A decidable $\kappa_\omega$-categorical theory with a non-recursive Ryll-Nardzewski function

by

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Abstract. According to the theorem of Ryll-Nardzewski, a complete first-order theory $T$ is $\kappa_\omega$-categorical if for each $n<\omega$, the set $S_n(T)$ of its $n$-types is finite. For such theories $T$, we call the function $n \rightarrow |S_n(T)|$ the Ryll-Nardzewski function of $T$. Weaskiewicz asked if the Ryll-Nardzewski function of a decidable theory is recursive. It is shown that this question has a negative answer. More specifically, for any Turing degree $a$ there is a function $G: \omega \rightarrow \omega$ of degree $b$ with the following property: whenever $\alpha$ is a degree such that $b$ is recursively enumerable in $a$, then there is a complete, $\kappa_\omega$-categorical theory of degree $\alpha$ whose Ryll-Nardzewski function is $G$.

According to the classic theorem of Ryll-Nardzewski [1], a complete first-order theory $T$ is $\kappa_\omega$-categorical iff, for each $n<\omega$, the set $S_n(T)$ of its $n$-types is finite. For such theories $T$ let us denote the function $n \rightarrow |S_n(T)|$ by $R_T$, which we call the Ryll-Nardzewski function of the theory $T$. The following question was posed by Waszkiewicz in [2]: Is the Ryll-Nardzewski function of a decidable $\kappa_\omega$-categorical theory always recursive? It is the purpose of this note to give a negative answer to this question (?). More generally, we consider a relativized version of Waszkiewicz's question: If the Turing degree of $T$ is $a$, then what are the possible Turing degrees $b$ of $R_T$? It is a straightforward matter to show that $b$ must be recursively enumerable in $a$. Our theorem shows that this is the only restriction. What is perhaps most surprising is that the Ryll-Nardzewski function of degree $b$ can be chosen independently of $a$.

Theorem. For any Turing degree $b$ there is a function $G: \omega \rightarrow \omega$ of degree $b$ with the following property: whenever $\alpha$ is a degree such that $b$ is recursively enumerable in $a$, then there is a complete $\kappa_\omega$-categorical theory of degree $\alpha$ in a language consisting of one binary relation symbol whose Ryll-Nardzewski function is $G$.

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(?) A negative answer was also given by E. Herrmann of Humboldt University in Berlin. His results, obtained independently of ours, are contained in a manuscript entitled "About Lindenbaum functions of $\kappa_\omega$-categorical theories of finite similarity type". The examples he gives are not, however, as extensive as ours.
Before embarking on the proof of the theorem, we note the following conventions. All theories which we use are in a countable language. All \( \kappa \)-categorical theories are assumed to be complete and deductively closed, so that the decision problem for such theories has the same degree as the theory itself. The set of finite subsets of a set \( X \) will be denoted by \( [X]^{\leq n} \). For a finite object \( X \), we denote its \( \text{Goedel number} \) by \( \ulcorner X \urcorner \). We identify a natural number with the set of its predecessors.

The cardinality of a set \( X \) will be denoted by \( |X| \). Finally, we will make a concession to expediency by frequently confounding symbols and their denotations.

The proof of the theorem requires the construction of a model. The actual model we construct will be for a finite language: the further improvement to a language with just one binary relation symbol is a routine exercise, and will consequently be omitted.

Let us start by considering functions \( f: \alpha \rightarrow \omega \) which are finite-one; that is, for each \( n < \omega \), the set \( \{ i < \omega : f(i) = n \} \) is finite. Our first goal is to construct a model \( \mathfrak{M}_f \). To accomplish this we let \( T_f \) be a theory in a language which consists of a unary predicate symbol \( U \), and for each \( i < \omega \) an \((i+1)\)-ary relation symbol \( P_i \) and an \( (f(i)+1)\)-ary relation symbol \( Q_i \). Let \( T_f \) be axiomatized by the following sentences (1)-(6):

1. \( \forall x_1 \ldots x_n \exists X \exists y (x_1 \neq x_2 \land \ldots \land x_n \neq x_1) \) whenever \( 1 \leq i < \omega \).
2. \( \forall x_1 \ldots x_n \exists P_i (x_1, \ldots, x_n) \rightarrow (x_{i+1} \in P_i (x_1, \ldots, x_n)) \) whenever \( i < \omega \) and \( \pi \) is a permutation of \( i+1 \).

Because of sentences (1) and (2) we can unambiguously refer to \( P(X) \), where \( X \) is a nonempty, finite set of variables. We will say that \( X \) is \textit{minimal} if \( P(X) \), and \( \not\exists Y(P(Y)) \) whenever \( Y \subseteq X \) and \( 0 \neq Y \neq X \). We will write \( \text{Min}(X) \) for \( X \) is minimal.

3. \( \forall X \exists Y, X = Y \land \forall x : (x \in X \rightarrow x \in Y) \).
4. \( \forall X, Y, (X \subseteq Y \land Y \subseteq X) \rightarrow (X = Y) \).

Lastly, we give a schema which defines the relations \( Q_i \).

6. \( \forall X, \forall x (x \in X \land x \not\in X \rightarrow \exists Y, \forall x, (x \in Y \rightarrow (x \in X \lor \exists y, (y \in X \land y \neq x)))) \).

Of course, sentences (3)-(6) are schemata. Notice that \( T_f \) is an \( \mathfrak{V}_3 \) theory, and as a set it has the same degree as \( f \).

It is readily observed that \( T_f \) has a model-completion \( T'_f \), and that \( T'_f \) is a complete, \( \kappa \)-categorical theory. Let \( B \) be a countable model of \( T'_f \).

We now form a new structure \( \mathfrak{M}_f \) in the following way. Let \( \mathfrak{M}_f \) be a collection of subsets \( M \subseteq B \) such that the following are satisfied:

(a) if \( M \in \mathfrak{M}_f \) and \( F \subseteq [M]^{\leq n} \), then \( \neg P(F) \);
(b) if \( F_1, F_2 \in [M]^{\leq n} \) are such that \( F_1 \cap F_2 = \emptyset \) and \( \neg P(F_1) \), then there is \( M \in \mathfrak{M}_f \) such that \( F_1 \subseteq M \) and \( F_2 \cap M = \emptyset \);
(c) if \( x_0, \ldots, x_n \in B \) and \( M_0, \ldots, M_n, N_0, \ldots, N_n \in \mathfrak{M}_f \) are all distinct and are such that \( P(M_0 \cup \ldots \cup M_n) \) for each \( i < n \), then there is \( y \in M_0 \cap \ldots \cap M_n \) for each \( i < n \) such that \( x_0 \in M_0 \) and \( x_0 \neq x_1 \).

The collection \( \mathfrak{M}_f \) can easily be constructed by a sort of forcing construction. We now define the structure \( \mathfrak{M}_f \) to be \( (B \cup \mathfrak{M}_f, \mathfrak{I}(B), \mathfrak{I}(\mathfrak{M}_f), E) \), and let \( T_f \) be \( \mathfrak{M}_f \). It is easily seen that \( T_f \) and \( f \) have the same degree. Here we use the fact that a complete theory is recursive in any of its axiomatizations. Also, it is easy to see that \( T_f \) is \( \kappa \)-categorical.

From the previous observation, we can even get an explicit method for evaluating \( R_{f}(\alpha) \). For a finite-one \( f: \alpha \rightarrow \omega \) let

\[ f^{*}(\alpha) = \{ i < \alpha : f(i) = n \} \]

and

\[ I_{f}(\alpha) = \{ i < \alpha : i + f(i) + 1 = n \} \]

Then there is a primitive recursive \( H: \alpha \times \alpha \rightarrow \omega \) (the choice of which does not depend on \( f \)) such that

\[ R_{f}(\alpha) = H(\alpha, I_{f}(\alpha)) \text{ or } n \leq \alpha \text{, for any finite-one } f: \alpha \rightarrow \omega \]

From (7) it follows that there is a primitive recursive sequence \( \langle M_{i} : i < \omega \rangle \) of functions \( M_{i} : \omega \rightarrow \omega \) with the following property. Whenever \( f: \alpha \rightarrow \omega \) is a finite-one function and \( G: \alpha \rightarrow \omega \) and \( n < \omega \) are such that \( f^{*}(\alpha) \leq G(i+1) \) for each \( i < n \), then

\[ M_{n}(G(n+1)) > R_{f}(\alpha) \]

Let \( \alpha \) be the identity function. The function \( R_{f}(\alpha) \) is a primitive recursive function with the following significant property. For any degree \( a \), if \( a \) is a finite set of degree \( a \), and let \( f: \alpha \rightarrow \omega \) be a monotonically increasing function. Then the theory \( T_f \) is such that both \( T_f \) and \( R_{f} \) have degree \( a \), and that \( R_{f}(a) \leq R_{f}(a) \) for all \( i < n \).

Let each \( B \subseteq \omega \) be a countable model of \( G^{a} \). Let \( B \) be the characteristic function of \( B \); that is, \( c_{B} : \omega \rightarrow [0, 1] \) is such that \( c_{B}(a) = 1 \) if \( a \in B \). We now define \( G^{a} \) so that \( G^{a}(0) = 1 \), and

\[ G^{a}(n+1) = c_{B}(n) + M_{n+1}(G^{a}(n+1)) + \sum_{i < n} c_{B}(i) M_{n}(G^{a}(i+1)) R_{f}(n+1) \]

It is clear that \( G^{a} \) and \( B \) have the same degree.
a degree such that \( b \) is recursively enumerable in \( a \). Let \( T' \) be such that the degree of \( T' \) and the degree of \( R_{T'} \) are both \( a \), and \( R_{T'} \preceq R_{T'}^a \). (Such a \( T' \) is described above.) There is a bijection \( F: \omega \to B \) of degree \( a \). For each \( f < \omega \) let \( B_f = \{ F(i): i < f \} \), and then let \( G = G^\alpha \). The following properties are all evident:

(i) If \( i < f < \omega \) and \( n < \omega \), then \( G(i) \preceq G(n) \).

(ii) \( \lim G_i = G \).

(iii) \( \langle G_i: i < \omega \rangle \) has degree \( a \).

We are next going to define a function \( f: \omega \to \omega \). Let us first make the following convention: if \( j < i < \omega \) and \( h: i \to \omega \), then \( R_{T_f}(h) = R_{T_f}(g) \), where

\[
f(h) = \begin{cases} h(n) & \text{if } n < i, \\ n + 1 & \text{if } n \geq i. \end{cases}
\]

Notice that if \( f: \omega \to \omega \) is any finite-one function which extends \( h \), then \( R_{T_f}(h) \preceq R_{T_f}(g) \). We will define \( f \) by induction by defining a sequence \( \langle f_i: i < \omega \rangle \) of functions \( f_i: i \to \omega \) such that \( f_{i+1} \supseteq f_i \), and then setting \( f = \bigcup \{ f_i: i < \omega \} \). Let

\[
f_{i+1}(i) = \mu n < i \{ G(n+1) \geq \sum_{j \in B_i} (n+1) R_{T_f}(f_j) R_{T_f}(n+1 - j) \}.
\]

It is clear from (7) that \( f \) is finite-one: a rough estimate for \( f^* \) is given by \( f^*(n) \equiv G(n+1) \).

The function \( f \) was chosen so as to satisfy the following identity, which we now verify:

\[
G(n) = \sum_{i < \omega} \binom{n}{i} R_{T_f}(f_i) R_{T_f}(n - f_i).
\]

It is clear from (7) and (10) that it suffices to show only that

\[
G(n) \succ \sum_{i < \omega} \binom{n}{i} R_{T_f}(f_i) R_{T_f}(n - f_i).
\]

But this is a consequence of (9) (keeping in mind both (8) and the fact that \( R_{T_f} \preceq R_{T_f}^a \)).

To complete the proof of the theorem, we construct the desired theory \( T \).

Let \( B \) be a countable model of \( T' \), and let \( T = \text{Th}(B, +) \). Here, \( B, + \) is the two-sorted structure which is the disjoint union of \( B_0 \) and \( B \). Since \( T' \) has degree \( a \) and \( T_f \) also does (since \( T_f \) has the same degree as \( R_{T_f} \)), it is clear that \( T \) has degree \( a \).

Finally, a moment’s reflection is enough to conclude from (11) that \( R_T = G \). This completes the proof of the theorem.

Remark 1. If it is clear from the proof of the theorem that if \( b = \emptyset \), then \( G \) can be chosen to be primitive recursive. Thus, there is a primitive recursive \( G \) such that for any degree \( a \) there is a \( \aleph_1 \)-categorical \( T \) of degree \( a \) such that \( R_T = G \).

Remark 2. From the manner in which the \( G \)'s were constructed it is easy to see that there is some primitive recursive \( G_0 \) such that for each \( G \) constructed in the proof, \( G(n) \equiv G_0(n) \) for each \( n < \omega \). It is, however, easy to find a decidable \( T \) such that \( R_T \) majorizes every recursive function. Let \( f: \omega \to \omega \) be a finite-one recursive function such that \( f^* \) majorizes every recursive function. Then \( T_f \) is recursive and \( R_T \) majorizes every recursive function.

We close with the following problem.

**Problem.** What is true if only finitely axiomatizable \( \Delta_0 \)-categorical theories are considered?


References


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