard modification is applied in the first or the second case. In the third case let $i = i_1$ or $i = i_2$ as in the definition of the standard modification. Since i_1, i_2 are minimal in $J \setminus \{1, \ldots, k(j_0)\}$ the 3-cell $\psi_{j_0}(T_i)$ is a tunnel in $M_{J'k} \cup \psi_{j_0}(T_i)$, and this tunnel can be closed by a disk D which does not intersect L. It is an immediate consequence of Lemma 3 (if we take $Q = M_{J'k} \cup \psi_{j_0}(T_i)$, $T = \psi_{j_0}(T_i)$) that K can be unknotted in $S^3 \setminus (M_{J'k} \cup \psi_{j_0}(T_i))$. But $M_{J'k'}$ is contained in $M_{J'k} \cup \psi_{j_0}(T_i)$, and the remark follows.

Lemma 4. *Let M be a compact 3-manifold in S^3 for which a boundary collar $\varkappa$ and a triangulation $\mathfrak{T}$ of* BdM *are given such that the manifold M' in $\varkappa(\mathrm{Bd}M \times I)$ is defined. Then, if a simple closed polygon K in $S^3 \setminus M$ can be unknotted in $S^3 \setminus M'$, it can be unknotted in $S^3 \setminus M$.*

Proof. Let J^*, k^* be as in Remark 6. It follows by repeated application of Remark 7 that K can be unknotted in $S^3 \setminus M_{J^*k^*}$. Since $J^* \neq \emptyset$ and $k^* \equiv 2^s$, the manifold $M_{J^*k^*}$ contains (for a suitable $t \in I$) the surface $\varkappa(\mathrm{Bd}M \times \{t\})$, and K can be unknotted in the complement of this surface which is parallel to BdM. Hence K can be unknotted in the complement of BdM and, since K lies in $S^3 \setminus M$, in the complement of M. This proves the lemma.

6. The curve X. To define the curve X we construct step by step a sequence $M^{(0)} \supseteq M^{(1)} \supseteq M^{(2)} \supseteq \ldots$ of compact polyhedral 3-manifolds in S^3: $M^{(0)}$ is the complement of the interior of a knotted solid torus in S^3. If $M^{(n)}$ is defined, we choose a triangulation $\mathfrak{T}^{(n)}$ of Bd$M^{(n)}$ with triangles of diameters smaller than 2^{-n-2} and a collar $\varkappa^{(n)}$ of Bd$M^{(n)}$ in $M^{(n)}$ such that $d(x, \varkappa^{(n)}(x, t)) < 2^{-n-2}$ for all $(x, t) \in \mathrm{Bd}M^{(n)} \times I$. These assumptions imply that for each triangle Δ_j of $\mathfrak{T}^{(n)}$ the 3-cell $\varkappa^{(n)}(\Delta_j \times I)$ is of diameter less than 2^{-n}. Then $M^{(n+1)}$ is the manifold $(M^{(n)})'$ which was described in Section 5. The set X is the intersection

$$X = \bigcap_{n=0}^{\infty} M^{(n)}.$$

We prove now that *each component of X has the strong arc pushing property and is therefore definable by thin cubes with handles.* Let X_c denote a component of X and let $M_c^{(n)}$ be the component of $M^{(n)}$ which contains X_c. We have to prove that for each polyhedral arc A and for each positive ε there is an ε-homeomorphism h of S^3 onto itself such that $h(A) \cap X_c = \emptyset$.

If we choose the index n large enough, we easily get an $\frac{1}{2}\varepsilon$-autohomeomorphism h_1 of S^3 such that the following holds: If Δ_j is a triangle of the triangulation $\mathfrak{T}^{(n)}$ of Bd$M^{(n)}$, then $h_1(A)$ does not intersect $\mathrm{Bd}\big((\varkappa^{(n)}(\Delta_j \times I)) \cap M^{(n+1)}\big)$. Now using Remark 5 of Section 5 we get another autohomeomorphism h_2 of S^3 which is the identity outside $\varkappa^{(n)}(\mathrm{Bd}M^{(n)} \times I)$ and on the boundary of each 3-cell $\varkappa^{(n)}(\Delta_j \times I)$ where Δ_j is a triangle of $\mathfrak{T}^{(n)}$ and which maps each of these 3-cells onto itself such that $h_2 h_1(A) \cap M_c^{(n+1)} = \emptyset$. If the index n was chosen sufficiently large, the cells $\varkappa^{(n)}(\Delta_j \times I)$ become small such that $h = h_2 h_1$ is an ε-autohomeomorphism of S^3.

This argument shows moreover that X_c does not locally disconnect S^3 and is therefore at most 1-dimensional (see [1], p. 208 or [2], IV, § 5.6). Of course X_c is at least 1-dimensional, and as a compact space each of whose components is 1-dimensional *X itself is 1-dimensional.*

Finally we show that *X has a knotted complement in S^3.* Since $S^3 \setminus M^{(0)}$ is a knotted open solid torus, there is a simple closed polygon K in $S^3 \setminus M^{(0)}$ which can not be unknotted in $S^3 \setminus M^{(0)}$. Looking at Lemma 4 we see that K can not be unknotted in $S^3 \setminus M^{(n)}$ for $n = 1, 2, \ldots$ Since an unknotting always proceeds on a compact set (the image of the unknotting homotopy), an unknotting of K in $S^3 \setminus X$ would be an unknotting in a set which has positive distance from X. Therefore, if K could be unknotted in $S^3 \setminus X$ it could be unknotted in $S^3 \setminus M^{(n)}$ for sufficiently large n, but this is impossible as weh ave seen above. So we have found a simple closed polygon in $S^3 \setminus X$ which can not be unknotted there.

7. Curves definable by cubes with handles but not by thin cubes with handles. Here we define a curve Y in S^3 which is definable by cubes with handles but not by thin cubes with handles and each of whose components is definable by thin cubes with handles.

Let F be a compact polyhedral 2-manifold in S^3 each of whose components has a non empty boundary, and let $\mathfrak{T}$ be a triangulation of F. Then F can be contracted by simple collapses onto a subset of the one-dimensional skeleton of $\mathfrak{T}$. (If Δ^n is an n-simplex of a simplicial complex $\mathfrak{T}$, and Δ^{n-1} is an $(n-1)$-dimensional face of Δ^n which is not a face of any other simplex in $\mathfrak{T}$, we say that $\mathfrak{T} \setminus \{\Delta^n, \Delta^{n-1}\}$ is obtained from $\mathfrak{T}$ by an elementary collaps.) This shows that any regular neighborhood of F in S^2 is also a regular neighborhood of a finite polyhedral graph and therefore a cube with handles (concerning regular neighborhood theory see [6], Chap. II). This proves the following lemma.

LEMMA 5. *Let X be a compact subset of S^3 such that for each $\varepsilon > 0$ there is a compact polyhedral 2-manifold F in S^3 each of whose components has a non empty boundary and a regular neighborhood N of F in S^3 which contains X and is contained in the ε-neighborhood of X. Then X is definable by cubes with handles.*

Looking at the property (G) in Section 1 we see that the property of being definable by thin cubes with handles is a local one; i.e., a curve X in S^3 is definable by thin cubes with handles if and only if each point of X has a compact neighborhood in X which is definable by thin cubes with handles.

Now we define the curve Y in S^3 by a construction which is very similar to the construction of the curve X in Section 6. The difference is that we replace the defining sequence $M^{(0)}, M^{(1)}, \ldots$ by a sequence $N^{(0)}, N^{(1)}, \ldots$ where $N^{(0)} = M^{(0)}$ and $N^{(n+1)}$ is obtained from $N^{(n)}$ in the same way as $M^{(n+1)}$ was obtained from $M^{(n)}$ with the single exception that for each component B of $\mathrm{Bd}\, N^{(n)}$ we choose a triangle Δ_B of $\mathfrak{T}^{(n)}$ in B and remove $\varkappa^{(n)}(\mathrm{Int}\, \Delta_B \times I) \cap (N^{(n)})'$ from $(N^{(n)})'$ to obtain $N^{(n+1)}$. These triangles Δ_B have to be selected such that the sets $\varkappa^{(n)}(\Delta_B \times I)$ do not intersect a neighborhood V of a point x of the curve X at which X is locally not definable by thin cubes with handles. (As pointed out above, such a point x must exist, but by the construction of X each point of X has this property.) Then $V \cap N^{(n)} = V \cap M^{(n)}$ for each index n, and if we define Y to be the curve $\bigcap_{n=0}^{\infty} N^{(n)}$, we have $Y \cap X \supseteq V$. This implies that Y can not be defined by thin cubes with handles. By Lemma 5 we see easily that Y can be defined by cubes with handles. To prove that each component of Y is definable by thin cubes with handles we have to proceed as in Section 6 where we proved the corresponding fact for the curve X.

Remark 8. This construction and especially Lemma 5 show that the example in [4] was unnecessarily complicated.

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