Finally, we have

(3.1) Lemma. *Let X be a locally compact separable metric space and let A be a first category subset of $F^n(X)$. Then there exists a dense countable subset B of X such that $F^n(B) \cap A = \emptyset$.*

Proof. Let $\{U_i\}$ be a countable basis for X. We will define B inductively. The first $n-1$ points of B will be chosen by induction as follows. By (4.3) choose $b_1 \in U_1$ such that $D(b_1, A)$ is a first category subset of $F^{n-1}(X)$. If $s<i-1$, assume that $b_1, \ldots, b_s$ have been chosen so that $b_j \in U_j$ and $D(b_j, A)$ is a first category subset of $F^{n-1}(X)$, $D(b_j, b_k, A)$ is a first category subset of $F^{n-2}(X)$ for $k<j$ and in general if $k_1, k_2, \ldots, k_l$, $l<j$ is a decreasing sequence of natural numbers with $k_1<j$, then $D(b_j, b_{k_1}, \ldots, b_{k_l}, A)$ is a first category subset of $F^{n-(l+1)}$. For $k = 1, \ldots, p$, let,

$$E_k = \bigcup \{D(b_{j_1}, \ldots, b_{jk}, A) |\ p \geqslant j_1 > j_2 \ldots > j_k \geqslant 1\} \subset F^{n-k}(X).$$

Note that by the above assumptions E_k is a first category subset of $F^{n-k}(X)$. Hence, by (4.3) we can chose $b_{p+1} \in U_{p+1}$ such that $D(b_{p+1}, A)$ and $D(b_{p+1}, E_k)$ are first category for $k = 1, \ldots, p$. It is easily verified that $\{b_1, \ldots, b_{p+1}\}$ satisfy the inductive hypothesis.

In order to define b_n we let the E_i's be as above, however, note that E_{n-1} is a first category subset of X. Therefore, using (4.3) again choose $b_n \in U_n - E_{n-1}$ such that $D(b_n, A)$ is a first category subset of $F^{n-1}(X)$ and $D(b_n, E_k)$ is a first category subset of $F^{n-(k+1)}(X)$ for $k<n-1$. Now we proceed as before if we assume $b_1, \ldots, b_p$ have been defined and $p>n$, we let E_i be defined as before but only for $i = 1, \ldots, n-1$ and by (4.3) we choose $b_{p+1} \in U_{p+1} - E_{n-1}$ so that $D(b_{p+1}, A_i)$ and $D(b_{p+1}, E_i)$ are first category for $i = 1, \ldots, n-2$.

Let $B = \{b_i\}_{i=1}^{\infty}$. It is clear from the construction that B is a dense countable subset of X. All that remains to be shown is that $F^n(B) \cap A = \emptyset$. Therefore assume that $(c_1, \ldots, c_n) \in F^n(B) \cap A$. For the sake of simplicity, we will assume that $(c_1, \ldots, c_n) = (b_{k_1}, \ldots, b_{k_n})$ where $k_1<k_2<\ldots<k_n$. This implies that

$$b_{k_n} \in D(b_{k_1}, \ldots, b_{k_{n-1}}, A) \subset E_{n-1}.$$

But b_{k_n} was chosen in $U_{k_n} - E_{n-1}$ and hence we have a contradiction.

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A counter-example concerning quasi-homeomorphisms of compacta

by

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Abstract. Two metric compacta X and Y are said to be *quasi-homeomorphic* if for every $\varepsilon>0$ there are two ε-mappings: f mapping X onto Y and g mapping Y onto X. A locally connected continuum X belongs to the class α if there is a $\delta>0$ such that no simple closed curve $C \subset X$ with $\operatorname{diam} C<\delta$ is a retract of X. We prove in the paper that there are two quasi-homeomorphic, 2-dimensional locally connected continua X and Y such that $X \in \alpha$ and $Y \notin \alpha$.

1. Introduction. Let X be a (metric) compactum and let Y be a topological space. A map $f\colon X \to Y$ is said to be an ε-*mapping* if $\operatorname{diam}(f^{-1}(y))<\varepsilon$ for every $y \in f(X)$. X is said to be Y-*like* if for every $\varepsilon>0$ there is an ε-mapping of X onto Y. Two compacta X and Y are said to be *quasi-homeomorphic* if X is Y-like and Y is X-like.

In a sequence of papers (cf. [3], [4], [5]) concerned with these notions we considered the following class α:

Definition 1. A locally connected compactum X *belong to the class* α if there is an $\varepsilon>0$ such that no simple closed curve $C \subset X$ with $\operatorname{diam} C<\varepsilon$ is a retract of X.

In [5] we proved the following theorem: *Let Y be a compact semi-*lc_1 *space in the homological sense, i.e. such that $i_*(H_1(A)) = 0$ for each compact subset A of Y with diameter less than a given $\delta>0$, where $H_1(A)$ is the first Čech homology group of A with integer coefficients and $i\colon A \to Y$ is the inclusion map. Then each locally connected compactum X which is Y-like belongs to the class α.*

In the same paper we raised the question whether the property α is a quasi-homeomorphism invariant. In the present paper we shall prove that this is not the case, i.e. that there exist two quasi-homeomorphic locally connected continua X and Y such that $X \in \alpha$ and $Y \notin \alpha$.

Given a compactum A, $H_n(A)$ will denote the nth Čech homology group of A with integer coefficients. It is well known (cf. [2], p. 6) that, if A is a retract of X and $i\colon A \to X$ is the inclusion map, then the group $i_*(H_n(A))$ is a direct summand of the group $H_n(X)$. If C is a simple closed curve, then it follows from the Bruschlinsky results (cf. [1] and [2], p. 526) that the algebraic conditions are also sufficient in order that C be a retract of X, i.e.:

(1.1) Let X be a compactum and let $C\subset X$ be a simple closed curve. Then C is a retract of X if and only if $i_*(H_1(C))$ is a direct summand of the group $H_1(X)$, where $i\colon C\to X$ is the inclusion map.

In Section 2, we present an auxiliary construction of polyhedra P_n, $n=1,2,\ldots$ and we prove some of their properties. By means of these polyhedra, we construct in Section 3 the compacta X and Y and prove their properties in question. Now, we shall start with some intuitive remarks concerning these constructions.

The polyhedra P_n, $n=1,2,\ldots$, and also the spaces X and Y will be 2-dimensional. It is easy to see by using the above-mentioned results of the paper [5] that the example cannot be 1-dimensional. For a fixed n, we shall construct P_n such that $H_1(P_n)$ will be a free cyclic group with a generator β whose carrier will be a simple closed curve S_0 forming the boundary of the unit square on the plane E^2. As the "first approximation" of P_n, we can imagine the polyhedron $P=S_0\times\langle 0,1\rangle\subset E^3$. As the "first approximation" of X we can imagine the closure of the set $X_0=\bigcup\limits_{n=1}^{\infty}h_n(P)$, where h_n is the linear homeomorphism mapping P onto the set $C_n\times\langle 0,1\rangle\subset E^3$, where C_n is the boundary (in E^2) of the rectangle $\{x\in E^2\colon 0\leqslant x_1\leqslant 1,\ 1/(n+1)\leqslant x_2\leqslant 1/n\}$. As the "first approximation" of Y we can imagine the closure of the set $Y_0=\bigcup\limits_{n=1}^{\infty}g_n(P)$, where g_n's are linear homeomorphisms mapping P onto subsets of E^3 such that $\lim\limits_{n\to\infty}\operatorname{diam}g_n(P)=0$ and that $g_n(P)\cap g_m(P)$ is a (PL) disk if $|n-m|=1$ and it is the empty set if $|n-m|>1$. Then $\overline{Y}_0\notin\alpha$, and no simple closed curve $C\subset\overline{X}_0$ with diameter smaller than 1 is a retract of $\overline{X}_0$; however, $\overline{X}_0$ is not locally connected.

In order to improve it, we must adjoin arcs joining the points of $\overline{X}_0$ whose x_3-coordinates are close to one another; and therefore, for a sufficiently large number n, the polyhedron P_n must contain (homologically non-trivial) simple closed curves with small diameters. Let $C\subset P_n$ be such a simple closed curve, let γ be a generator of the group $H_1(C)$ and let $i_*\colon H_1(C)\to H_1(P_n)$ denote the homomorphism induced by the inclusion $i\colon C\to P_n$. Notice that the relation $i_*(\gamma)=0$ cannot hold for "too many" curves C, because — if such was the case — either the group $H_1(P_n)$ would be trivial or a generator of this group would have a carrier with a small diameter. Consequently, since the simple closed curve C cannot be a retract of X (and therefore of P_n either), we shall construct P_n in such a way that $i_*(\gamma)$ (if not 0) will be a multiple of the generator β.

For any integer $m>1$, we shall construct a simple closed curve being a carrier of $m\beta$ by means of the so called "pseudo-projective" m-band. To construct it, consider the curve S_0 defined before and take a copy of the set $P=S_0\times\langle 0,1\rangle$, which we now assume to be disjoint from S_0. Let us identify $S_0\times(0)\subset P$ with S_0 by means of the map $p\colon S_0\times(0)\to S_0$, which is the covering projection of order m. The identification space $M=(P\oplus S_0)/p$ is called the *pseudo-projective m-band* and the simple closed curve $S_0\times(1)$ — which is assumed to be contained in M — is called the *boundary* of M. Evidently, this simple closed curve is a carrier of $m\beta$. We can assume that $S_0\subset E^2\subset E^5$ and that M is a polyhedron in E^5 containing S_0 in the way suggested in Figure 1 for the case $m=2$ (in this case, evidently, M is the Mobius band).

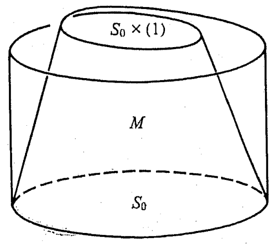

Fig. 1

The whole construction of the polyhedron P_n will be given in the space E^5. We do it in several steps. If, in a certain step, a polyhedron Q_1 has been constructed, in the next step we extend Q_1 by constructing a polyhedron Q_2 and identifying a connected subpolyhedron $Q_1'\subset Q_1$ with a connected subpolyhedron $Q_2'\subset Q_2$. We shall assume that the identification space also lies in E^5 and contains the images of Q_1 and Q_2 under the identification map as subpolyhedra. We shall identify these images with Q_1 and Q_2, respectively, if the construction makes it possible.

In calculating of the first homology group $H_1(Q_1\cup Q_2)$, where $Q_0=Q_1\cap Q_2$ is assumed to be a connected polyhedron (thus $\tilde{H}_0(Q_1\cap Q_2)=0$, $\tilde{H}_0$ being the reduced homology functor), we shall use the following formula:

(1.2) $H_1(Q_1\cup Q_2)=H_1(Q_1)\oplus H_1(Q_2)/\operatorname{Im}\psi$, where $\psi\colon H_1(Q_1\cap Q_2)\to H_1(Q_1)\oplus\oplus H_1(Q_2)$ is defined by $\psi(\gamma)=\big(i_{1*}(\gamma)-i_{2*}(\gamma)\big)$, $i_\mu\colon Q_1\cap Q_2\to Q_\mu$, $\mu=1,2$ being the inclusion maps.

This formula is an easy consequence of the Mayer–Vietoris exact sequence.

2. Construction of a polyhedron P_n. Given a positive integer n, we shall construct a polyhedron P_n, which will be crucial for the whole paper.

Let $p_1=2$, $p_2=3,\ldots$ denote the sequence of the prime numbers. Let $q=p_1\cdot p_2\cdot\ldots\cdot p_{2^n}$ and let $q_i=q/p_i$ for $i=1,2,\ldots,2^n$. The numbers $q_1,\ldots,q_{2^n}$ are relatively prime, and so there exist integers $m_1,\ldots,m_{2^n}$ such that

$$m_1q_1+\ldots+m_{2^n}q_{2^n}=1\,.$$

We define a sequence of integers $r_1,\ldots,r_{2^n}$ as a permutation of the integers $m_1q_1,\ldots,m_{2^n}q_{2^n}$. In Figure 2 we construct a (theoretically) infinite table, in whose rows these permutations are successively indicated for $n=1,2,\ldots$ The ith row of

$n=1$	m_1q_1				m_2q_2			
$n=2$	m_1q_1		m_3q_3		m_2q_2		m_4q_4	
$n=3$	m_1q_1	m_5q_5	m_3q_3	m_6q_6	m_2q_2	m_7q_7	m_4q_4	m_8q_8

Fig. 2

the table contains 2^i small rectangles. If the permutation appearing in the ith row is known, we construct the permutation in the $(i+1)$-th row in the following way: Let Q be a small rectangle appearing in the ith row. There are two small rectangles Q', Q'' in the $(i+1)$-th row, each with one side lying on a side of Q. Let Q' be the first of them. If Q contains m_jq_j, we put the same in Q'. Thus half of the rectangles appearing in the $(i+1)$-th row are filled in. The remaining ones are filled in successively by $m_{2^i+1}q_{2^i+1}, \ldots, m_{2^{i+1}}q_{2^{i+1}}$.

The permutation $r_1, \ldots, r_{2^n}$ is not essential in constructing the polyhedron P_n itself; however, it is essential in constructing X, which we now substantiate intuitively. The exact construction of X from the sequence $P_1, P_2, \ldots$ of polyhedra is similar to, what we called the "first approximation" in the Introduction. Assuming that the widths of the rows in Figure 2 converge to zero when i becomes infinite, the boundaries of the rectangles appearing in this figure can be assumed to be subsets of X. Moreover, for each small rectangle Q appearing in the ith row, $\mathrm{Bd}\,Q$ can be assumed to be a simple closed curve contained in P_i such that if m_jq_j appears in Q then $\mathrm{Bd}\,Q$ is a carrier of $m_jq_j\beta$, where β is a fixed generator of the group $H_1(P_i)$. Evidently, for any sequence $Q_1, Q_2, \ldots, Q_k$ of small rectangles appearing in the ith row, not containing all of those rectangles, there is a prime number $p_{j_0} \geqslant 2$ which is a common divisor of all integers m_jq_j appearing in $Q_1, Q_2, \ldots, Q_k$.

Now, consider a simple closed curve C lying in the closure of the union of the boundaries of all the rectangles in Figure 2, where we assume that the width of the rows converges to zero. As has been mentioned, we can assume that $C \subset X$. Let i_0 denote the first index i such that C intersects the boundary of a rectangle Q appearing in the ith row. Denote by π_{i_0} the (orthogonal) projection onto the lower side of the i_0th row. If the diameter of C is sufficiently small, then $i_0 > 1$ and, moreover, the sequence of all the small rectangles $Q_1, \ldots, Q_k$ appearing in the i_0th row and intersecting $\pi_{i_0}(C)$ does not contain all of those rectangles. Consequently, there is a prime number $p_{j_0} \geqslant 2$ with the property mentioned before. It follows from the construction of the table given in Figure 2 that, for each $i \geqslant i_0$ and for each small rectangle Q appearing in the ith row such that either $\mathrm{Bd}\,Q \cap C \neq \emptyset$ or Q lies in the bounded component of $E^2 \setminus C$, the prime number p_{j_0} is a divisor of the integer m_jq_j appearing in Q. Consequently, if γ is a generator of the group $H_1(C)$ and $i\colon C \to X$ is the inclusion map, then

$$i_*(\gamma) = p_{j_0}\gamma',$$

where $\gamma' \in H_1(X)$. Thus we conclude from (1.1) that C cannot be a retract of X, which is a necessary condition in order that $X \in \alpha$.

We shall define P_n in the Euclidean space E^5. First, let S_0 denote the boundary of the unit square in the plane $E^2 \subset E^5$. Divide this square into 2^n subrectangles as in Figure 3 and denote their boundaries successively by $S_1, S_2, \ldots, S_{2^n}$. Let us

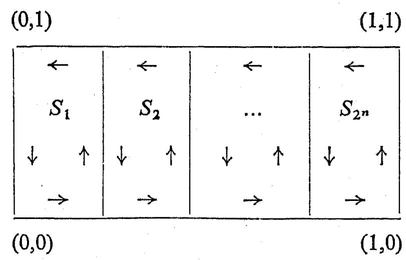

Fig. 3

orient coherently all S_i's, $0 \leqslant i \leqslant 2^n$, such that if the oriented S_i defines the cycle ξ_i, then $\xi_0 = \xi_1 + \xi_2 + \ldots + \xi_{2^n}$.

Constructing P_n, we shall first extend the polyhedron $\bigcup_{i=0}^{2^n} S_i$ so as to obtain the relations

$$[\xi_i] = r_i[\xi_0],$$

where the brackets denote the homology class of the cycle in P_n.

For this purpose consider the unit 3-cube Q in the 3-space $E^3 \subset E^5$ and let $Q' = [x \in E^5\colon (x_1, x_2, x_3) \in Q, 0 \leqslant x_4, x_5 \leqslant 2^{-(n+1)}]$. We shall construct P_n as a subset of Q'. By the phrase "a sufficiently small neighborhood of a set A" we shall mean the generalized ball on A (in Q') with radius $2^{-(n+1)}$. For each i, $1 \leqslant i \leqslant 2^n$, construct a pseudo-projective $|r_i|$-band M_i lying in a sufficiently small neighborhood (in Q') of the union of the lateral faces of Q such that $M_i \cap M_j = S_0$ if $i \neq j$ and $M_j \cap \bigcup_{i=0}^{2^n} S_i = S_0$. Denote by S_i' the oriented boundary simple closed curve of M_i such that if S_i' defines the cycle ξ_i' then

$$[\xi_i'] = |r_i|\,[\xi_0],$$

where the brackets denote the homology class of the cycle in $\bigcup_i M_i$. Next, extend M_i in a sufficiently small neighborhood (in Q') of the upper face of Q to obtain a pseudo-projective $|r_i|$-band M_i^* containing M_i whose boundary curve S_i^* (in the preceding sense) lies in a sufficiently small neighborhood of the orthogonal projection of S_i into the upper face of Q. We can assume that $M_i^* \cap M_j^* = S_0$ for $i \neq j$. Now, we extend M_i^* by a tube joining S_i^* with S_i and yielding the homology between the cycles defined by the oriented S_i^* and S_i, where we orient S_i^* coherently with the orientation of S_i' if $|r_i| = r_i$ or else in the opposite way.

We can assume that this tube lies in a sufficiently small neighborhood (in Q') of the rectangular solid tube joining in Q the rectangle bounded by S_i with its projection into the upper face of Q. Denote the union of M_i^* and of that tube by M_i^{**}. We can assume that the construction has been executed so that $M_i^{**} \cap M_j^{**} = S_0 \cup (S_i \cap S_j)$ for $i \neq j$.

Now, to improve the first homology group, we shall make some identifications in the polyhedron $\bigcup_{i=1}^{2^n} M_i^{**}$. Notice that the group $H_1(M_i^{**})$ has two generators. The carrier of the one different from $[\xi_0]$ is a (PL) simple closed curve $C_i^{**} \subset M_i^{**}$ which is the union of two simple arcs, disjoint except at the end-points and lying close to each other, one lying on M_i^* and the other on the tube $\overline{M_i^{**} \setminus M_i^*}$. Let us identify these arcs by using a (PL) homeomorphism between them. Denote by $\hat{M}_i$ the decomposition space of M_i^{**} obtained in this way. We can assume that $\hat{M}_i$ differs from M_i^{**} only on a small neighborhood of C_i^{**}.

By construction, $H_1(\hat{M}_i)$ is a free cyclic group with $[\xi_0]$ as a generator. Since $[\xi_i'] = |r_i|[\xi_0]$ and $r_1, \ldots, r_{2^n}$ is a permutation of $m_1 q_1, \ldots, m_{2^n} q_{2^n}$, we conclude from the definition of m_i's that $H_1(\bigcup_{i=1}^{2^n} \hat{M}_i)$ is also a free cyclic group with $[\xi_0]$ as a generator.

Next, as described in the Introduction, we shall complicate the construction to make X locally connected. First, we shall join $\hat{M}_i$'s on larger sets. For this purpose consider polyhedral disks $D_1' \subset M_1$, $D_i, D_i' \subset M_i$ for $1 < i < 2^n$, $D_{2^n} \subset M_{2^n}$, where each D_i (D_i') contains a large arc J contained in S_0 (i.e. such that $\operatorname{diam}(S_0 \setminus J) < 2^{-(n+1)}$), $D_i \cap D_i' = J \subset S_0$ and both D_i and D_i' lie close to the set

$$[x \in Q\colon (x_1, x_2) \in J, 0 \leqslant x_3 \leqslant 1]\,.$$

For $i = 1, \ldots, 2^n - 1$, identify disks D_i' and D_{i+1} using a (PL) homeomorphism between them mapping the arc J onto itself. Denote by $\hat{M}$ the polyhedron obtained from $\bigcup_i \hat{M}_i$ by these identifications.

The aim of the further completion of the construction is to ensure that the points of $\hat{M}$ whose x_3 coordinates are sufficiently close to one another could be joined by "small arcs" without changing at the same time the first homology group of $\hat{M}$ and without permitting small simple closed curves to yield cycles homological with ξ_0. In order not to change the first homology group we must take care not to add new 1-cycles independent of ξ_0 and not to permit different multiples of ξ_0 to be homological. In order not to permit small simple closed curves to yield cycles homological with ξ_0 one must take care that no set of small diameter contains carriers of multiples of ξ_0 whose greatest common divisor is equal to one.

For this purpose fix an index i and consider the previously constructed $|r_i|$-band M_i. One can construct polyhedral simple closed curves $C_j^{(i)}, j = 1, 2, \ldots, |r_i|$, (denoted by C_j if i is fixed), lying on M_i and defining cycles $z_j^{(i)}$ homological with ξ_0,

as in Figure 4. We can assume that C_j lies sufficiently close to the union of the boundary simple closed curve S_i' of M_i and of the vertical segment erected at the point $(0,0) \in S_0 \subset E^2$. Moreover, we assume that the ordering of the curves $C_1, \ldots, C_{|r_i|}$ is such that the successive curves (with indices ordered modulo $|r_i|$)

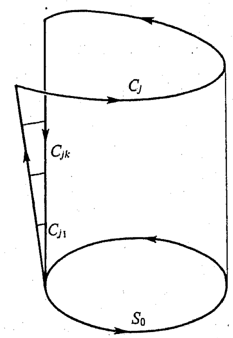

Fig. 4

have a "segment" in common (lying close to the vertical segment mentioned above) and that

$$\sum_{j=1}^{|r_i|} [z_j^{(i)}] = |r_i|[\xi_0]\,.$$

Let us replace the part of C_j, $j < |r_i|$, lying close to this vertical segment by 2^n smaller simple closed curves C_{jk}, $k = 1, 2, \ldots, 2^n$, by adding "segments" parallel to the plane $E^2 \subset E^5$, as suggested in Figure 4. We can assume that for a pair of successive curves C_j, C_{j+1} the added segments have one point in common and that the diameters of the simple closed curves C_{jk}, $k = 1, 2, \ldots, 2^n$, are equal. Moreover, we assume that the ordering of the curves C_{jk}, $k = 1, \ldots, 2^n$, is the following: C_{j1} is the curve which lies closest to the plane $E^2 \subset E^5$, C_{jk} and $C_{j,k+1}$ have a common segment and C_{j2^n} is the curve which lies closest to the plane $[x \in E^5\colon x_3 = 1$ and $x_i = 0$ for $i > 3]$.

Let us orient the curves C_{jk} coherently with the orientation of C_j, and let them define the cycles z_{jk}, $k = 1, 2, \ldots, 2^n$. Evidently, C_{jk}, z_{jk} depend on i, so we should rather use the notation $C_{jk}^{(i)}$, $z_{jk}^{(i)}$. We shall assume that the construction is executed so that $C_{jk}^{(i)} \cap C_{j'k'}^{(i')} \subset C_j^{(i)} \cap C_{j'}^{(i')}$ if $(i,j) \neq (i',j')$.

Now, to complete the construction of P_n, we shall extend the polyhedron

$$M' = \hat{M} \cup \bigcup [C_{jk}^{(i)}\colon 1 \leqslant i \leqslant 2^n, 1 \leqslant j < |r_i|, 1 \leqslant k \leqslant 2^n]$$

so as to obtain the relations

$$[z_{jk}^{(i)}] = [\xi_k]\,,$$

where the brackets denote the homology classes in P_n. At the same time, we must take care of the matters mentioned before the construction of $C_j^{(i)}$'s.

To do this, first construct a tube $T_{2^n}^{(1)}$ yielding the relations

$$[z_{12^n}^{(1)}] = [z_{22^n}^{(1)}] = \dots = [z_{|r_1|-1,2^n}^{(1)}] = [\xi_{2^n}]$$

and lying in a small neighborhood of the union of the upper face of Q and of the rectangular tube erected at in Q on S_{2^n}. For this purpose notice that the curves $C_j^{(1)}$, $1 \leqslant j < |r_1|$, divide in the natural way the band M_1 into subsets M_{1j}, $1 \leqslant j \leqslant |r_1|$, where the ordering of M_{1j}'s agrees with the ordering of $C_j^{(1)}$'s ($M_{1|r_1|}$ joins $M_{1,|r_1|-1}$ to M_{11}).

First, construct a part of the tube $T_{2^n}^{(1)}$ yielding the relation

$$[z_{12^n}^{(1)}] = [\xi_{2^n}]$$

and intersecting M_1 on a subset of M_{11} which is a (PL) disk constituting the union of the following three disks: The first and the third join a segment which is a component of $M_{11} \cap C_{12^n}^{(1)}$ to a segment which is a component of $S_{2^n} \cap \overline{M_{11} \setminus [x\colon x_1 = 1]}$ and lie close to the union of a part of the "boundary" arc of M_{11} and of a vertical segment erected at in a point of S_{2^n}. The second disk intersects the first and the third on "vertical segments" and contains the remaining part of $S_{2^n} \cap M_{11}$; it lies close to the set $[x \in Q\colon x_1 = 1]$. This part of the tube $T_{2^n}^{(1)}$ lies close to the disk just described, which is its intersection with M_{11}.

Next, we extend successively the constructed tube to obtain the relations:

$$[z_{12^n}^{(1)}] = [z_{22^n}^{(1)}] = \dots = [z_{|r_1|-1,2^n}^{(1)}]\,.$$

The first extension, yielding the first of these relations, is obtained by a (PL) tube with two "segments" lying respectively on the boundary simple closed curves of it identified by a (PL) homeomorphism, because $C_{12^n}^{(1)} \cap C_{22^n}^{(1)}$ is a "segment". This tube intersects M_1 on a disk constituting the union of two disks, one lying on M_{11} and containing $C_{12^n}^{(1)} \cap M_{11}$, the other lying on M_{12} and containing $C_{22^n}^{(1)} \cap M_{12}$. The tube lies in a small neighborhood of the disk just described. Further extensions of the constructed tube, yielding the subsequent homology relations, are built analogously. Finally, to obtain the whole tube $T_{2^n}^{(1)}$, we extend the constructed tube in a similar way joining the curve $C_{|r_1|-1,2^n}^{(1)}$ to a curve lying close to $C_{12^n}^{(1)}$ but such that this final extension is disjoint with $C_{12^n}^{(1)}$ and intersects M_1 on a subset of $M_{1,|r_1|-1} \cup M_{1|r_1|}$ which is the union of two disks intersecting $C_{|r_1|-1,2^n}^{(1)}$ on two arcs. We assume that the tube $T_{2^n}^{(1)}$ does not intersect M' except the sets described above, where we admit $M_1 \subset M'$.

Next, we extend the tube $T_{2^n}^{(1)}$ by (PL) tubes $T_{2^n}^{(2)}, \dots, T_{2^n}^{(2^n)}$, where $T_{2^n}^{(i)}$ yields the relations

$$[z_{|r_{i-1}|-1,2^n}^{(i-1)}] = [z_{12^n}^{(i)}] = \dots = [z_{|r_i|-1,2^n}^{(i)}]\,.$$

The tube $T_{2^n}^{(i)}$ lies in a small neighborhood of the boundary curve of M_i and it is constructed in the same way as the extensions of the first part of the tube $T_{2^n}^{(1)}$. It is

joined to the tube $T_{2^n}^{(i-1)}$ by a small tube joining the boundary simple closed curve of $T_{2^n}^{(i-1)}$ with $C_{12^n}^{(i)}$ and intersecting M' only on these curves. We can assume here that these simple closed curves intersect on a common arc obtained in the process of the identification of the disks $D'_{i-1} \subset \hat{M}_{i-1}$ and $D_i \subset \hat{M}_i$, and so a "meridional" disc on the joining tube is contracted to an arc.

Let us denote $T_{2^n} = \bigcup [T_{2^n}^{(i)}\colon 1 \leqslant i \leqslant 2^n]$.

In a similar way we construct the tube $T_{2^{n-1}}$ yielding the relations

$$[z_{j,2^{n-1}}^{(i)}] = [\xi_{2^{n-1}}] \quad \text{for} \quad 1 \leqslant i \leqslant 2^n,\ 1 \leqslant j \leqslant |r_i|-1$$

and lying in a small neighborhood of the tube T_{2^n}. We shall assume that $T_{2^n} \cap T_{2^{n-1}}$ is a disk containing all arcs of the form $C_{j2^n}^{(i)} \cap C_{j,2^{n-1}}^{(i)}$ and also the arc $S_{2^n} \cap S_{2^{n-1}}$, and intersecting M_i on the union of two arcs lying in the boundary of $T_{2^n} \cap M_i$. Moreover, we assume that the boundary simple closed curves of the tubes T_{2^n} and $T_{2^{n-1}}$ different from S_{2^n} and $S_{2^{n-1}}$ intersect on a common arc.

In the same way we construct the tubes $T_{2^{n-2}}, \dots, T_2, T_1$, where the first part of the tube T_1 has one "meridional" arc contracted to a point, because the curves $C_{11}^{(1)}$ and S_1 joined by it have the point $(0, 0) \in S_1 \subset E^2$ in common.

Finally, we define

$$P_n = M' \cup \bigcup_{j=1}^{2^n} T_j\,.$$

Now, let us examine the group $H_1(P_n)$. As we have seen, $H_1(\bigcup_i \hat{M}_i)$ is a free cyclic group with $[\xi_0]$ as a generator. It is evident that the construction of $\hat{M}$ from $\bigcup_i \hat{M}_i$ yields no change of the homology group, and so $H_1(\hat{M})$ is also a free cyclic group with the same generator.

Moreover, observe that each (PL) simple closed curve

$$C \subset \hat{M} \cap [x \in E^5\colon \tfrac{1}{8}i - \tfrac{1}{16} \leqslant x_1 \leqslant \tfrac{1}{8}(i+1) + \tfrac{1}{16}]$$

where i is one of $0, 1, \dots, 7$) such that the cycle z given by an orientation of C is not homologous to zero in $\hat{M}$ must lie in the union of some tubes $\overline{M_j^{**} \setminus M_j^*}$. Thus

$$[z] = \sum_j \varepsilon_j r_j [\xi_0]\,,$$

where $\varepsilon_j = \mp 1$ and the summation runs over some indices j such that

$$S_j \subset [x \in E^2 \subset E^5\colon \tfrac{1}{8}i - \tfrac{1}{16} \leqslant x_1 \leqslant \tfrac{1}{8}(i+1) + \tfrac{1}{16}]\,.$$

The further construction of P_n is performed by attaching the tubes T_j, $j = 1, \dots, 2^n$. Since the generator of the group $H_1(T_j)$ has been identified with $r_j[\xi_0]$ and since T_j intersects $\hat{M} \cup \bigcup_{i=1}^{j-1} T_i$ on a connected set, in the process of attaching the tubes we do not add new generators to the group $H_1(\hat{M})$.

The only incertainty whether we obtained a relation of the form

$$m_1[\xi_0] = m_2[\xi_0] \quad \text{with} \quad m_1 \neq m_2$$

arises by considering the curves $C_j^{(i)}$, $j = 1, \ldots, |r_i|$. For $j<|r_i|$ denote by $C_{j0}^{(i)}$ the (large) simple closed curve obtained during the construction of $C_{jk}^{(i)}$'s such that if $z_{j0}^{(i)}$ is the cycle given by a suitable orientation of $C_{j0}^{(i)}$ then $\sum_{k=0}^{2^n} z_{jk}^{(i)} = z_j^{(i)}$. Thus $\sum_{k=0}^{2^n} [z_{jk}^{(i)}] = [\xi_0]$ and, since $\sum_{k=1}^{2^n} [z_{jk}^{(i)}] = [\xi_0]$, we obtain $[z_{j0}^{(i)}] = 0$. $C_{|r_i|0}^{(i)}$ is defined similarly, but (considering $C_{|r_i|}^{(i)}$) we have

$$[z_{|r_i|0}^{(i)}] - (|r_i - 1)[\xi_0] = [\xi_0],$$

and therefore $[z_{|r_i|0}^{(i)}] = |r_i|[\xi_0]$.

On the other hand, since $\sum_{j=1}^{|r_i|} [z_j^{(i)}] = |r_i|[\xi_0]$, it follows from the definition of $z_{j0}^{(i)}$ that

$$\sum_{j=1}^{|r_i|} [z_{j0}^{(i)}] = |r_i|[\xi_0].$$

Since $[z_{j0}^{(i)}] = 0$ for $j<|r_i|$, we obtain $[z_{|r_i|0}^{(i)}] = |r_i|[\xi_0]$ again, and so our construction yields no relation between different multiples of $[\xi_0]$.

Thus we have proved that:

(2.1) $H_1(P_n)$ is a free cyclic group with generator $[\xi_0]$.

Using (1.1), we conclude that:

(2.2) The simple closed curve S_0 is a retract of P_n.

Now, given integers i_1, i_3 such that $0 \leqslant i_1, i_3 \leqslant 7$, consider a (PL) simple closed curve $C \subset P_n \cap Q'_{i_1 i_3}$, where

$$Q'_{i_1 i_3} = [x \in E^5\colon \tfrac{1}{8} i_j - \tfrac{1}{16} \leqslant x_j \leqslant \tfrac{1}{8}(i_j+1) + \tfrac{1}{16} \text{ for } j = 1 \text{ and } 3]$$

and such that the cycle z determined by an orientation of C is not homologous to zero in P_n. Observe that the cycle z can be represented as $z_1 + z_3$, where z_1 lies in $\hat{M} \cap Q'_{i_1 i_3}$ and z_3 lies in $\bigcup_{i=1}^{2^n} T_i \cap Q'_{i_1 i_3}$. By the preceding remarks concerning the cycles lying in $\hat{M}$, we have

$$[z_1] = \sum_k \mp r_k [\xi_0],$$

where the summation runs over some k, $1 \leqslant k \leqslant 2^n$, such that

$$S_k \subset [x \in E^2 \subset E^5\colon \tfrac{1}{8} i_1 - \tfrac{1}{16} \leqslant x_1 \leqslant \tfrac{1}{8}(i_1+1) + \tfrac{1}{16}]$$

(it could happen that the same k appears in the sum more than once).

Considering the position of the tubes T_i's in P_n, one sees that

$$[z_3] = \sum_l \mp r_l [\xi_0] + \sum_m \mp r_m [\xi_0],$$

where the first sum is of the same kind as the preceding one, but the second summation runs over some m such that

$$C_{1m}^{(1)} \subset [x \in E^5\colon \tfrac{1}{8} i_3 - \tfrac{1}{16} \leqslant x_3 \leqslant \tfrac{1}{8}(i_3+1) + \tfrac{1}{16}].$$

Considering the ordering of the curves $C_{1m}^{(1)}$ (or of the curves $C_{jm}^{(i)}$ for any fixed i, j with $1 \leqslant i \leqslant 2^n$, $1 \leqslant j < |r_i|$) and slightly modifying the construction if necessary, one sees that the second summation runs over some m (possibly with repetitions) such that $1 \leqslant m \leqslant 2^n$ and

$$S_m \subset [x \in E^2 \subset E^5\colon \tfrac{1}{8} i_3 - \tfrac{1}{16} \leqslant x_1 \leqslant \tfrac{1}{8}(i_3+1) + \tfrac{1}{16}].$$

Thus we have proved that

$$[z] = \sum_p \mp r_p [\xi_0],$$

where the summation runs over some p (possibly with repetitions) such that $1 \leqslant p \leqslant 2^n$ and

$$S_p \subset [x \in E^2 \subset E^5\colon x_1 \in [\tfrac{1}{8} i_1 - \tfrac{1}{16}, \tfrac{1}{8}(i_1+1) + \tfrac{1}{16}] \cup [\tfrac{1}{8} i_3 - \tfrac{1}{16}, \tfrac{1}{8}(i_3+1) + \tfrac{1}{16}]].$$

The sum of the lengths of these intervals is equal to $\frac{1}{2}$. It is easy to see from the remarks concerning the sequence $r_1, \ldots, r_{2^n}$ that for $n \geqslant 3$ there is a prime number $p_j > 1$ such that

$$p_j | r_p$$

for all indices p appearing in the representation of $[z]$ as the sum described above. Moreover, one sees from the remarks concerning $r_1, \ldots, r_{2^n}$ that the number p_j can be chosen so that it does not depend on n, but only on i_1 and i_3.

Consequently, we have proved that:

(2.3) If C is a (PL) simple closed curve lying in $P_n \cap Q'_{i_1 i_3}$ where $n \geqslant 3$, such that the cycle z given by an orientation of C is not homologous to zero in P_n, then there is a prime number $p_j > 1$ (depending only on i_1 and i_3 and not on n) such that $[z] = p_j \cdot m[\xi_0]$, where m is an integer and $[\xi_0]$ is the generator of $H_1(P_n)$.

3. Construction of two quasi-homeomorphic compacta X and Y, $X \in \alpha$, $Y \notin \alpha$. To construct X, consider first the cube $Q = [x \in E^3 \subset E^5\colon 0 \leqslant x_i \leqslant 1 \text{ for } i = 1, 2, 3]$ and let Q_k denote the subcube $[x \in E^3 \subset E^5\colon 0 \leqslant x_1, x_3 \leqslant 1,\ 1/(k+1) \leqslant x_2 \leqslant 1/k]$, $k = 1, 2, \ldots$ Let $Q'_k = [x \in E^5\colon (x_1, x_2, x_3) \in Q_k, 0 \leqslant x_4, x_5 \leqslant 2^{-(n+1)}]$. There is a linear homeomorphism h_k mapping the polyhedron P_k constructed in Section 2 onto a subset of Q'_k, where we assume that h_k changes only the x_2-coordinate of a point (without changing the orientation of the x_2-axis). Moreover, we assume that $h_k(P_k) \cap h_{k+1}(P_{k+1})$ coincides with the disk $R_k = Q_k \cap Q_{k+1}$ and that $h_k(P_k)$ intersects the sets $Q'_k \cap Q'_{k+1}$ and $Q'_k \cap Q'_{k-1}$ only on R_k and R_{k-1}.

Divide linearly in the standard way the square R_k onto 4^k squares and erect a segment $I_{kl} \subset Q'_k$ from the centre of the lth square, $l = 1, \ldots, 4^k$. We assume that the length of I_{kl} is equal to $1/2^{k+2}$ and that $I_{kl} \cap \bigcup_{k=1}^{\infty} h_k(P_k) = I_{kl} \cap R_k$ is one end-point of I_{kl}. Moreover, we assume that $I_{kl} \setminus R_k \subset Q'_k \setminus (Q'_{k-1} \cup Q'_{k+1})$.

Define

$$X = \bigcup_{k=1}^{\infty} [h_k(P_k) \cup \bigcup_{l=1}^{4^k} I_{kl}] \cup [x \in Q\colon x_2 = 0] .$$

It is easy to see from the construction that X is a locally connected continuum. We shall now prove that $X \in \alpha$. By (2.1), the group $H_1(P_k)$ is a free cyclic group. Let β_k denote the image of a generator of this group under the homomorphism induced by the map $i \circ h_k\colon P_k \to X$, where $i\colon h_k(P_k) \to X$ is the inclusion map. By the construction of X, the group $H_1(X)$ consists of elements of the form $\sum_{k=1}^{\infty} n_k \beta_k$, where each n_k is an integer.

Observe that there is an $\varepsilon_0 > 0$ so small that for each simple closed curve $C \subset X$ with $\operatorname{diam} C < \varepsilon_0$ and with $\gamma \neq 0$, where γ is the image of a generator of the group $H_1(C)$ under its natural homomorphism into $H_1(X)$, there are two integers i_1, i_3, $0 \leqslant i_1, i_3 \leqslant 7$, such that $C \subset [x \in X\colon 0 \leqslant x_2 \leqslant \frac{1}{3}] \cap Q'_{i_1 i_3}$ (where $Q'_{i_1 i_3} = [x \in E^5\colon \frac{1}{8} i_j - \frac{1}{16} \leqslant x_j \leqslant \frac{1}{8}(i_j+1) + \frac{1}{16}$ for $j = 1$ and $3]$). Thus γ is of the form

$$\gamma = \sum_{k=3}^{\infty} n_k \beta_k ,$$

where $n_k \beta_k$ has a representative lying in the intersection of $h_k(P_k)$ with the subset of X just described. It follows from (2.3) and from the definitions of h_k's that there is a prime number p_{j_0} such that $p_{j_0} | n_k$ for all k.

Consequently, $\gamma = p_{j_0} \gamma'$, where $\gamma' \in H_1(X)$ and therefore the subgroup of $H_1(X)$ generated by γ is not a direct summand of the group $H_1(X)$. We conclude from (1.1) that C cannot be a retract of X and thus we have proved that

(3.1) $$X \in \alpha .$$

Now, let us construct Y. Denote by $\hat{Q}$ the polyhedron in $E^3 \subset E^5$ constituting the convex hull of the set $[x \in E^3 \subset E^5\colon x_2 = 1, 0 \leqslant x_1, x_3 \leqslant 1] \cup (0)$, where 0 is the point with all coordinates equal to 0. Let $\hat{Q}_k = [x \in \hat{Q}\colon 1/k+1 \leqslant x_2 \leqslant 1/k]$ and let $\hat{Q}'_k = [x \in E^5\colon (x_1, x_2, x_3) \in \hat{Q}_k, 0 \leqslant x_4, x_5 \leqslant 2^{-(k+1)}]$. There is a linear homeomorphism φ_k mapping the cube Q'_k onto $\hat{Q}'_k$, which maps the vertices accordingly to the natural ordering of their coordinates. Then $\varphi_k | R_k = \varphi_{k+1} | R_k$. Let

$$Y = \bigcup_{k=1}^{\infty} [\varphi_k \circ h_k(P_k) \cup \bigcup_{l=1}^{4^k} \varphi_k(I_{kl})] \cup (0) .$$

It is evident that Y is a locally connected continuum. Now, observe that

(3.2) Y does not belong to the class α.

Indeed, since $h_k(P_k) \cap h_{k+1}(P_{k+1}) = R_k$ and φ_k is a homeomorphism it is easy to see from the construction that, for each k, the set $\varphi_k \circ h_k(P_k)$ is a retract of Y. It follows from (2.2) that $\varphi_k \circ h_k(S_0)$ is a retract of $\varphi_k \circ h_k(P_k)$. Consequently, the simple closed curve $\varphi_k \circ h_k(S_0)$ is a retract of Y. One sees from the construction that $\operatorname{diam}(\varphi_k \circ h_k(S_0)) \leqslant \operatorname{diam} Q'_k \underset{k\to\infty}{\to} 0$, and therefore (3.2) is proved.

Finally, let us prove that:

(3.3) X and Y are quasi-homeomorphic.

Given an $\varepsilon > 0$, we must find two ε-mappings $f_\varepsilon\colon X \xrightarrow{\text{onto}} Y$ and $g_\varepsilon\colon Y \xrightarrow{\text{onto}} X$.

To construct f_ε, choose a number $k_1 > 1$ such that $3/k_1 + 2^{-(k_1+1)} < \varepsilon$ and denote by π_1 the orthogonal projection of $\bigcup [Q'_k | k \geqslant k_1]$ onto the disk $R_{k_1} = Q_{k_1} \cap Q_{k_1+1}$.

Consider now the set

$$Y_0 = \bigcup_{l=1}^{4^{k_1}} \varphi_{k_1}(I_{k_1 l}) \cup \bigcup_{k > k_1} [\varphi_k \circ h_k(P_k) \cup \bigcup_{l=1}^{4^k} \varphi_k(I_{kl})] \cup (0) .$$

One sees from the construction that Y_0 can be divided into subsets Y_l, $l = 1, \ldots, 4^{k_1}$, such that each Y_l is a locally connected continuum, Y_l intersects $\varphi_{k_1}(R_{k_1})$ on a (rectangular) subdisk with the centre at the point $\varphi_{k_1}(I_{k_1 l} \cap R_{k_1})$, contains $\varphi_{k_1}(I_{k_1 l})$ and intersects none of the other intervals $\varphi_{k_1}(I_{k_1 l'})$. Moreover, we assume that Y_l and $Y_{l'}$ intersect if and only if the disks $Y_l \cap \varphi_{k_1}(R_{k_1})$ and $Y_{l'} \cap \varphi_{k_1}(R_{k_1})$ intersect.

Since Y_l is a locally connected continuum, there is a mapping ψ_l of the interval $I_{k_1 l}$ onto Y_l. Using the fact that Y_l is arcwise connected and modifying ψ_l if necessary, we can assume that the point $\psi_l(I_{k_1 l} \cap R_{k_1})$ is equal to $\varphi_{k_1}(I_{k_1 l} \cap R_{k_1})$.

Now, define $f_\varepsilon\colon X \to Y$ by the formula:

$$f_\varepsilon(x) = \begin{cases} \varphi_k(x) & \text{if } k < k_1 \text{ and } x \in h_k(P_k) \cup \bigcup_{l=1}^{4^k} I_{kl} \text{ or if } k = k_1 \text{ and } x \in h_{k_1}(P_{k_1}), \\ \varphi_{k_1} \circ \pi_1(x) & \text{if } x \in \bigcup_{k > k_1} [h_k(P_k) \cup \bigcup_{l=1}^{4^k} I_{kl}], \\ \psi_l(x) & \text{if } x \in I_{k_1 l}, \text{ where } 1 \leqslant l \leqslant 4^{k_1} . \end{cases}$$

It is easy to see from the definition that f_ε is a map of X onto Y. If $y \in Y$ then the set $f_\varepsilon^{-1}(y)$ is either a point or a subset of the set $\pi_1^{-1}(F)$, where F is the union of all disks of the form $\varphi_{k_1}^{-1}(Y_l \cap \varphi_{k_1}(R_{k_1})) = R_{k_1} \cap \varphi_{k_1}^{-1}(Y_l)$ intersecting one of them. The disks $Y_l \cap \varphi_{k_1}(R_{k_1})$ can be constructed so that $\operatorname{diam} F \leqslant 2^{-(k+1)}$ and therefore it follows from the choice of k_1 that f_ε is an ε-mapping.

To construct g_ε: $Y \to X$, first find a number $k_2 > 1$ such that $\operatorname{diam}(\bigcup_{k \geqslant k_2} \hat{Q}'_k) < \varepsilon$. Denote by π_2 the linear projection of $\bigcup_{k \geqslant k_2} \hat{Q}'_k \cup (0)$ onto the disk $\hat{Q}_{k_2} \cap \hat{Q}_{k_2+1} = \varphi_{k_2}(R_{k_2})$.

Since $\bigcup_{k>k_2} h_k(P_k) \cup \bigcup_{k\geqslant k_2} \bigcup_{l=1}^{4^k} I_{kl} \cup [x\in Q\colon x_2 = 0]$ is a locally connected continuum, there is a mapping ψ of the interval $\varphi_{k_2}(I_{k_2 1})$ onto it such that the image by ψ of the point $p = \varphi_{k_2}(I_{k_2 1} \cap R_{k_2})$ is equal to the point $\varphi_{k_2}^{-1}(p) = I_{k_2 1} \cap R_{k_2}$.

Let us define $g_\varepsilon\colon Y \to X$ by the formula:

$$g_\varepsilon(y) = \begin{cases} \varphi_k^{-1}(y) & \text{if } k<k_2 \text{ and } y \in \varphi_k \circ h_k(P_k) \cup \bigcup_{l=1}^{4^k} \varphi_k(I_{kl}) \text{ or if} \\ & \qquad k = k_2 \text{ and } y \in \varphi_{k_2} \circ h_{k_2}(P_{k_2}), \\ \varphi_{k_2}^{-1} \circ \pi_2(y) & \text{if } y \in \bigcup_{k>k_2} [\varphi_k \circ h_k(P_k) \cup \bigcup_{l=1}^{4^k} \varphi_k(I_{kl})] \cup \bigcup_{l=2}^{4^{k_2}} \varphi_{k_2}(I_{k_2 l}), \\ \psi(y) & \text{if } y \in \varphi_{k_2}(I_{k_2 1}). \end{cases}$$

It follows from the definitions of π_2 and ψ that g_ε is a map of Y onto X. Since each φ_k is a homeomorphism, we infer that for every $x \in X$ either $g_\varepsilon^{-1}(x)$ is a point or $g_\varepsilon^{-1}(x)$ is a subset of $\bigcup_{k\geqslant k_2} \hat{Q}_k' \cup (0)$, whence $\operatorname{diam}[g_\varepsilon^{-1}(x)] < \varepsilon$. Thus g_ε is the desired ε-mapping. This concludes the proof of (3.3), and therefore the following theorem is proved:

THEOREM. *There exist two quasi-homeomorphic locally connected continua X and Y such that $X \in \alpha$ and $Y \notin \alpha$.*

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A wildly embedded 1-dimensional compact set in S^3 each of whose components is tame

by

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Abstract. A compact set X in the 3-Sphere S^3 is said to be *definable by cubes with handles*, if $X = \bigcap_{i=0}^{\infty} H_i$ where each H_i is a compact polyhedral 3-manifold in S^3 whose components are cubes with handles (i.e. regular neighborhoods of connected finite polyhedral graphs in S^3), and $H_{i+1} \subseteq \operatorname{Int} H_1$. If X is a curve (i.e. 1-dimensional) and if these cubes with handles can be chosen thin in the sense that for each $\varepsilon>0$ there is an index i_0 such that for $i \geqslant i_0$ the retraction of H_1 to the corresponding graph is an ε-retraction, X is called *definable by thin cubes with handles*. Each of these two properties of X is equivalent to some further geometrically reasonable tameness conditions of the embedding $X \subseteq S^3$. In the following paper examples of curves in S^3 are constructed with components which are definable by thin cubes with handles such that these curves themselves are not definable by cubes with handles or are definable by cubes with handles but not by thin cubes with handles.

1. Introduction. For topological embeddings of compact sets in manifolds several conditions were introduced in order to distinguish tame embeddings from wild ones. Here we are concerned with topological embeddings of curves X (i.e. compact sets each of whose components is a 1-dimensional continuum) in the euclidian 3-space or — what is almost the same — in the 3-sphere S^3. In this case the following conditions among others have proved to be useful (we prefer embeddings in S^3 for technical reasons).

(A) *X is definable by cubes with handles* if there is a sequence $H_1, H_2, \ldots$ of compact polyhedral manifolds in S^3 each of whose components is a cube with handles such that $H_{i+1} \subseteq \operatorname{Int} H_i$ $(i = 1, 2, \ldots)$ and $X = \bigcap_{i=1}^{\infty} H_i$. (A *cube with handles* is a connected 3-manifold which is the union of a finite number of closed 3-cells $Z_1, \ldots, Z_n$ such that $Z_i \cap Z_j$ is empty or a disk on $\operatorname{Bd} Z_i \cap \operatorname{Bd} Z_j$, and no three of the cells have a common point.)

(B) *X is definable by thin cubes with handles* if for each $\varepsilon>0$ the cubes with handles in (A) can be replaced by ε-thin cubes with handles. (An ε-thin cube with handles is a cube with handles for which the cells $Z_1, \ldots, Z_n$ in the definition above can be chosen with diameters smaller than ε.)