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Quadratic forms over fields with $u = q/2 < +\infty$

by

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Abstract. This paper determines the quadratic form structure for fields K of characteristic not two with $q = |\dot{K}/\dot{K}^2| < \infty$ and $u = q/2$ where u is the u-invariant for K. Some results are shown for elements $a \in K$ which have the property that the quadratic form $(1, a)$ represents only two square classes in K. These elements play a central role in fields with $u = q/2$. An iterated power series extension of K is a field of the type $K((x_1)) \dots ((x_r))$ where $K((x_1))$ is the field of formal power series over K. If the stufe, s, of K is at least four and if $u = q/2 < \infty$, then it is shown that K is equivalent to the 2-adic numbers or an iterated power series extension of them. If $s = 1$ or 2, K is equivalent to one of three types of fields or iterated power series extensions thereof. Unfortunately it is not known whether these three types of fields exist.

1. If K is a non-formally real field of characteristic different from two, let $Q(K) = \dot{K}/\dot{K}^2$, $q = |Q(K)|$, s be the level (Stufe) of K, and u be the u-invariant. The goal of this paper is to determine all possible quadratic form structures over fields with $u = q/2 < +\infty$. The structures over fields with $u = q$ are completely known — see [1], [8]. Moreover, Elman and Lam [4] have shown that if $u < q$, then $u \leqslant q/2$. We will obtain precise characterizations for fields with $u = q/2$ and $s \geqslant 4$, but the cases $s = 1, 2$ have only been determined up to the existence of certain special types of fields.

Some additional notation: $(a_1, \dots, a_n)$ denotes the diagonalized quadratic form $\sum_{i=1}^{n} a_i X_i^2$; for a quadratic form φ, $G(\varphi)$ denotes the elements of $Q(K)$ represented by φ; $G(\varphi) = \{a_1, \dots, a_n\}$ is an abbreviation for $G(\varphi) = \{a_1\dot{K}^2, \dots, a_n\dot{K}^2\}$ while $G(\varphi) = \langle a_1, \dots, a_n\rangle$ means $G(\varphi)$ is the group in $Q(K)$ generated by the independent elements $a_1\dot{K}^2, \dots, a_n\dot{K}^2$; and finally $a \in G(\varphi)$ means $a\dot{K}^2 \in G(\varphi)$. Two fields K, F are equivalent with respect to quadratic forms if there is an isomorphism $t\colon Q(K) \to Q(F)$ such that $t(-1) = -1$ and $t[G(a_1, \dots, a_n)] = G[t(a_1), \dots, t(a_n)]$ for all n and $a_i \in \dot{K}$. It was shown in [1] that $n = 2$ suffices. Also pointed out in [1] was that two fields are equivalent if and only if the Witt rings are isomorphic. Thus equivalent fields have the same quadratic form structures and conversely. The fields we will classify for $u = q/2$ will be determined up to this equivalence.

One further concept that is required is Kaplansky's radical, R [5]. One formulation for R is $\{a \in \dot{K} \mid G(1, -a) = Q(K)\}$, and it is a subgroup of $\dot{K}$ containing $\dot{K}^2$.

Properties of R were discussed in [2], and all fields with $|\dot{K}/R| \leqslant 8$ were found up to equivalence in [3]. It was demonstrated in [2] that fields with $u = |\dot{K}/R|$ act akin to fields with $u = q$. So we only have to be concerned with the case $u = q/2$ and $R = \dot{K}^2$.

2. This section contains some general results about what elements particular binary quadratic forms represent over arbitrary non-formally real fields. Recall that $x \in G(1, y)$ if and only if $-y \in G(1, -x)$. If $\varphi \oplus \psi$ is the direct sum of two forms φ, ψ, then

$$G(\varphi \oplus \psi) = \bigcup_{b \in G(\psi)} G[\varphi \oplus (b)] = \bigcup_{a \in G(\varphi),\, b \in G(\psi)} G(a, b).$$

The last two remarks as well as the following lemma will be used frequently in the remainder of the paper.

LEMMA. *Let a, b be non-zero elements out of any field K. Then*

$$G(1, a) \cap G(1, b) \subseteq G(1, -ab).$$

Proof. If $x \in G(1, a) \cap G(1, b)$, then $x \in G(1, a)$ and $x \in G(1, b)$. Thus $-a, -b \in G(1, -x)$ and so is ab. Hence $x \in G(1, -ab)$.

Our first theorem is concerned with fields which have an element x satisfying the sort of minimum condition (on the number of elements represented by a binary form) $G(1, x) = \langle x \rangle$.

THEOREM 1. *Suppose for some positive integer r that $2^r \times (1)$ is not universal over a nonreal field K. If $G(1, x) = \langle x \rangle$ for some $x \in \dot{K}$, then $G(1, ax) = \langle ax \rangle$ for all $a \in G[2^r \times (1)]$.*

Proof. It follows easily by induction that $G(1, x) = \langle x \rangle$ implies

$$G[(k \times (1)) \oplus (x)] = G[k \times (1)] \cup G(1, x).$$

So in particular

(1) $$G[(2^r \times (1)) \oplus (x)] = G[2^r \times (1)] \cup G(1, x).$$

On the other hand

$$G[(2^r \times (1)) \oplus (x)] = \bigcup_{\alpha \in G[2^r \times (1)]} G(\alpha, x).$$

So for $a \in G[2^r \times (1)]$, we have

(2) $$G[(2^r \times (1)) \oplus (x)] = G(1, x) \cup G(a, x) \cup \ldots$$

For both (1) and (2) to hold, we must have $G(a, x) - \{x\} \subseteq G[2^r \times (1)]$ or equivalently, $G(1, ax) - \{ax\} \subseteq aG[2^r \times (1)]$. But $G[2^r \times (1)]$ is a group (see [6, Satz 2]), so $aG[2^r \times (1)] = G[2^r \times (1)]$. If $G(1, ax) \neq \langle ax \rangle$, then there would be $b, bax \in G(1, ax) - \langle ax \rangle$. Hence, $b \cdot (bax)$ and $ax \in G[2^r \times (1)]$. And this means $x \in G[2^r \times (1)]$. But, if this were the case, $G[(2^r \times (1)) \oplus (x)] = G[2^r \times (1)]$; and this contradicts the facts that K is nonreal and $G[2^r \times (1)] \neq Q(K)$. So $G(1, ax) = \langle ax \rangle$.

COROLLARY. *If K is a nonreal field, then for $x \in \dot{K}$, $G(1, x) = \langle x \rangle$ if and only if $G(1, -x) = \langle -x \rangle$.*

Proof. Since K is nonreal, it has a finite level s. By Pfister [6], s is a power of 2. If $s \times (1)$ is not universal, the corollary follows from the theorem. If $s \times (1)$ is universal, then by Elman and Lam [4, Corollary 3.6], $|G(\varphi)| \geqslant 2 \dim \varphi$ for all anisotropic φ of dimension greater than one. Hence, $G(1, x) = \langle x \rangle$ would be impossible in this case.

Another result we can obtain from Theorem 1 and the corollary concerns the set $G(1, 1)$. This is turn gives another characterization of nonreal fields whose Witt rings are finite of maximal order with respect to a fixed q.

PROPOSITION 1. *If K is a field with $u < \infty$ and $u < q$, then $G(1, 1) \neq \langle -1 \rangle$.*

Proof. Let us suppose $G(1, 1) = \langle -1 \rangle$. Then $s = 2$. By [1, Theorem 5.13], there is an $a \in \dot{K}(-\dot{K}^2)$ such that $|G(1, a)| > 2$. Consider the group $G(1, 1, a, a)$. $G(1, 1) = \langle -1 \rangle$ implies

$$G(1, 1, a, a) = \bigcup_{\alpha, \beta \in G(1,1)} (\alpha, a\beta) = \pm G(1, a) \cup \pm G(1, -a);$$

and $\langle -1 \rangle G(1, a), \langle -1 \rangle G(1, -a)$ are subgroups of $G(1, 1, a, a)$. However, it is an elementary fact that no group is the union of two subgroups neither of which contains the other. So a contradiction will be reached if we can show neither $\langle -1 \rangle G(1, a)$ nor $\langle -1 \rangle G(1, -a)$ contains the other.

By the lemma, $G(1, a) \cap G(1, -a) \subseteq G(1, 1) = \langle -1 \rangle$. So $G(1, a) \cap G(1, -a) = \{1\}$ since $-1 \in G(1, a)$ would imply $-a \in G(1, 1)$ which clearly is not the case. This means if $G(1, a) \subseteq \langle -1 \rangle G(1, -a)$, then $G(1, a) - \{1\} \subseteq -G(1, -a)$. Suppose $b \in G(1, a) \cap -G(1, -a)$. Then $b \in G(1, a)$ and $a \in G(1, b)$. Thus, $ab \in G(1, a) \cap \cap G(1, b) \subseteq G(1, -ab)$, and $-1 \in G(1, -ab)$. Hence $ab \in G(1, 1)$ and we must have $b = a$ or $-a$ (modulo $\dot{K}^2$). But $b = -a$ is eliminated since $-1 \notin G(1, a)$. However $|G(1, a)| > 2$, so $G(1, a) - \{1\} \nsubseteq -G(1, -a)$. Combining the above, we obtain $G(1, a) \nsubseteq \langle -1 \rangle G(1, -a)$ so obviously $\langle -1 \rangle G(1, -a)$ cannot contain $\langle -1 \rangle G(1, a)$.

By the last corollary $G(1, -a) \neq \langle -a \rangle$. The same argument as above now shows $\langle -1 \rangle G(1, a)$ cannot contain $\langle -1 \rangle G(1, -a)$. The desired contradiction is established, and the proof is complete.

As an immediate corollary to the proposition, we get another characterization of fields with $s = 2$ and $u = q < \infty$. See also [1, Theorem 5.13] and [8].

COROLLARY. *Let K be a field with $s = 2$ and $u < \infty$. Then $u = q$ if and only if $G(1, 1) = \langle -1 \rangle$.*

One type of field where $G(1, x) = \langle x \rangle$ occurs is $K((x))$, the field of formal power series over the field K. Just as in the case $u = q$, these fields will figure prominently in the answer for $u = q/2$.

3. In this section, we will assume throughout K is a nonformally real field with $u = q/2 < +\infty$ and $s \geqslant 4$. By the corollary to Theorem 1 in [2] and by Proposition 2 [2], $R = \dot{K}^2$.

Let φ be a u-dimensional anisotropic form over K. Write $\varphi \cong \beta_1 \oplus \ldots \oplus \beta_r \oplus \varphi_0$ where this is Elman and Lam's β-decomposition of φ [4]. That is, $\beta_i = (x_i, y_i)$ with $\beta_i \oplus \beta_i \cong (1, -1, 1, -1)$ and φ_0 has dimension 0 or 1 or $\varphi_0 \oplus \varphi_0$ is anisotropic. Since $u = q/2$ and $s \geqslant 4$, we must have $\dim \varphi_0 = 0$ or $2\varphi_0$ anisotropic here. From the proofs of Theorems 2.4 and 2.8 in [4], $x_i \notin y_i \dot{K}^2$, $\pm\{x_i, y_i\} \subseteq G(x_i, y_i)$ for $1 \leqslant i \leqslant r$ and $G(\beta_1), \ldots, G(\beta_r), G(\varphi_0), -G(\varphi_0)$ are pairwise disjoint. If $\varphi_0 = (z_1, \ldots, z_k)$, this implies $q \geqslant 4r + 2k = 2(2r+k) = 2u = q$. Thus $|G(\beta_i)| = 4$ for all i and $|G(\varphi_0)| = k$. It now follows that the z_i must be in different square classes for if not, $\varphi_0 \cong (z, z, \ldots)$. By Pfister's proof of Satz 18(d) [7], $|G(1, 1)| \geqslant s$. This combined with Kneser's Theorem (see 2.1 of [4]) and $s \geqslant 4$ would yield $|G(\varphi_0)| \geqslant k+2$. In exactly the same fashion, we see $|G(z_i, z_j)| = 2$ for all $1 \leqslant i < j \leqslant k$ or equivalently $|G(1, z_i z_j)| = 2$. At any rate $Q(K) = \pm\{x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_k\}$. Finally we note that $2\beta_i$ being isotropic gives $(x_i, y_i) \cong -(x_i, y_i)$ and $(x_i, x_i) \cong -(y_i, y_i)$ for $1 \leqslant i \leqslant r$.

PROPOSITION 2. *If $(x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_k)$ is the β-decomposition of the above φ and if $s \geqslant 4$, then $G(x_i, y_i, z_j) \subseteq \pm\{x_i, y_i, z_j\}$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant k$.*

Proof. $s \geqslant 4$ implies $|G(1, 1)| \geqslant 4$. Let $\psi = \beta_1 \oplus \ldots \oplus \beta_r$. Now $-z_1 \in G(\varphi)$ so $-z_1 = a + b$ where $a \in G(\psi)$, $b \in G(\varphi_0)$. If $b \in z_i \dot{K}^2$ for $i \neq 1$, then $-a \in G(z_1, z_i)$ and φ would be isotropic. Thus $b \in z_1 \dot{K}^2$ and it follows that $G(\psi) \cup \{\pm z_1\} \subseteq G[\psi \oplus (z_1)]$. We would like to show $G(\psi) \cup \{\pm z_1\} = G[\psi \oplus (z_1)]$.

Recall the above $a \in G(\psi)$. We may suppose $a = x_1$, and then obtain $-x_1 \in G(z_1, z_1)$ from $-z_1 = a + b$. Consider $G(x_1, y_1, z_1)$. Clearly this must be a subset of $G(x_1, y_1) \cup \{\pm z_1\} \cup \{z_2, \ldots, z_k\}$. Suppose $z_2 \in G(x_1, y_1, z_1)$. As above $-z_2 = a_1 + b_1$ where $a_1 \in G(\psi)$, $b_1 \in z_2 \dot{K}^2$. Moreover, $a_1 \in G(x_1, y_1)$ for otherwise $a_1 \in G(x_i, y_i)$, $i \neq 1$; and φ would represent $z_2 - z_2 = 0$ non-trivially. Also as above, $-a_1 \in G(z_2, z_2)$. So $(z_1, z_1, z_2, z_2) \cong -(x_1, x_1, a_1, a_1)$. But $2\varphi_0$ being anisotropic and $2\beta_1$ being isotropic now show $a_1 \notin -x_1 \dot{K}^2$, $y_1 \dot{K}^2$. However, if $a_1 \in x_1 \dot{K}^2$, then $-z_2 \in G(x_1, z_2)$ and $-z_1 \in G(x_1, z_1)$ give $-1 \in G(1, x_1 z_2) \cap G(1, x_1 z_1)$; and this intersection is contained in $G(1, -z_1 z_2)$ by the lemma. Therefore $|G(1, -z_1 z_2)| > 2$ but this contradicts $|G(1, z_1 z_2)| = 2$ and the corollary to Theorem 1. The only other possibility is for $a_1 \in -y_1 \dot{K}^2$.

Now if some other z_i, say z_3, is also in $G(x_1, y_1, z_1)$, then $a_2 \in -y_1 \dot{K}^2$ where $-z_3 = a_2 + b_2$. But then $-z_2 \in G(-y_1, z_2)$, $-z_3 \in G(-y_1, z_3)$ give

$$-1 \in G(1, -y_1 z_2) \cap G(1, -y_1 z_3) \subseteq G(1, -z_2 z_3).$$

This yields the same contradiction as above. So

$$G(x_1, y_1, z_1) \subseteq \{\pm x_1, \pm y_1, \pm z_1, z_2\}$$

if $z_2 \in G(x_1, y_1, z_1)$. In this case, $(x_1, y_1, z_1) \cong (z_2, u, v)$ where $uv \in x_1 y_1 z_1 z_2 \dot{K}^2$ and $u, v \in \{\pm x_1, \pm y_1, \pm z_1, z_2\}$. The possibilities for uv are

$$\pm\{1, x_1 y_1, x_1 z_1, y_1 z_1, x_1 z_2, y_1 z_2, z_1 z_2\}.$$

Clearly $uv \neq -1$; and $uv \neq 1$ for if so, $x_1 y_1 \in z_1 z_2 \dot{K}^2$ and then $2 = |G(1, z_1 z_2)| = |G(1, x_1 y_1)| = |G(x_1, y_1)| = 4$. Contradiction. The other possibilities also all lead to contradictions of the types $x_1 = \pm y_1$ and $z_i = \pm x_1$ or $\pm y_1$, $i = 1, 2$. Thus $z_2 \notin G(x_1, y_1, z_1)$ cannot occur and $G(x_1, y_1, z_1) = \pm\{x_1, y_1, z_1\}$.

Now consider $G(x_2, y_2, z_1)$. Clearly (x_2, y_2, z_1) can only represent elements from $G(x_2, y_2) \cup \{\pm z_1\} \cup \{z_2, \ldots, z_k\}$. Suppose $z_2 \in G(x_2, y_2, z_1)$. There is an i such that $-z_2 \in G(x_i, y_i, z_2)$. In fact $i = 2$ or else φ will be isotropic. By the above then, $G(x_2, y_2, z_2) = \pm\{x_2, y_2, z_2\}$. Moreover, $z_2 \in G(x_2, y_2, z_1)$ if and only if

$$-z_1 \in G(x_2, y_2, -z_2) = G(-x_2, -y_2, -z_2) = -G(x_2, y_2, z_2).$$

So $z_1 \in G(x_2, y_2, z_2)$. This cannot happen so we must have $G(x_2, y_2, z_1) \subseteq \pm\{x_2, y_2, z_1\}$.

Hence $G(x_i, y_i, z_1) \subseteq \pm\{x_i, y_i, z_1\}$ for $1 \leqslant i \leqslant r$. Actually in the above, it is easy to show $G(x_2, y_2, z_1) = \{\pm x_2, \pm y_2, z_1\}$, but this will not be needed. Of course the same argument applies to any z_j so the proof is complete.

THEOREM 2. *Let K be a field with $u = q/2 < +\infty$ and $s \geqslant 4$. Then $s = 4$ and $|G(1, 1)| = 4$.*

Proof. Suppose instead that $|G(1, 1)| \geqslant 8$, and consider φ again. We can scale φ so as to put it in the form $\varphi = (1, y_1, x_2, y_2, \ldots, x_r, y_r, z_1, \ldots, z_k)$. Clearly

$$G(1, x_i) \subsetneq G(1, y_1, x_i, y_i) = \pm\{1, y_1, x_i, y_i\} \quad \text{for} \quad 2 \leqslant i \leqslant r.$$

Now $y_1 \notin G(1, x_i)$ for if so, $(1, y_1, x_i, y_i) \cong (y_1, y_1, y_i, x_i y_1)$ and $|G(1, 1)| \geqslant 8$ yields $|G(1, y_1, x_i, y_i)| \geqslant 10$. Similarly $y_i \notin G(1, x_i)$; and φ anisotropic gives $-y_1, -y_i \notin G(1, x_i)$. Thus $G(1, x_i) \subseteq \langle -1, x_i \rangle$. Since $(x_i, y_i) \cong -(x_i, y_i)$, the same technique leads to $G(1, -x_i) \subseteq \langle -1, x_i \rangle$. By the corollary to Theorem 1, $|G(1, x_i)| > 2$ if and only if $|G(1, -x_i)| > 2$. So if $G(1, x_i) = \langle -1, x_i \rangle$, so does $G(1, -x_i)$ and by the lemma $-1 \in G(1, 1)$. This contradicts $s \geqslant 4$. Hence $G(1, x_i) = \langle x_i \rangle$, $G(1, -x_i) = \langle -x_i \rangle$, $2 \leqslant i \leqslant r$. Similarly $G(1, y_i) = \langle y_i \rangle$, $G(1, -y_i) = \langle -y_i \rangle$, $2 \leqslant i \leqslant r$.

We also have $G(1, z_j) \subsetneq G(1, y_1, z_j) \subseteq \pm\{1, y_1, z_j\}$ by Proposition 2 for $1 \leqslant j \leqslant k$. Now $\pm y_1 \notin G(1, z_j)$ since $\pm y_1 z_j \notin \pm\{1, y_1, z_j\}$. So $G(1, z_j) \subseteq \langle -1, z_j \rangle$, $1 \leqslant j \leqslant k$. Furthermore,

$$G(1, -z_j) \subsetneq G(1, y_1, -z_j) = G(-1, -y_1, -z_j) \subseteq \pm\{1, y_1, z_j\}.$$

Again $G(1, -z_j) \subseteq \langle -1, z_j \rangle$, and it follows as before that $G(1, z_j) = \langle z_j \rangle$, $G(1, -z_j) = \langle -z_j \rangle$, $1 \leqslant j \leqslant k$.

Since $a \in G(1, 1)$ if and only if $-1 \in G(1, -a)$, we may conclude $G(1, 1) \cap \pm\{x_2, y_2, \ldots, x_r, y_r, z_1, \ldots, z_k\} = \emptyset$. Thus $G(1, 1) \subseteq \langle -1, y_1 \rangle$, and this contradicts $|G(1, 1)| \geqslant 8$. Hence $s \leqslant 4$ (and so $s = 4$) and $|G(1, 1)| = 4$.

Having reached the conclusion that $s \leqslant 4$, we would like to characterize all such fields with $s = 4$. It will turn out that they are all equivalent to power series field extensions of the 2-adic numbers.

Again let $\varphi = (x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_k)$ be a u-dimensional form with its β-decomposition. By the proof of the last theorem, $|G(1, \pm x_i z_j)| = 2$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant k$. Notice that only Proposition 2 and not the assumption $|G(1,1)| \geqslant 8$ was needed to prove $|G(1, \pm z_j)| = 2$ in that theorem. From $2\beta_1$ isotropic, we obtain $x_1, -y_1 \in G(x_1, x_1)$. But there are two other square classes in $G(x_1, x_1)$ too — denote them $a\dot{K}^2$, $b\dot{K}^2$. Since $|G(1, \pm x_1 z_j)| = 2$, $a, b \notin \pm G(z_1, \ldots, z_k)$. Also $a, b \notin \{-x_1, y_1\}$ since $s = 4$. So $a, b \in G(x_2, y_2, \ldots, x_r, y_r)$ and we may assume $a = x_2$. Then $G(x_1, x_1) = \{x_1, -y_1, x_2, b\}$ and since $G(1,1) = \{1, -x_1 y_1, x_1 x_2, b x_1\}$ is a group, $b x_1 \in -x_2 y_1 \dot{K}^2$. Moreover, $x_1 x_2 \in G(1,1)$ implies $\langle -1, x_1 x_2 \rangle \subseteq G(1, -x_1 x_2)$. So by Theorem 1's corollary, $|G(1, x_1 x_2)| \geqslant 4$. Also

$$G(1, x_1 x_2) = x_1 G(x_1, x_2) \subsetneq x_1 G(x_1, y_1, x_2, y_2) = \pm\{1, x_1 y_1, x_1 x_2, x_1 y_2\}.$$

However, $-x_1 y_1, -x_1 y_2 \notin G(1, x_1 x_2)$ since φ is anisotropic. If $-1 \in G(1, x_1 x_2)$ then $-1 \in G(1, x_1 x_2) \cap G(1, -x_1 x_2) \subseteq G(1,1)$ — contradiction to $s = 4$. Thus $G(1, x_1 x_2) = \{1, x_1 y_1, x_1 x_2, x_1 y_2\}$, and in particular, $x_1 x_2 \in y_1 y_2 \dot{K}^2$ from which follows $(x_1, x_2) \cong (y_1, y_2)$. Finally, we may conclude $(x_1, y_1, x_2, y_2) \cong (x_1, x_2, x_1, x_2) \cong (x_1, x_1, x_1, x_1)$.

Now consider (x_3, y_3). As above $x_3, -y_3, a_1, b_1 \in G(x_3, x_3)$. Suppose $a_1 \in G(x_1, y_1, x_2, y_2)$. If $a_1 \in G(x_1, y_1)$, then as before

$$(x_1, y_1, x_3, y_3) \cong (x_1, x_1, x_1, x_1).$$

Hence $(x_1, y_1, x_2, y_2, x_3, y_3) \cong (x_1, x_1, x_1, x_1, x_2, y_2)$ with

$$G(x_2, y_2) \subsetneq G(x_1, x_1, x_1, x_1) = -G(x_1, x_1, x_1, x_1)$$

and φ would be isotropic. So $a_1 \notin G(x_1, y_1)$ and similarly $a_1 \notin G(x_2, y_2)$. In fact, then, $a_1 \notin G(x_1, y_1, x_2, y_2)$. Therefore $a_1, b_1 \in G(x_4, y_4, \ldots, x_r, y_r)$; we can assume $a_1 = x_4$ and as above $(x_3, y_3, x_4, y_4) \cong (x_3, x_3, x_3, x_3)$. We can continue this process to obtain $\varphi \cong (x_1, x_1, x_1, x_1, \ldots, x_t, x_t, x_t, x_t, z_1, \ldots, z_k)$ where $t = r/2$.

Notice that from the last paragraph, it follows that $|G(1,1,1,1)| = 8$. From Elman and Lam [4, Lemma 3.2], $|G(1,1,1)| \geqslant 6$. So we may assume $G(1,1,1,1) = \langle -1, a, b \rangle$, $G(1,1) = \langle a, b \rangle$, and $G(1,1,1) \supseteq \langle a, b \rangle \cup \{-a, -b\}$. We then have $G(1, a) \subseteq G(1,1,1) \subsetneq \langle -1, a, b \rangle$. The corollary to Theorem 1 and $-1 \in G(1, -a)$ yield $|G(1, a)| \geqslant 4$. From $(1,1,1,1) \simeq (1,1,a,a)$ being anisotropic, we get $G(1, a) \cap -G(1, a) = \varnothing$. So $G(1, a) = \langle a, b \rangle$ or $\langle a, -b \rangle$. However, if it were $\langle a, b \rangle$, then $G(1, a) \cap -G(1,1) = \varnothing$ and $(1,1,1,a)$ would be anisotropic — contradiction to $-a \in G(1,1,1)$. Consequently $G(1, a) = \langle a, -b \rangle$, and similarly $G(1, b) = \langle -a, b \rangle$. From these and $G(1, ab) \subseteq G(1,1,1)$, we also get $G(1, ab) = \langle -a, -b \rangle$.

$\langle -1, a \rangle \subseteq G(1, -a) \subseteq G(1,1,1,1)$. But $G(1, -a) \neq \langle -1, a, b \rangle$ for if so, $b \in G(1, -a) \cap G(1,1) \subseteq G(1, a)$. Thus $G(1, -a) = \langle -1, a \rangle$, and in the same fashion $G(1, -b) = \langle -1, b \rangle$. Finally $\langle -1, ab \rangle \subseteq G(1, -ab) \subseteq G(1,1,a) \subseteq G(1,1,1,1)$ and $ab \notin G(1, -b)$ implies $b \notin G(1, -ab)$ which gives $G(1, -ab) = \langle -1, ab \rangle$. In particular, notice, this gives $G(1,1,1) = \langle a, b \rangle \cup \{-a, -b, -ab\}$.

We have now shown $|G(1, x)| = 4$ and $G(1, x) \subseteq G(1,1,1,1)$ for all $x \in G(1,1,1,1) - (-\dot{K}^2)$. Next we show every anisotropic $\varphi = (y_1, y_2, y_3, y_4)$ with $y_i \in G(1,1,1,1)$ represents $G(1,1,1,1)$. From the above calculations and the fact that $G(\varphi) = \bigcup G(\alpha, \beta)$, $\alpha \in G(y_1, y_2)$, $\beta \in G(y_3, y_4)$, it follows that $G(\varphi) \subseteq G(1,1,1,1)$. If three y_i are in the same square class, then $|G(1,1,1)| = 7$ implies $|G(\varphi)| \geqslant 8$ and hence $G(\varphi) = G(1,1,1,1)$. If two y_i are in the same square class, then $\varphi \simeq (y, y, y_3, y_4)$. We may assume $G(y, y) \cap G(y_3, y_4) = \varnothing$ or else φ has three like entries. But then $|G(\varphi)| \geqslant |G(y, y)| + |G(y_3, y_4)| = 8$ and so $G(\varphi) = G(1,1,1,1)$. Finally, if φ contains no like entries, we may assume $G(y_1, y_2) \cap \cap G(y_3, y_4) = \varnothing$ and the reasoning is the same. An immediate consequence of this result and $-1 \in G(1,1,1,1)$ is that any five-dimensional form with coefficients out of $G(1,1,1,1)$ is isotropic.

The 2-adic numbers Q_2 have $Q(Q_2) = \langle -1, 2, -3 \rangle$. Moreover, $G(1,1) = \langle 2, -3 \rangle$, $G(1, -2) = \langle -1, 2 \rangle$, $G(1, 3) = \langle -1, -3 \rangle$, $G(1, 2) = \langle 2, 3 \rangle$, $G(1, -3) = \langle -2, -3 \rangle$, $G(1, 6) = \langle -1, 6 \rangle$, and $G(1, -6) = \langle -2, 3 \rangle$. Thus $\sigma\colon Q(Q_2) \to Q(K)$ defined by $\sigma(-1) = -1$, $\sigma(2) = a$, $\sigma(-3) = b$ and extended homomorphically satisfies $\sigma G(\alpha, \beta) = G(\sigma(\alpha), \sigma(\beta))$ for all $\alpha, \beta \in Q(Q_2)$.

Let us return to φ and scale it so we may assume

$$\varphi = (1, 1, 1, 1, x_2, x_2, x_2, x_2, \ldots, x_t, x_t, x_t, x_t, z_1, \ldots, z_k).$$

Consider $x \in G(x_i, x_i, x_i, x_i)$, $2 \leqslant i \leqslant t$. Then $(x_i, x_i, x_i, x_i) \cong (x, x, x, x)$ and φ anisotropic implies $G(1, x) \cap -G(1,1,1) = G(1, x) \cap -G(x, x, x) = \varnothing$. Hence

$$G(1, x) \cap G(1,1,1,1) = \{1\}, \quad G(1, x) \cap G(x, x, x, x) = \{x\};$$

and $G(1, x) \subseteq G(1,1,1,1,x,x,x,x) = G(1,1,1,1) \cup G(x,x,x,x)$ then gives $G(1, x) = \langle x \rangle$.

We now want to prove there are no z_j in φ. From previous work,

$$G(1,1,1,1,\ldots,x_t,x_t,x_t,x_t) = G(1,1,1,1) \bigcup_{i=2}^{t} x_t G(1,1,1,1).$$

The z_j then must lie in the $q/8 - t = k/4$ cosets of $G(1,1,1,1)$ left over. Since no 5 can come from one such coset (or φ would be isotropic), 4 must come from each coset. If (z_1, z_2, z_3, z_4) is one such set, then it represents 8 elements from $Q(K)$ while earlier it was seen (z_1, z_2, z_3, z_4) represents only 4. This contradiction shows $k = 0$.

If $q(K) = 2^{r+3}$ and if $Q_2((y))$ denotes the field of formal power series over Q_2, it is now clear that any isomorphic extension of $\sigma\colon Q(Q_2) \to K$ to

$$Q[Q_2((y_1))((y_2)) \ldots ((y_r))]$$

satisfies the hypothesis of Proposition 2.2 of [1]. Hence σ is an equivalence map and Theorem 3 follows.

THEOREM 3. *Let K be a field with $u = q/2$, $s \geqslant 4$, and $q = 2^{r+3}$. Then K is equivalent with respect to quadratic forms to $Q_2((x_1))((x_2)) \dots ((x_r))$ where Q_2 is the 2-adic numbers and $Q_2((x_1))$ denotes the field of formal power series over Q_2.*

4. The cases for $s = 1, 2$ are complicated by some unanswered questions concerning the radical R of a field. From the discussion preceding Theorem 4 in [2] and by the same techniques used for Proposition 5.15 in [1], it is easily seen that there are only three possible fields (up to equivalence) with $u = q/2$ and $R \neq \dot{K}^2$. These fields all have $|R/\dot{K}^2| = 2$ and correspond to $s = 2$ with $-1 \in R$, $s = 2$ with $-1 \notin R$, and $s = 1$. Let us call these fields respectively types 1, 2, and 3. Although each type is unique, the problem is that their existence is not determined for any $q \geqslant 8$. In fact there is only one known example of a field with radical $R \neq \dot{K}, \dot{K}^2$ (see [2]). It turns out that all fields with $u = q/2 \geqslant 8$ and $s = 1, 2$ have quadratic form structures which are identical to what the structures would be for iterated formal power series extensions of fields of types 1, 2, 3 above or (4) or Theorem 6.11 [1]. Theorem 6.11 classifies all nonreal fields with $q = 8$, and in fact (4) is a power series extension of a field with $u = 2$, $q = 4$.

The basic idea in this section is the same as in the last one — namely to write down a particular form of an anisotropic u-dimensional φ and read off what the value sets for all binary forms must be. Instead of using a β-decomposition for $s = 2$, we try to write a φ in the form $(x_1, x_1, \dots, x_r, x_r)$. For $s = 1$, an element a is found so $\varphi = (x_1, ax_1, \dots, x_r, ax_r)$. Since the work is similar to Section 3, the proofs will just be sketched.

First assume K has $s = 2$ and that there is some anisotropic u-dimensional form with like entries in some diagonalization. Let

$$\varphi = (1, 1, x_2, x_2, \dots, x_r, x_r, z_1, \dots, z_k)$$

be such a form with r maximal. We want to show $2r = u$. Clearly $G(1, 1)$, $G(x_i, x_i)$, $2 \leqslant i \leqslant r$, $\{z_i\}_{i=1}^k$, and $\{-z_i\}_{i=1}^k$, are pairwise disjoint sets. Thus $|G(1, 1)| = 4$. Let $G(1, 1) = \langle -1, a\rangle$. Using an argument with $G(1, 1)$ similar to the discussion just preceding Theorem 3, we find $(z_1, \dots, z_k) \cong (y_{11}, y_{21}, \dots, y_{1l}, y_{2l})$ where y_{1j}, y_{2j}, $1 \leqslant j \leqslant l$, belong to the same coset of $G(1, 1)$. Thus $(y_{1j}, y_{2j}) \simeq b_j(1, a)$ or $b_j(1, -a)$. But then it follows $G(1, a) = G(1, 1)$ or $G(1, -a) = G(1, 1)$, and $\varphi' = (1, 1, x_2, x_2, \dots, x_r, x_r, b_1, b_1, \dots, b_l, b_l)$ is anisotropic. Hence $2r = u$. Moreover, for $\varphi = (1, 1, x_2, x_2, \dots, x_r, x_r)$, $G(1, \pm x_i) \subsetneq G(1, 1, x_i, x_i)$ so $|G(1, \pm x_i)| = 2$ or 4. The quadratic form structure is determined by how many of these are 2.

THEOREM 4. *Let K be a field with $u = q/2 < +\infty$ and $s = 2$ and suppose K contains an anisotropic u-dimensional form with like entries in some diagonalization. Then K is equivalent to a field of type 2, a field with $u = 2$, $q = 4$, or an iterated power series extension of one of these.*

Proof. Let $\varphi = (1, 1, x_2, x_2, \dots, x_r, x_r)$ be as above and recall $G(1, 1) = \langle -1, a\rangle$. Suppose $|G(1, x_i)| = 2$ for $2 \leqslant i \leqslant r$. Then by Theorem 1, $G(1, x) = \langle x\rangle$ for all $x \notin \langle -1, a\rangle$. Also then $G(1, a) = G(1, -a) = \langle -1, a\rangle$. Thus K is a field with $u = 2$, $q = 4$ or power series extension thereof.

Now suppose $|G(1, x_2)| = 4$. From $-1 \notin G(1, x_2)$ and

$$G(1, x_2) \subsetneq G(1, 1, x_2, x_2)$$

comes $G(1, x_2) = \langle a, x_2\rangle$ or $\langle -a, x_2\rangle$. By changing $-a$ to a if necessary, we can assume $G(1, x_2) = \langle a, x_2\rangle$. From $G(1, a) \subseteq G(1, 1, x_2)$, we get $G(1, a) = \langle -1, a\rangle$. Using these, the lemma, Theorem 1 and its corollary, we obtain $G(1, -x_2) = \langle a, -x_2\rangle$, $G(1, ax_2) = \langle a, x_2\rangle$, and $G(1, -ax_2) = \langle a, -x_2\rangle$.

If any other $|G(1, x_i)| = 4$, then $G(1, x_i) = \langle a, x_i\rangle$ and the value sets for $(1, \pm ax_i)$ and $(1, -x_i)$ are as above too. We rewrite

$$\varphi = (1, 1, x_2, x_2, \dots, x_k, x_k, y_1, y_1, \dots, y_l, y_l)$$

so $|G(1, x_i)| = 4$, $2 \leqslant i \leqslant k$ and $|G(1, y_j)| = 2$, $1 \leqslant j \leqslant l$. Then

$$G(1, -a) = \langle -1, a\rangle \cdot \{1, x_2, \dots, x_k\}.$$

If $l = 0$, then K is equivalent to a field of type 2. If $l \neq 0$, K is equivalent to a power series extension of a field F of type 2 with $a \in R(F)$ and $q(F) = 4k$. This completes the proof.

THEOREM 5. *Let K be a field with $u = q/2 < +\infty$ and $s = 2$ and suppose K contains no anisotropic u-dimensional form with like entries in any diagonalization. Then K is equivalent to a field of type 1 or an iterated power series extension of such a field.*

Proof. Let $\varphi = (1, x_2, \dots, x_u)$ be anisotropic. Note $Q(K) = \pm\{1, x_2, \dots, x_u\}$. Also $G(1, x_2) \cap \pm\{x_3, \dots, x_u\} = \emptyset$ shows $|G(1, x_2)| \leqslant 4$. This holds for all $G(1, x_i)$. If $|G(1, x_i)| = 2$, $2 \leqslant i \leqslant u$, then $|G(1, -x_i)| = 2$ and so $|G(1, 1)| = 2$. By the corollary to Proposition 1, $u = q$. This contradiction means $|G(1, x_i)| = 4$ for some i.

Write $\varphi = (1, x_2, \dots, x_k, y_1, \dots, y_l)$ where $|G(1, x_i)| = 4$ and $|G(1, y_j)| = 2$. It is easy to see $G(1, \pm x_i) = \langle -1, x_i\rangle$, $2 \leqslant i \leqslant k$. It also follows that $G(1, 1) = \pm\{1, x_2, \dots, x_k\}$. If $l = 0$, then K is equivalent to a field of type 1. If $l \neq 0$, K is equivalent to a power series extension of such a field.

Fields with $s = 1$ are done in a similar manner by showing there exists an x such that $|G(1, x)| = 4$ and $\varphi = (1, x, x_2, xx_2, \dots, x_r, xx_r)$.

THEOREM 6. *Let K be a field with $u = q/2 < +\infty$ and $s = 1$. Then K is equivalent to a field of type 3, a field with $u = 2$, $q = 4$, or an iterated power series extension of one of these.*

The role of power series extensions was central for fields with both $u = q$ and $u = q/2$. This concept along with the radical proved to be the keys in classifying all such fields. It would appear that elements x satisfying $|G(1, x)| = 2$ or q deserve more study.

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Undefinable ordinals and the rank hierarchy

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Abstract. This paper shows that certain definability properties concerning the ordinals α and β are equivalent to the property of $\langle R\alpha, \varepsilon\rangle$ being a proper elementary substructure of $\langle R\beta, \varepsilon\rangle$.

1. Introduction. This note was motivated by [5]. Section 2 starts by answering a question from [5] and then it gives a number of conditions involving undefinable ordinals, each of which is equivalent to $R\alpha \prec R\beta$ (where $R\alpha \prec R\beta$ means $\langle R\alpha, \varepsilon\rangle$ is a proper elementary substructure of $\langle R\beta, \varepsilon\rangle$).

Most of our notation is standard but $\mathrm{Df}(x, y)$ is the set of those elements of x which are definable in $\langle x, \varepsilon\rangle$ using a first order ε formula with parameters from $y \cap x$. Also, $\mathrm{Df}(x) = \mathrm{Df}(x, \varphi)$ and $\bar{\bar{x}}$ is the cardinality of x.

It is well known that $V = L$ implies the existence of certain definable well orderings and we shall make use of this fact in the following form (see Theorem 4.11 of [4], for instance).

THEOREM 1. *Suppose that* $V = L$ *holds and that* $\beta \geqslant \omega$. *Then there is an* ε *formula* φ *with two free variables such that* $\{\langle x, y\rangle \mid \varphi^{R\beta+1}(x, y)\}$ *is a well ordering of* $R\beta+1$.

2. Results. The following notions were introduced in [5]. An ordinal α $(\epsilon\, x)$ is said to be *inconceivable* in x if $\alpha \notin \mathrm{Df}(x, \alpha)$, *strongly inconceivable* in x if $\beta \geqslant \alpha \to \beta \notin \mathrm{Df}(x, \alpha)$ and *weakly inconceivable* in x if it is inconceivable, but not strongly inconceivable in x. Then Theorem 2.4 (i) of [5] gives

$$R\beta \models \mathrm{ZF} \to (\alpha \text{ is strongly inconceivable in } R\beta \to R\alpha \prec R\beta),$$

and Rucker asks if this result can be proved without assuming $R\beta \models \mathrm{ZF}$. More precisely, he asks "If x is a model of Z and there is an $\alpha \in x$ such that α is strongly inconceivable in x, then is x a model of ZF?"

Theorem 2 shows that the answer to Rucker's question is no, in general, as there is an α which is strongly inconceivable in $R\omega_1$ and $R\omega_1$ is not a model of ZF. However, Theorem 3 shows that if $V = L$ holds, then we get a positive answer to Rucker's question when $x = R\beta$ and β is a singular ordinal.

THEOREM 2. *If* β *is a regular ordinal* $> \omega$, *then there is an* $\alpha < \beta$ *such that* α *is strongly inconceivable in* $R\beta$.