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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Countable dense homogeneity and n-homogeneity

by

Gerald S. Ungar (Cincinnati, Ohio)

Abstract. The main results of this paper are that for compact metric spaces n-homogeneity and countable dense homogeneity are equivalent and countable dense homogeneity is hereditary on dense open sets.

1. Introduction. In [2] Bennet defined the concept of countable dense homogeneity. He noted that all manifolds are countable dense homogeneous and proved that the universal curve (or any strongly locally homogeneous locally compact separable metric space) is countable dense homogeneous. He also showed that a connected countable dense homogeneous first countable space is homogeneous. However, all other questions concerning the relation of countable dense homogeneity to other types of homogeneity were left unanswered.

In this paper we show that for compact metric spaces countable dense homogeneity is equivalent to strong n-homogeneity for all n, and hence, by results in [4] it is equivalent to n-homogeneity for all n. We also show that the two concepts are almost equivalent for locally compact separable metric spaces and that countable dense homogeneity is hereditary on dense open sets.

Before we can start, we need the following definitions, notations and preliminary facts.

(2.1) Definition. A space X is *countable dense homogeneous* if given any two countable dense subsets A and B of X, there exists a homeomorphism h of X onto itself such that $h(A) = B$.

(2.2) Definition. A space X is *strongly n-homogeneous* if given any two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of distinct points of X, there exists a homeomorphism h of X onto itself such that $h(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

(2.3) Definition. If X is a locally compact separable metric space, let $H(X)$ (and when there is no ambiguity H) denote the group of homeomorphisms of X onto itself with the topology induced on $H(X)$ by considering it as a subspace of the group of homeomorphisms of the one point compactification of X onto itself with the compact open topology. It is well-known [1] that $H(X)$ is a separable metric topological group with a complete metric.

(2.4) Definition. Let X be a topological space. Define

$$F^n(X) = \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \text{ iff } i = j\}.$$

This is sometimes called the *n-th configuration space of X*. We will denote a point $(a_1, \dots, a_n) \in F_n(X)$ by $\hat{a}$. It should be noted that if X is also a locally compact separable metric space then X and $F_n(X)$ have complete metrics.

(2.5) Notation. If X is a locally compact separable metric space, we will frequently use the fact that H acts continuously on $F^n(X)$ by h, $(x_1, \dots, x_n) \to (h(x_1), \dots, h(x_n))$. The point $(h(x_1), \dots, h(x_n))$ will be denoted by $h(x_1, \dots, x_n)$ or by $h(\hat{x})$ whichever is more convenient at the time. We will also denote the orbit of $\hat{x}$ by $H(\hat{x})$. It should be noted that X is strongly n-homogeneous iff the above action is transitive.

We also need the following theorem of Effros [3] and some consequences which follow it.

(2.6) Theorem (Effros). *Let (G, X) be a transformation group with both G and X complete separable metric spaces. Then the following are equivalent:*

(1) *For each $x \in X$, the map $gG_x \to gx$ of G/G_x onto Gx is a homeomorphism.*

(2) *Each orbit is of second category in itself.*

(3) *Each orbit is a G_δ in X.*

(2.7) Remark. As the author pointed out in [4] (2.6) (1) is equivalent to the following: For each $x \in X$ the map $T_x\colon G \to Gx$ defined by $T_x(g) = gx$ is open onto Gx.

Piecing all of the above together we get the following: (Again see [4] for more details).

(2.8) Theorem. *If X is a strongly n-homogeneous compact metric space, then for each $\hat{x} \in F_n(X)$ the map $T_{\hat{x}}\colon H \to F_n(X)$ (defined as in (2.7)) is an open map onto $F_n(X)$.*

(2.9) Notation. If A is a subset of a topological space, we will use $\overline{A}$ to denote the closure of A and A^0 to denote the interior of A.

Before we could prove the main theorem we need the following lemma which may be of interest by itself.

(3.1) Lemma. *Let X be a locally compact separable metric space and let A be a first category subset of $F^n(X)$. Then there exists a dense countable subset B of X such that $F^n(B) \cap A = \emptyset$.*

We will postpone the proof of this until the last section due to the fact that it is long and there have already been enough preliminaries.

(3.2) Theorem. *If X is a locally compact countable dense homogeneous separable metric space such that no finite set separates X, then X is strongly n-homogeneous for all n.*

Proof. Let H act on $F^n(X)$ as in (2.5). We will show that the hypothesis of the theorem implies that this action has only one orbit and hence, by the last sentence of (2.5), X will be strongly n-homogeneous.

The major steps of the proof are as follows:

(1) There are at most a countable number of orbits.

(2) Every orbit is a second category subset of $F^n(X)$ (and hence of itself).

(3) For each $\hat{x} \in F^n(H)$, $H(\hat{x}) \subset \overline{H(\hat{x})}^{\circ}$.

(4) For each $\hat{x} \in F^n(X)$, $\overline{H(\hat{x})}$ is the union of orbits and hence, if $\hat{x}$ and $\hat{y} \in F^n(X)$ either $\overline{H(\hat{x})} = \overline{H(\hat{y})}$ or $\overline{H(\hat{x})} \cap \overline{H(\hat{y})} = \emptyset$.

(5) $\overline{H(\hat{x})}$ is open.

(6) Conclusion.

In order to prove (1), let A be a dense countable subset of X and let $\hat{x} = (x_1, \dots, x_n) \in F^n(X)$. Then $B = A \cup \{x_1, \dots, x_n\}$ is also a dense countable subset of X and hence, there exists a homeomorphism h of X onto itself such that $h(B) = A$. However this implies that $h(\hat{x}) \in F^n(A)$ which is countable. Therefore every orbit intersects the countable set $F^n(A)$ and so there are only countably many orbits.

In order to prove (2), let $H(\hat{x})$ be a first category orbit. Then by (3.1) there exists a dense countable subset A of X such that $H(\hat{x}) \cap F^n(A)$ is empty. This is a contradiction to what occurred in (1).

In order to prove (3), we should note that for each $\hat{x} \in F^n(X)$, $H(\hat{x}) \cap \overline{H(\hat{x})}^{\circ} \neq \emptyset$. Therefore, let $\hat{y} \in H(\hat{x}) \cap \overline{H(\hat{x})}^{\circ}$ and let $\hat{z}$ be any point of $H(\hat{x})$. Then there exist homeomorphisms h and $g \in H$ such that $h(\hat{x}) = \hat{y}$ and $g(\hat{x}) = \hat{z}$. However, we then have that $\hat{z} = gh^{-1}(\hat{y}) \in gh^{-1}\overline{H(\hat{x})}^{\circ} = \overline{gh^{-1}H(\hat{x})}^{\circ} = \overline{H(\hat{x})}^{\circ}$. (The first equality follows since gh^{-1} acts like a homeomorphism of $F_n(X)$ onto itself.) Therefore, we have that $H(\hat{x}) \subset \overline{H(\hat{x})}^{\circ}$ as desired.

In order to prove (4), we will show if $H(\hat{x}) \cap \overline{H(\hat{y})} \neq \emptyset$, then $\overline{H(\hat{x})} = \overline{H(\hat{y})}$. First, let $\hat{z} \in H(\hat{x}) \cap \overline{H(\hat{y})}$ and let $\hat{w}$ be any point of $H(\hat{x})$. Then there exist homeomorphisms $h, g, h_i \in H$ such that $h(\hat{x}) = \hat{z}$, $g(\hat{x}) = \hat{w}$ and $\{h_i(\hat{y})\}$ converges to $\hat{z}$. Therefore $\{gh^{-1}h_i(\hat{y})\}$ converges to $\hat{w}$ and hence $\hat{w} \in \overline{H(\hat{y})}$. Therefore, $H(\hat{x}) \subset \overline{H(\hat{y})}$ and so $\overline{H(\hat{x})} \subset \overline{H(\hat{y})}$.

I claim $H(\hat{y}) \cap \overline{H(\hat{x})} \neq \emptyset$ since if $H(\hat{y}) \cap \overline{H(\hat{x})} = \emptyset$, then $H(\hat{y}) \cap \overline{H(\hat{x})}^{\circ} = \emptyset$ which implies that $\overline{H(\hat{y})} \cap \overline{H(\hat{x})}^{\circ} = \emptyset$. However this is impossible since $\overline{H(\hat{x})}^{\circ} \subset \overline{H(\hat{x})} \subset \overline{H(\hat{y})}$. Therefore, $H(\hat{y}) \cap \overline{H(\hat{x})} \neq \emptyset$ so as in the first paragraph of (4) we have $\overline{H(\hat{y})} \subset \overline{H(\hat{x})}$ and therefore $\overline{H(\hat{x})} = \overline{H(\hat{y})}$.

From the above we finish the proof of (4) by noting that if $\hat{z} \in \overline{H(\hat{x})} \cap \overline{H(\hat{y})}$, then $\overline{H(\hat{x})} = \overline{H(\hat{z})} = \overline{H(\hat{y})}$.

We obtain (5) from (4) and (3) by noting that for $\hat{x} \in F^n(X)$,

$$\overline{H(\hat{x})} = \bigcup \{H(\hat{y}) \mid \hat{y} \in \overline{H(\hat{x})}\} \subset \bigcup \{\overline{H(\hat{y})}^{\circ} \mid \hat{y} \in \overline{H(\hat{x})}\}$$
$$\subset \bigcup \{\overline{H(\hat{y})} \mid \hat{y} \in \overline{H(\hat{x})}\} = \overline{H(\hat{x})}.$$

We get the conclusion (6) by noting that $F_n(X)$ is the union of all the orbits and hence, of all the orbit closures. Therefore by (5) either $F^n(X)$ is not connected or every orbit is dense. By (3.9) of [4] it follows that $F^n(X)$ is connected and hence, every orbit is dense. Using (2) and (2.6) we have that every orbit is a dense G_δ subset of $F_n(X)$. However, since there are at most countably many orbits we get that there is exactly one orbit as desired.

The next theorem gives a converse to (3.2).

(3.3) THEOREM. *If X is a strongly n-homogeneous locally compact metric space for all n, then X is countable dense homogeneous.*

Proof. Let A and B be countable dense subsets of X, let d be a complete metric on H, and let $N(h, \varepsilon)$ denote $\{h \in H|\ d(g, h) < \varepsilon\}$. Let $c_1 = a_1$ and $d_1 = b_1$. Since X is homogeneous, there exists a homeomorphism h_1 of X onto itself such that $h_1(c_1) = d_1$. Let $c_2 = a_2$ and note by (2.8), $N(h_1, 1)(c_1, c_2)$ is open in $F^2(X)$ and it contains $(d_1, h_1(c_2))$. Let d_2 be the first b_i such that $b_i \neq d_1$ and $(d_1, b_i) \in N(h_1, 1)(c_1, c_2)$. Therefore, there exists a homeomorphism $h_2 \in N(h_1, 1)$ such that $h_2(c_1, c_2) = (d_1, d_2)$ and there exists $\varepsilon_2 > 0$ such that $\varepsilon_2 < \frac{1}{2}$ and $\overline{N(h_2, \varepsilon_2)} \subset N(h_1, 1)$. Let d_3 be the first b_i such that $b_i \neq d_1$ and $b_i \neq d_2$. Also note that since H is a topological group, there exists a δ_2 such that if $g \in N(h_2^{-1}, \delta_2)$ then $g^{-1} \in N(h_2, \varepsilon_2)$. Hence, using (2.8) again we have that $N(h_2^{-1}, \delta_2)(d_1, d_2, d_3)$ is an open subset of $F^3(X)$ and contains $(c_1, c_2, h^{-1}(d_3))$. Therefore, let c_3 be the first element of A such that $d \neq c_1$ and $d \neq c_2$ and $(c_1, c_2, c_3) \in N(h_2^{-1}, \delta_2)(d_1, d_2, d_3)$. Therefore there exists $g_3 \in N(h_2^{-1}, \delta_2)$ such that $g_3(d_1, d_2, d_3) = (c_1, c_2, c_3)$. Let $h_3 = g_3^{-1}$. Then by the choice of δ_2, $h_3 \in N(h_2, \varepsilon_2)$ and $h_3(c_1, c_2, c_3) = (d_1, d_2, d_3)$. We then get $\varepsilon_3 > 0$ such that $\varepsilon_3 < \frac{1}{3}$ and $\overline{N(h_3, \varepsilon_3)} \subset N(h_2, \varepsilon_2)$. We can continue this process inductively, however, since it is a standard type of argument, we will just note that we get a sequence of homeomorphisms $\{h_i\}$ and numbers ε_i such that $0 < \varepsilon_i < 1/i$ and $\overline{N(h_{i+1}, \varepsilon_{i+1})} \subset N(h_i, \varepsilon_i)$. Since d is a complete metric, there exists a homeomorphism $h \in \bigcap N(h_i, \varepsilon_i)$ and by construction $h(A) = B$.

The above proof will also work for the following:

(3.4) THEOREM. *If X is a locally compact separable metric space which strongly n-homogeneous for all n and U is a dense open subset of X, then U is countable dense homogeneous.*

Proof. All one needs to do to modify the proof of (3.3) is note that $F^{n+1}(U)$ is dense and open in $F^{n+1}(X)$, and hence each time a point of $N(h_n, \varepsilon_n)(c_1, \dots, c_{n+1})$ must be chosen, it could be found in $F^{n+1}(U)$.

The last theorem together with (3.2) gives a partial answer to the question asked by B. Fitzpatrick and H. Cook whether countable dense homogeneity is G-hereditary.

(3.5) THEOREM. *Let X be a locally compact countable dense homogeneous separable metric space such that no finite set separates X. Let U be a dense open subset of X. Then U is countable dense homogeneous and n-homogeneous for all n.*

Proof. By (3.2) X is n-homogeneous for all n and by (3.3) U is countably dense homogeneous and by (3.2) again U is n-homogeneous for all n.

(3.6) COROLLARY. *Let X be a connected compact metric space other than the circle. Then X is n-homogeneous for all n iff X is countably dense homogeneous.*

Proof. This follows from (3.2) and (3.3) of this paper and (3.11) of [4].

4. Proof of (3.1). First we need the following notation

(4.1) NOTATION. (a) Define ${}_nP_i\colon F^{n+1}(X) \to F^n(X)$ by

$${}_nP_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

(b) If $b \in X$ and $S \subset F^m(X)$ $(m > 1)$ let $C(i, b, S) = \{(x_1, \dots, x_m) \in S|\ x_i = b\}$ and let $D(i, b, S) = {}_{m-1}P_iC(i, b, S)$ and let $D(b, S) = \bigcup_i D(i, b, S)$ and let

$$D(b_1, \dots, b_k, S) = D\Big(b_1\Big(D\Big(b_2\big(D \dots (D(b_k, S))\big)\Big)\Big)\Big).$$

Essentially, $C(i, b, S)$ is a section of S, $D(i, b, S)$ is the projection of this section, $D(b, S)$ is the union of all these projections and $D(b_1, \dots, b_k, S)$ does this process k times, each time lowering the "dimension".

(4.2) LEMMA. *Let X be a locally compact separable metric space and let A be a first category subset of $F^n(X)$ $(n > 1)$. Then $B = \{x \in X|\ D(x, A)$ is not first category$\}$ is first category.*

Proof. Assume that B is second category. Since A is first category we could write $A = \bigcup A_i$ such that $\overline{A_i}^{\circ} = \emptyset$. If $b \in B$, then $D(b, A) = \bigcup_{i=1}^{n} D(i, b, A)$ $= \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{n} D(i, b, A_j)$ is second category, hence there exists i_b, j_b such that $\overline{D(i_b, b, A_{j_b})}^{\circ} \neq \emptyset$. Let $\{U_i\}$ be a countable base for $F^n(X)$ and let U_b be the first element of this base which is contained in $\overline{D(i_b, b, A_{j_b})}^{\circ}$. Let $B_i = \{b \in B|\ U_b = U_i\}$. Then $\bigcup B_i = B$ and since B is second category, there exists N such that B_N is second category. Let $B_N(i, j) = \{b \in B_N|\ i_b = i$ and $j_b = j\}$. Again $\bigcup B_N(i, j) = B_N$, so there exists I, J such that $B_N(I, J)$ is second category. In other words, if $b \in B_N(I, J)$, then $\overline{D(I, b, A_J)}^{\circ} \supset U_N$ which contains a basic open set of the form $V_1 \times \dots \times V_{I-1} \times V_I \times \dots \times V_N$ where each of the V_j's are open in X. It is easily seen that this implies that $V_1 \times \dots \times V_{I-1} \times B_N(I, J) \times V_{J+1} \times \dots \times V_n \subset \bar{A}_J$ which in turn implies that $V_1 \times \dots \times V_{I-1} \times \overline{B_N(I, J)} \times V_{J+1} \times \dots \times V_n \subset \bar{A}_J$. This is a contradiction since $V_1 \times \dots \times V_{I-1} \times \overline{B_N(I, J)} \times V_{I+1} \times \dots \times V_n$ has non-empty interior.

(4.3) LEMMA. *Let X be a locally compact space. Let A_i, $i = 2, \dots, n$ be first category subsets of $F^i(X)$ and let B be a second category subset of X. Then there exists $b \in B$ such that $D(b, A_i)$ is first category for each $i = 2, \dots, n$.*

Proof. Let $C_i = \{x \in X|\ D(x, A_i)$ is not first category$\}$. By (4.2), C_i is first category and hence $C = \bigcup C_i$ is first category. Since B is second category, $B \cap \tilde{C} \neq \emptyset$. If $b \in B \cap \tilde{C}$, then $D(b, A_i)$ is a first category subset of $F^i(X)$ as desired.

Finally, we have

(3.1) LEMMA. *Let X be a locally compact separable metric space and let A be a first category subset of $F^n(X)$. Then there exists a dense countable subset B of X such that $F^n(B) \cap A = \emptyset$.*

Proof. Let $\{U_i\}$ be a countable basis for X. We will define B inductively. The first $n-1$ points of B will be chosen by induction as follows. By (4.3) choose $b_1 \in U_1$ such that $D(b_1, A)$ is a first category subset of $F^{n-1}(X)$. If $s<i-1$, assume that $b_1, \ldots, b_s$ have been chosen so that $b_j \in U_j$ and $D(b_j, A)$ is a first category subset of $F^{n-1}(X)$, $D(b_j, b_k, A)$ is a first category subset of $F^{n-2}(X)$ for $k<j$ and in general if $k_1, k_2, \ldots, k_l$, $l<j$ is a decreasing sequence of natural numbers with $k_1<j$, then $D(b_j, b_{k_1}, \ldots, b_{k_l}, A)$ is a first category subset of $F^{n-(l+1)}$. For $k = 1, \ldots, p$, let,

$$E_k = \bigcup \{D(b_{j_1}, \ldots, b_{jk}, A) |\ p \geqslant j_1 > j_2 \ldots > j_k \geqslant 1\} \subset F^{n-k}(X).$$

Note that by the above assumptions E_k is a first category subset of $F^{n-k}(X)$. Hence, by (4.3) we can chose $b_{p+1} \in U_{p+1}$ such that $D(b_{p+1}, A)$ and $D(b_{p+1}, E_k)$ are first category for $k = 1, \ldots, p$. It is easily verified that $\{b_1, \ldots, b_{p+1}\}$ satisfy the inductive hypothesis.

In order to define b_n we let the E_i's be as above, however, note that E_{n-1} is a first category subset of X. Therefore, using (4.3) again choose $b_n \in U_n - E_{n-1}$ such that $D(b_n, A)$ is a first category subset of $F^{n-1}(X)$ and $D(b_n, E_k)$ is a first category subset of $F^{n-(k+1)}(X)$ for $k<n-1$. Now we proceed as before if we assume $b_1, \ldots, b_p$ have been defined and $p>n$, we let E_i be defined as before but only for $i = 1, \ldots, n-1$ and by (4.3) we choose $b_{p+1} \in U_{p+1} - E_{n-1}$ so that $D(b_{p+1}, A_i)$ and $D(b_{p+1}, E_i)$ are first category for $i = 1, \ldots, n-2$.

Let $B = \{b_i\}_{i=1}^{\infty}$. It is clear from the construction that B is a dense countable subset of X. All that remains to be shown is that $F^n(B) \cap A = \emptyset$. Therefore assume that $(c_1, \ldots, c_n) \in F^n(B) \cap A$. For the sake of simplicity, we will assume that $(c_1, \ldots, c_n) = (b_{k_1}, \ldots, b_{k_n})$ where $k_1<k_2<\ldots<k_n$. This implies that

$$b_{k_n} \in D(b_{k_1}, \ldots, b_{k_{n-1}}, A) \subset E_{n-1}.$$

But b_{k_n} was chosen in $U_{k_n} - E_{n-1}$ and hence we have a contradiction.

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A counter-example concerning quasi-homeomorphisms of compacta

by

H. Patkowska (Warszawa)

Abstract. Two metric compacta X and Y are said to be *quasi-homeomorphic* if for every $\varepsilon>0$ there are two ε-mappings: f mapping X onto Y and g mapping Y onto X. A locally connected continuum X belongs to the class α if there is a $\delta>0$ such that no simple closed curve $C \subset X$ with $\operatorname{diam} C<\delta$ is a retract of X. We prove in the paper that there are two quasi-homeomorphic, 2-dimensional locally connected continua X and Y such that $X \in \alpha$ and $Y \notin \alpha$.

1. Introduction. Let X be a (metric) compactum and let Y be a topological space. A map $f\colon X \to Y$ is said to be an *ε-mapping* if $\operatorname{diam}(f^{-1}(y))<\varepsilon$ for every $y \in f(X)$. X is said to be *Y-like* if for every $\varepsilon>0$ there is an ε-mapping of X onto Y. Two compacta X and Y are said to be *quasi-homeomorphic* if X is Y-like and Y is X-like.

In a sequence of papers (cf. [3], [4], [5]) concerned with these notions we considered the following class α:

DEFINITION 1. A locally connected compactum X *belong to the class* α if there is an $\varepsilon>0$ such that no simple closed curve $C \subset X$ with $\operatorname{diam} C<\varepsilon$ is a retract of X.

In [5] we proved the following theorem: *Let Y be a compact semi-*lc_1 *space in the homological sense, i.e. such that $i_*(H_1(A)) = 0$ for each compact subset A of Y with diameter less than a given $\delta>0$, where $H_1(A)$ is the first Čech homology group of A with integer coefficients and $i\colon A \to Y$ is the inclusion map. Then each locally connected compactum X which is Y-like belongs to the class α.*

In the same paper we raised the question whether the property α is a quasi-homeomorphism invariant. In the present paper we shall prove that this is not the case, i.e. that there exist two quasi-homeomorphic locally connected continua X and Y such that $X \in \alpha$ and $Y \notin \alpha$.

Given a compactum A, $H_n(A)$ will denote the nth Čech homology group of A with integer coefficients. It is well known (cf. [2], p. 6) that, if A is a retract of X and $i\colon A \to X$ is the inclusion map, then the group $i_*(H_n(A))$ is a direct summand of the group $H_n(X)$. If C is a simple closed curve, then it follows from the Bru-