Collectionwise normality and extensions of continuous functions

by

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Abstract. A subset A of a space X is *P-embedded* (resp. *P^τ-embedded*) if every continuous pseudometric (resp. pseudometric of weight $\leqslant \tau$) on A is continuously extendable onto X. It is known that a T_1-space X is collectionwise normal (resp. τ-collectionwise normal) if its every closed subset is P-embedded (resp. P^τ-embedded) in X.

In this paper we characterize P- and P^τ-embeddability by means of extendability of certain continuous functions. The results obtained here give affirmative answers to the questions raised by Aló and Sennott. We also derive a number of corollaries characterizing collectionwise normality and τ-collectionwise normality, including the well-known theorem of Tamano and recent theorems of Rudin and Starbird.

In order to prove the above mentioned results, we generalize the Kowalsky universality theorem, by showing, that if M is an absolute retract for metrizable spaces and has weight τ, then $M^{\aleph_0}$ is universal for the class of metrizable spaces of weight $\leqslant \tau$.

§ 1. Introduction. A pseudometric ϱ defined on X is of *weight* τ if the topology induced on X by ϱ has weight τ. A subset A of a space X is *P-embedded* (resp. *P^τ-embedded*) if every continuous pseudometric (resp. pseudometric of weight $\leqslant \tau$) defined on A is continuously extendable onto X (Arens [3] and Shapiro [17]).

The main aim of this paper is to give new characterizations of P- and P^τ-embeddability and derive from them the known characterizations of collectionwise normality due to Tamano [20], Rudin [16] and Starbird [18]. In Section 1 we introduce necessary notions and symbols. Section 2 is devoted to a generalization of the Kowalsky universality theorem and in Section 3 we formulate and prove the above mentioned characterizations of P- and P^τ-embeddability.

This paper is closely related to our previous paper [15], though the results obtained here are not based on the results from [15] (see also [21] and [22]).

We adopt the notation and terminology from [7]. In particular, $w(X)$ denotes the *weight* of X and $J(\tau)$ stands for the *hedgehog with* τ *spikes* (τ always denotes an *infinite* cardinal number). A T_1-space X is *τ-collectionwise normal* if every discrete family $\{F_\alpha\}_{\alpha<\tau}$ of closed subsets of X can be separated by disjoint open sets. Clearly, normality is equivalent to $\aleph_0$-collectionwise normality. By I and E we denote the unit interval and the real line respectively. A subset A of X is *C-embedded* (resp.

C^*-embedded) if every continuous function $f\colon A\to E$ (resp. $f\colon A\to I$) is continuously extendable onto X. A metrizable space M is an *absolute retract for metrizable spaces* if every continuous mapping $f\colon F\to M$ of a closed subset F of a metrizable space X into M is continuously extendable onto X. It is known that every (retract of *a*) Banach space is an absolute retract for metrizable spaces (Dugundji [5]).

Let M and Z denote a metric and a compact space respectively. By $C(Z, M)$ we denote the metric space of all continuous mappings $f\colon Z\to M$ with the topology of uniform convergence. It is known that if M is a Banach space, then also $C(Z, M)$ is a Banach space and that $w(C(Z, I)) = \aleph_0\cdot w(Z)$.

For an arbitrary space X the exponential mapping $\Lambda\colon M^{X\times Z}\to C(Z, M)^X$, defined by $[\Lambda(f)(x)](z) = f(x, z)$, for $x\in X$, $z\in Z$ and $f\in M^{X\times Z}$, establishes a one-to-one correspondence between continuous mappings $f\colon X\times Z\to M$ and continuous mappings $g\colon X\to C(Z, M)$ (see [6]; Chapter XII, Theorems 3.1, 5.3 and 8.2).

§ 2. **Universal metric spaces.** It is well-known that the space $J(\tau)$ is an absolute retract for metrizable spaces. Therefore the following theorem, which will be useful in the sequel, is a generalization of the Kowalsky theorem, stating that the space $J(\tau)^{\aleph_0}$ is universal for the class of metrizable spaces of weight $\leqslant\tau$ (see [7]; Theorem 4.4.7).

THEOREM 1 (Generalized Kowalsky's theorem). *Let M be an absolute retract for metrizable spaces, which has weight τ. The space $M^{\aleph_0}$ is universal for the class of metrizable spaces of weight $\leqslant\tau$.*

Moreover, if M is non-compact, then every completely metrizable space of weight $\leqslant\tau$ can be embedded into $M^{\aleph_0}$ as a closed subspace.

Proof. Let X be a metrizable space of weight $\leqslant\tau$. Choose a base $\{B_s\}_{s\in S}$ of X such that $|S|\leqslant\tau$, $S = \bigcup_{n<\omega} S_n$, $S_n\cap S_m = \emptyset$ for $n\neq m$ and the family $\{B_s\}_{s\in S_n}$ is discrete in X for every $n<\omega$. Take two different points a, b which do not belong to S. Since $w(M) = \tau$, there exists in M a disjoint family $\{V_t\colon t\in S\cup\{a, b\}\}$ of non-empty open sets (see [4]; Theorem 4). Choose points $y_t\in V_t$, put $G_n = \bigcup_{s\in S_n} B_s$ and let $G_n = \bigcup_{m<\omega} F_{nm}$, where F_{nm} are closed in X. Since M is an absolute retract for metrizable spaces, the mapping $f_n\colon \bigcup_{s\in S_n}\bar{B}_s\to M$ defined by $f_n(\bar{B}_s)\subset\{y_s\}$ for $s\in S_n$ can be continuously extended to the mapping $\tilde{f}_n\colon X\to M$. Similarly, the mapping $f_{nm}\colon F_{nm}\cup(X\setminus G_n)\to M$ defined by $f_{nm}(F_{nm})\subset\{y_a\}$ and $f_{nm}(X\setminus G_n)\subset\{y_b\}$ can be continuously extended to the mapping $\tilde{f}_{nm}\colon X\to M$.

Let $g_{nm} = \tilde{f}_n\Delta\tilde{f}_{nm}\colon X\to M^2$ be the diagonal of mappings $\tilde{f}_n$ and $\tilde{f}_{nm}$. It suffices to show that the diagonal mapping $\mathop{\Delta}_{(n,m)\in\omega^2} g_{nm}\colon X\to(M^2)^{\aleph_0}\simeq M^{\aleph_0}$ is a homeomorphic embedding. To this end, we shall prove that the family $\{g_{nm}\}_{(nm)\in\omega^2}$ separates points from closed sets (see [7]; The Diagonal Lemma, p. 78). Let F be a closed subset of X and take $x\in X\setminus F$. There exist $n, m<\omega$ and an $s\in S_n$ such that $x\in B_s\subset X\setminus F$ and $x\in F_{nm}$. We have $g_{nm}(x) = (y_s, y_a)\in V_s\times V_a$, but if $z\in F$ then either $z\notin G_n$ and $g_{nm}(z) = (y, y_b)$ for some $y\in M$, or there exists an $s'\in S_n$ such that $s\neq s'$, $z\in B_{s'}$ and consequently $g_{nm}(z) = (y_{s'}, y)$ for some $y\in M$. In any case, $g_{nm}(z)\notin V_s\times V_a$ since $y_b\notin V_a$ and $y_{s'}\in V_s$. This proves the first part of the theorem.

New, let us assume additionally that M is non-compact and that X is completely metrizable. By the first part of the theorem, X is embeddable as a G_δ-subspace into $M^{\aleph_0}$. Hence, by [10]; § 21, XIII, Corollary X is embeddable as a closed subspace into $M^{\aleph_0}\times E^{\aleph_0}$. Therefore, to complete the proof it is enough to show, that E is embeddable as a closed subspace into $M^{\aleph_0}$.

Let Z denote the set of integers [1]. Define

$$I_k = [2k, 2k+1],\quad J_k = [2k+1, 2k+2],$$
$$A = \bigcup\{I_k\colon k\in Z\}\quad\text{and}\quad B = \bigcup\{J_k\colon k\in Z\}.$$

Choose a closed and discrete subset $F = \{a_k\colon k\in Z\}$ of M and let $g\colon E\to M^{\aleph_0}$ be an arbitrary embedding of E into $M^{\aleph_0}$. Since M is an absolute retract for metrizable spaces, the perfect mappings $u\colon A\to M$, and $v\colon B\to M$ defined by $u(I_k) = \{a_k\}$ and $v(J_k) = \{a_k\}$ can be continuously extended to the mappings $\tilde{u}\colon E\to M$ and $\tilde{v}\colon E\to M$ respectively. Let $h = \tilde{u}\Delta\tilde{v}\colon E\to M^2$ be the diagonal of $\tilde{u}$ and $\tilde{v}$. Since $h|A$ and $h|B$ are perfect mappings and $E = A\cup B$, the mapping h is also perfect. The diagonal mapping $f = g\Delta h\colon E\to M^{\aleph_0}\times M^2\simeq M^{\aleph_0}$ is clearly a closed embedding of E into $M^{\aleph_0}$ (cf. [7]; The Diagonal Lemma, p. 78). ■

Remark 1. As observed by H. Toruńczyk, not every absolute retract for metrizable spaces M of weight $\tau>\aleph_0$ contains a copy of $J(\tau)$. ■

§ 3. **Characterizations of P^τ-embedded subsets.** The following proposition is a slight modification of the result of Aló and Sennott [1]. The short proof of this fact has been communicated to the author by R. Pol.

PROPOSITION 1. *A subset A of a space X is P^τ-embedded if and only if every continuous mapping $f\colon A\to B$ into an arbitrary Banach space B of weight $\leqslant\tau$ is continuously extendable onto X.*

Proof. Assume that A is P^τ-embedded in X and let $f\colon A\to B$ be a continuous mapping into a Banach space B of weight $\leqslant\tau$. Define a pseudometric ϱ on A by putting $\varrho(a, a') = \|f(a)-f(a')\|$, where $a, a'\in A$ and $\|\cdot\|$ denotes a norm in B. Clearly the weight of ϱ is $\leqslant\tau$, so that we can find a continuous extension $\tilde{\varrho}$ of ϱ onto X. Let $X_{\tilde{\varrho}}$ denote the set X with the topology induced by the pseudometric $\tilde{\varrho}$, let F be the closure of A in $X_{\tilde{\varrho}}$ and consider F with the topology of the subspace of $X_{\tilde{\varrho}}$. If $\{a_n\}_{n=1}^\infty$ is a sequence of points of A, $x\in X$ and $\lim_{n\to\infty}\tilde{\varrho}(a_n, x) = 0$, then the sequence $\{f(a_n)\}_{n=1}^\infty$ satisfies Cauchy's condition in B. We deduce easily that there exists a continuous extension $\bar{f}\colon F\to B$ of f onto F. By the Dugundji theorem [5] we can find a continuous extension $\tilde{f}\colon X_{\tilde{\varrho}}\to B$ of $\bar{f}$ onto $X_{\tilde{\varrho}}$. Clearly $\tilde{f}$ is the required extension of f.

(1) The short proof of the second part of Theorem 1 is due to R. Pol.

Assume now that X satisfies the second condition of the proposition and let ϱ be a continuous pseudometric of weight $\leqslant\tau$ defined on A. Denote by M the set A with the *metric* topology induced in the obvious way by ϱ and let $f\colon A\to M$ be a natural continuous projection. We can assume that M is isometrically embedded in a Banach space B of weight $\leqslant\tau$. By our assumption, we can find a continuous extension $\tilde{f}\colon X\to B$ of f into B. The continuous pseudometric $\tilde{\varrho}$ defined by $\tilde{\varrho}(x,x') = ||\tilde{f}(x)-\tilde{f}(x')||$, where $x,x'\in X$ and $||\cdot||$ denotes a norm in B, is the required extension of ϱ. ■

THEOREM 2. *For a cardinal number τ and a subset A of X the following conditions are equivalent*:

(i) *A is P^τ-embedded in X.*

(ii) *Every continuous mapping $f\colon A\to M$ into an arbitrary absolute retract for metrizable spaces M, which is complete and has weight $\leqslant\tau$, is continuously extendable onto X.*

(iii) *Every continuous mapping $f\colon A\to J(\tau)$ is continuously extendable onto X.*

(iv) *There exists a non-compact absolute retract for metrizable spaces M of weight τ such, that every continuous mapping $f\colon A\to M$ is continuously extendable onto X.*

Proof. (i)⇒(ii). Let $f\colon A\to M$ be a continuous mapping. We can assume that M is embedded as a closed subspace into a Banach space B of weight $\leqslant\tau$. By Proposition 1 there exists a continuous extension $\tilde{f}\colon X\to B$ of f into B. The composition $\hat{f} = r\circ\tilde{f}\colon X\to M$, where $r\colon B\to M$ denotes a continuous retraction, is a continuous extension of f.

Since $J(\tau)$ is an absolute retract for metrizable spaces, which is complete and has weight τ, the implications (ii)⇒(iii) and (iii)⇒(iv) are obvious.

(iv)⇒(i). Let $f\colon A\to B$ be a continuous mapping into a Banach space B of weight $\leqslant\tau$. By Proposition 1, it suffices to prove that f admits a continuous extension onto X. Assume (iv). By Theorem 1 B is embeddable as a closed subset into $M^{\aleph_0}$. By our assumption, there exists a continuous extension $\tilde{f}\colon X\to M^{\aleph_0}$ of f into $M^{\aleph_0}$. The composition $\hat{f} = r\circ\tilde{f}\colon X\to B$, where $r\colon M^{\aleph_0}\to B$ is a retraction, is a required extension of f. ■

Remark 2. (a) From [15]; Corollary 4 we infer that the condition (ii) of Theorem 2 is equivalent to the following:

(ii)* every continuous mapping $f\colon A\to M$ into an arbitrary metrizable absolute retract for τ-collectionwise normal spaces M is continuously extendable onto X.

Therefore, by Corollary 2 below, a metrizable space M is an absolute retract for τ-collectionwise normal spaces if and only if every continuous mapping $f\colon A\to M$ from a P^τ-embedded subset A of an arbitrary space X is continuously extendable onto X.

(b) It follows easily from ([15]; Theorem 1) that the condition (ii) is also equivalent to:

(ii)** every continuous mapping $f\colon A\to P$ into an arbitrary Čech-complete absolute retract for paracompact p-spaces P satisfying $l(P)\leqslant\tau$, is continuously extendable into X.

(c) The equivalence of (i) and (ii) has been first proved by Morita [12]. ■

COROLLARY 1 (Gantner [8]). *A subset A of X is C-embedded if and only if it is $P^{\aleph_0}$-embedded.* ■

COROLLARY 2 (Gantner [9]; for collectionwise normal spaces Shapiro [17]). *A T_1-space X is τ-collectionwise normal if and only if its every closed subset is P^τ-embedded in X.*

Proof. This follows immediately from Theorem 2 and the fact that a T_1-space X is τ-collectionwise normal if and only if every continuous mapping $f\colon F\to J(\tau)$ from a closed subset F of X into $J(\tau)$ is continuously extendable onto X (see Engelking [7]; Problem 5.E; for an outline of proof see [15]; Fact 5).

Remark 3. For every τ there exists a perfectly normal τ-collectionwise normal space, which is not τ^+-collectionwise normal ([14]; Theorem 1; see also [15]; Lemma 3). We infer from Corollary 2 that for every τ there exists a perfectly normal space X and its closed subset, which is P^τ-embedded in X but not P^{τ^+}-embedded. ■

THEOREM 3. *For a cardinal number τ and a subset A of X the following conditions are equivalent*:

(i) *A is P^τ-embedded in X.*

(ii) *$A\times Z$ is P^τ-embedded in $X\times Z$, where Z is an arbitrary compact space of weight $\leqslant\tau$.*

(iii) *$A\times I^\tau$ is C^*-embedded in $X\times I^\tau$.*

(iv) *There exists a compact space Z of weight τ such, that $A\times Z$ is C^*-embedded in $X\times Z$.*

Proof. (i)⇒(ii). Let $f\colon A\times Z\to B$ be a continuous mapping into a Banach space B of weight $\leqslant\tau$. By Proposition 1 is suffices to find a continuous extension of f onto $X\times Z$. The mapping $\Lambda(f)\colon A\to C(Z,B)$ is continuous (see § 1) and since $C(Z,B)$ is a Banach space of weight $\leqslant\tau$, we infer from Proposition 1 that there exists a continuous extension $g\colon X\to C(Z,B)$ of $\Lambda(f)$ onto X. The continuous mapping $\tilde{f} = \Lambda^{-1}(g)\colon X\times Z\to B$ is a required extension of f.

In virtue of Corollary 1, implications (ii)⇒(iii) and (iii)⇒(iv) are obvious.

(iv)⇒(i). As a retract of the Banach space $C(Z,E)$, the non-compact space $C(Z,I)$ of weight τ is an absolute retract for metrizable spaces. By Theorem 2 it is enough to prove that every continuous mapping $f\colon A\to C(Z,I)$ is continuously extendable onto X. The mapping $\Lambda^{-1}(f)\colon A\times Z\to I$ is continuous and, by our assumption, there exists a continuous extension $g\colon X\times Z\to I$ of $\Lambda^{-1}(f)$ onto $X\times Z$. The continuous mapping $\tilde{f} = \Lambda(g)\colon X\to C(Z,I)$ is the required extension of f. ■

Remark 4. (a) The equivalence of (i) and (ii) has been first proved by Aló and Sennott [2]. They also raised the problem of the equivalence of conditions, (i), (iii) and (iv).

(b) In [11] Morita proved that a space X is normal and τ-paracompact if and only if $X\times I^{\tau}$ is normal. ■

Theorem 3 and Corollary 2 imply

COROLLARY 3 (Starbird [18]). *Let Z be a compact space of weight τ. A T_1-space X is τ-collectionwise normal if and only if $F\times Z$ is C^*-embedded in $X\times Z$ for every closed subset F of X.* ■

Since every closed subset of a normal space is C^*-embedded we have

COROLLARY 4 (Rudin [16]). *Let Z be a compact space of weight τ. If $X\times Z$ is normal, then X is τ-collectionwise normal.* ■

THEOREM 4. *For a subset A of a completely regular space X the following conditions are equivalent:*

(i) *A is P-embedded in X.*

(ii) *$A\times Z$ is P-embedded in $X\times Z$, for every compact Z.*

(iii) *$A\times\beta X$ is C^*-embedded in $X\times\beta X$.*

(iv) *There exists a compact space Z of weight $\geqslant w(A)$ such, that $A\times Z$ is C^*-embedded in $X\times Z$.*

Proof. The implication (i)⇒(ii) follows from Theorem 3 and the implications (ii)⇒(iii) and (iii)⇒(iv) are obvious.

(iv)⇒(i). From Theorem 3 we infer that A is P^{τ}-embedded in X, where $\tau = w(Z)\geqslant w(A)$. This implies that A is P-embedded in X since every continuous pseudometric on A has clearly weight $\leqslant w(A)$. ■

Remark 5. (a) The equivalence of (i) and (ii) has been first proved by Aló and Sennott [2]. They also raised the problem of the equivalence of conditions (i), (iii) and (iv). The equivalence of conditions (i) and (iii) has been also conjectured by R. Blair.

(b) Tamano [19] proved that a space X is paracompact if and only if $X\times\beta X$ is normal. ■

Theorem 4 and Corollary 2 imply

COROLLARY 5 (Tamano [20]). *A T_1-space X is collectionwise normal if and only if $F\times\beta X$ is C^*-embedded in $X\times\beta X$ for every closed subset F of X.* ■

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Added in proof. Prof. S. Neder pointed out that Theorem 1 actually implies the following: The space $M^{\aleph_0}$ is isometrically universal for the class of metrizable spaces of weight $\leqslant\tau$, provided M is a non-compact absolute retract for metric spaces of weight τ and an appropriate metric on $M^{\aleph_0}$ is chosen. ■

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