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Some structure theorems for inverse ω-semigroups *

by

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Abstract. This paper gives various conditions for an inverse ω-semigroup to be an ω-semilattice of ω-groups, and characterizes the effectivity conditions and the structure that must be placed on a collection of ω-groups in order that they form an ω-semilattice of ω-groups. It is also shown by means of counterexamples that the effectivity conditions mentioned above are the best possible ones.

§ 1. Introduction. Let ε stand for the set of non-negative integers (*numbers*), V for the class of all subcollections of ε (*sets*), Λ for the set of isols, and Ω for the class of all recursive equivalence types (RET). By a *function*, we mean a mapping from a subset of ε into a subset of ε. If f is a function, δf and ϱf stand for domain of f and range of f respectively. The relation of inclusion is denoted by $\subset$, α *recursively equivalent* to β by $\alpha \simeq \beta$, for sets α and β, and the RET of α by $\mathrm{Req}(\alpha)$. The concepts of an ω-semigroup and an inverse ω-semigroup were introduced in [3] and that of an ω-group in [7]. The purpose of this paper is to see how to put ω-groups together to form inverse ω-semigroups. Theorems T1, T2 and T5 tell when an inverse ω-semigroup is an ω-semilattice of ω-groups, and Theorem T6 tells us exactly how to put ω-groups together in order to form an inverse ω-semigroup which is an ω-semilattice of these ω-groups. Examples are given following T6 to show that the effectivity conditions of T6 are the best possible. Finally T3 demonstrates that there are continuum many non ω-isomorphic regressive isolic semigroups that are inverse semigroups but not inverse ω-semigroups.

§ 2. Basic concepts and notations. The reader of this paper is assumed to be familiar with the notation and basic results of [3], [4], [6], [7]. The meaning of the concept, "an element y can be effectively found given $x_1, \ldots, x_n$ such that $P(x_1, \ldots, x_n, y)$" can be found on p. 602 of [3].

NOTATION. For the rest of the paper an *ω-semigroup* will be an ordered pair, (α, p), where $\alpha \subset \varepsilon$ and p is a semigroup operation on $\alpha \times \alpha$ which can be extended to a partial recursive function of two variables. We usually denote $p(x, y)$, by $x \cdot y$ or just xy, for $x, y \in \alpha$.

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Remark. We recall from semigroup theory that a *semilattice* is a commutative idempotent semigroup, i.e. a commutative semigroup in which every element is an idempotent. Also, a semigroup S is a *semilattice of groups* if S is a disjoint union of group $\{G_e|\ e \in Y\}$ where Y is a semilattice and such that if $x \in G_a$ and $y \in G_b$ then $x \cdot y \in G_{a \cdot b}$.

DEFINITION. A semigroup S is a *semilattice of ω-groups* if S is a semilattice of groups and the groups in question are each ω-groups.

DEFINITION. A ω-semigroup S is an *ω-semilattice of ω-groups* if S is a semilattice of ω-groups $\{G_e|\ e \in Y\}$ indexed by an ω-semilattice Y and there are functions f and g (each with a partial recursive extension) such that for each $x \in S$, $f(x)$ is the unique $e \in Y$ such that $x \in G_e$ and $g(x)$ is the group inverse, x^{-1}, of x in G_e.

Remark. (i) Let S be an ω-semigroup and $a \in S$. We recall from [3] that $i \in S$ is an *ω-regular left (right) unit of a*, if given a, we can effectively find i and $x \in S$ such that $i \cdot a = a$ and $a \cdot x = i$ ($a \cdot i = a$ and $x \cdot a = i$). Also we recall that i is an *ω-regular two sided unit of a* if it is both an ω-regular left unit of a and an ω-regular right unit of a; and that each $a \in S$ may have at most one ω-regular two sided unit.

(ii) We recall from [3] that an ω-semigroup S is an *inverse ω-semigroup* if for each $a \in S$ there exists a unique $b \in S$ (denoted a^{-1}) such that $a \cdot b \cdot a = a$, $b \cdot a \cdot b = b$ and b is an ω-inverse of a. In other words, an inverse ω-semigroup is an inverse semigroup in which for each $a \in S$ we can uniformly effectively find a^{-1}.

(iii) If S is an inverse ω-semigroup and e is an idempotent of S, from [3] we remember that

$$G_e = \{x \in S|\ e \text{ is an } \omega\text{-regular two sided unit of } x\}$$

is the largest ω-group in S with e as its identity.

§ 3. Decomposing inverse ω-semigroups.

Remark. The purpose of this section is to give various characterizations for an inverse ω-semigroup to be an ω-semilattice of ω-groups.

LEMMA L1 [5, II, p. 41]. *An inverse semigroup S is a union of groups iff the regular left and right units of each element of S are equal.*

LEMMA L2 [5, I, p. 129]. *The condition that a semigroup S be a semilattice of groups is equivalent to the conjunction of any two of the following conditions*:

(1) *S is a union of groups.*

(2) *S is an inverse semigroup.*

(3) *Every one-sided ideal of S is a two sided ideal of S.*

THEOREM T1. *Let S be an inverse ω-semigroup. Then S is a semilattice of groups iff S is an ω-semilattice of ω-groups.*

Proof. The only if part is immediate. So, suppose S is an inverse ω-semigroup that is a semilattice of groups. Then S is a disjoint union of groups $\{H_e|\ e \in Y\}$, for a semilattice Y, and by L1, the regular left and right units of each element of S are equal. However, by [8, p. 79], for each $a \in S$, $a \cdot a^{-1}$ is the only regular right unit of a and $a^{-1} \cdot a$ is the only regular left unit of a. Thus $a \cdot a^{-1} (= a^{-1} \cdot a)$ is the unique regular two-sided unit of a. Since the only idempotents in S are the identities of each H_e, the set of idempotents of S is of the form $\{a \cdot a^{-1}|\ a \in S\}$. But in an inverse ω-semigroup the idempotents commute, thus $\{a \cdot a^{-1}|\ a \in S\}$ is an ω-semilattice and it easily follows we can assume $Y = \{a \cdot a^{-1}|\ a \in S\}$. (Note: Y may not be ω-isomorphic to $\{a \cdot a^{-1}|\ a \in S\}$ but it is enough that it is isomorphic to it.) Also, since S is an inverse ω-semigroup, a^{-1} is the ω-inverse of a; from which it follows that Y is the set of ω-regular two-sided units of elements of S. Thus for each $e \in Y$, $H_e = G_e = \{x \in S|\ e$ is an ω-regular two sided unit of $x\}$, is an ω-group, and since for each $a \in S$ we can effectively find a^{-1} and $a \cdot a^{-1}$ we see that $S = \bigcup \{G_e|\ e \in Y\}$ is an ω-semilattice ω-groups.

COROLLARY 1. *Let S be an inverse ω-semigroup. Then S is an ω-semilattice of ω-groups iff for each $a \in S$, the ω-regular left and ω-regular right units of a are equal.*

Proof. Use L1, L2, and then T1.

COROLLARY 2. *If every element of an inverse ω-semigroup S commutes with its ω-inverse then S is an ω-semilattice of ω-groups.*

COROLLARY 3. *If S is a commutative inverse ω-semigroup, then S is an ω-semilattice of ω-groups.*

COROLLARY 4. *If S is an ω-semigroup then S is an ω-semilattice of ω-groups iff S is an inverse ω-semigroup and either*

(1) *S is a union of groups or*

(2) *every one-sided ideal of S is a two-sided ideal of S.*

Remark. If S is an ω-semigroup which is a group but not an ω-group, then S is a union of groups and an inverse semigroup and every one-sided ideal of S is a two-sided ideal of S; but S is not an inverse ω-semigroup nor an ω-semilattice of ω-groups. However, if S is a periodic ω-semigroup then things are nicer.

THEOREM T2. *If S is a periodic ω-semigroup then S is an ω-semilattice of periodic ω-semigroups iff S is a semilattice of groups.*

Proof. The if part is immediate. If S is a semilattice of groups $\{G_e|\ e \in Y\}$ and S is periodic then by T1 of [3] each of these groups G_e is an ω-group in which an inverse of an element $a \in G_e$ is just a power of a. It follows that S is an inverse ω-semigroup and by T_1, S is an ω-semilattice of periodic ω-groups.

COROLLARY 1. *If S is an isolic semigroup then S is an ω-semilattice of isolic groups iff S is a semilattice of groups.*

COROLLARY 2. *If S is an isolic (periodic ω-) semigroup then S is an ω-semilattice of isolic (periodic ω-) groups iff any two of the following three conditions hold*:

(i) *S is an inverse semigroup.*

(ii) *S is a union of groups.*

(iii) *Every one-sided ideal of S is a two-sided ideal of S.*

Corollary 3. *If S is a commutative isolic (periodic ω-) semigroup then S is an ω-semilattice of isolic (periodic ω-) groups iff either* (i) *is an inverse semigroup or* (ii) *S is a union of groups.*

Proof. Condition (iii) of Corollary 2 holds since S is commutative.

Corollary 4. *If S is a commutative isolic (periodic ω-) semigroup then S is an inverse ω-semigroup iff S is an inverse semigroup.*

Remark. We know from [3] that if S is an isolic semigroup then:

(i) S is a group iff S is an isolic group.

(ii) S is a right group iff S is an isolic right group.

Now by Corollary 4 we see that if S is also commutative then S is an inverse semigroup iff S is an inverse ω-semigroup. The question arises as to how much more general S can get and still allow us to effectively get inverses. The following theorem is relevant to this question.

Theorem T3. *There exist continuum many non ω-isomorphic regressive isolic semigroups which are inverse semigroups but not inverse ω-semigroups.*

Proof. Recall from [3] that $I(\varepsilon)$, the symmetric inverse ω-semigroup on ε, is the coded set of one-to-one finite partial functions from ε into ε under the multiplication of composition of functions. The semigroup we want to consider is a subsemigroup of $I(\varepsilon)$ using functions with one point domains and a nonstandard coding. Partition ε into two element sets, $\varepsilon = \{\{2n, 2n+1\} |\ n \in \varepsilon\}$ and consider four functions, $f_{4n+1}, f_{4n+2}, f_{4n+3}, f_{4n+4}$ on each two element set. Let $f_{4n+1} = \{(2n, 2n)\}$, $f_{4n+2} = \{(2n+1, 2n+1)\}$ $f_{4n+3} = \{(2n, 2n+1)\}$, $f_{4n+4} = \{(2n+1, 2n)\}$. Finally let f_0 be the empty function. The collection $\{f_i |\ i \in \varepsilon\}$ with the normal coding is a subsemigroup of $I(\varepsilon)$, with the property that $f_{4n+j} \circ f_{4m+k} = f_0$ if $m \neq n$ and $1 \leqslant j, k \leqslant 4$. In other words, to examine multiplication in this subsemigroup it suffices to consider how $\{f_{4n+1}, f_{4n+2}, f_{4n+3}, f_{4n+4}\}$ intermultiply. This is easy to see and is left to the reader. It follows that $\{f_i |\ i \in \varepsilon\}$ forms an inverse semigroup. The idea of the proof is to choose a coding of $\{f_i |\ i \in \varepsilon\}$ which gives an effective multiplication but not effective inverses. To this end, let a_n be a T-retraceable function, that is a retraceable function with the property that for each partial recursive function p of one variable there is an $m \in \varepsilon$ such that $n \geqslant m$ implies $p(a_n) < a_{n+1}$ (where $p(a_n) < a_{n+1}$ also holds if $p(a_n)$ is undefined). We recall that ϱa is a retraceable immune set and $\mathrm{Req}(\varrho a)$ is called a *T-regressive isol.* Let a_i be the code of f_i, for all $i \in \varepsilon$; and let $A = (\varrho a, *)$, where $a_i * a_j = a_k$ iff $f_i \circ f_j = f_k$. Since a_n is retraceable, it is easy to check that multiplication in A is effective. Thus A is a regressive isolic semigroup. Now, suppose there were a partial recursive function p such that $p(a_i) = a_i^{-1}$, for all $a_i \in \varrho a$, then there is an $m \in \varepsilon$ such that, $n \geqslant m$ implies $p(a_n) < a_{n+1}$. But, for all $k \in \varepsilon$, $a_{4k+3}^{-1} = a_{4k+4}$. So, for all $k \in \varepsilon$, $p(a_{4k+3}) = a_{4k+4}$. But if $4k+3 \geqslant m$ then $p(a_{4k+3}) < a_{4k+4}$. Hence, we have a contradiction and we conclude no such p exists. Thus A is not an inverse ω-semigroup. The fact that there are continuum many non ω-isomorphic such semigroups A follows from the fact that there are continuum many T-regressive isols [9].

Corollary. *There exists a cosimple regressive semigroup which is an inverse semigroup but not an inverse ω-semigroup.*

Proof. There exists a cosimple T-retraceable set.

Lemma L3 [5, II, p. 41]. *Let S be an inverse semigroup with a finite set E of idempotents. If E forms a chain (under its natural ordering), then S is a union of groups.*

Remarks. (i) The natural ordering on E is given by

$$e \leqslant f \quad \text{iff} \quad e \cdot f = f \cdot e = e \quad \text{for all } e, f \in E.$$

Also, recall that since S is an inverse semigroup, E is commutative, i.e. E is a semilattice.

(ii) We would like to extend L3 to the case where E is isolic. This is of interest, since it is known that L3 is not true in general if E is infinite.

(iii) Let e, a, b be distinct numbers, and let C be the semigroup generated by a and b, $\langle a, b\rangle$, such that $e \cdot a = a \cdot e = a$, $e \cdot b = b \cdot e = b$, $a \cdot b = e$, $b \cdot a \neq e$. Then by [5, I, p. 44], every element of C is uniquely expressible in the form $b^m \cdot a^n$, for $m, n \in \varepsilon$, $(a^0 = b^0 = e)$ and C is called the *bicyclic semigroup* (*generated by a and b*). Thus $C = \{b^m \cdot a^n |\ m, n \in \varepsilon\}$. We call e the *identity of C.*

Definition. Let a and b be distinct numbers. Code C as follows. The code of $b^m \cdot a^n \in C$ is $j(j(b, m), j(a, n))$, where $j(x, y)$ is the standard recursive pairing function from ε^2 onto ε. Then we call $C_{\langle a, b\rangle} = \{j(j(b, m), j(a, n)) |\ m, n \in \varepsilon\}$, with the appropriate associated multiplication, the *bicyclic ω-semigroup* (*generated by a and b*).

Remark. For any $a, b \in \varepsilon$, it is easy to see that $C_{\langle a,b\rangle}$ is an r.e. semigroup.

Lemma L4 [5, I, p. 45]. *If φ is a homomorphism of a bicyclic semigroup C into a semigroup S, then either φ is an isomorphism of C into S, or else $\varphi(C)$ is a cyclic group.*

Theorem T4. *If φ is an ω-homomorphism of a bicyclic ω-semigroup $C_{\langle a,b\rangle}$ into an ω-semigroup S, then either φ is an ω-isomorphism of $C_{\langle a,b\rangle}$ into S, or else $\varphi(C_{\langle a,b\rangle})$ is a cyclic ω-group.*

Proof. The proof follows from T30(iii) of [3], L4 and the fact that $C_{\langle a,b\rangle}$ is an r.e. semigroup, and is left to the reader.

Remark. (i) By using [5, I, p. 80] we see that the set of idempotents of $C_{\langle a,b\rangle}$ is $E_{\langle a,b\rangle} = \{j(j(b, n), j(a, n)) |\ n \in \varepsilon\}$, that $E_{\langle a,b\rangle}$ is recursive and clearly that $E_{\langle a,b\rangle} \simeq \varepsilon$. Furthermore, $E_{\langle a,b\rangle}$ is inverse order ω-isomorphic to $\{\varepsilon, \leqslant\}$, i.e. if $e_n = j(j(b, n), j(a, n))$ for $n \in \varepsilon$, then $e_n \leqslant e_m$ (under the natural ordering of idempotents) iff $m \leqslant n$.

(ii) Also if S is an ω-semigroup and $a, b \in S$ such that C is the bicyclic semigroup generated by a and b then $\{b^n \cdot a^n |\ n \in \varepsilon\}$, the set of idempotents of C, is an infinite r.e. subset of S.

LEMMA L5. *$C_{\langle a,b\rangle}$ is an inverse ω-semigroup.*

Proof. The proof follows from the fact that an inverse of $j(j(b,n),j(a,m))$ is $j(j(b,m),j(a,n))$ and can hence be effectively found, and the fact that C is an inverse semigroup by [5, I, p. 80].

Remark. We are now ready to generalize L3.

THEOREM T5. *Let S be an inverse ω-semigroup with an isolic set E of idempotents. If E forms a chain (under its natural ordering) then S is an ω-semilattice of ω-groups.*

Proof. Let $a\in S$, $e=a\cdot a^{-1}$, $f=a^{-1}\cdot a$. Then, as in the proof of L3, $e\cdot a=a\cdot f=a$ and $a^{-1}\cdot e=f\cdot a^{-1}=a^{-1}$. By hypothesis, either $e\leqslant f$ or $f\leqslant e$, say the latter. Then, $e\cdot f=f\cdot e=f$. So, $a\cdot e=(a\cdot f)\cdot e=a\cdot(f\cdot e)=a\cdot f=a$ and similarly $e\cdot a^{-1}=a^{-1}$. Hence $e\cdot a=a\cdot e=a$ and $e\cdot a^{-1}=a^{-1}\cdot e=a^{-1}$. Now if $e=f$ then by Corollary 1 to T1, we are done. Hence assume $e\neq f$. Thus $a\cdot a^{-1}=e$, $a^{-1}\cdot a\neq e$ and by above e is the identity of $C_{\langle a,a^{-1}\rangle}$. But $E_{\langle a,a^{-1}\rangle}$ is an infinite r.e. set of idempotents in $C_{\langle a,a^{-1}\rangle}$ and also $\{a^{-n}\cdot a^{n}|\ n\in\varepsilon\}$ is an infinite r.e. subset of E. However, this contradicts the fact that E is isolic. Thus $e=f$ and we are done.

Remark. By L5, $C_{\langle a,b\rangle}$ is an inverse ω-semigroup with its set of idempotents forming a chain. However $E_{\langle a,b\rangle}$ is not isolic and furthermore $C_{\langle a,b\rangle}$ is not a union of groups.

EXAMPLE. (An S satisfying hypotheses of T5). Let τ be a regressive immune set, where $\tau=\varrho t$. Then, let $G=\langle\beta,\odot\rangle$ be an ω-group with e the identity of G. Put $\alpha=\{j(t_n,x)|\ n\in\varepsilon, x\in G\}$. Define $S=(\alpha,\cdot)$ where $j(t_n,x)\cdot j(t_m,y)=j(t_{\max(n,m)},x\odot y)$. Thus we see that the set of idempotents of S, $E=\{j(t_n,e)|\ n\in\varepsilon\}$. Furthermore, for $m,n\in\varepsilon$, $j(t_n,e)\cdot j(t_m,e)=j(t_m,e)\cdot j(t_n,e)=j(t_m,e)$ iff $m\geqslant n$. Thus the natural ordering on E is isomorphic to $\geqslant$ on ε. Hence E forms a chain under its natural ordering. Since $\mathrm{Req}(E)=\mathrm{Req}(\tau)$, E is isolic. Finally, since $j(t_n,x^{-1})$ is the unique inverse in S of $j(t_n,x)$, we see that S is an inverse ω-group.

§ 4. The structure of ω-semilattices of ω-groups.

Remark. The purpose of this section is to find the best possible effective analogue of the following theorem.

LEMMA L6 [5, I, p. 128]. *Let Y be any semilattice, and to each element a of Y assign a group G_a such that G_a and G_b are disjoint iff $a\neq b$ in Y. To each pair of elements $a,b\in Y$ such that $a>b$, assign a homomorphism $\varphi_{a,b}$ of G_a into G_b such that if $a>b>c$ then $\varphi_{b,c}\circ\varphi_{a,b}=\varphi_{a,c}$. Let $\varphi_{a,a}$ be the identity automorphism of G_a. Let $S=\bigcup\{G_a|\ a\in Y\}$ and define the product of $x_a,y_b\in S(x_a\in G_a,y_b\in G_b)$ by $x_a\cdot y_b=\varphi_{a,c}(x_a)\cdot\varphi_{b,c}(y_b)$, where c is the product of a and b in Y.*

Then S is an inverse semigroup which is a union of groups. Conversely, every such semigroup can be constructed in this manner.

LEMMA L7 [5, I, p. 127]. *Let S be an inverse semigroup which is a union of groups, then every idempotent of S is in the center of S.*

Remark. The analogue we want is the following theorem.

THEOREM T6. *Let Y be an ω-semilattice and to each $a\in Y$ assign an ω-group G_a such that G_a and G_b are disjoint if $a\neq b$. Set $S=\bigcup\{G_a|\ a\in Y\}$ and suppose the function f from S onto Y such that $x\in G_a\leftrightarrow f(x)=a$ has a partial recursive extension. Also suppose if $a\in Y$ we can effectively find the multiplication and inverse functions of G_a. To each pair $a,b\in Y$, $a>b$, effectively assign a homomorphism $\varphi_{a,b}$ of G_a into G_b (where $\varphi_{a,b}$ has a partial recursive extension), such that if $a>b>c$ then $\varphi_{b,c}\circ\varphi_{a,b}=\varphi_{a,c}$. Let $\varphi_{a,a}$ be the identity automorphism of G_a. Define the product of elements of S, $x_a\in G_a$, $y_b\in G_b$ by: $x_a\cdot y_b=\varphi_{a,c}(x_a)\cdot\varphi_{b,c}(y_b)$, where $c=a\cdot b$, in Y, and where the product is in G_c.*

Then S is an ω-semilattice of ω-groups. Furthermore, every ω-semilattice of ω-groups, indexed by its set of idempotents E, can be constructed in the above manner.

Proof. By L6, S is a semilattice of groups. Hence to show S is an ω-semilattice of ω-groups, it suffices by T1 to show S is an inverse ω-semigroup. By T28 of [3] this can be accomplished by showing S has an effective multiplication, effective inverses, and any two idempotents of S commute. Since S is a semilattice of groups we need only show the first two conditions.

Given $x,y\in S$, set $a=f(x)$ and $b=f(y)$, where f has a partial recursive extension. Then $x\in G_a$, $y\in G_b$ and we can effectively get $c\in Y$, where $c=a\cdot b$ in Y. But $a>c$ and $b>c$ (in the natural ordering on Y) and we can effectively get, from a and b, the homomorphisms with partial recursive extensions, $\varphi_{a,c}$ and $\varphi_{b,c}$. Thus $x\cdot y=\varphi_{a,c}(x)\cdot\varphi_{b,c}(y)$ where the multiplication is in G_c. Since given c, we can effectively find the multiplication in G_c, it follows that given $x,y\in S$, we can effectively find the product of x and y in S.

To find the inverse of $x\in S$ we need only find its group inverse. But given $x\in S$, $x\in G_a$ where $a=f(x)$, and from a we can effectively find the group inverse of x. Hence S has effective inverses. It follows that S is an ω-semilattice of ω-groups.

Conversely, if S is an ω-semilattice of ω-groups indexed by E, then $S=\bigcup\{G_a|\ a\in E\}$ where E is an ω-semilattice and there are functions f and g (with partial recursive extensions) such that for each $x\in S$, $f(x)=a\leftrightarrow x\in G_a$ and $g(x)$ is the group inverse, x^{-1}, of x in G_a. (Note: $f(x)=x\cdot x^{-1}$). If suffices to consider the condition of effectively assigning a homomorphism $\varphi_{a,b}$ of G_a into G_b for each $a,b\in E$ such that $a>b$. To this end, if $a>b$ (in E) then assign the mapping $\varphi_{a,b}$ defined by $\varphi_{a,b}(x)=x\cdot e_b$, where $x\in G_a$, e_b is the identity of G_b and the multiplication is in S. Now $\varphi_{a,b}$ is a homomorphism, since if $x,y\in G_a$, then using L7, $\varphi_{a,b}(x)\cdot\varphi_{a,b}(y)=(x\cdot e_b)\cdot(y\cdot e_b)=(x\cdot y)(e_b\cdot e_b)=(xy)\cdot e_b=\varphi_{a,b}(x\cdot y)$. Furthermore, the only idempotent in G_b is e_b so $e_b=b$. Hence if follows that given $a,b\in E$ we can effectively find a homomorphism $\varphi_{a,b}$ with a partial recursive extension. The remaining three conditions concerning the homomorphisms are straightforward to check providing we keep in mind that E is contained in the center of S.

Remark. The restriction in the second part of T6 that the ω-semilattice be indexed by E (or equivalently any set ω-isomorphic to E) is necessary as indicated by the next example.

EXAMPLE. Consider a special case of the ω-semilattice of ω-groups in the example following T5. Let $G = \{0\}$ be the trivial ω-group. Thus $\alpha = \{j(t_n, 0)|\ n \in \varepsilon\}$ and $S = (\alpha, \cdot)$. Now S can be considered to be an ω-semilattice of ω-groups indexed by $Y = \varepsilon$ with dual ordering $\geqslant'$, since given $j(t_n, 0) \in S$ we can effectively find n. However, if $m \geqslant n$, i. e. $n \geqslant' m$, then we have $\varphi_{n,m}$ defined by $\varphi_{n,m}(j(t_n, 0)) = j(t_n, 0)\cdot j(t_m, 0) = j(t_m, 0)$. It follows that if there is an effective procedure for getting $\varphi_{n,m}$ from m and n then letting $n = 0$, we have an effective way of getting t_m from m, for any m. This contradicts the fact that T is a regressive immune set. Thus this ω-semilattice cannot be gotten by the construction in T6.

Remark. It might seem that some of the effectivity conditions in T6 might be unnecessary. However, the following examples will show that all of the following effectivity conditions of T6 are independent of one another, in the sense that if any one is false, S is not even an inverse ω-semigroup (except that if (ii) is false then (iv) or (v) are vacuously false, and if (vi) is false (vii) is irrelevant).

(i) Y is an ω-semilattice.

(ii) $\{G_a|\ a \in Y\}$ is a class of ω-groups.

(iii) f has a partial recursive extension.

(iv) Given $a \in Y$, we can effectively find the multiplication in G_a.

(v) Given $a \in Y$, we can effectively find the inverse function of G_a.

(vi) For each $a > b$ in Y, $\varphi_{a,b}$ has a partial recursive extension.

(vii) Given $a, b \in Y$, $a > b$, we can effectively find $\varphi_{a,b}$.

EXAMPLES. (A) (Independence of (i)). Let τ be a regressive immune set and $\tau = \rho t$. Define $t_m \cdot t_n = t_{lcm(m,n)}$, where $lcm(m, n) =$ least common multiple of m and n. This multiplication cannot be effective. For, if it were, then given t_3 we could effectively generate all of τ, by using the regressiveness of τ and the fact that $lcm(n, n-1) > n$ for all $n \geqslant 3$. So letting $Y = \tau$ with the above multiplication, we see Y is a semilattice but not an ω-semilattice. Now, for each $n \in \varepsilon$, let $G_n = \{j(t_n, 0\})$, the trivial group. Then $\{G_n|\ t_n \in Y\}$ is a class of ω-groups and (ii) is true. The function f in this case is just $f(j(t_n, 0)) = t_n$ and clearly has a partial recursive extension. Thus (iii) is satisfied. Since all G_n are trivial, (iv) and (v) are true. In Y, $t_m > t_n$ iff m divides n, $(m|n)$. So if $t_m > t_n$, define φ_{t_m,t_n} to be the trivial homomorphism from G_m onto G_n. Thus φ_{t_m,t_n} satisfies the conditions of the theorem and given $t_m, t_n \in Y$ we can effectively get φ_{t_m,t_n}. So (vi) and (vii) are true. But (i) is false since multiplication in Y is not effective. Thus (i) is independent. Also we see that S is not an ω-semigroup, since if multiplication in S were effective, then since $j(t_m, 0)\cdot j(t_n, 0) = j(t_{lcm(m,n)}, 0)$ in S, the multiplication in Y would be effective.

(B) (Independence of (iii)). We note first that given a regressive immune set α there exists a retraceable immune set β such that $\alpha \simeq \beta$ and if $\beta = \rho b$ then $i \leqslant j \leftrightarrow b_i | b_j$. For if $\alpha = \rho a$ define b by, $b_0 = 2^{a_0}$ and for all $n \geqslant 1$, $b_n = 2^{a_0}\cdot 3^{a_1} \ldots p_n^{a_n}$, where $p_0 = 2$ and $p_n = n$th odd prime. Also since we can always assume $a_0 = 1$ we can always get $b_0 = 2$.

Now to continue the example. Let α, β be retraceable immune sets such that $\mathrm{Req}(\alpha) < \mathrm{Req}(\beta)$. Hence $\mathrm{Req}(\alpha) \leqslant_* \mathrm{Req}(\beta)$. Also let $\alpha = \rho a$, $\beta = \rho b$ and α have the property that $i \leqslant j \leftrightarrow a_i | a_j$. Since $\mathrm{Req}(\alpha) \leqslant_* \mathrm{Req}(\beta)$, there is a function h, with a partial recursive extension such that $h(a_n) = b_n$, for $n \in \varepsilon$. Because $\mathrm{Req}(\alpha) < \mathrm{Req}(\beta)$, $\alpha \not\simeq \beta$, and so h^{-1} does not have a partial recursive extension. Let G_n be a copy of the infinite cyclic group with generator z encoded by:

$$
\begin{aligned}
&\vdots\\
z^m &\leftrightarrow j(b_n, 2m),\\
&\vdots\\
z^2 &\leftrightarrow j(b_n, 4),\\
z &\leftrightarrow j(b_n, 2),\\
e &\leftrightarrow j(b_n, 0),\\
z^{-1} &\leftrightarrow j(b_n, 1),\\
z^{-2} &\leftrightarrow j(b_n, 3),\\
&\vdots\\
z^{-m} &\leftrightarrow j(b_n, 2m-1),\\
&\vdots
\end{aligned}
$$

Let $Y = \alpha$ and $a_i \cdot a_j = \max(a_i, a_j)$. Thus in the natural ordering of Y, $a_i < a_j \leftrightarrow a_i \cdot a_j = a_i \leftrightarrow a_j | a_i \leftrightarrow j < i$. Now for $a_j < a_i$ in Y, define φ_{a_j,a_i} by,

$$\varphi_{a_j,a_i}(j(b_j, 2m)) = j\left(b_i, \left(\frac{a_i}{a_j}\cdot 2m\right)\right), \quad m \geqslant 0,$$

and

$$\varphi_{a_j,a_i}(j(b_j, 2m-1)) = j\left(b_i, \left(\frac{a_i}{a_j}\cdot 2m\right)-1\right), \quad m \geqslant 1.$$

It can be shown that φ_{a_j,a_i} is a homomorphism and that it satisfies the necessary conditions in T6. Now given a_i and a_j, $a_j > a_i$ we can effectively get a_i/a_j and using h, we can effectively get b_j and b_i, i.e. we can effectively get φ_{a_j,a_i}. Thus (vi) and (vii) are satisfied. Also (i), (ii), (iv) and (v) are clearly satisfied. It remains to see that (iii) is false. But if (iii) is true then given b_i we can get $j(b_i, 0)$ and hence $f(j(b_i, 0)) = a_i$, i.e. we get a partial recursive extension for h^{-1}, which is not possible. So (iii) is false. Again we also get that multiplication in S is not effective. Since, if it were, given b_n we could form the product $j(b_n, 0)\cdot j(b_0, 2)$. But using the fact that $a_0 = 2$, we get

$$
\begin{aligned}
j(b_n, 0)\cdot j(b_0, 2) &= \varphi_{a_n,a_n\cdot a_0}(j(b_n, 0))\cdot \varphi_{a_0,a_n\cdot a_0}(j(b_n, 2))\\
&= \varphi_{a_n,a_n}(j(b_n, 0))\cdot \varphi_{a_0,a_n}(j(b_0, 2))\\
&= j(b_n, 0)\cdot j(b_n, a_n) = j(b_n, a_n).
\end{aligned}
$$

Thus given b_n we can get a_n which is not possible. It follows that multiplication in S is not effective and S is not an ω-semigroup.

(C) (Independence of (iv)). Let t be a strictly increasing total recursive function such that $0 \notin \varrho t$. Recall that $p_n = n$th odd prime, where $p_0 = 2$. Set

$$\alpha_0 = \{2^{t_0}\cdot x|\ (x = 1) \text{ or } (x = p_{i_1}\cdot p_{i_2}\cdot \ldots \cdot p_{i_k}, \text{ where } k, i_1, i_2, \ldots, i_k \in \varepsilon - \{0\}, \text{ and } i_1, i_2, \ldots, i_k \text{ are all distinct})\}.$$

For $2^{t_0}\cdot x$, $2^{t_0}\cdot y \in \alpha_0$, define m_0 by $m_0(2^{t_0}\cdot x, 2^{t_0}\cdot y) = 2^{t_0}\cdot z$, where z is the product of all primes that appear as factors in exactly one of x and y. Let $G_0 = (\alpha_0, m_0)$. We see that G_0 is an ω-group, where 2^{t_0} is the identity and all elements of G_0 are of order 2. Now order α_0 by:

$$2^{t_0}, 2^{t_0}\cdot p_1, 2^{t_0}\cdot p_2, 2^{t_0}\cdot p_2 p_1, 2^{t_0}\cdot p_3, 2^{t_0}\cdot p_3 p_1, 2^{t_0}\cdot p_3 p_2, 2^{t_0}\cdot p_3 p_2 p_1, 2^{t_0}\cdot p_4, \ldots,$$

so that each time a new prime, p_n, is introduced we repeat all the 2^{n-1} terms that preceed p_n with p_n added as a factor to these terms. Let r_n be the recursive function which enumerates α_0 in the preceeding order, where $r_0 = 2^{t_0}$. Now let h be a non-recursive total function such that $0, 1, 2, 3, \notin \varrho h$. For each $n \geqslant 1$, define χ_n to be the cyclic permutation $(r_1, r_2, \ldots, 2^{t_0}\cdot p_{h(n)})$. Also for each $n \geqslant 1$, let

$$\alpha_n = \{2^{t_n}\cdot x|\ 2^{t_0}\cdot x \in \alpha_0\}$$

and define ψ_n, a one-to-one map from α_n onto α_0 by $\psi_n(2^{t_n}\cdot x) = 2^{t_0}x$. Then for $n \geqslant 1$ and $a, b \in \alpha_n$, let

$$m_n(a, b) = \psi_n^{-1}\chi_n^{-1} m_0\big(\chi_n\psi_n(a), \chi_n\psi_n(b)\big).$$

It can be shown that m_n is a group multiplication, and since χ_n and ψ_n have partial recursive extensions, it follows that $G_n = (\alpha_n, m_n)$ is an ω-group of elements of order 2, for all $n \in \varepsilon$. For $t_m, t_n \in \varrho t$ define $t_m\cdot t_n = \max(t_m, t_n)$. Thus $(\varrho t, \cdot)$ is an ω-semilattice. Furthermore, if we define for $t_m > t_n$, $\varphi_{t_m, t_n}(2^{t_m}\cdot x) = 2^{t_n}$, then φ_{t_m, t_n} satisfies the conditions of T6 and $S = \bigcup \{G_n|\ t_n \in Y\}$ is a semilattice of ω-groups satisfying conditions (i), (ii), (iii), (v), (vi), (vii). However we claim that given $t_n \in Y$, we cannot effectively find the multiplication m_n. For if we could, we will show it would follow that h is recursive. To this end, consider the following:

(1) If $2 < k < h(n)$,

$$\begin{aligned} m_n(2^{t_n}\cdot p_k, 2^{t_n}\cdot p_k\cdot p_1) &= \psi_n^{-1}\chi_n^{-1} m_0\big(\chi_n\psi_n(2^{t_n}p_k), \chi_n\psi_n(2^{t_n}p_k p_1)\big) \\ &= \psi_n^{-1}\chi_n^{-1} m_0\big(\chi_n(2^{t_0}p_k), \chi_n(2^{t_0}p_k p_1)\big) \\ &= \psi_n^{-1}\chi_n^{-1} m_0(2^{t_0}p_k p_1, 2^{t_0}p_k p_2) \\ &= \psi_n^{-1}\chi_n^{-1}(2^{t_0}p_2 p_1) = \psi_n^{-1}(2^{t_0}p_2) = 2^{t_n}p_2. \end{aligned}$$

(2) If $k = h(n)$

$$\begin{aligned} m_n(2^{t_n}p_k, 2^{t_n}p_k\cdot p_1) &= \psi_n^{-1}\chi_n^{-1} m_0\big(\chi_n\psi_n(2^{t_n}p_k), \chi_n\psi_n(2^{t_n}p_k p_1)\big) \\ &= \psi_n^{-1}\chi_n^{-1} m_0\big(\chi_n(2^{t_0}p_k), \chi_n(2^{t_0}p_k p_1)\big) \\ &= \psi_n^{-1}\chi_n^{-1} m_0(2^{t_0}p_1, 2^{t_0}p_k p_1) \\ &= \psi_n^{-1}\chi_n^{-1}(2^{t_0}p_k) = \psi_n^{-1}(2^{t_0}\cdot p_{k-1}\cdot p_{k-2}\cdot \ldots \cdot p_2 p_1) \\ &= 2^{t_n}\cdot p_{k-1}\cdot p_{k-2}\cdot \ldots \cdot p_2 p_1. \end{aligned}$$

Thus we have, for all $n \in \varepsilon$,

$$h(n) = k \leftrightarrow 3 | m_n(2^{t_n}p_k, 2^{t_n}p_k p_1).$$

Hence if (iv) were true, h would be a recursive function. Therefore, (iv) is false and multiplication in S is not effective.

It is of interest to note that S is recursive as a set.

(D) (Independence of (v)). Let α be an r.e. set which is not recursive and $a(x)$ be a one-to-one recursive function ranging over α. Assume without loss of generality that $a(0) = 0$ and $a(1) = 1$. For each number $n > 1$ we define a cyclic group G_n as follows:

(a) If $n \in \alpha$, $(\exists k)[a(k) = n]$. Let G_n be a cyclic group of order k,

$$\{x, \ldots, x^{k-1}, x^k = e\}$$

and encoded by:

$$\begin{gathered} e \leftrightarrow j(n, 0), \\ x \leftrightarrow j(n, 2), \\ x^2 \leftrightarrow j(n, 4), \\ \vdots \\ x^{k-1} \leftrightarrow j(n, 2k-2). \end{gathered}$$

(b) If $n \notin \alpha$, G_n is an infinite cyclic group with generator z and encoded by:

$$\begin{gathered} \vdots \\ z^m \leftrightarrow j(n, 2m), \\ \vdots \\ z^2 \leftrightarrow j(n, 4), \\ z \leftrightarrow j(n, 2), \\ e \leftrightarrow j(n, 0), \\ z^{-1} \leftrightarrow j(n, 1), \\ z^{-2} \leftrightarrow j(n, 3), \\ \vdots \\ z^{-m} \leftrightarrow j(n, 2m-1), \\ \vdots \end{gathered}$$

Let Y be the ω-semilattice $\varepsilon-\{0,1\}$ under $\leqslant$, i.e. $m\cdot n = \min(m,n)$ for $m,n\in\varepsilon-\{0,1\}$. Also if $n>m$, let $\varphi_{n,m}(j(n,i)) = j(m,0)$. It follows using T2 of [3] that $S = \bigcup\{G_n|\ n\in Y\}$ is a semilattice of ω-groups satisfying conditions (i), (ii), (iii), (iv), (vi), (viii). However by construction, $n\in\alpha\leftrightarrow$ the inverse of $j(n,2)$ is not $j(n,1)$. Thus if (v) were true, α would be recursive. Thus (v) is independent and S is not an inverse ω-semigroup.

(E) (Independence of (vi)). Let α be an infinite set and let Y be the ω-semilattice on $\{1,2\}$ with $1\leqslant 2$. We recall from [1] that $P(\alpha)$ is the ω-group of codes of the finite permutations of α. Let $\alpha_1 = \{j(1,x)|\ x\in P(\alpha)\}$ and $\alpha_2 = \{j(2,x)|\ x\in P(\alpha)\}$. For $i=1,2$, define multiplication, $*$, on α_i by $j(i,x)*j(i,y) = j(i,x\cdot y)$ where $x\cdot y$ is multiplication in $P(\alpha)$. Then $G_1 = (\alpha_1,*)$ and $G_2 = (\alpha_2,*)$ are ω-groups. By P6 of [3], let ψ be an automorphism of $P(\alpha)$ which does not have a partial recursive extension. Define $\varphi_{2,1}$ from G_2 onto G_1 by $\varphi_{2,1}\ (j(2,x)) = j(1,\psi(x))$, for $x\in P(\alpha)$. We see that $S = G_1\cup G_2$ clearly satisfies conditions (i) through (v). Recall that k and l are the recursive functions such that $j(k(n),l(n)) = n$. Since $\psi(x) = l\varphi_{2,1}(j(2,x))$ and ψ does not have a partial recursive extension we have $\varphi_{2,1}$ does not have a partial recursive extension. Thus condition (vi) is false. Also multiplication in S is not effective since for all $x\in P(\alpha)$,

$$\begin{aligned} l[j(1,1)\cdot j(2,x)] &= l[\varphi_{1,1}(j(1,1)) * \varphi_{2,1}j(2,x)] \\ &= l[j(1,1)*j(1,\psi(x))] \\ &= lj(1,\psi(x)) = \psi(x). \end{aligned}$$

Thus S is not an ω-semigroup.

(F) (Independence of (vii)). Let τ be as in Example (A). Let $P(\varepsilon)$ be as in Example (E). For each $n\in\varepsilon$, let $\alpha_n = \{j(t_n\cdot x)|\ x\in P(\varepsilon)\}$ and $G_n = (\alpha_n, m_n)$, where $m_n(j(t_n,x), j(t_n,y)) = j(t_n, x\cdot y)$, with $x\cdot y$ being multiplication in $P(\varepsilon)$. Also, let $Y=\varepsilon$ with multiplication $m\cdot n = lcm(m,n)$. For $m>n$ in Y (i.e. $m|n$), let $\varphi_{m,n}(j(t_m,x)) = j(t_n,x)$, for all $x\in P(\varepsilon)$. It is straightforward to show that $S = \bigcup\{G_n|\ n\in\varepsilon\}$ satisfies conditions (i) through (vi). We observe that multiplication in S takes the form:

$$\begin{aligned} j(t_n,x)\cdot j(t_m,y) &= m_p\big(\varphi_{n,p}(j(t_n,x)),\ \varphi_{m,p}(j(t_m,y))\big) \\ &= m_p(j(t_p,x), j(t_p,y)) \\ &= j(t_p, x\cdot y) \quad \text{where } p = lcm(m,n)\,. \end{aligned}$$

As in Example (A), if given $m,n\in\varepsilon$, $m>n$, we could effectively find $\varphi_{m,n}$, then from $j(t_3,1)$ we could effectively get $j(t_n,1) = \varphi_{3,n}(j(t_3,1))$ for all n such that $3|n$. Thus, by the regressiveness of τ, from t_3 we could effectively get all of τ. This contradicts the fact that τ is immune. So condition (vii) is false. By the same argument as in Example (A) we can show that multiplication in S is not effective and hence that S is not an ω-semigroup.

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