[10] W. Marek and M. Srebrny, *Gaps in constructible universe*, Ann. Math. Logic 6 (1974), pp. 359–394.
[11] — — *There is no minimal model of* Z^-, Z. Math. Logik. 21 (1975), pp. 225–228.
[12] P. Zbierski, *Models for higher order arithmetics*, Bull. Acad. Polon. Sci. 19 (1971), pp. 557–562.

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A decidable $\aleph_0$-categorical theory with a non-recursive Ryll-Nardzewski function

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Abstract. According to the theorem of Ryll-Nardzewski, a complete first-order theory T is $\aleph_0$-categorical iff for each $n<\omega$, the set $S_n(T)$ of its n-types is finite. For such theories T, we call the function $n\to|S_n(T)|$ the Ryll-Nardzewski function of T. Waszkiewicz asked if the Ryll-Nardzewski function of a decidable theory is recursive. It is shown that this question has a negative answer. More specifically, for any Turing degree $\boldsymbol{b}$ there is a function $G\colon \omega\to\omega$ of degree $\boldsymbol{b}$ with the following property: whenever $\boldsymbol{a}$ is a degree such that $\boldsymbol{b}$ is recursively enumerable in $\boldsymbol{a}$, then there is a complete, $\aleph_0$-categorical theory of degree $\boldsymbol{a}$ whose Ryll-Nardzewski function is G.

According to the classic theorem of Ryll-Nardzewski [1], a complete first-order theory T is $\aleph_0$-categorical iff, for each $n<\omega$, the set $S_n(T)$ of its n-types is finite. For such theories T let us denote the function $n\to|S_n(T)|$ by R_T, which we call the Ryll-Nardzewski function of the theory T. The following question was posed by Waszkiewicz in [2]: Is the Ryll-Nardzewski function of a decidable $\aleph_0$-categorical theory always recursive? It is the purpose of this note to give a negative answer to this question (1). More generally, we consider a relativized version of Waszkiewicz's question: If the Turing degree of T is $\boldsymbol{a}$, then what are the possible Turing degrees $\boldsymbol{b}$ of R_T? It is a straightforward matter to show that $\boldsymbol{b}$ must be recursively enumerable in $\boldsymbol{a}$. Our theorem shows that this is the only restriction. What is perhaps more surprising is that the Ryll-Nardzewski function of degree $\boldsymbol{b}$ can be chosen independently of $\boldsymbol{a}$.

THEOREM. *For any Turing degree* $\boldsymbol{b}$ *there is a function* $G\colon \omega\to\omega$ *of degree* $\boldsymbol{b}$ *with the following property: whenever* $\boldsymbol{a}$ *is a degree such that* $\boldsymbol{b}$ *is recursively enumerable in* $\boldsymbol{a}$, *then there is a complete* $\aleph_0$-*categorical theory of degree* $\boldsymbol{a}$ *in a language consisting of one binary relation symbol whose Ryll-Nardzewski function is* G.

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(1) A negative answer was also given by E. Herrmann of Humboldt University in Berlin. His results, obtained independently of ours, are contained in a manuscript entitled "About Lindenbaum functions of $\aleph_0$-categorical theories of finite similarity type". The examples he gets are not, however, as extensive as ours.

Before embarking on the proof of the theorem, we note the following conventions. All theories which we use are in a countable language. All $\aleph_0$-categorical theories are assumed to be complete and deductively closed, so that the decision problem for such theories has the same degree as the theory itself. The set of finite subsets of a set X will be denoted by $[X]^{<\omega}$. For a finite object X, we denote its Gödel number by $\ulcorner X \urcorner$. We identify a natural number with the set of its predecessors. The cardinality of a set X will be denoted by $|X|$. Finally, we will make a concession to expediency by frequently confounding symbols and their denotations.

The proof of the theorem requires the construction of a model. The actual model we construct will be for a finite language: the further improvement to a language with just one binary relation symbol is a routine exercise, and will consequently be omitted.

Let us start by considering functions $f\colon \omega\to\omega$ which are finite-one; that is, for each $n<\omega$, the set $\{i<\omega : f(i) = n\}$ is finite. Our first goal is to construct a model $\mathfrak{A}_f$. To accomplish this we let T'_f be a theory in a language which consists of a unary predicate symbol U, and for each $i<\omega$ an $(i+1)$-ary relation symbol P_i and an $(f(i)+1)$-ary relation symbol Q_i. Let T'_f be axiomatized by the following sentences (1)–(6):

(1) $\forall \bar{x}\big(P_i(x_0, \ldots, x_i) \to x_0 \neq x_1\big)$, whenever $1 \leqslant i < \omega$.

(2) $\forall \bar{x}\big(P_i(x_0, \ldots, x_i) \to P_i(x_{\pi(0)}, \ldots, x_{\pi(i)})\big)$, whenever $i<\omega$ and π is a permutation of $i+1$.

Because of sentences (1) and (2) we can unambiguously refer to $P(X)$, where X is a nonempty, finite set of variables. We will say that X is *minimal* if $P(X)$, yet $\neg P(Y)$ whenever $Y \subseteq X$ and $0 \neq Y \neq X$. We will write $\mathrm{Min}(X)$ for X is minimal.

(3) $$\forall X, Y\big(P(X) \wedge X \subseteq Y \to P(Y)\big),$$

(4) $$\forall X\big(\mathrm{Min}(X) \wedge |X \cap U| = n \to |X-U| = f(n)+1\big),$$

(5) $$\forall X, Y\big(\mathrm{Min}(X) \wedge \mathrm{Min}(Y) \wedge X-U = Y-U \to |X| = |Y|\big).$$

Lastly, we give a schema which defines the relations Q_n.

(6) $$\forall X\Big(Q_n(X) \leftrightarrow \big(X \cap U = 0 \wedge \exists Y(\mathrm{Min}(Y) \wedge |Y| = n+f(n)+1 \wedge Y-U = X)\big)\Big).$$

Of course, sentences (3)-(6) are schemata. Notice that T'_f is an $\forall\exists$ theory, and as a set it has the same degree as f.

It is readily observed that T'_f has a model-completion T''_f, and that T''_f is a complete, $\aleph_0$-categorical theory. Let $\mathfrak{B}$ be a countable model of T''_f.

We now form a new structure $\mathfrak{A}_f$ in the following way. Let $\mathfrak{M}$ be a collection of subsets $M \subseteq B$ such that the following are satisfied:

(a) if $M \in \mathfrak{M}$ and $F \in [M]^{<\omega}$, then $\neg P(F)$;

(b) if $F_1, F_2 \in [B]^{<\omega}$ are such that $F_1 \cap F_2 = \emptyset$ and $\neg P(F_1)$, then there is $M \in \mathfrak{M}$ such that $F_1 \subseteq M$ and $F_2 \cap M = \emptyset$;

(c) if $x_0, \ldots, x_k \in B$ and $M_0, \ldots, M_m, N_0, \ldots, N_n \in \mathfrak{M}$ are all distinct and are such that $\sim P\big((\{x_0, \ldots, x_{k-1}\} \cap M_i) \cup \{x_k\}\big)$ for each $i \leqslant m$, then there is $y \in (M_0 \cap \ldots \cap M_m) - (N_0 \cup \ldots \cup N_n)$ such that $\langle x_0, \ldots, x_k\rangle$ and $\langle x_0, \ldots, x_{k-1}, y\rangle$ realize the same type in $\mathfrak{B}$.

The collection $\mathfrak{M}$ can easily be constructed by a sort of forcing construction. We now define the structure $\mathfrak{A}_f$ to be $(B \cup \mathfrak{M}, U, \mathfrak{M}, E)$, and let $T_f = \mathrm{Th}(\mathfrak{A}_f)$. It is easily seen that T_f and f have the same degree. Here we use the fact that a complete theory is recursive in any of its axiomatizations. Also, it is easy to see that T_f is $\aleph_0$-categorical. For, let us form an expansion of $\mathfrak{A}_f$ by adjoining the relations P_i and Q_i for $i<\omega$. Then we get the structure $\mathfrak{A}^*_f = (\mathfrak{A}_f, P_i, Q_i)_{i<\omega}$, which is a definable expansion of $\mathfrak{A}_f$, and which admits the elimination of quantifiers. Thus T_f is $\aleph_0$-categorical (using the fact that f is finite-one).

From the previous observation, we can even get an explicit method for evaluating $R_{T_f}(n)$. For a finite-one $f\colon \omega\to\omega$ let

$$f^*(n) = |\{i<\omega\colon f(i) = n\}|$$

and

$$I_n(f) = \{i<\omega\colon i+f(i)+1 = n\}.$$

Then there is a primitive recursive $H\colon \omega\times\omega\to\omega$ (the choice of which does not depend on f) such that

(7) $R_{T_f}(n+1) = H\big(\ulcorner\langle I_m(f)\colon m \leqslant n\rangle\urcorner, \ulcorner f^*|n\urcorner\big) + f^*(n)$, for any finite-one $f\colon \omega\to\omega$.

From (7) it follows that there is a primitive recursive sequence $\langle M_i\colon i<\omega\rangle$ of functions $M_i\colon \omega\to\omega$ with the following property. Whenever $f\colon \omega\to\omega$ is a finite-one function and $G\colon \omega\to\omega$ and $n<\omega$ are such that $f^*(i) \leqslant G(i+1)$ for each $i<n$, then

(8) $$M_n\big(\ulcorner G|(n+1)\urcorner\big) > R_{T_f}(n).$$

Let $\mathrm{id}\colon \omega\to\omega$ be the identity function. The function $R_{T_{\mathrm{id}}}$ is a primitive recursive function with the following significant property. For any degree $\boldsymbol{a}$ let $A \subseteq \omega$ be an infinite set of degree $\boldsymbol{a}$, and let $f\colon \omega\to A$ be a monotonic bijection. Then the theory T_f is such that both T_f and R_{T_f} have degree $\boldsymbol{a}$, and that $R_{T_f}(n) \leqslant R_{T_{\mathrm{id}}}(n)$ for all $n<\omega$.

For each $B \subseteq \omega$ we are going to define a function $G^B\colon \omega\to\omega$. Let c_B be the characteristic function of B; that is, $c_B\colon \omega\to\{0, 1\}$ is such that $c_B(n) = 1$ iff $n \in B$. We now define G^B so that $G^B(0) = 1$, and

(9) $$G^B(n+1) = c_B(n) + M_{n+1}\big(\ulcorner (G^B|(n+1)) \cup \{\langle n+1, 1\rangle\}\urcorner\big) + \sum_{i=0}^{n} \binom{n+1}{i} M_i\big(\ulcorner G^B|(i+1)\urcorner\big) R_{T_{\mathrm{id}}}(n+1-i).$$

It is clear that G^B and B have the same degree.

Now let us fix a degree $\boldsymbol{b}$, and let $B \subseteq \omega$ be a fixed infinite set of degree $\boldsymbol{b}$. We will write G for G^B, and show that G satisfies the condition of the theorem. Let $\boldsymbol{a}$ be a degree such that $\boldsymbol{b}$ is recursively enumerable in $\boldsymbol{a}$. Let T' be such that the degree of T' and the degree of $R_{T'}$ are both $\boldsymbol{a}$, and $R_{T'} \leqslant R_{T_{\mathrm{id}}}$. (Such a T' is described above.) There is a bijection $F\colon \omega \to B$ of degree $\boldsymbol{a}$. For each $j<\omega$ let $B_j = \{F(i)\colon i<j\}$, and then let $G_j = G^{B_j}$. The following properties are all evident:

(i) If $i<j<\omega$ and $n<\omega$, then $G_i(n) \leqslant G_j(n)$.

(ii) $\lim_j G_j = G$.

(iii) $\langle G_j\colon j<\omega\rangle$ has degree $\boldsymbol{a}$.

We are next going to define a function $f\colon \omega\to\omega$. Let us first make the following convention: if $j \leqslant i < \omega$ and $h\colon i \to \omega$, we then set $R_{T_h}(n) = R_{T_f}(n)$, where

$$f(n) = \begin{cases} h(n) & \text{if} \quad n<i, \\ n+1 & \text{if} \quad n \geqslant i. \end{cases}$$

Notice that if $f\colon \omega\to\omega$ is any finite-one function which extends h, then $R_{T_h}(j) \leqslant R_{T_f}(j)$ We will define f by induction by defining a sequence $\langle f_i\colon i<\omega\rangle$ of functions $f_i\colon i\to\omega$ such that $f_{i+1} \supseteq f_i$, and then setting $f = \bigcup \{f_i\colon i<\omega\}$. Let

$$f_{i+1}(i) = (\mu n<i)\Big[G_i(n+1) > \sum_{j=0}^{n+1} \binom{n+1}{j} R_{T_{f_i}}(j) R_{T'}(n+1-j)\Big]. \tag{10}$$

It is clear from (7) that f is finite-one: a rough estimate for f^* is given by $f^*(n) \leqslant G(n+1)$.

The function f was chosen so as to satisfy the following identity, which we now verify:

$$G(n) = \sum_{j=0}^{n} \binom{n}{j} R_{T_f}(j) R_{T'}(n-j). \tag{11}$$

It is clear from (7) and (10) that it suffices to show only that

$$G(n) \geqslant \sum_{j=0}^{n} \binom{n}{j} R_{T_f}(j) R_{T'}(n-j).$$

But this is a consequence of (9) (keeping in mind both (8) and the fact that $R_{T'} \leqslant R_{T_{\mathrm{id}}}$).

To complete the proof of the theorem, we construct the desired theory T. Let $\mathfrak{B}$ be a countable model of T', and let $T = \mathrm{Th}(\mathfrak{A}_f + \mathfrak{B})$. Here, $\mathfrak{A}_f + \mathfrak{B}$ is the two-sorted structure which is the disjoint union of $\mathfrak{A}_f$ and $\mathfrak{B}$. Since T' has degree $\boldsymbol{a}$ and T_f also does (since T_f has the same degree as R_T), it is clear that T has a degree $\boldsymbol{a}$. Finally, a moment's reflection is enough to conclude from (11) that $R_T = G$. This completes the proof of the theorem.

Remark 1. It is clear from the proof of the theorem that if $\boldsymbol{b} = \boldsymbol{0}$, then G can be chosen to be primitive recursive. Thus, there is a primitive recursive G such that for any degree $\boldsymbol{a}$ there is an $\aleph_0$-categorical T of degree $\boldsymbol{a}$ such that $R_T = G$.

Remark 2. From the manner in which the G's were constructed it is easy to see that there is some primitive recursive G_0 such that for each G constructed in the proof, $G(n) \leqslant G_0(n)$ for each $n<\omega$. It is, however, easy to find a decidable T such that R_T majorizes every recursive function. Let $f\colon \omega\to\omega$ be a finite-one recursive function such that f^* majorizes every recursive function. Then T_f is recursive and R_{T_f} majorizes every recursive function.

We close with the following problem.

PROBLEM. What is true if only finitely axiomatizable $\aleph_0$-categorical theories are considered?

Added in proof. Herrmann's article has now appeared. (Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), pp. 17–21.) A similar result was obtained independently by M.C. Venning in his Ph. D. thesis (Cornell Univ., 1976).

References

[1] C. Ryll-Nardzewski, *On the categoricity in power $\leqslant\aleph_0$*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 7 (1959), pp. 545–548.

[2] J. Waszkiewicz, *On cardinalities of algebras of formulas for ω_0-categorical theories*, Colloq. Math. 27 (1973), pp. 7–11.

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