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DEPARTMENT OF MATHEMATICAL SCIENCES
NEW MEXICO STATE UNIVERSITY
Las Cruces, New Mexico

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS
CAMBRIDGE UNIVERSITY
Cambridge, England

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Exact sequences of pairs in commutative rings

by

R. Kiełpiński, D. Simson and **A. Tyc** (Toruń)

Abstract. Let R be a commutative ring with unit and let M be an R-module. We say that a pair (u, v), $u, v \in R$, is *M-exact* if the sequence $M \xrightarrow{u} M \xrightarrow{v} M \xrightarrow{u} M$ is exact. A sequence of pairs $(\boldsymbol{u}, \boldsymbol{v}) = ((u_1, v_1), \ldots, (u_n, v_n))$ is *M-exact* if the pair (u_i, v_i) is $M/(u_1, \ldots, u_{i-1})M$-exact for $i = 1, \ldots, n$.

In the paper we investigate the full subcategory $E_R(\boldsymbol{u}, \boldsymbol{v})$ of R-Mod consisting of all R-modules M such that $(\boldsymbol{u}, \boldsymbol{v})$ is M-exact and rings R such that $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$ and the Jacobson radical $J(R)$ of R is generated by elements $u_1, \ldots, u_n$.

Introduction. Section 1 contains definitions, examples and preliminary results. A homological characterization of modules from $E_R(\boldsymbol{u}, \boldsymbol{v})$ is given provided $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$.

In Section 2 we study conditions which ensure the projectivity or the injectivity of a module from the category $E_R(\boldsymbol{u}, \boldsymbol{v})$ under the assumption that $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$ and $J(R) = (u_1, \ldots, u_n)$. Our main result says that in this case $\mathrm{Inj}_R = E_R(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{Proj}_R$ iff R is artinian, or equivalently, iff R is noetherian and $E_R(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{Fl}_R$ where Fl_R, Inj_R and Proj_R denote the classes of all flat, injective and projective R-modules, respectively.

Section 3 is devoted to the study of local rings R whose maximal ideals are generated by elements $u_1, \ldots, u_n$ such that $(u_1, u_1^{t_1}), \ldots, (u_n, u_n^{t_n})$ is an R-exact sequence of pairs for some natural numbers $t_1, \ldots, t_n$. It is proved that such a ring is R always artinian of the length $(t_1+1)(t_2+1) \ldots (t_n+1)$ and that the associated graded algebra $\mathrm{gr}(R)$ is of the same type.

Throughout this paper R denotes a commutative ring with identity element and $J(R)$ is the Jacobson radical of R. If X is a subset of R and M is an R-module, we set $\mathrm{Ann}_M X = \{m \in M,\ Xm = 0\}$.

§ 1. Exact sequences of pairs and the category $E_R(\boldsymbol{u}, \boldsymbol{v})$.

DEFINITION 1.1 Let M be a module over a commutative ring R. A pair (u, v) of elements of R is *M-exact* if $uvM = 0$ and the left complex

$$M(u, v):\ \ldots \to M \xrightarrow{v} M \xrightarrow{u} M \xrightarrow{v} M \xrightarrow{u} M \to 0$$

is acyclic (i.e. $H_jM(u, v) = 0$ for $j = 1, 2, \ldots$). A sequence of pairs

$$(\boldsymbol{u}, \boldsymbol{v}) = ((u_1, v_1), \ldots, (u_n, v_n)), \quad u_i, v_i \in R,$$

is M-exact (or equivalently the module M is $(\boldsymbol{u}, \boldsymbol{v})$-exact) if (u_{i+1}, v_{i+1}) is M_i-exact for $i = 0, 1, \ldots, n-1$ where

$$M_0 = M \quad \text{and} \quad M_i = M/(u_1M+\ldots+u_iM) \quad \text{for} \quad i \geqslant 1.$$

$(\boldsymbol{u}, \boldsymbol{v})$ is said to be *exact* if it is R-exact.

EXAMPLES. 1. If $e_1, \ldots, e_n$ are orthogonal idempotents of a ring R such that $e_1+\ldots+e_n = 1$, then $(e_1, 1-e_1), \ldots, (e_n, 1-e_n)$ is an exact sequence of pairs in R.

2. If $a_1, \ldots, a_n$ is an h-regular sequence in the sense of [9] and if the height h_i of a_i is finite for each $i = 1, 2, \ldots, n$, then the sequence $(a_1, a_1^{h_1-1}), \ldots, (a_n, a_n^{h_n-1})$ is exact.

3. Let $R = k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_1Y_1, \ldots, X_nY_n)$ where k is a ring. It is not difficult to check that the sequence $(x_1, y_1), \ldots, (x_n, y_n)$ is exact where x_i and y_i are the residue classes of X_i and Y_i, respectively.

4. Let $T = \bigoplus_{j\in J} \{t_j\}$ be a direct sum of finite cyclic groups and let A be a commutative ring such that $mA = A$ if m is the order of an element of T. Moreover, let R be the group ring $A[T]$ of T with coefficients in A and let us consider elements

$$\varepsilon_j = \frac{1}{m_j}(1+t_j+\ldots+t_j^{m_j-1}), \quad \delta_j = 1-\varepsilon_j$$

where m_j is the order of t_j, $j \in J$. Then the sequence $(\delta_{j_1}, \varepsilon_{j_1}), \ldots, (\delta_{j_n}, \varepsilon_{j_n})$ is exact for any $j_1, \ldots, j_n \in J$ (see [1], p. 244).

For a given sequence $(\boldsymbol{u}, \boldsymbol{v})$ of pairs in R we define $E_R(\boldsymbol{u}, \boldsymbol{v})$ as the full subcategory of R-Mod consisting of all $(\boldsymbol{u}, \boldsymbol{v})$-exact R-modules. Obviously, $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$ iff $(\boldsymbol{u}, \boldsymbol{v})$ is R-exact.

PROPOSITION 1.2. *The category $E_R(\boldsymbol{u}, \boldsymbol{v})$ is closed under direct sums, direct summands, products, direct limits and localizations with respect to multiplicative subsets of R. If*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of R-modules and two of them belong to $E_R(\boldsymbol{u}, \boldsymbol{v})$, then the third also belongs to $E_R(\boldsymbol{u}, \boldsymbol{v})$ and the sequence

$$0 \to M_i' \to M_i \to M_i'' \to 0$$

is exact for $i = 1, \ldots, n$.

Proof. The first part is an easy exercise. To prove the second assertion consider the exact sequence of left complexes

$$0 \to M'(u_1, v_1) \to M(u_1, v_1) \to M''(u_1, v_1) \to 0.$$

By the assumption two of them are acyclic. Since for any R-module N $H_nN(u_1, v_1) = H_{n+2}N(u_1, v_1)$ for $n \geqslant 1$ and $H_0N(u_1, v_1) = N_1$, the long homology sequence arguments imply that the third complex is also acyclic and that

$$0 \to M_1' \to M_1 \to M_1'' \to 0$$

is exact. An easy induction gives the required result.

PROPOSITION 1.3. *If M is an R-module and $(u_1, v_1), \ldots, (u_n, v_n)$ is an M-exact sequence of pairs, then*

(a) $\mathrm{Ann}_M(v_1v_2\ldots v_iR) = u_1M+\ldots+u_iM$,

(b) $\mathrm{Ann}_M(u_1R+\ldots+u_iR) = v_1v_2\ldots v_iM$

for any $i = 1, 2, \ldots, n$.

Proof. For $i = 1$ the equality (a) immediately follows from the definition of an M-exact pair. Assume that (a) holds for some $i<n$. If $m \in M$, then $v_1v_2\ldots v_{i+1}m = 0$ if and only if $v_{i+1}m \in u_1M+\ldots+u_iM$. Hence by the M_i-exactness of the pair (u_{i+1}, v_{i+1}) (a) holds also for $i+1$. Equality (b) may be proved in a similar way.

Now we shall give a homological characterization of modules from the category $E_R(\boldsymbol{u}, \boldsymbol{v})$ assuming that $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$. First we prove the following technical result.

LEMMA 1.4. *Let (u, v) be an exact pair in R and let M be an R-module. For $\overline{M} = M/uM$ and $\overline{R} = R/uR$ the following conditions are equivalent:*

(1) *(u, v) is M-exact.*

(2) *$\mathrm{Tor}_n^R(\overline{R}, M) = 0$ for $n \geqslant 1$.*

(3) *$\mathrm{Tor}_n^R(N, M) = \mathrm{Tor}_n^{\overline{R}}(N, \overline{M})$ for $n \geqslant 1$ and any $\overline{R}$-module N.*

(4) *$\mathrm{Ext}_R^n(\overline{R}, M) = 0$ for $n \geqslant 1$.*

(5) *$\mathrm{Ext}_R^n(N, M) = \mathrm{Ext}_{\overline{R}}^n(N, \overline{M})$ for $n \geqslant 1$ and any $\overline{R}$-module N.*

Proof. (3)→(2) and (5)→(4) are obvious. By the assumption $R(u, v)$ is a projective resolution of the R-module $\overline{R}$. Hence

$$\mathrm{Tor}_n^R(\overline{R}, M) = H_n(R(u, v)\otimes_R M) = H_n(M(u, v)),$$

which shows that the statements (1) and (2) are equivalent. The proof of (1)↔(4) is similar.

(4)→(5). By [4, XVI, § 5] the natural ring epimorphism $R \to \overline{R}$ induces a spectral sequence

$$E_2^{pq} = \mathrm{Ext}_{\overline{R}}^p(N, \mathrm{Ext}_R^q(\overline{R}, M)) \underset{p}{\Rightarrow} \mathrm{Ext}_R^n(N, M),$$

which gives (5) because

$$\mathrm{Ext}_R^0(\overline{R}, M) = \mathrm{Hom}_R(\overline{R}, M) = vM = \overline{M}.$$

Implication (2)→(3) may be proved in a similar way. The lemma follows.

We are now able to prove

THEOREM 1.5. *Let $(\boldsymbol{u}, \boldsymbol{v})$ be an exact sequence of pairs in a ring R and let M be an R-module. Then the following conditions are equivalent*:

(i) $M \in E_R(\boldsymbol{u}, \boldsymbol{v})$.

(ii) $\operatorname{Tor}_m^R(R_i, M) = 0$ *for all* $m>0$ *and* $i = 1, 2, \dots, n$.

(iii) $\operatorname{Ext}_R^m(R_i, M) = 0$ *for all* $m>0$ *and* $i = 1, 2, \dots, n$.

Proof. (i)→(iii). Since (u_j, v_j) is R_{j-1}-exact and M_{j-1}-exact for $j = 1, 2, \dots, n$, by Lemma 1.4 we have

$$\operatorname{Ext}_R^m(R_i, M) = \operatorname{Ext}_{R_1}^m(R_i, M_1) = \dots = \operatorname{Ext}_{R_i}^m(R_i, M_i) = 0, \quad m \geq 1, \ i \leq n,$$

as required.

(iii)→(i). Since $\operatorname{Ext}_R^m(R_1, M) = 0$, by Lemma 1.4 the pair (u_1, v_1) is M-exact. Suppose that the sequence $(u_1, v_1), \dots, (u_j, v_j)$ is M-exact for $1 \leq j < n$. Since it is R-exact by the assumption, Lemma 1.4 yields

$$\operatorname{Ext}_{R_j}^m(R_{j+1}, M_j) = \operatorname{Ext}_{R_{j-1}}^m(R_{j+1}, M_{j-1}) = \dots = \operatorname{Ext}_R^m(R_{j+1}, M) = 0.$$

Then by Lemma 1.4 again the pair (u_{j+1}, v_{j+1}) is M_j-exact and therefore the sequence $(u_1, v_1), \dots, (u_{j+1}, v_{j+1})$ is M-exact. This proves the inductive step and hence (i) follows.

The equivalence (i)↔(ii) may be proved in a similar way.

As an immediate consequence we have

COROLLARY 1.6. *If $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$, then any flat and any injective R-module belongs to $E_R(\boldsymbol{u}, \boldsymbol{v})$.*

COROLLARY 1.7. *Let $f\colon R \to S$ be a ring homomorphism such that S is either flat or injective as an R-module. If $(u_1, v_1), \dots, (u_n, v_n)$, is an R-exact sequence of pairs, then the sequence $(fu_1, fv_1), \dots, (fu_n, fv_n)$ is S-exact.*

Suppose $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$. It follows from Corollary 1.6 that a morphism f in the category $E_R(\boldsymbol{u}, \boldsymbol{v})$ is a monomorphism (resp. epimorphism) if and only if it is injective (resp. surjective). Hence, by Proposition 1.2, $E_R(\boldsymbol{u}, \boldsymbol{v})$ is an additive category in which every monomorphism has a cokernel and every epimorphism has a kernel. In the next section we give an example which shows that in general $E_R(\boldsymbol{u}, \boldsymbol{v})$ is not closed under inverse limits and therefore is not abelian.

We end this section by a short discussion of the exactness of a sequence of pairs $(\boldsymbol{u}, \boldsymbol{v})$ with the property $u_i v_i = 0$ for $i = 1, 2, \dots, n$. Observe that any h-regular sequence (Example 2) has this property.

THEOREM 1.8. *Suppose that $(\boldsymbol{u}, \boldsymbol{v}) = ((u_1, v_1), \dots, (u_n, v_n))$ is a sequence of pairs in R such that $u_i v_i = 0$ for $i = 1, 2, \dots, n$. Then $(\boldsymbol{u}, \boldsymbol{v})$ is exact if and only if the left complex*

$$R^{(j)} = R(u_1, v_1) \otimes_R R(u_2, v_2) \otimes_R \dots \otimes_R R(u_j, v_j)$$

is a projective resolution of the R-module $R_j = R/(u_1, \dots, u_j)$ for any $j \leq n$.

Proof. We apply arguments from [1], p. 244. Fix $1 < j \leq n$ and suppose that the complex $R^{(j-1)}$ is acyclic. By [4, XV, § 6] there is a spectral sequence such that $E_{pq}^0 = R_q^{(j-1)} \otimes_R R(u_j, v_j)_p$. Our assumption yields

$$E_{pq}^1 = \begin{cases} R_{j-1} \otimes_R R(u_j, v_j)_p & \text{for } q = 0, \\ 0 & \text{for } q \geq 1 \end{cases}$$

and it is clear that d_{p0}^1 is induced by the differential of $R(u_j, v_j)$. Then $E_{pq}^2 = 0$ for $q \geq 1$ and therefore

$$H_m R^{(j)} = E_{m0}^2 = H_m E_{*0}^1 = H_m(R_{j-1} \otimes_R R(u_j, v_j))$$

for any $m \geq 0$. Consequently, if $R^{(j-1)}$ is acyclic, then the complex $R^{(j)}$ is acyclic if and only if the pair (u_j, v_j) is R_{j-1}-exact. Using this fact we can prove the theorem by an easy induction on n, which we leave to the reader.

§ 2. Injectivity, projectivity and $(\boldsymbol{u}, \boldsymbol{v})$-exactness.

In this section we look for conditions which ensure either the injectivity or the projectivity of modules from $E_R(\boldsymbol{u}, \boldsymbol{v})$ whenever $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$.

Our main result requires the following technical fact.

LEMMA 2.1. *Suppose that $f\colon M \to N$ is an R-homomorphism of $(\boldsymbol{u}, \boldsymbol{v})$-exact R-modules M and N, and put $\overline{K} = K/uK$ for any R-module K. Then*

(a) *if f is an essential monomorphism (resp. minimal epimorphism), then so is the induced map $\bar{f}\colon \overline{M} \to \overline{N}$.*

(b) *f is an isomorphism whenever so is $\bar{f}$ and one of the conditions below is satisfied:*

(i) *the element u is nilpotent,*

(ii) *$u \in J(R)$, M is finitely generated and N is finitely presented.*

Proof. (a) Assume that f is an essential monomorphism. Then, by Proposition 1.2, $\bar{f}$ is a monomorphism, and hence f and $\bar{f}$ may be regarded as inclusions. Let $\bar{x}$ be a non-zero element of $\overline{N}$. Then $x \notin uN = \operatorname{Ann}_N v$ and hence $vx \neq 0$. Since $M \subset N$ is essential, there exists an $r \in R$ such that $0 \neq rvx \in M$. But $u(rvx) = 0$ implies $rvx = vm$ for a certain $m \in M$, or equivalently $v(rx - m) = 0$. Consequently, $rx - m \in uN$ and therefore $0 \neq r\bar{x} = \bar{m} \in \overline{M}$ since $vrx \neq 0$. This shows that $\bar{f}$ is essential.

Now if f is a minimal epimorphism, then, by Proposition 1.2, $\bar{f}$ is an epimorphism and $\operatorname{Ker} \bar{f} = \operatorname{Ker} f / u(\operatorname{Ker} f)$, which implies $\bar{f}$ is minimal.

(b) Assume that (ii) is satisfied. Since $\bar{f}$ is an isomorphism, we have $\operatorname{Im} f + uN = N$ and $\operatorname{Ker} f = u(\operatorname{Ker} f)$. By [3, I, § 2, Lemma 9] $\operatorname{Ker} f$ is finitely generated and therefore f is an isomorphism by the Nakayama Lemma. The proof in case (i) is similar.

PROPOSITION 2.2 *Let $(\boldsymbol{u}, \boldsymbol{v}) = ((u_1, v_1), \dots, (u_n, v_n))$ be an exact sequence of pairs in R such that $u_1, \dots, u_n$ are nilpotent and let M be an R-module. Then*

(a) *M is injective if and only if M is $(\boldsymbol{u}, \boldsymbol{v})$-exact and M_n is an injective R_n-module.*

(b) *M is projective if and only if M is $(\boldsymbol{u}, \boldsymbol{v})$-exact, M_n is R_n-projective and M has a projective cover.*

Proof. (a) The general case follows from the case $n = 1$ by an easy induction. Assume $n = 1$. If M is injective then by Corollary 1.6 it is (u_1, v_1)-exact and, by Lemma 1.4, M_1 is R_1-injective. Conversely, assume that M is (u_1, v_1)-exact and that M_1 is R_1-injective. Now if $f\colon M \to Q$ is an injective envelope of the R_1-module M, then by Lemmas 1.4 and 2.1 the induced map $f_1\colon M_1 \to Q_1$ is an injective envelope of the R_1-module M_1. Hence f_1 is an isomorphism and it follows from Lemma 2.1 that f is an isomorphism. Assertion (b) may be proved in a similar way.

We now prove the main result of this section.

THEOREM 2.3. *Let R be a commutative ring with an exact sequence of pairs $(\boldsymbol{u}, \boldsymbol{v}) = ((u_1, v_1), \dots, (u_n, v_n))$ such that $J(R) = (u_1, \dots, u_n)$. The following conditions are equivalent:*

(a) *R is artinian,*

(b) $\mathrm{Proj}_R = E_R(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{Inj}_R$,

(c) *R is noetherian and* $\mathrm{Fl}_R = E_R(\boldsymbol{u}, \boldsymbol{v})$,

(d) *R is quasi-Frobenius,*

where Proj_R, Inj_R and Fl_R denote the classes of all projective, injective and flat R-modules, respectively.

Proof. The implication (a)→(b) follows from Proposition 2.2 and (b)→(d)→(a) is a consequence of Theorem 5.3 in [6]. Hence, in view of [2], (b) implies (c). Finally, if (c) is satisfied, then by Corollary 1.6 every injective R-module is flat and it follows from [8, Proposition 4.2] that R is self-injective. Consequently, R is a quasi-Frobenius ring and the proof is complete.

Suppose we are given an exact sequence of pairs $(\boldsymbol{u}, \boldsymbol{v})$ in a ring R such that $E_R(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{Fl}_R$. If R is coherent, then by [7, Theorem 5] the category $E_R(\boldsymbol{u}, \boldsymbol{v})$ is closed under inverse limits if and only if $\mathrm{w.gl.dim}\, R \leqslant 2$.

Now let $S = Z_2[X_1, \dots, X_n]/(X_1^2, \dots, X_n^2)$ and consider the following exact sequence of pairs $(\boldsymbol{u}, \boldsymbol{v}) = ((\overline{X}_1, \overline{X}_1), \dots, (\overline{X}_n, \overline{X}_n))$ in S where $\overline{X}_i$ is the residue class of X_i. Then $\mathrm{w.gl.dim}\, S$ is infinite and, by Theorem 2.3, $E_S(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{Fl}_S$. It follows that $E_S(\boldsymbol{u}, \boldsymbol{v})$ is not closed under inverse limits.

We now give an example of a quasi-Frobenius local ring A without any exact pair (u, v) with $u, v \in J(A)$.

EXAMPLE 5. Let $A = K[X, Y, Z]/(X^2, Y^2, Z^2, X(Y-Z), Y(Z-X), Z(X-Y))$ where K is a field of characteristic $\neq 2$. It is clear that A is a local artinian ring and the elements $1, x, y, z, xy$ form a basis of A over K where x, y, z denote the residue classes of X, Y, Z, respectively. Consider a K-linear function $\varphi\colon A \to K$ defined by $\varphi(k_1 + k_2 x + k_3 y + k_4 z + k_5 xy) = k_1 + k_2 + k_3 + k_4 + k_5$, $k_i \in K$. It is easy to check that the kernel of φ contains no non-zero ideals of A and therefore, by Theorem 61.3 in [5], A is a Frobenius K-algebra. We now verify that there is no exact pair (u, v) in A with $u, v \in J(A)$. Suppose, on the contrary, that (u, v) is such an exact pair in A. Then the sequence $0 \to vA \to A \to uA \to 0$ is exact and consequently $l(vA) + l(uA) = l(A) = 5$ where $l(M)$ denotes the length of an A-module M. But this is impossible since, as can easily be shown, $l(aA) \leqslant 2$ for any $a \in J(A)$.

§ 3. Exact local rings.

DEFINITION 3.1. A commutative local ring R is called *exact* if there exists an exact sequence of pairs $(\boldsymbol{u}, \boldsymbol{v}) = ((u_1, v_1), \dots, (u_n, v_n))$ in R such that its unique maximal ideal m is generated by elements $u_1, \dots, u_n$. If $v_1 = u_1^{h_1'}, \dots, v_n = u_n^{h_n'}$ for certain integers $h_i' > 0$, then R is called an *h-exact ring.*

Throughout this section R denotes a commutative local ring and m is its unique maximal ideal.

LEMMA 3.1. *Let R be an exact local ring (not necessarily noetherian) with an exact sequence of pairs $(\boldsymbol{u}, \boldsymbol{v})$ such that $m = (u_1 \dots u_n)$. Then $Rv_1 v_2 \dots v_n$ is a unique minimal ideal of R.*

Proof. Let us denote by v the product $v_1 v_2 \dots v_n$. By Proposition 1.3 $Rv = R/m$ and therefore Rv is a minimal ideal. Now if $I \neq 0$ is a minimal ideal in R, then $\mathrm{Ann}_R I = m$ and hence $Rv = \mathrm{Ann}_R m \supset I$. Then the minimality of Rv yields $I = Rv$, which completes the proof.

LEMMA 3.2. *Let R be an h-exact local ring with an exact sequence of pairs $(u_1, u_1^{h_1'}), \dots, (u_n, u_n^{h_n'})$ such that $m = (u_1, \dots, u_n)$. Then R is artinian and its length $l(R)$ is equal to $(h_1'+1)(h_2'+1) \dots (h_n'+1)$.*

Proof. Since $R \supset Ru_1 \supset Ru_1^2 \supset \dots \supset Ru_1^{h_1'} \supset (0)$ and there are isomorphisms $R/Ru_1 \cong Ru_1^i/Ru_1^{i+1}$ for $i = 1, \dots, h_1'$, we have $l(R) = (h_1'+1)\, l(R/Ru_1)$. Hence a simple induction gives the required result.

We now prove the following useful technical result.

LEMMA 3.3. *Suppose R is a local ring, $m = (u_1, \dots, u_n)$, $\bar{u}_i^{h_i} = 0$ in the ring $R_{i-1} = R/(u_1, \dots, u_{i-1})$ and put $h_i' = h_i - 1$. Then the sequence of pairs $(u_1, u_1^{h_1'}), \dots, (u_n, u_n^{h_n'})$ is R-exact if and only if $u_1^{h_1'} u_2^{h_2'} \dots u_n^{h_n'} \neq 0$.*

Proof. Assume $u_1^{h_1'} \dots u_n^{h_n'} \neq 0$. It is not difficult to show that any element x of m can be expressed as a sum $\sum a_{i_1 \dots i_n} u_1^{i_1} \dots u_n^{i_n}$ with $a_{i_1 \dots i_n} \notin m$ and $i_k \leqslant h_k'$ for $1 \leqslant k \leqslant n$, and thus also in the form

$$x = a u_1^{h_1'} + \sum a_{i_1 \dots i_n} u_1^{i_1} \dots u_n^{i_n}$$

with $a_{i_1 \dots i_n} \notin m$, $i_1 < h_1'$ and $i_k \leqslant h_k'$ for $k \geqslant 2$. Suppose that $xu_1 = 0$ and let us denote by Γ the set of all tuples $\langle i_1, \dots, i_n \rangle$ such that $a_{i_1 \dots i_n} \neq 0$ in the above expression of x. If Γ is non-empty and $\langle i_1', \dots, i_n' \rangle$ is its minimal element in the lexicographic order, then

$$0 = x u_1^{h_1' - i_1'} \dots u_n^{h_n' - i_n'} = a_{i_1' \dots i_n'} u_1^{h_1'} \dots u_n^{h_n'},$$

which is a contradiction. Hence Γ is empty and therefore $x = au_1^{h_1'}$, which shows that $\mathrm{Ann}_R u_1 = (u_1^{h_1'})$. Furthermore, a simple computation shows that $\mathrm{Ann}_R u_1^{h_1'} = (u_1)$, which proves that the pair $(u_1, u_1^{h_1'})$ is R-exact. By our assumption $\bar{u}_2^{h_2'} \dots \bar{u}_n^{h_n'} \neq 0$ with $\bar{u}_i = u_i + (u_1) \in R_1$. Then the sufficiency of the lemma follows by an easy induction. The converse implication is a consequence of Lemma 3.1.

We are now able to prove the main result of this section:

THEOREM 3.4. *If R is an h-exact local ring, then the associated graded algebra* $\mathrm{gr}(R)$ *is also an h-exact local ring.*

Proof. Suppose that $(u_1, u_1^{h_1'}), \dots, (u_n, u_n^{h_n'})$ is an exact sequence of pairs in R such that $m = (u_1, \dots, u_n)$. Then for $s = h_1' + \dots + h_n'$

$$\mathrm{gr}(R) = R/m + m/m^2 + \dots + m^s$$

because $m^{s+1} = 0$ by an easy computation. Let us denote by $\bar{u}_i$ the element $u_i + m^2 \in \mathrm{gr}(R)$. Then, applying Lemma 3.3, we conclude that

$$\bar{u}_1^{h_1'} \dots \bar{u}_n^{h_n'} = u_1^{h_1'} \dots u_n^{h_n'} + m^{s+1} \neq 0,$$

and therefore $(\bar{u}_1, \bar{u}_1^{h_1'}), \dots, (\bar{u}_n, \bar{u}_n^{h_n'})$ is a $\mathrm{gr}(R)$-exact sequence of pairs. The theorem is proved.

We now give an example of an exact local artinian ring which is not h-exact.

EXAMPLE 6. Let K be a field such that $(-1)^{1/2} \notin K$ and let

$$R = K[x, y]/(xy, x^2 - y^2).$$

It is easy to see that the elements $1, \bar{x}, \bar{y}, d = \bar{x}^2$ form a K-basis of R, R is local with maximal ideal $m = (\bar{x}, \bar{y})$, $m^2 = (d)$, $m^3 = (0)$ and $l(R) = 4$. Moreover, it is easy to verify that the sequence of pairs $(\bar{x}, \bar{y})$, $(\bar{y}, \bar{y})$ is exact. We now prove that there is no exact pair of the form $(u, u^{h'})$ with $u \in m$. Assume, on the contrary, that $(u, u^{h'})$, $u \in m$, is an exact pair. Since $m^3 = (0)$, h' is either 1 or 2. Suppose that $h' = 1$ and let $u = t_1\bar{x} + t_2\bar{y} + t_3 d$, $t_i \in K$. Then $0 = u^2 = (t_1^2 + t_2^2)d$ implies $t_1 = t_2 = 0$ since $(-1)^{1/2} \notin K$. Hence $u = t_3 d$ and therefore $\mathrm{Ann}_R u = m$, which contradicts the exactness of the pair (u, u). Now suppose that $h' = 2$. Since the exactness of the pair (u, u^2) implies $R/Ru \simeq Ru/Ru^2 \simeq Ru^2$, we have $4 = l(R) = 3l(R/(u))$ and we again obtain a contradiction. Consequently R is an exact but not h-exact local artinian ring.

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INSTITUTE OF MATHEMATICS, NICHOLAS COPERNICUS UNIVERSITY
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

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