Further conclusions on functional completeness

by

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Abstract. This is a continuation of an earlier article by the author devoted to finding examples of locally complete algebras. Necessary and sufficient conditions for local completeness are given in terms of transitivity, binary operations with a zero or a unit, or both, and in terms of other notions first formulated here. These yield numerous examples involving groups, quasigroups, rings, fields and the real numbers $\boldsymbol{R}$. For example, the algebra $\langle \boldsymbol{R}; b, u, c(c \in \boldsymbol{R})\rangle$ is locally complete if either (0) b is addition and u is multiplication (already known from the Lagrange interpolation formula), or (1) b is addition and $u(x) = x^2$, or (2) b is addition and $u(x) = x^+$, or (3) b is multiplication and $u(x) = x+1$, or (4) b is multiplication and $u(x) = 1/(1-x)$ with $1/0 = 0$, or (5) b is multiplication and u interchanges 0 and 1, leaving the remaining elements fixed.

Introduction

This paper is introduced to the reader by first, an examination of its consequences in several highly concrete examples, then a summary of its general results about locally complete algebras, and finally some reasons for studying these algebras.

The interpolation polynomials of Lagrange tell us that in the real numbers a polynomial of degree n can always be fitted to pass through $n+1$ points of an arbitrary function. For a function φ of one argument to be matched at the three points a, b and c, Lagrange's polynomial is

$$\frac{\varphi(a)}{(a-b)(a-c)}(x-b)(x-c)+\frac{\varphi(b)}{(b-a)(b-c)}(x-a)(x-c)+\frac{\varphi(c)}{(c-a)(c-b)}(x-a)(x-b).$$

An understanding of why this polynomial agrees with φ at a, b, c reveals how to construct such a formula fitting a function of any number of arguments at any finite number of points. Observe that these formulas involve only repeated addition and multiplication of the variables and constants. Can we weaken these operations and still equal an arbitrary function at a finite number of points?

Yes, and here are some of the many ways of doing this. As one example we shall show that addition and squaring (together with the constants) suffice. Consider how we might match this function of two arguments,

$$\varphi(x, y) \equiv \tfrac{3}{2}x+\tfrac{1}{2}y+\tfrac{1}{2},$$

at the three points $\langle 0, 0\rangle$, $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$. A formula in terms of addition and squaring is

$$(x+\tfrac{1}{4})^2+(y-\tfrac{1}{4})^2+\tfrac{3}{8}\,.$$

As another example, we may get by with only addition and the operation of taking positive part (plus the constants); our function φ is touched at the three given points by

$$(x+x-\tfrac{1}{2})^+ + (y-\tfrac{1}{2})^+ + \tfrac{1}{2}\,.$$

In these examples we do not expect the reader to see how to construct the matching functions; this will become clear only in the later sections when we give detailed proofs.

More bases are obtained when, instead, multiplication is retained while the power of addition is reduced or modified. We give three examples. Let the three functions ε_1, ε_2 and ε_3 of one argument be defined by

$$\varepsilon_1(x) \equiv x+1,$$

$$\varepsilon_2(x) \equiv \frac{1}{1-x}, \text{ with the understanding } \varepsilon_2(1) \equiv 0,$$

$\varepsilon_3 \equiv (01)$, i.e., ε_3 interchanges 0 and 1 while leaving the other elements fixed.

We shall derive from our general theory that multiplication together with the constants and any one of the ε_i has the matching property. At the three points mentioned, functional agreement with φ is given by any of these compositions:

$$\tfrac{1}{2}\cdot\varepsilon_1(3x)\cdot\varepsilon_1(y)\,,$$

$$\tfrac{1}{2}\cdot\varepsilon_2(\tfrac{3}{4}x)\cdot\varepsilon_2(\tfrac{1}{2}y)\,,$$

$$\tfrac{1}{2}\cdot\varepsilon_3(4x)\cdot\varepsilon_3(2y)\,.$$

Let us remark that as the name implies, the Lagrange interpolation polynomial approximates a given smooth function φ in an interval in a certain, well-known sense. In what sense, if any, our novel formulas approximate a given function we do not investigate. Since in the general theory the underlying set will have no topological structure, we only consider agreement at a finite number of points.

These five examples are most easily and systematically obtained as consequences of the general theory of local completeness to which we now turn. Recall that the *locally complete algebras* of Foster [4] are those universal algebras of at least two elements in which every function of a finite number of arguments is a local polynomial; and a *local polynomial* is a function such that every restriction of it to a finite set can be extended to a polynomial. In other words, in a locally complete algebra every function of a finite number of arguments on the underlying set can be matched at a finite number of elements by some polynomial (a composition of the operations of the algebra). The purpose of this paper is to continue the program begun by the author [8] of finding examples of locally complete algebras.

We now describe the contents of this article. Recall the classical result of Słupecki [22] that a finite algebra of at least three elements is complete iff every unary function is a monomial and there is a surjective binomial depending on both arguments. We strengthen this theorem by greatly reducing the number of unary functions required while demanding more of the binomial.

To this end, I introduce three notions: *singularity*, *transitivity*, and *neatness*; precise definitions will come later. Singular algebras come up as exceptional cases in theoretical results; they have much local symmetry and appear in applications only as elementary Abelian 2-groups or not at all. The concept of (j, k)-transitivity ensures the presence of monomials which take any j distinct elements into k others in a prescribed way. The first principal result is that low order $(3, 2)$-transitivity yields local completeness of nonsingular algebras in which there is a multiplication with a unit. The local completeness of the first two examples mentioned earlier follows from this.

The second principal result is that for nonsingular algebras, $(3, 2)$-transitivity also yields local completeness when there is a 0-neat multiplication, i.e., every three elements are found in some two-by-two subtable of the multiplication table. Many applications flow from this. Any null-group (Foster [4]) together with a transitive group of permutations is locally complete. From this follows the local completeness of the last three examples given earlier. Any finite algebra with a multiplication with a zero and a unit is complete if there is a product of all the non-zero elements of the algebra which is equal to one. Finally, for any integer $k\geqslant 3$, we shall give necessary and sufficient conditions for an algebra to be locally complete when it generates a k-fold transitive group of permutations; this extends to infinite algebras some results of Salomaa [18], Rosenberg [14] and Schofield [20].

Reasons for studying locally complete algebras are numerous. Mathematically, every locally complete algebra gives rise to a representation theorem analogous to Stone's [23] theorem for Boolean algebras. This was discovered by Foster [4] and more neatly formulated by Hu [5] and Kelenson [7] as follows. Any algebra which locally satisfies the identities of a locally complete algebra is a subdirect power of the latter. We say that one algebra *locally satisfies the identities* of a second algebra if for any finite subset M of the first there is a finite subset N of the second such that any identity on N is an identity on M.

One very important application of functionally complete algebras is to multi-valued logic. Various paradoxes or, more correctly, inaccuracies in interpretation can be eliminated. For example, the law of the excluded middle need not hold in a logic of three values. Using a three-valued logic, Turquette [24] clarifies some of the antinomies of quantum mechanics. In particular, he shows how Heisenberg's notion of "necessary uncertainty" can be interpreted naturally in Łukasiewicz's [12] three-valued logic (its completeness is discussed in Knoebel [8]). However, it should be mentioned that some algebras corresponding to modal logics are not complete.

The curious reader who is interested in the relationship between propositional calculi and their corresponding algebras is referred to Łukasiewicz and Tarski [12], Rosser and Turquette [24], and Chang [3].

A third reason for the study of complete algebras is their possible application in the design of digital computers. Here, constants represent constant voltages, currents, or the like, which are readily available; the operations of the algebra are the basic circuits; and thus functional completeness guarantees that all possible switching functions can be built out of the basic circuits. Physical realizations of some likely operations and their synthesis into switching networks are discussed in Santos and Arango [19], Lowenschuss [11], and Vranesic and Hamacher [25].

We adopt in this paper the conventions and definitions of Knoebel [8], including the assumption that all algebras have at least two elements.

Singularity

Before getting to the heart of this article, we must deal with the nasty notion of singularity, which is present in the theoretical results, but arises in practice only in elementary Abelian 2-groups and zero-rings on them. In this section we find out when certain common algebras are singular.

The quadruple $a, b, c, d \in A^n$ of n-tuples is a *quartet* if for each $i<n$, no one of the a_i, b_i, c_i, d_i is distinct from the remaining three; in other words, the a_i, b_i, c_i, d_i are all equal or split into two pairs of equal elements. For example, if $A \equiv \{0, 1, \ldots, 5\}$ and $n \equiv 3$, then the quadruple of triplets $a \equiv \langle 0, 2, 3\rangle$, $b \equiv \langle 0, 3, 3\rangle$, $c \equiv \langle 1, 2, 3\rangle$ and $d \equiv \langle 1, 3, 3\rangle$ is a quartet; whereas the quadruple a, b, c and $\langle 1, 3, 4\rangle$ is not. A function $\varphi\colon A^n \to A$ is *singular* if for every quartet $a, b, c, d \in A^n$, we have that $\varphi a = \varphi b$ implies $\varphi c = \varphi d$, i.e., for every quartet $a, b, c, d \in A^n$, the elements $\varphi a, \varphi b, \varphi c, \varphi d$ are all distinct, all equal, or form two pairs of equal elements. An algebra is *singular* if each polynomial of it is singular. Note that all constants ($n = 0$) and unary functions ($n = 1$) are singular.

The following are multiplication tables for some singular binary functions φ on the set $\{0, 1, 2, 3\}$.

	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

	0	1	2	3
0	0	0	1	1
1	0	0	1	1
2	2	2	3	3
3	2	2	3	3

	0	1	2	3
0	0	0	2	0
1	1	1	3	1
2	1	1	3	1
3	1	1	3	1

	0	1	2	3
0	2	1	2	2
1	2	1	2	2
2	1	2	1	1
3	1	2	1	1

From the study of these examples, we may, for binary functions, simplify the definition of singularity to two dimensions: a binary function φ is singular iff for all $a, b \in A^2$ the values of φ take on one of the following patterns:

$$\begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & x \\ b_0 & x & x \end{array} \qquad \begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & x \\ b_0 & y & y \end{array} \qquad \begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & y \\ b_0 & x & y \end{array} \qquad \begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & y \\ b_0 & y & x \end{array} \qquad \begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & y \\ b_0 & z & w \end{array}$$

where $x, y, z, w \in A$ are presumed to be distinct. A complimentary result is that a binary function φ is nonsingular iff for some $a, b \in A^2$ the values of φ take on one of the remaining possible patterns:

$$\begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & x \\ b_0 & x & y \end{array} \qquad \begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & x \\ b_0 & y & z \end{array} \qquad \begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & y \\ b_0 & x & z \end{array} \qquad \begin{array}{c|cc} & a_1 & b_1 \\ \hline a_0 & x & y \\ b_0 & z & x \end{array}$$

Binary functions occur in everyday algebras, and when working with them, we will often have these patterns in mind.

The following propositions and corollaries assert that various common algebras are nonsingular.

PROPOSITION 1. *The following algebras and functions are nonsingular:*

(i) *any semilattice,*

(ii) *any nonconstant binary function* $\cdot\colon A^2 \to A$ *with a zero* 0, *i.e.*,

$$a\cdot 0 = 0 = 0\cdot a \quad (a \in A),$$

(iii) *any binary function* $\cdot\colon A^2 \to A$ *with a unit* 1, *i.e.*,

$$a\cdot 1 = a = 1\cdot a \quad (a \in A),$$

for which there exists an $a \in A$ *such that* $a\cdot a \neq 1$.

Proof. (i) Let a and b be two comparable elements of the semilattice (we always assume at least two elements per algebra). Consider the quartet $\{a, b\} \times \{a, b\} = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle\}$. On these pairs the semilattice operation takes the values a, a, a, b or a, b, b, b, depending on which of a and b is larger, and these form one of the patterns listed for nonsingular binomials.

(ii) There are $a, b \in A$ such that $ab \neq 0$. Consider the quartet $\{0, a\} \times \{0, b\}$.

(iii) Consider the quartet $\{1, a\} \times \{1, a\}$ where $aa \neq 1$. ■

The concepts of *loop*, *quasigroup* and *isotopy*, which appear in the following proposition and its corollary are defined in Kurosh [9, Chap. 2].

PROPOSITION 2. *A loop is singular iff it is an elementary Abelian 2-group.*

Proof. → Let $\mathfrak{A} \equiv \langle A; \cdot\rangle$ be a singular loop with unit 1. By part (iii) of the previous proposition, $aa = 1$ for all $a \in A$. Since a loop is solvable, it suffices to show $\mathfrak{A}$ is associative, and we do this in several steps. Firstly $\cdot$ is commutative, for if not, $ab \neq ba$ for some $a, b \in A$, and the quartet $\{a, b\} \times \{a, b\}$ yields a contradiction to singularity since $aa = 1 = bb$. Secondly, for all $a, b, c, d \in A$, we have $ac = bd$ iff $ad = bc$, by looking at the quartet $\{a, b\} \times \{c, d\}$. Thirdly, $a(ab) = b$ for all $a, b \in A$, since $1(ab) = ab$ iff $1b = a(ab)$. Now let a, b, c be any elements of A. By commutativity and the third point, $a(ab) = b = c(bc)$, and by the second point applied to this and commutativity, $a(bc) = c(ab) = (ab)c$.

← In an elementary Abelian 2-group, for any quartet $\{a, b\} \times \{c, d\}$, $ac = bd$ iff $ad = bc$, and this insures singularity. ■

COROLLARY. *A quasigroup is singular iff it is isotopic to an elementary Abelian 2-group.*

Proof. Every quasigroup is isotopic to a loop (Kurosh [9, Chap. 2]). and isotopy preserves singularity. ■

COROLLARY. *A ring* $\mathfrak{R} \equiv \langle R; +, \cdot\rangle$ *is singular iff*

(i) $\mathfrak{R}$ *has characteristic* 2, *and*

(ii) $\mathfrak{R}$ *is a zero-ring.*

Proof. Use the previous two propositions and the easily proven fact that in a zero-ring $\mathfrak{R} \equiv \langle R; +, \cdot\rangle$, every polynomial of $\mathfrak{R}$ is a polynomial of just $\langle R; +, 0\rangle$. ■

There is a characterization, not needed in this article, of singular functions emphasizing the local symmetry that is enforced. Consider the function $f\colon A^n \to A$. We restrict our attention to n two-element subsets $B_0, \ldots, B_{n-1}$ of A. Each B_i is given the structure of the two-element group Z_2, and the cartesian product $P \equiv B_0 \times \ldots \times B_{n-1}$ the structure of the product group. Although there are two choices for the group structure on each B_i and even more for the product, our use of these groups is not affected by these alternatives. Define the kernel of $f|P$ to be the equivalence relation $\{\langle a, b\rangle \in P^2 |\ fa = fb\}$. Then it can be shown that f is singular iff for each n-fold product P of two-element subsets, the kernel of $f|P$ is a congruence of the product group.

The import of this section is that most algebras are nonsingular.

Transitivity

In this section we define (j, k)-transitivity and show that the low orders of transitivity are usually equivalent. Then this is used to obtain the first main result, characterizing local completeness in algebras with a unit. As an application we convert Archimedean linearly ordered groups into locally complete algebras.

An algebra $\mathfrak{A} \equiv \langle A; o_0, o_1, \ldots\rangle$ is (j, k)-*transitive* if every unary function whose domain and range are subsets of A of no more than j and k elements, respectively, can be extended to a monomial. The algebra $\mathfrak{A}$ is (ω, k)-*transitive* if $\mathfrak{A}$ is (j, k)-transitive for all finite j. Similarly (ω, ω)-*transitivity* is defined; it is equivalent to saying that every unary function is a local monomial. Here are two examples with cyclic groups in which all constants are added as operations: Z_4 is $(\omega, 1)$-transitive, and Z_5 is both $(2, 2)$- and $(\omega, 1)$-transitive. The special case of $(2, 2)$-transitivity is just the two-point property of Hu [5]. For our proofs we need the following theorem of Knoebel [8].

THEOREM 0. *An algebra* $\mathfrak{A}$ *with at least three elements is locally complete iff*

(i) $\mathfrak{A}$ *has a surjective local polynomial depending on at least two arguments, and*

(ii) $\mathfrak{A}$ *is* (ω, ω)-*transitive.*

We use this now to weaken condition (ii) to $(3, 2)$-transitivity at the expense of the local polynomial of condition (i).

THEOREM 1. *An algebra is locally complete iff*

(i) $\mathfrak{A}$ *has a local binomial* $\cdot$ *with unit* 1,

(ii) $\mathfrak{A}$ *is* $(3, 2)$-*transitive, and*

(iii) $\mathfrak{A}$ *is nonsingular.*

The proof proceeds by establishing four useful lemmas.

LEMMA 1. *Suppose* $\mathfrak{A}$ *is* $(2, 2)$-*transitive. Then* $\mathfrak{A}$ *is nonsingular iff* $\mathfrak{A}$ *has a nonsingular binomial.*

Proof. → Let φ be a nonsingular polynomial of n arguments. By nonsingularity, there is a quartet $a, b, c, d \in A^n$ for which

$$\varphi a = \varphi b \neq \varphi c \neq \varphi d\,.$$

On the basis of $(2, 2)$-transitivity, we may assume elements 0,1 such that

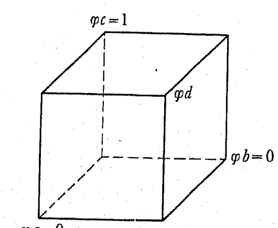
$$\varphi a = \varphi b = 0,$$
$$\varphi c = 1,$$
$$a_i = 0 \qquad (i<n),$$
$$b_i, c_i, d_i \in \{0, 1\} \qquad (i<n).$$

The arguments of φ are naturally divided into four groups according to the values of b_i, c_i and d_i, viz.:

Group:	I	II	III	IV
a_i	0	0	0	0
b_i	0	1	1	0
c_i	0	0	1	1
d_i	0	1	0	1

Assume each group of arguments is nonempty; when not, the proof is easily modified.

In each group, the arguments of φ are to be identified. The resulting argument of group I is replaced by a monomial taking 0 and 1 both to 0, and the new argument is then identified with the argument of group II. What results is a trinomial whose values on triples of 0's and 1's are best illustrated by the accompanying cube. Now



this cube has the property that the trinomial is nonsingular either on the bottom face or on the quartet consisting of c, d and their antipodes. In either case, a nonsingular binomial is easily manufactured from the trinomial by substituting into its arguments appropriate monomials obtained via $(2,2)$-transitivity. ∎

The following lemma, stated without proof, is due to Yablonskiĭ [26].

LEMMA 2. *If $\mathfrak{A}$ has a surjective local polynomial dependent on at least two arguments and $|A|\geqslant 3$, then there is a quartet and a polynomial taking at least three values on the quartet.*

Although insignificant in appearance, the next lemma is crucial in that it allows us to map a finite set onto two of its elements.

LEMMA 3. *Suppose $\mathfrak{A}$ is $(3,2)$-transitive, and M is a finite subset of A containing* 0 *and* 1. *Then there is a monomial γ for which*

$$\gamma(M)\subseteq\{0,1\},$$
$$\gamma(0)=0,$$
$$\gamma(1)=1.$$

Proof. Using $(3,2)$-transitivity, we induct on the size of M. ∎

In the final lemma of our series, we lift the algebra $\mathfrak{A}$ from $(3,2)$-transitivity to $(\omega,2)$-transitivity.

LEMMA 4. *If $\mathfrak{A}$ has a surjective local polynomial dependent on at least two arguments and $|A|\geqslant 3$, then the following are equivalent:*

(i) $\mathfrak{A}$ *is* $(3,2)$-*transitive and nonsingular,*

(ii) $\mathfrak{A}$ *is* $(4,2)$-*transitive,*

(iii) $\mathfrak{A}$ *is* $(\omega,2)$-*transitive.*

Proof. (iii)→(ii). Trivial.

(ii)→(i). By Lemma 2 and $(4,2)$-transitivity we establish nonsingularity.

(i)→(iii). Assume M is a finite subset of A containing 0 and 1. By Lemma 1, there is a nonsingular binomial. From this, by applying $(3,2)$-transitivity, we can construct a binomial $\cdot$ such that

$$0\cdot 0=0\cdot 1=1\cdot 0=0,$$
$$1\cdot 1=1.$$

Next we shall find polynomials δ_m $(m\in M)$ such that

$$\delta_m(m)=1,$$
$$\delta_m(x)=0\quad (x\in M\setminus\{m\}).$$

Applying $(2,2)$-transitivity to the function of Lemma 3, we find polynomials γ_m^n $(n,m\in M)$ such that when $n\neq m$,

$$\gamma_m^n(M)\subseteq\{0,1\},$$
$$\gamma_m^n(n)=0,$$
$$\gamma_m^n(m)=1.$$

Then

$$\delta_m=\prod_{n\in M\setminus\{m\}}\gamma_m^n,$$

where $\prod$ is the iteration of the binomial $\cdot$.

To prove $(\omega,2)$-transitivity, we must extend any $\varepsilon\colon M\to\{0,1\}$ to a monomial. The required extension (when ε is not the constant 1) is

$$\prod_{\varepsilon(m)=0}\delta_m',$$

where $'$ is a monomial interchanging 0 and 1. ∎

Proof of Theorem 1. ← First assume $|A|\geqslant 3$. In view of Theorem 0, it suffices to show that $\mathfrak{A}$ is (ω,ω)-transitive. To that end suppose $\varepsilon\colon M\to A$ and M is finite; we must find a monomial which extends ε. For each $m\in A$, by $(\omega,2)$-transitivity from Lemma 4, there is a monomial δ_m such that

$$\delta_m(x)=\begin{cases}\varepsilon(x) & \text{if}\quad x=m,\\ 1 & \text{if}\quad x\in M\setminus\{m\}.\end{cases}$$

Then $\prod_{m\in M}\delta_m$ is the required monomial $\cdot$.

Now suppose $|A|=2$. By Lemma 1 there is a nonsingular binomial; with the help of $(2,2)$-transitivity we obtain from it a binomial with zero and unit. It is well known in the theory of Boolean algebras that such a binary operation together with the nontrivial permutation generate by composition all other functions. ∎

In view of Lemma 4, this theorem has the alternative form: an algebra is locally complete iff it has a local binomial with a unit and is $(4,2)$-transitive. In contrast, it is possible to weaken $(3,2)$-transitivity by strengthening the binomial: an algebra is locally complete iff it has a local binomial with both a zero and a unit and is $(2,2)$-transitive (Knoebel [8, Theorem 2]).

For the following corollary and succeeding results, $\mathfrak{A}^{\#}$ is the algebra $\mathfrak{A}$ with all constants added as operations.

COROLLARY. *Let $\mathfrak{F}\equiv\langle F;+,\cdot\rangle$ be a field not of characteristic two, and let ^{2} denote the operation of squaring. Then the algebra $\mathfrak{A}\equiv\langle F;+,{}^2\rangle^{\#}$ is locally complete.*

Proof. In view of Theorem 1 and Proposition 1 (iii), it is sufficient to prove that $\mathfrak{A}$ is $(3,2)$-transitive. (Note that $+$ has the unit 0, and is nonsingular since $\mathfrak{F}$ is not of characteristic two.) If a, b and c are distinct elements of F, then the monomial $[*-(b+c)/2]^2$ takes b and c into the same element while leaving a distinct. This is so since a quadratic equation in a field can have no more than two roots. Thus the problem is reduced to showing that $\mathfrak{A}$ is $(2,2)$-transitive. To this end, assume $a\neq b$. Then the monomial

$$\left[*+\frac{c-d-a^2+b^2}{2(a-b)}\right]^2-\left[\frac{c-d+(a-b)^2}{2(a-b)}\right]^2+c$$

takes a into c and b into d. ∎

This verifies the first example given at the beginning of the introduction.

While one might be tempted to base a proof of this corollary on the Lagrange interpolation formula, it will not work; for not all of the usual field polynomials are polynomials of $\mathfrak{A}$. For example, in a field $\mathfrak{F}$ of characteristic 0, any nonconstant monomial of $\mathfrak{A}$ must have a leading term with positive integral coefficient and exponent a power of two.

We apply Theorem 1 to Archimedean linearly ordered groups, which, by a theorem of Hölder (see Kurosh [9, p. 268]), are isomorphic to subgroups of the multiplicative group of positive real numbers in its natural ordering.

PROPOSITION 3. *Let $\mathfrak{G} \equiv \langle G; \cdot, \leqslant \rangle$ be an Archimedean linearly ordered group. Let / be defined by $x/y = x \cdot y^{-1}$, and $^{+}$: $G \to G$ by*

$$x^{+} \equiv \begin{cases} 1 & \text{if} \quad x<1\,, \\ x & \text{if} \quad x \geqslant 1\,. \end{cases}$$

Then the algebra $\mathfrak{A} \equiv \langle G; /, {}^{+} \rangle^{\#}$ is locally complete.

Proof. Clearly from the binomial / we may recover $\cdot$, which has the unit 1; and so we turn to Theorem 1 for a proof. We prove successively that $\mathfrak{A}$ is nonsingular, $(2,2)$-transitive, and $(3,2)$-transitive.

Nonsingularity is established by examining the values of the group multiplication on the quartet $\{1, g\} \times \{1, g\}$, where $1 \neq g \in G$.

Now let us find for arbitrary elements $a \neq b$ and c, d of A a monomial ε such that $\varepsilon a = c$ and $\varepsilon b = d$. Since it is possible to interchange two elements by application of $^{-1}$ and a suitable translation, we may assume without loss of generality that $a<b$ and $c \leqslant d$. With two more translations, we may also assume $a = 1 = c$. There is a positive integer n such that

$$b^n > d\,,$$

since $\mathfrak{G}$ is Archimedean. We transform a into c and b into d by raising to the nth power, translating by $b^{-n} \cdot d$, and taking positive part; viz.:

$$\begin{array}{ccccccc} & \overset{*^n}{} & & \overset{*\cdot b^{-n}\cdot d}{} & & \overset{*^{+}}{} & \\ 1 & \mapsto & 1^n = 1 & \mapsto & b^{-n}\cdot d & \mapsto & 1\,, \\ b & \mapsto & b^n & \mapsto & d & \mapsto & d\,. \end{array}$$

Thus $\mathfrak{A}$ is $(2,2)$-transitive.

To prove $(3,2)$-transitivity it suffices, in view of $(2,2)$-transitivity, to show that for any three distinct elements a, b, c there is a monomial effecting the following transformation

$$a \mapsto a, \qquad b \mapsto b, \qquad c \mapsto a\,.$$

There are six cases according to how the a, b, c are ordered. If a and c are both less than b, then the positive part of a suitable translation will merge a with c while keeping b distinct; the $(2,2)$-transitivity already established finishes these two cases. If a and c are both greater than b, then inversion leads to the previous cases.

With the inverse operation $^{-1}$ present, it will suffice to consider only one of the remaining two cases, say $a<b<c$. By the preceding, there are monomials

$$g\colon a \mapsto a;\ b \mapsto b;\ c \mapsto b\,,$$

and

$$h\colon a \mapsto b;\ b \mapsto b;\ c \mapsto c\,.$$

Then the monomial $f \equiv g/h$ changes the ordering so that both fa and fc are less than $fc = 1$, and thus we have reduced this case to a previous one. ■

This verifies the second example of the introduction.

Neatness

The notion of neatness evolves naturally from the idea that there may be more distinct elements in some small subtable of the multiplication table of a function than one would have a right to expect. As Lemma 2 demonstrates, even rather weak conditions insure some degree of neatness. This fortuitous fact leads to interesting theoretical results. Many applications are developed in both this section and the next.

An algebra $\mathfrak{A}$ is *k-neat* ($k = 0, 1, 2$, or 3) if there are distinct elements $e_1, \ldots, e_k \in A$ such that for all $e_{k+1}, \ldots, e_3 \in A$ there exist a surjective polynomial φ and a quartet a_0, a_1, a_2, a_3 for which $e_1 = \varphi a_1$, $e_2 = \varphi a_2$ and $e_3 = \varphi a_3$. An equivalent but more expansive definition is developed by considering the set T of all triples φa_1, φa_2, φa_3 where φ is a surjective polynomial and the a_0, a_1, a_2, a_3 form a quartet. Then the four degrees of neatness become:

$\mathfrak{A}$ is 0-*neat* if $A \times A \times A \subseteq T$;

$\mathfrak{A}$ is 1-*neat* if $\{e_1\} \times A \times A \subseteq T$ for some $e_1 \in A$;

$\mathfrak{A}$ is 2-*neat* if $\{\langle e_1, e_2 \rangle\} \times A \subseteq T$ for some distinct $e_1, e_2 \in A$;

$\mathfrak{A}$ is 3-*neat* if $\langle e_1, e_2, e_3 \rangle \in T$ for some distinct $e_1, e_2, e_3 \in A$.

For the sequel it should be noted that without loss of generality, we can always take all the e_1, e_2, e_3 in the original definition to be distinct since otherwise, if some of them were the same, they could be obtained by a trivial polynomial acting on a degenerate quartet.

By way of example, quasigroups are 0-neat. As another example, an algebra $\mathfrak{A}$ with a binomial $\cdot$ and a unit 1 is 1-neat; for setting $e_1 = 1$ in the definition, we see that $\{1, e_2\} \cdot \{1, e_3\} \supseteq \{1, e_2, e_3\}$ for arbitrary $e_2, e_3 \in A$. Lemma 2 ensures that most everyday algebras are 3-neat. Note that if $\mathfrak{A}$ is k-neat and $k \leqslant k' \leqslant 3$, then $\mathfrak{A}$ is k'-neat.

An algebra $\mathfrak{A}$ is *k-fold transitive* if there is a set P of monomials which is, under composition, a k-fold transitive group of permutations on A. On the basis of Ledermann's [10] definition of k-fold transitive group (there called *k-ply transitive*), if an algebra $\mathfrak{A}$ is k-fold transitive, then it is j-fold transitive for all $j<k$; also $\mathfrak{A}$ is k-fold transitive if $k > |A|$ and $\mathfrak{A}$ is already $|A|$-fold transitive.

We are now prepared to state the principal theorem, which relates all the notions introduced in this paper.

THEOREM 2. *Let* $k = 0, 1, 2$ *or* 3. *Then an algebra* $\mathfrak{A}$ *is locally complete if*

(i) $\mathfrak{A}$ *is nonsingular,*

(ii) $\mathfrak{A}$ *is* $(3, 2)$-*transitive,*

(iii) $\mathfrak{A}$ *is* k-*neat, and*

(iv) $\mathfrak{A}$ *is* k-*fold transitive.*

Proof. Let us assume $|A| \geqslant 3$ since the two-element case is classical. We may replace conditions (iii) and (iv) by 0-neatness for the following reason. Let $e_1, \ldots, e_k$ satisfy the definition of k-neatness. We must show that any three distinct elements $f_1, f_2, f_3 \in A$ are the functional values of some surjective polynomial on some quartet. By k-fold transitivity, there is a permutation π which is also a monomial of $\mathfrak{A}$ such that $\pi(e_i) = f_i$ $(i = 1, \ldots, k)$. Define $e_i \equiv \pi^{-1}(f_i)$ $(i = k+1, \ldots, 3)$. By k-neatness there is a surjective polynomial φ such that the elements e_1, e_2, e_3 are values of φ on some quartet. Then f_1, f_2, f_3 are values of $\pi \cdot \varphi$ on the same quartet. Thus $\mathfrak{A}$ is 0-neat.

We now assume $\mathfrak{A}$ is nonsingular, $(3, 2)$-transitive and 0-neat, and has at least three elements. We want to show that $\mathfrak{A}$ is (ω, ω)-transitive. By Lemma 4, $\mathfrak{A}$ is $(\omega, 2)$-transitive. By induction on j, we shall show that $\mathfrak{A}$ is (ω, j)-transitive for all finite j. Assume $\mathfrak{A}$ is (ω, j)-transitive for some $j \geqslant 2$. Let $\varepsilon\colon A \to A$ have a range of no more than $j+1$ elements $e_1, \ldots, e_{j+1}$. Our object is to show that ε is a local monomial. There is a surjective polynomial φ whose values on some quartet a_0, a_1, a_2, a_3 contains e_1, e_2, e_3 (assume $e_1 = \varphi a_1$, $e_2 = \varphi a_2$, and $e_3 = \varphi a_3$). Let a_i be a preimage of e_i $(i = 4, \ldots, j+1)$. Since the cardinality of the set of gth components of the quartet a_0, a_1, a_2, a_3 is at most 2, the cardinality of the set of gth components of all the a_i is at most j. Thus by (ω, j)-transitivity there exist local monomials $\delta_0, \ldots, \delta_{n-1}$ such that $\varepsilon = \varphi(\delta_0, \ldots, \delta_{n-1})$, which proves that ε is also a local monomial. Hence $\mathfrak{A}$ is (ω, ω)-transitive.

We finish by Theorem 0. ∎

In the statement of Theorem 2 (as well as in succeeding results) we could strengthen the implication to logical equivalence by localizing all notions, i.e., by replacing the term polynomial by local polynomial, wherever it occurs. However, this would be an unnecessary complication as such a local version is a trivial consequence of our present one.

Here are a couple of applications of Theorem 2 to semilattices. Recall from Proposition 1 that all semilattices are nonsingular.

COROLLARY. *If the algebra* $\mathfrak{A} \equiv \langle A; \wedge, \ldots\rangle$ *is 2-fold transitive and* $\langle A; \wedge\rangle$ *is a semilattice with a zero and at least one atom, then* $\mathfrak{A}$ *is locally complete.*

Proof. That $\wedge$ has a zero and $\mathfrak{A}$ is 1-fold transitive insures that $\mathfrak{A}$ is $(\omega, 1)$-transitive, i.e., all constants are available locally. If a is an atom of $\langle A; \wedge\rangle$, then the function $a \wedge *$ takes on exactly two values, and hence $\wedge$ is 2-neat. Likewise, on account of 2-fold transitivity, the function $a \wedge *$, along with $\wedge$, compels $\mathfrak{A}$ to be $(3, 2)$-transitive. Theorem 2 with $k = 2$ can now be applied. ∎

PROPOSITION 4. *If for some* $k \geqslant 2$, *the algebra* $\mathfrak{A} \equiv \langle A; \wedge, \ldots\rangle$ *is* (k, k)-*transitive and* $\langle A; \wedge\rangle$ *is a semilattice with at most* $k-1$ *pairwise incomparable elements, then* $\mathfrak{A}$ *is locally complete.*

Proof. For this, the proof of Theorem 2 must be modified. As in the proof of the previous corollary, $\mathfrak{A}$ is $(\omega, 1)$-transitive. Assume for the moment that $|A| \geqslant 4$ and $k \geqslant 3$.

First we prove by induction on j that $\mathfrak{A}$ is (ω, j)-transitive for $j \leqslant k$. By Lemma 4, $\mathfrak{A}$ is $(\omega, 2)$-transitive. Assume $2 \leqslant j < k$ and $\mathfrak{A}$ is (ω, j)-transitive. We must show that any $\varepsilon\colon F \to B$ is a monomial when F, B are finite subsets of A and $|B| = j+1$. Since $\mathfrak{A}$ is (k, k)-transitive, we may without loss of generality fix B as follows. By the 3-neatness of $\mathfrak{A}$, there are subsets B, C, $D \subseteq A$ for which $B \subseteq C \wedge D$, $|B| = j+1$, and $|C| = j = |D|$. We can obtain the monomial ε as a composition of $\wedge$ into whose arguments are substituted monomials found from the induction hypothesis.

Let us next show that for any set $B \subseteq A$ of $k+1$ elements there are sets C, D, each of k elements, such that $B \subseteq C \wedge D$. Since there at most $k-1$ pairwise incomparable elements, one can find distinct a, b, c, $d \in B$ satisfying the relations

$$a < c \quad \text{and} \quad b < d,$$

or the relation

$$a < b < c.$$

Set $B' \equiv B \setminus \{a, b, c, d\}$. Then

$$B \subseteq (B' \cup \{a, c, d\}) \wedge (B' \cup \{b, c, d\}).$$

Now we finish by entering the proof of Theorem 2 at the induction step when $j \geqslant k$.

This demonstration can be modified *ad hoc* to accommodate the excluded cases of $|A| \leqslant 3$ or $k = 2$. ∎

If $\mathfrak{A}$ is k-fold transitive, when is $\mathfrak{A}$ locally complete? The answer, almost always. From Theorem 2 we shall find the exceptional cases when $k \geqslant 3$. When $k \leqslant 2$, we can find a formulation of the exceptions only when A is finite, and for this we use a theorem of Rosenberg [16].

First we need some definitions. The algebra $\mathfrak{A}$ is *affine with respect to an Abelian group* $\mathfrak{G} \equiv \langle A; +\rangle$ if every n-ary polynomial φ can be written

$$\varphi(x_0, \ldots, x_{n-1}) = \varepsilon_0 x_0 + \ldots + \varepsilon_{n-1} x_{n-1} + a$$

for some endomorphisms ε_i of $\mathfrak{G}$ and some $a \in A$. The function $\varphi\colon A^n \to A$ is *metamonic* if $\varphi x = \varphi y$ implies $x_i = y_i$ for some $i = 0, \ldots, n-1$. The algebra $\mathfrak{A}$ is said to be *special* if A can be written as a product $B \times H_1 \times \ldots \times H_m$ $(m \geqslant 1)$ with $|H_1| = \ldots = |H_m| \geqslant 3$ such that for each operation $o\colon A^n \to A$ of $\mathfrak{A}$ and for each $i = 1, \ldots, m$, the projection of o to H_i either is not onto or depends only on the value of one of the arguments of o on one of the factors $H_1, \ldots, H_m$.

PROPOSITION 5. *Let $\mathfrak{A}$ be a k-fold transitive algebra with a surjective polynomial depending on at least two arguments.*

(i) *If $k = 1$ and A is finite, then $\mathfrak{A}$ is not complete iff*
(A) *$\mathfrak{A}$ has a nontrivial automorphism with cycles of equal prime length, or*
(G) *$\mathfrak{A}$ is affine with respect to some elementary Abelian group, or*
(S) *$\mathfrak{A}$ is not simple, or*
(Z) *$\mathfrak{A}$ is special.*

(ii) *If $k = 2$ and A finite, then $\mathfrak{A}$ is not complete iff*
(A′) *$|A| = 2$ and $\mathfrak{A}$ has the nontrivial automorphism, or*
(G) *$\mathfrak{A}$ is affine with respect to some elementary Abelian group.*

(iii) *If $k = 3$, then $\mathfrak{A}$ is not locally complete iff*
(G′) *$\mathfrak{A}$ is affine with respect to some elementary Abelian 2-group or an isomorphic image of Z_3, or*
(M) *every operation of $\mathfrak{A}$ is metamonic.*

(iv) *If $k \geqslant 4$, then $\mathfrak{A}$ is not locally complete iff*
(G″) *$\mathfrak{A}$ is affine with respect to some isomorphic image of Z_2, Z_3 or Z_2^2, or*
(M) *every operation of $\mathfrak{A}$ is metamonic.*

Note that in (i) on account of the 1-fold transitivity, any congruence must have equivalence classes of equal size. Observe that in (iii) and (iv) condition (M) includes (A′), and so the latter is omitted. Note also that condition (M) occurs only if A is infinite or $|A| = 2$.

There are several antecedents of this theorem when A is finite, especially for $k \geqslant 3$. Salomaa [18], as the culmination of a sequence of papers, proved that $\mathfrak{A}$ is complete when $k \geqslant 3$ and the cardinality of A is greater than 5 and not a power of 2. The exceptions when A is a power of two were completely characterized by Schofield [20]. This line of thought leads to the concept of *basic set* — a set of operations on A for which completeness is attained whenever there is adjoined to the set any other surjective operation dependent on at least two arguments. Basic groups of permutations were characterized by Blohina and Kudrjavcev [2]; and basic semigroups of unary operations have been studied by Mal'cev [13], Zaharova [27] and Baĭramov [1].

There is the related result of Rousseau [17] that a finite algebra $\langle A; f\rangle$ with a single operation is complete iff there are no nontrivial subalgebras, automorphisms, or congruences. An improvement and variations on Rousseau's result are given by Rosenberg [15] and Schofield [21]. These are all derived from the comprehensive theorem of Rosenberg [14] mentioned earlier.

Proof. For each part, conditions (A), (G), (S), etc. clearly imply incompleteness, so we need only consider the other direction of implication. The first two parts are derived from a result of Rosenberg [16], with which we shall assume the reader is familiar.

(i) It is easy to see that 1-fold transitivity rules out conditions (1) and (5) of Rosenberg. The only remaining condition of Rosenberg not listed in our proposition is his condition (6), and this is special on account of the 1-fold transitivity.

(ii) Rosenberg's conditions (1), (4), (5) and (6) are eliminated by 2-fold transitivity. The only possibility for condition (2) is the rather trivial (A′).

(iii) We shall assume that $\mathfrak{A}$ is 3-fold transitive, has a surjective polynomial φ depending on at least two arguments, is not locally complete, and is not metamonic; and we shall show that $\mathfrak{A}$ is affine with respect to some elementary Abelian 2-group or an isomorphic image of Z_3. Without loss of generality, assume φ depends on all its arguments, and $0, 1 \in A$. For the moment assume $|A| \geqslant 4$.

We first show that A is $(3, 2)$-transitive. Since $\mathfrak{A}$ is not metamonic, there are a nonconstant polynomial $\varphi\colon A^n \to A$ and elements $z, u \in A^n$ such that $\varphi z = \varphi u$ but $z_i \neq u_i$ for all $i<n$. Without loss of generality, assume $z = \langle 0, \ldots, 0\rangle$ and $u = \langle 1, \ldots, 1\rangle$. Set $B \equiv \{0, 1\}$ and $C \equiv A \setminus \{0, 1\}$. Let us call $a, b, c \in A^n$ a *true triple* if for each $i<n$, the a_i, b_i and c_i are distinct and $\varphi a \neq \varphi b = \varphi c$. If there is a true triple, then by 3-fold transitivity we are finished. With no true triples at hand, by way of contradiction, we successively realize that for any $c \in C^n$, $\varphi c = \varphi z = \varphi u$; that for any $b \in B^n$, $\varphi b = \varphi c$ since $|C| \geqslant 2$; and that for any other $a \in A^n$ there are $b \in B^n$ and $c \in C^n$ such that $b_i \neq a_i \neq c_i$ $(i<n)$, and hence $\varphi a = \varphi b = \varphi c$. That φ is constant is a contradiction.

By $(3, 2)$-transitivity, Lemma 2 and Theorem 2, $\mathfrak{A}$ is singular.

Define a ternary relation R on A as follows: $R(a, b, c)$ if there is a polynomial φ taking the values $1, a, b, c$ on some quartet. We shall show R is a loop operation with unit 1.

To this end we first show R is a partial function. Suppose $R(a, b, c)$ and $R(a, b, c')$ and $c \neq c'$. By Lemma 3 there is a function ε taking a, b, c, c' to $\{0, 1\}$ and separating c and c'. Since $\mathfrak{A}$ is singular it must be that ε sums to zero on the values on each quartet, viz.:

$$\varepsilon 1 + \varepsilon a + \varepsilon b + \varepsilon c = 0,$$

$$\varepsilon 1 + \varepsilon a + \varepsilon b + \varepsilon c' = 0,$$

where addition is modulo two. Since $\varepsilon c \neq \varepsilon c'$, we have a contradiction.

Next we show R is a function, i.e., for all $a, b \in A$ there is a $c \in A$ such that $R(a, b, c)$. This follows from Lemma 2 and 3-fold transitivity. We are now justified in writing $a \cdot b = c$ for $R(a, b, c)$. This same argument also shows $\cdot$ is solvable in each argument.

Clearly $a \cdot 1 = a = 1 \cdot a$ since $R(a, 1, a)$ and $R(1, a, a)$ by singularity. Therefore, $\langle A; \cdot\rangle$ is a loop.

We want to show $\cdot$ is singular. This is equivalent to showing $ab = cd$ implies $ac = bd$ for all $a, b, c, d \in A$. And this is demonstrated in the manner of a previous argument by finding a monomial ε taking all elements in sight into $\{0, 1\}$ and separating ac from bd if they are unequal. Summing over the ε-values of all pertinent quartets, we find that the inequality $ab \neq cd$ must be just when $ac \neq bd$. So $\cdot$ is singular.

By Proposition 2, $\langle A; \cdot\rangle$ is an elementary Abelian 2-group. All that remains to be shown is that each polynomial of $\mathfrak{A}$ is affine with respect to $\langle A; \cdot\rangle$. First note that if $a, b, c, d \in A$ are the values of a polynomial on a quartet then $ab = cd$, for otherwise inequality leads by another separation argument to a contradiction, as in the previous paragraphs. By definition of $\cdot$, the elements 1, x, y and xy are the values of some polynomial on some quartet, and thus so are $\varepsilon 1$, εx, εy and $\varepsilon(xy)$. Then we have

$$\varepsilon(x\cdot y)\cdot\varepsilon 1 = \varepsilon x\cdot\varepsilon y = [\varepsilon x\cdot\varepsilon 1]\cdot[\varepsilon y\cdot\varepsilon 1];$$

hence $\varepsilon *\cdot\varepsilon 1$ is an endomorphism; and therefore, ε is the sum of an endomorphism and the constant $\varepsilon 1$. So any monomial is affine. For any binomial φ, we have by an analogous argument

$$\varphi(x_0, x_1) = \varphi(x_0, 1)\cdot\varphi(1, x_1)\cdot\varphi(1, 1)$$

where each of the factors is affine. We continue by induction on n to prove that any n-ary polynomial is affine.

We must still consider the cases when $|A| \leqslant 3$. For $|A| = 3$ we may develop an *ad hoc* argument. It is easier, however, to apply Rosenberg's theorem and realize that only his quasilinear ($\equiv$ affine) case can be 3-fold transitive.

For $|A| = 2$, we construct an *ad hoc* argument. We show that any polynomial φ is affine with respect to Z_2. Suppose there are $a, b \in A^n$ which differ only at the ith spot, and for which $\varphi a = \varphi b$. If φ depends on the ith argument, there are $c, d \in A^n$, differing only at the ith coordinate, for which $\varphi c \neq \varphi d$. Applying 2-fold transitivity if necessary, we obtain a binomial $\cdot$ with zero and unit. By a classical result of Boolean algebra, $\mathfrak{A}$ is complete, which contradicts our assumptions. Therefore, $\varphi a \neq \varphi b$ for any $a, b \in A^n$ which differ only at one essential argument of φ, and it is easy to show that this characterizes the affinity of φ.

(iv) We assume that $\mathfrak{A}$ is at least 4-fold transitive, has a surjective polynomial depending on at least two arguments, is not locally complete, and is not metamonic. We may proceed as in case (iii), proving everything here that was proven there. However when $|A| \geqslant 5$ a contradiction arises in proving that R is a partial function. For consider a quartet on which some polynomial takes at least three values (Lemma 2). It must really take four values since $\mathfrak{A}$ is singular. By 4-fold transitivity there are two polynomials agreeing on three elements of the quartet, taking the value 1 on one of them and differing on the fourth. Thus there are $(a, b, c) \in R$ for which the element c is not uniquely determined by a and b. This contradiction to (iii) restricts our attention to when $|A| \leqslant 4$, and this was already considered.

Divisibility

There are various ways in which the idea of divisibility in semigroups may be generalized to less structured algebras. We propose one approach which leads to some new, sufficient conditions for local completeness. Assume that $\cdot$ is a binomial of the algebra $\mathfrak{A}$. Define the relation $R \subseteq A^2$ by

aRb if there is an x such that $a\cdot x = b$ or $x\cdot a = b$.

Let $|$ be the ancestral closure of R, i.e.,

$a|b$ if there are $x_1, x_2, \ldots, x_n$ such that $aRx_1, x_1Rx_2, \ldots, x_nRb$.

In a commutative ring these concepts coincide with the usual notion of divisibility.

PROPOSITION 6. *The algebra $\mathfrak{A}$ is locally complete if*

(i) *$\mathfrak{A}$ has a binomial $\cdot$ with zero 0 such that for all $a, b \in A$ if $a \neq 0$ then $a|b$,*

(ii) *$\mathfrak{A}$ is 1-fold transitive, and*

(iii) *$\mathfrak{A}$ is 1-neat.*

Proof. On the basis of Theorem 2 this is true if $\mathfrak{A}$ is $(3, 2)$-transitive. We show first $(2, 2)$-transitivity. In view of 1-fold transitivity, it suffices to find for any $a, b \in A$ with $a \neq 0$ a monomial γ which takes 0 into 0 and a into b. Since $a|b$, there are an $(n+1)$-ary polynomial φ composed only from $\cdot$ and elements $c_1, \ldots, c_n$ for which

$$b = \varphi(a, c_1, c_2, \ldots, c_n)\,.$$

Let $\pi_1, \ldots, \pi_n$ be monomials satisfying $\pi_i a = c_i$ for $i = 1, \ldots, n$. Then the monomial

$$\gamma \equiv \varphi(*, \pi_1 *, \pi_2 *, \ldots, \pi_n *)$$

takes 0 into 0 and a into b.

Now for $(3, 2)$-transitivity. For arbitrary a, b, c with $a, b \neq 0$, we shall find a monomial ε for which $\varepsilon 0 = c$ and $\varepsilon a = 0 = \varepsilon b$. Obviously $\cdot$ is surjective, and hence there are $x, y \in A$ such that $x\cdot y = c$. From the first paragraph, there are monomials δ_1, δ_2 such that $\delta_1 0 = x$, $\delta_1 a = 0 = \delta_2 b$, and $\delta_2 0 = y$. Then the required monomial is

$$\varepsilon \equiv \delta_1 *\cdot\delta_2 *\,. \quad \blacksquare$$

This proposition leads to a curious criterion for completeness in finite algebras.

COROLLARY. *Suppose $\mathfrak{A}$ is a finite, 1-fold transitive algebra having a binomial $\cdot$ with a zero 0 and a unit 1. Assume that there is an iterated product $\prod_i a_i = 1$ in which each nonzero element of A occurs at least once. Then $\mathfrak{A}$ is complete.*

Proof. Since a binomial with unit is 1-neat, it is sufficient to show that $a|b$ for all nonzero b and a in A. Since a occurs in the product $\prod_i a_i$ we have

$$a|(\prod_i a_i\cdot b) = 1\cdot b = b\,. \quad \blacksquare$$

Recalling the Cayley representation theorem that every group gives rise to a transitive group of permutations, we have the following result.

COROLLARY. *Let $\mathfrak{R} \equiv \langle R; +, \cdot\rangle$ be a finite unitary ring. If there is an iterated product equal to* 1 *containing all nonzero elements, then* $\mathfrak{R}^{\#}$ *is complete.*

A more restricted concept than divisibility is that of solvability. We show how this notion can be combined with 1-fold transitivity to yield some locally complete algebras which are generalizations of fields.

PROPOSITION 7. *The algebra $\mathfrak{A}$ is locally complete if*

(i) *$\mathfrak{A}$ is 1-fold transitive; and*

(ii) *there exists a binomial $\cdot$ with a zero* 0 *such that for all $a, b \in A$, if $a \neq 0$, then there exists an $x \in A$ such that $a \cdot x = b$.*

Proof. We shall use Proposition 6. We need only show that $\mathfrak{A}$ is 1-neat. We do this by proving that for all $b, c \in A$ there are $x, y, z \in A$ such that

$$0\cdot y = 0, \quad x\cdot y = b, \quad x\cdot z = c.$$

We achieve this by setting $x \neq 0$ and finding y and z from the equations $x\cdot y = b$ and $x\cdot z = c$. ■

It follows immediately that any field (with all constants adjoined) is locally complete (cf. Foster [4, p. 44], Knoebel [8], and the corollary to Theorem 1 for different proofs). We now turn to various generalizations of fields.

An algebra $\mathfrak{G} \equiv \langle G; \cdot\rangle$ with one binary operation $\cdot$ is a *null-group* if there are $0, 1 \in G$ such that 0 is a zero of $\cdot$ and $\langle G\setminus\{0\}; \cdot\rangle$ is a group with identity 1. Foster [4, p. 47] first defined this notion and has shown that if $\mathfrak{G} \equiv \langle G; \cdot\rangle$ is a null-group, g^{-} is the $\cdot$-inverse of g with $0^{-} = 0$, $^{\cap}$ is any permutation on the set G with $0^{\cap} = 1$, and $^{\cup}$ is its inverse, then the algebra $\mathfrak{A} \equiv \langle G; \cdot, ^{\cap}, ^{\cup}, ^{-}\rangle^{\#}$ is locally complete. We shall give an improvement of this result.

COROLLARY. *Let $\mathfrak{G} \equiv \langle G; \cdot, \pi\rangle$ be an algebra such that*

(i) *$\langle G; \cdot\rangle$ is a null-group with zero* 0, *and*

(ii) *π is a permutation for which $\pi(0) \neq 0$.*

Then $\mathfrak{G}^{\#}$ is locally complete.

Proof. Let $P \equiv \{*\cdot g \mid g \in G\} \cup \{\pi\}$. It is easily shown that P generates a 1-fold transitive group of permutations, and so by Proposition 7, the result follows. ■

Let us now consider the last three concrete examples of the introduction. Recall that these were algebras on the real numbers whose operations were multiplication, the constants and a unary operation ε_i, varying from example to example, given by

$$\varepsilon_1(x) = x+1;$$

$$\varepsilon_2(x) = \begin{cases} 1/(x-1) & \text{if} \quad x \neq 1, \\ 0 & \text{if} \quad x = 1; \end{cases}$$

$$\varepsilon_3(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x = 1, \\ x & \text{otherwise.} \end{cases}$$

Clearly each ε_i is a permutation displacing 0, and so the corollary applies.

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Exact sequences of pairs in commutative rings

by

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Abstract. Let R be a commutative ring with unit and let M be an R-module. We say that a pair (u, v), $u, v \in R$, is M-*exact* if the sequence $M \xrightarrow{u} M \xrightarrow{v} M \xrightarrow{u} M$ is exact. A sequence of pairs $(\boldsymbol{u}, \boldsymbol{v}) = ((u_1, v_1), \ldots, (u_n, v_n))$ is M-*exact* if the pair (u_i, v_i) is $M/(u_1, \ldots, u_{i-1})M$-exact for $i = 1, \ldots, n$.

In the paper we investigate the full subcategory $E_R(\boldsymbol{u}, \boldsymbol{v})$ of R-Mod consisting of all R-modules M such that $(\boldsymbol{u}, \boldsymbol{v})$ is M-exact and rings R such that $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$ and the Jacobson radical $J(R)$ of R is generated by elements $u_1, \ldots, u_n$.

Introduction. Section 1 contains definitions, examples and preliminary results. A homological characterization of modules from $E_R(\boldsymbol{u}, \boldsymbol{v})$ is given provided $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$.

In Section 2 we study conditions which ensure the projectivity or the injectivity of a module from the category $E_R(\boldsymbol{u}, \boldsymbol{v})$ under the assumption that $R \in E_R(\boldsymbol{u}, \boldsymbol{v})$ and $J(R) = (u_1, \ldots, u_n)$. Our main result says that in this case $\mathrm{Inj}_R = E_R(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{Proj}_R$ iff R is artinian, or equivalently, iff R is noetherian and $E_R(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{Fl}_R$ where Fl_R, Inj_R and Proj_R denote the classes of all flat, injective and projective R-modules, respectively.

Section 3 is devoted to the study of local rings R whose maximal ideals are generated by elements $u_1, \ldots, u_n$ such that $(u_1, u_1^{t_1}), \ldots, (u_n, u_n^{t_n})$ is an R-exact sequence of pairs for some natural numbers $t_1, \ldots, t_n$. It is proved that such a ring is R always artinian of the length $(t_1+1)(t_2+1) \ldots (t_n+1)$ and that the associated graded algebra $\mathrm{gr}(R)$ is of the same type.

Throughout this paper R denotes a commutative ring with identity element and $J(R)$ is the Jacobson radical of R. If X is a subset of R and M is an R-module, we set $\mathrm{Ann}_M X = \{m \in M, Xm = 0\}$.

§ 1. Exact sequences of pairs and the category $E_R(\boldsymbol{u}, \boldsymbol{v})$.

Definition 1.1 Let M be a module over a commutative ring R. A pair (u, v) of elements of R is M-*exact* if $uvM = 0$ and the left complex

$$M(u, v): \ldots \to M \xrightarrow{v} M \xrightarrow{u} M \xrightarrow{v} M \xrightarrow{u} M \to 0$$