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The axiom of choice for linearly ordered families

by

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Abstract. We study the statement (1) Any linearly ordered family of non-empty sets has a choice function. (1) implies AC in ZF but not in ZF without foundation. We show that a weaker form of (1), namely "every family of non-empty sets indexed by $P(\omega)$ has a choice function", does not imply AC even in ZF; in fact it is consistent with the existence of a partition of $P(\omega)$ without a choice function. We study further properties of the model used to prove this, and also of Feferman's model.

§ 1. The axiom of choice for linearly ordered families is the following statement.

(1) Any linearly ordered family of non-empty sets has a choice function.

We prove in this paper that in the presence of the axiom of foundation, (1) implies AC, the axiom of choice. However this is false in set theory without the axiom of foundation. (1)+AC is therefore an example of what Pincus [8, pp. 740–741] calls a "non-transferable" consistency, i.e. it holds in an appropriate Fraenkel–Mostowski model (where the axiom of foundation may be violated) but not in any model of Zermelo–Fraenkel (ZF) set theory.

Our interest in this proposition was prompted by a question of A. Zalc. She asked whether (2) implies (3), where (2) and (3) are as follows.

(2) Every family of non-empty sets indexed by $P(\omega)$ has a choice function.

(3) Every partition of $P(\omega)$ into non-empty subsets has a choice function.

That the answer is "no" follows from consideration of one of the models $\mathfrak{N}_1$ of [12], of "Feferman type". We thought at first that it would be enough to consider Feferman's original model, $\mathfrak{N}$ [2]. However it turns out that both (2) and (3) are false there, the reason being connected with the fact that (1)→AC in ZF. (2) follows from (1), of course, but not conversely, as we shall show.

We include further information about $\mathfrak{N}$ (and similar results hold for the other models discussed in [12]). We show that for each ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}, 2^{\aleph_0}$, and moreover that for any set X of $\mathfrak{N}$ which can be linearly ordered, there is an α such that $|X| \leqslant 2^{\aleph_\alpha}$. This is a "Kinna–Wagner ordering principle" for orderable sets. In fact the proof will show that this conclusion holds for any set such that

$$[X]^2 = \{x \subseteq X\colon |x| = 2\}$$

has a choice function. This state of affairs can be paraphrased as "sets of reals in $\mathfrak{N}$ are well-behaved but partitions of reals are not". We conclude by showing that for any infinite set X in $\mathfrak{N}$ there is a mapping from X onto $2\times X$.

§ 2. THEOREM 1. *In* ZF *set theory, the axiom of choice for linearly ordered families implies the full axiom of choice.*

Proof. In fact we only need to assume the axiom of choice for families indexed by a set of sets of ordinals. By Rubin and Rubin [9, p. 77] it is enough to show that $P(\alpha)$ can be well-ordered for each ordinal α.

Let X be the set of all 1-1 well-ordered sequences of members of $P(\alpha)$. For each $x\in X$, A_x is the set of members of $P(\alpha)$ not in the domain of x. Suppose that $P(\alpha)$ cannot be well-ordered. Then A_x is non-empty for each $x\in X$. Let $\varkappa$ be the least ordinal which cannot be mapped 1-1 into $P(\alpha)$. It is clear that X can be mapped 1-1 into a subset of $P(\alpha\times\varkappa)$, so by applying the given version of the axiom of choice, $\{A_x\colon x\in X\}$ has a choice function. This provides a well-ordering of $P(\alpha)$, giving the desired contradiction.

THEOREM 2. *If* FM(= ZF *without the axiom of foundation*) *is consistent, then so is* FM+¬AC+*the axiom of choice for linearly ordered families.*

Proof. We construct a Fraenkel–Mostowski model as follows. Suppose that $\mathfrak{M}$ is a model of FM+AC+ "there are $\aleph_1$ urelemente". Here we can take "urelemente" to be sets x such that $\{x\} = x$, or alternatively, objects which are not sets but which can be members of sets. We adopt the latter approach. The reader is referred to Felgner's book [3] for more details.

U is the set of urelemente, and G is the group of all permutations of U. For each $\sigma\in G$, the action of σ on arbitrary $x\in\mathfrak{M}$ is defined in the natural way, i.e. $\sigma x = \{\sigma y\colon y\in x\}$. We then let

$$H(x) = \{\sigma\in G\colon \sigma x = x\} \quad\text{and}\quad K(x) = \{\sigma\in G\colon (\forall y\in x)\sigma y = y\}.$$

$\mathfrak{F}$ is the filter of subgroups of G generated by $\{K(A)\colon A\subseteq U, |A|\leqslant\aleph_0\}$. Since $K(A\cup B) = K(A)\cap K(B)$, $\mathfrak{F}$ is actually the set of all subgroups of G containing some such $K(A)$. $\mathfrak{N}$ is the Fraenkel–Mostowski model determined by U, G, and $\mathfrak{F}$. Thus $\mathfrak{N} = \{x\in\mathfrak{M}\colon x\subseteq\mathfrak{N}\wedge H(x)\in\mathfrak{F}\}$. ($\mathfrak{N}$ is used for this model only in § 2. Later on we use $\mathfrak{N}$ for Feferman's model).

That $\mathfrak{N}$ is a model of FM is proved in [3, pp. 52–55]. Certainly $U\in\mathfrak{N}$, and in [13, Theorem 5] we showed that the axiom of choice is false in $\mathfrak{N}$. In fact U is an uncountable set which is not the disjoint union of two uncountable sets. It remains to show that the axiom of choice for linearly ordered families is true in $\mathfrak{N}$.

Let $\langle X, \leqslant\rangle$ be a linearly ordered set and $\{A_x\colon x\in X\}$ a family of non-empty sets, both in $\mathfrak{N}$. Then for some countable subset B of U, $H(\langle X,\leqslant\rangle)\cap H(\{\langle x, A_x\rangle\colon x\in X\})\supseteq K(B)$. We show that in fact $K(X)\supseteq K(B)$, i.e. $H(x)\supseteq K(B)$ for every $x\in X$. If not, $H(x)\cap K(B)$ is a proper subgroup of $K(B)$. It is easy to see that $K(B)$ is generated by the elements of finite order. In fact it is generated by the elements of order 2. Hence there is $\sigma\in K(B)-H(x)$ of order 2. As $\sigma x\neq x$, and $\sigma x\in X$ (as $K(B)\subseteq H(\langle X,\leqslant\rangle)$), $x<\sigma x$ (or $\sigma x<x$). Since σ preserved $<$ this gives $\sigma x<x$ ($x<\sigma x$ respectively), a contradiction.

It is now clear that $K(B)$ fixes any well-ordering of X lying in $\mathfrak{M}$, since it fixes X pointwise, so X can be well-ordered in $\mathfrak{N}$. In fact this argument shows that any set X of $\mathfrak{N}$ such that $[X]^2 = \{x\subseteq X\colon |x| = 2\}$ has a choice function can be well-ordered.

Now let C be a countably infinite subset of $U-B$. We claim that every A_x has a member $f(x)$ such that $H(f(x))\supseteq K(B\cup C)$. The choice of the function f is made in $\mathfrak{M}$, of course, but it follows at once that $H(f)\supseteq K(B\cup C)$ and so $f\in\mathfrak{N}$ as desired. For let $y\in A_x$ be arbitrary, and let $H(y)\supseteq K(B\cup C_1)$, where we may take C_1 to be countably infinite and disjoint from B. Let σ be a member of $K(B)$ mapping C_1 onto C. As σ fixes x and $\{\langle x, A_x\rangle\colon x\in X\}$ it also fixes A_x, and hence $\sigma y\in A_x$. Let $f(x) = \sigma y$. Then

$$H(f(x)) = H(\sigma y) = \sigma H(y)\sigma^{-1} \supseteq \sigma K(B\cup C_1)\sigma^{-1} = K(B\cup C)$$

as desired.

This proof is similar to Solovay's proof that in Feferman's model, the axiom of choice for well-ordered families holds. We could use a countable set of urelemente if we wished (obtaining in fact an elementarily equivalent model), but taking $|U| = \aleph_1$ makes the argument slightly simpler. Two of the steps of the proof of Theorem 2 can be carried out in Feferman's model, i.e.

(i) Any set X such that $[X]^2$ has a choice function can be mapped 1-1 into $P(\alpha)$, some ordinal α.

(ii) Any well-ordered family of non-empty sets has a choice function.

It is the linking step

(iii) For any ordinal α, $P(\alpha)$ can be well-ordered, which fails there.

§ 3. We consider two models "of Feferman type", $\mathfrak{N}$ and $\mathfrak{N}_1$, $\mathfrak{N}$ is Feferman's model [2] and $\mathfrak{N}_1$ is one of its modifications described for example in [12]. Let $\mathfrak{M}$ be a countable transitive model of ZFC+V = L. We take as set of conditions P or P_1 the set of all maps in $\mathfrak{M}$ from a finite subset of $\omega\times\omega$ ($\omega_2\times\omega$ respectively) into 2, partially ordered by extension. If $\mathfrak{F}$ ($\mathfrak{F}_1$) is an $\mathfrak{M}$-generic subset of P (P_1) we obtain the Cohen extension $\mathfrak{M}[\mathfrak{F}]$ (or $\mathfrak{M}[\mathfrak{F}_1]$), and $\mathfrak{N}(\mathfrak{N}_1)$ is a submodel of this. Let $F = \bigcup\mathfrak{F}$, $F_1 = \bigcup\mathfrak{F}_1$. Then F, F_1 are maps from $\omega\times\omega$, $\omega_2\times\omega$ into 2. For each $n\in\omega$, $\alpha\in\omega_2$, let $a_n(l) = F(n, l)$, $a_\alpha(l) = F_1(\alpha, l)$. Then $\mathfrak{N}$ is the submodel of $\mathfrak{M}[\mathfrak{F}]$ consisting of all its members which are hereditarily ordinal definable over $\{a_n\colon n\in\omega\}$, i.e. using only finitely many parameters from this set, and $\mathfrak{N}_1$ is the submodel of $\mathfrak{M}[\mathfrak{F}_1]$ consisting of its members which are hereditarily ordinal definable over $\{\langle a_\beta\colon \beta<\alpha\rangle\colon \alpha<\omega_2\}$.

Now let I denote the ideal of finite subsets of ω, $P(\omega)/I$ the corresponding factor of $P(\omega)$, and $[A]$ the equivalence class of $A\in P(\omega)$ modulo I. The following is implicit in [2]. See also the account of Feferman's model given in [3, pp. 160–166].

THEOREM 3. *In $\mathfrak{N}$ and $\mathfrak{N}_1$ there is no choice function for $\{\{[A], [\omega - A]\}: A \in P(\omega)\}$. Hence $P(\omega)$ has no non-principal prime ideal, the axiom of choice for sets of pairs fails, and there is a partition of $P(\omega)$ into non-empty subsets with no choice function.*

Proof. We just observe how the final statement follows. Let Π be the partition given by $\Pi = \{[A] \cup [\omega - A]: A \in P(\omega)\}$. Thus each member of Π is the union of two complementary members of $P(\omega)/I$. If Π has a choice function then so does $\{\{[A], [\omega - A]\}: A \in P(\omega)\}$.

It follows from the considerations of § 2 and the fact that in $\mathfrak{N}$ any set is ordinal definable from a real that not every family of non-empty sets indexed by $P(\omega)$ has a choice function. In fact it is easy to give a specific counter-example, which turns out to be the most important case. We say that $x \in 2^{\omega}$ is *Cohen $\mathfrak{M}$-generic* if $\{x \restriction n: n \in \omega\}$ is an $\mathfrak{M}$-generic subset of $2^{<\omega}$. For each $a \in 2^{\omega}$, let $G_a = \{x \in 2^{\omega}: x$ is Cohen $\mathfrak{M}[a]$-generic$\}$. Then $X = \{G_a: a \in 2^{\omega}\}$ is a family of non-empty sets (in $\mathfrak{N}$) indexed by 2^{ω}. If f were a choice function for X in $\mathfrak{N}$, we should have f $\mathfrak{M}[a]$-definable in $\mathfrak{M}[\mathfrak{F}]$ for some real a. But then $f(a)$ would also be $\mathfrak{M}[a]$-definable in $\mathfrak{M}[\mathfrak{F}]$, and hence by arguments of Levy [5, 6] would lie in $\mathfrak{M}[a]$, contrary to $f(a)$ $\mathfrak{M}[a]$-generic.

LEMMA. *In $\mathfrak{N}_1$, $X = \{G_a: a \in 2^{\omega}\}$ has a choice function.*

Proof. We use the fact that $\langle a_\alpha: \alpha < \omega_1 \rangle \in \mathfrak{N}_1$. Let $f(a) = a_\alpha$ for the least α such that $a_\alpha \in G_a$. We just have to show that there is such an α. But this is clear, for if x is any countable set of ordinals in $\mathfrak{M}[\mathfrak{F}_1]$, by Levy [6, Lemma 4] there is a countable subset A of ω_2 such that $x \in \mathfrak{M}[\mathfrak{F}_1 \restriction A]$ where $\mathfrak{F}_1 \restriction A = \mathfrak{F}_1 \cap (A \times \omega \times 2)$.

THEOREM 4. *In $\mathfrak{N}_1$ any family of non-empty sets indexed by $P(\omega)$ has a choice function.*

Proof. Let $X = \{A_x: x \in 2^{\omega}\}$ be the given family. Then for some $\alpha_0 < \omega_2$, X is ordinal definable in $\mathfrak{M}[\mathfrak{F}_1]$ from $\langle a_\alpha: \alpha < \alpha_0 \rangle$. Without loss of generality we suppose that $\alpha_0 = 0$. Given x we show how to choose a member of A_x. It is enough to show that A_x has a member a which is ordinal definable from $\langle x, f(x) \rangle$, where f is the function given by the lemma, since we may then choose the first such in the canonical $\langle x, f(x) \rangle$ — ordinal definable well-ordering of sets ordinal definable from $\langle x, f(x) \rangle$. But A_x has a member b which is ordinal definable from $\langle x, y \rangle$ for some $\mathfrak{M}[x]$-generic y. In fact for any $b \in A_x$ there is such a y (applying Levy's lemma again and dovetailing the countably many $\mathfrak{M}[x]$-generic reals resulting). This must be forced by a finite initial segment of y, and hence by altering y to lie in $\mathfrak{M}[x][f(x)]$ we can find the desired a.

COROLLARY. *If* ZF *is consistent, then so is*

ZF + *every family of non-empty sets indexed by $P(\omega)$ has a choice function.*
+ *not every partition of $P(\omega)$ into non-empty subsets has a choice function,*

§ 4. This section contains results about the cardinals of $\mathfrak{N}$, Feferman's model, defined in § 3. These could be extended to the other models of [12]. Firstly we prove a result about power sets.

THEOREM 5. *In $\mathfrak{N}$, $2^{\aleph_\alpha} = \aleph_{\alpha+1} \cdot 2^{\aleph_0}$.*

Proof. That $2^{\aleph_\alpha} \geqslant \aleph_{\alpha+1} \cdot 2^{\aleph_0}$ is clear, so we just have to establish the reverse inequality. Let A be a subset of ω_α in $\mathfrak{N}$. Then for some $a \in 2^{\omega}$, A is ordinal definable from a in $\mathfrak{M}[\mathfrak{F}]$, and hence by Levy [5], $A \in \mathfrak{M}[a]$. We show that a may be taken to lie in $\mathfrak{M}[A]$. When this is established the inequality $2^{\aleph_\alpha} \leqslant \aleph_{\alpha+1} \cdot 2^{\aleph_0}$ follows thus. Each $A \in P(\omega_\alpha)$ is mapped to $\langle \beta, a \rangle$, where a is the first member of $\mathfrak{M}[A] \cap 2^{\aleph_0}$ in some $\mathfrak{M}[A]$-definable well-ordering of $\mathfrak{M}[A]$ such that $\mathfrak{M}[a] = \mathfrak{M}[A]$, and A is the βth member of the canonical $\mathfrak{M}[a]$-definable well-ordering of $\mathfrak{M}[a] \cap P(\omega_\alpha)$ of order-type $\omega_{\alpha+1}$.

We may suppose that a is Cohen $\mathfrak{M}$-generic. Then as $A \in \mathfrak{M}[a]$ there is a label $\underline{A}$ for A in this extension. Let b be the set of all $\sigma \in 2^{<\omega}$ which are compatible with the interpretation of $\underline{A}$ as A. Thus

$$b = \{\sigma \in 2^{<\omega}: (\forall \beta \in \omega_\alpha)(\sigma \Vdash \beta \in \underline{A} \to \beta \in A \wedge \sigma \Vdash \beta \notin \underline{A} \to \beta \notin A)\}.$$

Then $b \in \mathfrak{M}[A]$ and $A = \{\beta: (\exists \sigma \in b)\sigma \Vdash \beta \in \underline{A}\}$, giving $A \in \mathfrak{M}[b]$. This shows that $\mathfrak{M}[A] = \mathfrak{M}[b]$, and of course b can be coded as a member of 2^{ω}.

THEOREM 6. *Let X be any set in $\mathfrak{N}$ such that $[X]^2$ has a choice function (in particular, any X which can be linearly ordered). Then $|X| \leqslant 2^{\aleph_\alpha}$, some α.*

Proof. Without loss of generality we suppose that X and a choice function f for $[X]^2$ are ordinal definable in $\mathfrak{M}[\mathfrak{F}]$. Fix a well-ordering in $\mathfrak{M}$ of the labels for members of $\mathfrak{M}[\mathfrak{F}]$. For each $x \in X$ we let $\theta(x) = \langle \underline{x}, A \rangle$, where $\underline{x}$ is the least label in this well-ordering such that x is denoted by $\underline{x}$ in some $\mathfrak{M}[\mathfrak{F}']$ for an $\mathfrak{M}$-generic subset $\mathfrak{F}'$ of P satisfying $\mathfrak{M}[\mathfrak{F}] = \mathfrak{M}[\mathfrak{F}']$, and A is the set of all members of P lying in some such $\mathfrak{F}'$. These are the members of P "compatible" with the interpretation of $\underline{x}$ as x.

It is clear that $\theta(x) \in \mathfrak{N}$ and θ is ordinal definable in $\mathfrak{M}[\mathfrak{F}]$, so also $\theta \in \mathfrak{N}$. The range of θ is not actually a set of sets of ordinals, but whenever $\theta(x) = \langle \underline{x}, A \rangle$, A is a subset of $\mathfrak{M}$, and this is clearly enough. What remains to be verified is that θ is 1-1.

Suppose that $\theta(x) = \theta(y)$ where $x \neq y$. Let $\theta(x) = \langle \underline{x}, A \rangle$. There must be $\mathfrak{M}$-generic subsets $\mathfrak{F}'$, $\mathfrak{F}''$ of P such that $\mathfrak{M}[\mathfrak{F}] = \mathfrak{M}[\mathfrak{F}'] = \mathfrak{M}[\mathfrak{F}'']$ and x, y are the denotations of $\underline{x}$ in $\mathfrak{M}[\mathfrak{F}']$ and $\mathfrak{M}[\mathfrak{F}'']$ respectively. Now we work with the complete Boolean algebra $\boldsymbol{B}$ associated with P. There is an automorphism π of $\boldsymbol{B}$ in $\mathfrak{M}$ taking $\mathfrak{F}'$ to $\mathfrak{F}''$. To see this, let $\underline{F}$ be a label for $\mathfrak{F}'$ in $\mathfrak{M}[\mathfrak{F}'']$, and let $\pi(b) = [\![b \in \underline{F}]\!]$. Then π is in $\mathfrak{M}$ a complete homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{B}$. By adjusting the choice of $\underline{F}$ it can be made 1-1, i.e. an automorphism. Clearly π maps $\mathfrak{F}'$ to $\mathfrak{F}''$.

Let G be the group of automorphisms of $\boldsymbol{B}$ in $\mathfrak{M}$ which fix $\theta(x)$ (i.e. those which fix A). Thus $G \subseteq \mathfrak{M}$, but we do not necessarily have $G \in \mathfrak{M}$. A is a subtree of P (regarding P as, say, the Cantor tree $2^{<\omega}$), and so it is easy to see that G is generated by the elements of order 2. Hence we may write $\pi = \pi_1 \pi_2 \ldots \pi_n$ where for each i, $\pi_i \theta(x) = \theta(x)$ and $\pi_i^2 = 1$. As $\underline{x}$ denotes x in $\mathfrak{M}[\mathfrak{F}']$, $\pi \underline{x}$ denotes y. As $x \neq y$ there is i such that $\pi_i\pi_{i+1}\ldots\pi_n\underset{\sim}{x}$ and $\pi_{i+1}\ldots\pi_n\underset{\sim}{x}$ denote different elements of $\mathfrak{M}[\mathfrak{F}']$. Also if z is the denotation of $\pi_{i+1}\ldots\pi_n\underset{\sim}{x}$ in $\mathfrak{M}[\mathfrak{F}']$, then $\theta(z) = \langle \underset{\sim}{x}, A\rangle$, since $\pi_j(A) = A$, all j.

This shows that there is an $\underset{\sim}{x}$ (equal to the previous $\pi_{i+1}\ldots\pi_n\underset{\sim}{x}$) and an automorphism π of B (equal to the previous π_i) such that $\pi^2 = 1$, $\underset{\sim}{x}$ and $\pi\underset{\sim}{x}$ denote different elements x, y say of $\mathfrak{M}[\mathfrak{F}']$, and $\pi\theta(x) = \theta(x)$. Since X is ordinal definable in $\mathfrak{M}[\mathfrak{F}]$, x, y are in X. Suppose without loss of generality that $f\{x, y\} = x$. Then there must be $\sigma \in \mathfrak{F}'$ forcing $f\{\underset{\sim}{x}, \pi\underset{\sim}{x}\} = \underset{\sim}{x}$. Hence $\pi\sigma \Vdash f\{\underset{\sim}{x}, \pi\underset{\sim}{x}\} = \pi\underset{\sim}{x}$. As $\pi\sigma \in \pi\mathfrak{F}'$, and y is denoted by $\underset{\sim}{x}$ in $\mathfrak{M}[\pi\mathfrak{F}']$, $\pi\sigma \in A'$, where $\theta(y) = \langle \underset{\sim}{x}, A'\rangle$. But $\theta(x) = \theta(y)$. Hence there is an $\mathfrak{M}$-generic subset $\mathfrak{F}'''$ of P containing $\pi\sigma$ such that x is the denotation of $\underset{\sim}{x}$ in $\mathfrak{M}[\mathfrak{F}''']$. Thus also y is the denotation of $\pi\underset{\sim}{x}$. This gives $f\{x, y\} = y$, which is a contradiction.

COROLLARY. *Let X be any set of $\mathfrak{N}$ such that $[X]^2$ has a choice function. Then either X is finite or for some α, β, $|X| = \aleph_\alpha + \aleph_\beta \cdot 2^{\aleph_0}$ or $|X| = \aleph_\alpha$.*

Proof. Suppose that $X \subseteq \omega_\alpha \times 2^\omega$. Let $A = \{\delta\colon |X \cap (\{\delta\} \times 2^\omega)| = 2^{\aleph_0}\}$. By [12, Theorem 3.2] any set of reals of $\mathfrak{N}$ has cardinal $2^{\aleph_0}$ or $\leqslant \aleph_1$. Thus if $A = \emptyset$, the result follows at once using the axiom of choice for well-ordered families. If $A \neq \emptyset$, let $\aleph_\beta = \max(\aleph_0, |A|)$.

THEOREM 7. *If X is an infinite set of $\mathfrak{N}$ there is a mapping from X onto $2 \times X$.*

Proof. As in the proof of Theorem 6 we may let $X = \bigcup_{\alpha<\varkappa} \{\alpha\} \times X_\alpha$, where each X_α is a partition of a subset of 2^ω. We show that there is a family $\{Y_\alpha\colon \alpha < \varkappa\}$ of subsets of 2^ω such that for each α, X_α and Y_α can each be mapped onto the other.

If $|X_\alpha| \leqslant \aleph_1$ we just let Y_α be a subset of 2^ω of the same cardinal as X_α. Otherwise a repetition of the methods of [12] shows that X_α has a subset of cardinality $2^{\aleph_0}$. So we let $Y_\alpha = 2^\omega$. Now by the axiom of choice for well-ordered families we may choose families of maps $\{f_\alpha\colon \alpha<\varkappa\}$, $\{g_\alpha\colon \alpha<\varkappa\}$ such that for each α, f_α maps X_α onto Y_α and g_α maps Y_α onto X_α. Letting $Y = \bigcup\{\{\alpha\} \times Y_\alpha\colon \alpha<\varkappa\}$ we obtain, by piecing together the f_α, g_α, maps f, g from X onto Y and Y onto X respectively. Clearly $|Y| \leqslant |\varkappa| \cdot 2^{\aleph_0}$, so the result follows from the previous corollary.

Sageev [10] and Halpern and Howard [4] have given independent (negative) solutions to the long-standing problem; does $\forall x(x \text{ infinite} \to x = 2x)$ imply the axiom of choice? We conjecture that $\forall x(x \text{ infinite} \to x = 2x)$ holds also in Feferman's model. What we have shown above would of course follow at once, and can be written $\mathfrak{N} \models \forall x(x \text{ infinite} \to 2x \leqslant^* x)$ in the notation of Lindenbaum and Tarski [7]. It is clearly enough to show that $\forall x(x \text{ infinite and } x \leqslant^* 2^{\aleph_0} \to x = 2x)$, where $x \leqslant^* 2^{\aleph_0}$ means that x is the cardinal of a partition of 2^ω, and a useful test case seems to be to take $x = |\Pi|$, where Π is the set of degrees of non-constructibility of members of 2^ω. We do not know whether $|\Pi| = 2|\Pi|$.

To conclude, we make a remark about Solovay's model $\mathfrak{N}_2$ [11] (in which every set of reals is Lebesgue measurable). There are many similarities between $\mathfrak{N}$ and $\mathfrak{N}_2$. In particular Theorems 5 and 6 are true in $\mathfrak{N}_2$. However the proofs of Theorem 7 and the corollary to Theorem 6 break down since the axiom of choice for well-ordered families is false there. In fact if X_α is the set of all real numbers which code a well-ordering of type α, $\{X_\alpha\colon \alpha<\omega_1\}$ has no choice function in $\mathfrak{N}_2$.

Added in proof. A considerable simplification may be made in the proof of Theorem 6 by using the following result previously unknown to the author: If $\mathfrak{M}[\mathfrak{F}'] = \mathfrak{M}[\mathfrak{F}'']$ then there is an automorphism π of B in $\mathfrak{M}$ taking $\mathfrak{F}'$ to $\mathfrak{F}''$ such that π^2 is the identity.

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