Let $\bar{q}$ be the unique lifting of q to E_A such that $\bar{q}(0) = e$. In fact $\bar{q}$ can be represented by the path

$$Q(t)(s) = q(st).$$

Clearly Q is a path in $E_{\langle A,H\rangle}$ and $x = \bar{q}(1) = Q(1) \in E_{\langle A,H\rangle}$ completing the proof.

Finally, we define a Q-subgroup $G \subseteq \pi_1(\mathscr{E}(B), b)$ to be costandard if the corresponding $\mathscr{E}$-covering space is costandard as follows.

III. 36. DEFINITION. Suppose $G \subseteq \pi_1(\mathscr{E}(B), b)$ is a Q-subgroup and $p\colon E \to \mathscr{E}(B)$ is the corresponding $\mathscr{E}$-covering space. Let $e \in E$ be as above. G is said to be *costandard* if

(i) $p\colon E \to \mathscr{E}(B)$ is costandard and

(ii) using the notation of Definition III. 17, ψ may be chosen so that $\psi(e)$ is standard.

The results of this section can then be summarized by

III. 37. THEOREM. *Equivalence classes of overlay structures $p\colon E \to B$ with distinguished points e are in bijective correspondence with costandard, Q-subgroups of $\pi_1(\mathscr{E}(B), b)$.*

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Collectionwise normality and absolute retracts

by

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Abstract. In this paper we present a generalization and a unification of the classical theorems concerning absolute retracts for metrizable spaces, which were proved in the early fifties by Arens, Dowker, Hanner and Michael.

To illustrate possible applications of the obtained results we derive from them a generalization of Borsuk's homotopy extension theorem, which is a slight strengthening and reformulation of the recent result of Morita and Starbird.

§ 1. Introduction. Let Q be an arbitrary class of topological spaces. A space $X \in Q$ is an *absolute retract for the class Q* (briefly: an AR(Q)-space) if for every space $Z \in Q$ containing X as a closed subspace there exists a continuous retraction of Z onto X. A topological space X is an *absolute extensor for the class Q* (an AE(Q)-space) if for every space $Z \in Q$, its closed subspace F and continuous mapping $f\colon F \to X$ there exists a continuous extension $\bar{f}\colon Z \to X$ of f onto Z.

Note, that in the definition of an absolute extensor we do not require X to be a member of Q. A space $X \in Q$ is an *absolute neighbourhood retract for the class Q* (an ANR(Q)-space) if for every space $Z \in Q$ containing X as a closed subspace there exists a neighbourhood U of X in Z and continuous retraction of U onto X. A topological space X is an *absolute neighbourhood extensor for the class Q* (an ANE(Q)-space) if for every space $Z \in Q$, its closed subspace F and a continuous mapping $f\colon F \to X$ there exists a neighbourhood U of F in Z and a continuous extension $\bar{f}\colon U \to X$ of f onto U.

Absolute retracts for normal spaces are called briefly *absolute retracts* (AR-spaces). Similarly absolute extensors (AE-spaces), absolute neighbourhood retracs (ANR-spaces) and absolute neighbourhood extensors (ANE-spaces) are defined.

FACT 1 (The Tietze–Urysohn theorem). *The real line E and the unit interval $I = [0, 1]$ are absolute extensors.* ∎

One can easily check that if a space X belongs to Q and is an AE(Q)-space (resp. an ANE(Q)-space), then X is an AR(Q)-space (resp. an ANR(Q)-space). It turns out that in "good" classes of spaces the inverse implication holds.

FACT 2 (Hanner [12], Michael [15]). *If Q denotes the class of normal* (*resp. metrizable*; *resp. compact*) *spaces and if X belongs to Q, then:*

(a) *X is an* AR(Q)*-space iff X is an* AE(Q)*-space.*

(b) *X is an* ANR(Q)*-space iff X is an* ANE(Q)*-space.* ■

FACT 3 (Dugundji [8]). *Convex subspaces of locally convex linear topological spaces are absolute extensors for metrizable spaces.*

In particular, normed linear spaces are absolute retracts for metrizable spaces. ■

Fact 3 and the Kuratowski–Wojdysławski theorem (see Borsuk [5]; Theorem III. 8.1) imply:

FACT 4. *Any metrizable space M_0 can be embedded as a closed subspace into an absolute retract for metrizable spaces M.*

Moreover, one may assume that $w(M) = \aleph_0 \cdot w(M_0)$ and that M is complete if so is M_0. ■

Assume that Q and S are two classes of topological spaces, $Q \subset S$ and that X is an absolute retract for the class Q. A natural question arises: what conditions have to be satisfied in order that X be an absolute retract for the class S? For Q denoting the class of metrizable spaces several important theorems partially answering this question have been obtained in the early fifties by Arens [1], Dowker [7], Hanner [11], [12] and Michael [15]. It is the aim of this paper to present a generalization and a unification of these results.

The paper is organized as follows. In Section 2 main theorems dealing with AR(Q)- and AE(Q)-spaces are proved. Some of their consequences are also derived. In Section 3 we formulate the counterparts of these results for ANR(Q)- and ANE(Q)-spaces. Proofs are very similar and therefore are omitted. To illustrate possible applications of the obtained results, in Section 4 we derive from them a generalization of Borsuk's homotopy extension theorem, which is a slight strengthening of the recent result of Morita [17] and Starbird [20].

We shall denote by M, C and P the classes of metrizable, compact and paracompact p-spaces respectively. Let us recall (cf. Arhangel'skiĭ [2] and Morita [16]), that a space X is a *paracompact p-space* (= *paracompact M-space*) if X is a closed subspace of the cartesian product $M \times C$ of a metrizable space M and a compact space C, or — which is equivalent — if X is an inverse image of a metrizable space under a perfect mapping.

Letters τ and $\varkappa$ always denote infinite cardinal numbers. We adopt the terminology and notation from [10], in particular $J(\tau)$ stands for the hedgehog with τ spikes and $w(X)$ denotes the weight of the space X. A T_1-space is *τ-collectionwise normal* if every discrete family $\{F_\alpha\}_{\alpha<\tau}$ of its closed subsets can be separated by disjoint open subsets. Clearly $\aleph_0$-collectionwise normality is equivalent to normality. A space is *perfectly τ-collectionwise normal* if it is *perfect* (= open subsets are F_σ-sets) and τ-collectionwise normal. The *Lindelöf number* $l(X)$ of the space X is defined as the smallest cardinal number $\varkappa$ such that every open covering of X contains a subcovering of cardinality $\leqslant \varkappa$. Obviously, X has the Lindelöf property if and only if $l(X) \leqslant \aleph_0$.

The following facts will be useful in the sequel.

FACT 5 (see Engelking [10]; Problem 5. E). *$J(\tau)$ is an absolute extensor for τ-collectionwise normal spaces.*

Outline of proof. The set $J(\tau)$ can be represented as the union $\bigcup_{\alpha<\tau} (I \times \{\alpha\})$ with the points $\{(0, \alpha)\}_{\alpha<\tau}$ identified. Let $f\colon F \to J(\tau)$ be a continuous mapping of a closed subspace F of a τ-collectionwise normal space Z. Consider the function $g\colon J(\tau) \to I$ defined for $\alpha<\tau$ and $t \in I$ by $g\big((t, \alpha)\big) = t$ and let $G\colon Z \to I$ be a continuous extension of the mapping $g \circ f$. The family $\{F_\alpha\}_{\alpha<\tau}$, where $F_\alpha = f^{-1}((0, 1] \times \{\alpha\})$, is discrete in $G^{-1}((0, 1])$ and therefore we can find a disjoint family $\{U_\alpha\}_{\alpha<\tau}$ of open subsets of Z such that $F_\alpha \subset U_\alpha$. The function $h\colon (F \cup Z \setminus \bigcup_{\alpha<\tau} U_\alpha) \to I$ defined by $h|F = g \circ f$ and $h(Z \setminus \bigcup_{\alpha<\tau} U_\alpha) \subset \{0\}$ can be extended continuously to the function $H\colon Z \to I$. The mapping $F\colon Z \to J(\tau)$, where $F(z) = \big(H(z), \alpha\big)$ for $z \in U_\alpha$ and $F(z) = (0, \alpha)$, when $z \notin \bigcup_{\alpha<\tau} U_\alpha$ is a required continuous extension of f. ■

FACT 6 (Lisica [14]). *An absolute (neighbourhood) retract for metrizable spaces is an absolute (neighbourhood) extensor for paracompact p-spaces.*

We shall give a short proof of this fact.

Proof. Assume that X is an AR(M)-space. In case of X being an ANR(M)-space the proof is similar. Let $f\colon F \to X$ be a continuous mapping from a closed subspace F of a paracompact p-space Z into X and let $\psi\colon Z \to M$ be a perfect mapping of Z onto a metrizable space M. There exists τ such that $X \subset J(\tau)^{\aleph_0}$ (see [10]; Theorem 4.4.7) and — by Fact 5 — a continuous extension $\varphi\colon Z \to J(\tau)^{\aleph_0}$ of f into $J(\tau)^{\aleph_0}$. Since ψ is perfect the diagonal mapping $g = \varphi \Delta \psi\colon Z \to J(\tau)^{\aleph_0} \times M$ defined by $g(z) = \big(\varphi(z), \psi(z)\big)$ is perfect and therefore the subset $K = g(F)$ of $J(\tau)^{\aleph_0} \times M$, is closed. Denote by π the projection of $J(\tau)^{\aleph_0} \times M$ onto $J(\tau)^{\aleph_0}$. As X is an AE(M)-space (see Fact 2) the mapping $\pi|K\colon K \to X$ has a continuous extension $h\colon J(\tau)^{\aleph_0} \times M \to X$ onto $J(\tau)^{\aleph_0} \times M$. The composition $f = h \circ g\colon Z \to X$ is a required extension of f. ■

The last fact is obvious.

FACT 7. *An absolute (neighbourhood) retract for compact spaces is an absolute (neighbourhood) extensor for normal spaces.* ■

§ 2. Collectionwise normality and absolute retracts. The proposition below explains the relation between absolute retracts for paracompact p-spaces and absolute retracts for metrizable (resp. compact) spaces.

PROPOSITION 1. *A topological space X is an* AR(P)*-space if and only if X is a retract of a product $M \times C$, where M is an* AR(M)*-space and C is an* AR(C)*-space.*

Moreover, one may assume that $w(M) \leqslant l(X)$, $C = I^\varkappa$ for some $\varkappa$ and that M is Čech-complete if so is X.

Proof. Assume that X is an infinite AR(P)-space. There exist a metric space M_0 and a compact space C_0 such that X is a closed subspace of $M_0 \times C_0$. Denote by $\pi\colon M_0 \times C_0 \to M_0$ the projection parallel to C_0. We can clearly assume that $\pi(X) = M_0$. The mapping π and its restriction $\pi|X\colon X \to M_0$ are perfect and therefore

$w(M_0) = l(M_0) \leqslant l(X)$ and M_0 is Čech-complete if so is X (cf. [10]; Theorem 4.1.6 and Problem 3. Y). By Fact 4 the space M_0 can be embedded as a closed subspace into an AR(M)-space M which is complete if so is M_0 and satisfies $w(M) = \aleph_0 \cdot w(M_0)$. Consequently, we have $w(M) = \aleph_0 \cdot w(M_0) \leqslant l(X)$. Clearly C_0 can be embedded as a closed subspace into $C = I^\varkappa$ for some $\varkappa$. Since X is a closed subspace of $M \times C$, there exists a retraction of $M \times C$ onto X.

Assume now that X is a retract of $M \times C$, where M is an AR(M)-space and C is an AR(C)-space and let $r\colon M \times C \to X$ be the retraction. As a closed subspace of $M \times C$ the space X belongs to $\boldsymbol{P}$. It follows that it suffices to prove that X is an AE($\boldsymbol{P}$)-space. Let $f\colon F \to X$ be a continuous mapping of a closed subspace F of a paracompact p-space Z into X. By Facts 6 and 7 the space $M \times C$ is an absolute extensor for paracompact p-spaces and therefore there exists a continuous extension $g\colon Z \to M \times C$ of f into $M \times C$. The composition $\tilde{f} = r \circ g\colon Z \to X$ is a required continuous extension of f. ∎

PROPOSITION 2. *A paracompact p-space is an* AR($\boldsymbol{P}$)*-space if and only if it is an* AE($\boldsymbol{P}$)*-space.*

Proof. We have to prove only that an AR($\boldsymbol{P}$)-space is an AE($\boldsymbol{P}$)-space and this is a consequence of Proposition 1 and the second part of its proof. ∎

PROPOSITION 3. *Every paracompact p-space is embeddable as a closed subspace into an absolute retract for paracompact p-spaces.*

Proof. Let X be a paracompact p-space. There exists a metrizable space M_0 and a compact space C_0 such that X is a closed subspace of $M_0 \times C_0$. By Fact 4 M_0 is embeddable as a closed subspace into an AR(M)-space M and clearly C_0 is embeddable into $I^\varkappa$ for some $\varkappa$. By Proposition 1, X is a closed subspace of an AR($\boldsymbol{P}$)-space $M \times I^\varkappa$. ∎

THEOREM 1. *For an absolute retract for paracompact p-spaces X and a cardinal number τ the following conditions are equivalent:*

(i) *X is Čech-complete and $l(X) \leqslant \tau$.*

(ii) *X is an absolute retract for τ-collectionwise normal spaces.*

(iii) *X is an absolute extensor for τ-collectionwise normal spaces.*

(iv) *X is a closed subspace of $J(\tau)^{\aleph_0} \times I^\varkappa$, for some $\varkappa$.*

Our proof of Theorem 1 (and also of Theorem 2) depends heavily on the methods developed by Arens [1], Dowker [7], Hanner [11], [12] and Michael [15]. First of all we shall prove three lemmas.

LEMMA 1. *Every completely metrizable space of weight $\leqslant \tau$ is embeddable as a closed subspace into $J(\tau)^{\aleph_0}$.*

Proof. By the Kowalsky theorem (see [10]; Theorem 4.4.7) X can be embedded as a G_δ-subspace into $J(\tau)^{\aleph_0}$. It follows from ([13]; § 21, XIII, Corollary), that in order to prove our lemma it suffices to show that the real line E is embeddable as a closed subspace into $J(\aleph_0)^2$, since then X would be a closed subspace of $J(\tau)^{\aleph_0} \times J(\aleph_0)^{\aleph_0}$ and hence a closed subspace of $J(\tau)^{\aleph_0}$.

The set $J(\aleph_0)$ can be considered as the union $\bigcup_{n=-\infty}^{+\infty} (I \times \{n\})$, with the points $\{(0, n)\}_{n=-\infty}^{+\infty}$ identified. Any point $s \in E$ can be represented in the form $s = k + t$, where k is an even (resp. odd) integer and $|t| \leqslant 1$. Define continuous mappings $f, g\colon E \to J(\aleph_0)$ by putting $f(2n+t) = (1-|t|, n)$ and $g(2n+1+t) = (1-|t|, n)$, where n is an arbitrary integer and $|t| \leqslant 1$. One can easily check that the diagonal mapping $f \Delta g\colon E \to J(\aleph_0)^2$ is a homeomorphic embedding of E onto a closed subspace of $J(\aleph_0)^2$. ∎

Remark 1. More general result, with $J(\tau)$ replaced by an arbitrary non-compact AR(M)-space of weight τ is proved in ([19]; Theorem 1). ∎

LEMMA 2. *If a paracompact p-space X is an absolute retract for paracompact spaces, then X is Čech-complete.*

Proof. Assume that X satisfies the conditions of the lemma and let Y be the set βX with the topology obtained from the topology of βX by means of making the points from $\beta X \setminus X$ isolated (cf. [10]; Example 5.1.2).

The space Y is paracompact. Indeed, let $\mathfrak{U} = \{U_s\}_{s \in S}$ be an open covering of Y and $V_s = U_s \cap X$. By the paracompactness of X the open covering $\{V_s\}_{s \in S}$ of X admits an open refinement $\mathfrak{G} = \bigcup_{n<\omega} \mathfrak{G}_n$, such that the family $\mathfrak{G}_n = \{G_{ns}\}_{s \in S}$ is discrete in X and $G_{ns} \subset V_s$. Take open subsets H_{ns} of βX such that $H_{ns} \cap X = G_{ns}$ and $H_{ns} \subset U_s$.

It follows from the density of X in βX that the family $\mathfrak{H}_n = \{H_{ns}\}_{s \in S}$ consists of disjoint sets. We shall show that $\mathfrak{H}_n$ is discrete in Y. If $y \in Y \setminus X$ then $\{y\}$ is a neighbourhood of y intersecting at most one element of $\mathfrak{H}_n$. Take $y \in X$ and let V be a neighbourhood of y in X intersecting at most one element of $\mathfrak{G}_n$. If H is open in βX and $H \cap X = V$, then by the density of X in βX the set H intersects at most one element of $\mathfrak{H}_n$. The family $\bigcup_{n<\omega} \mathfrak{H}_n \cup \{\{y\} |\ y \in Y \setminus \bigcup_{n<\omega} \bigcup \mathfrak{H}_n\}$ is an open and σ-discrete refinement of $\mathfrak{U}$. This shows that Y is paracompact.

By our assumptions, there exists a continuous retraction $r\colon Y \to X$ of Y onto X and families $\mathfrak{G}_n$, $n<\omega$, of open subsets of βX such that $\bigcup \mathfrak{G}_n \supset X$ for $n<\omega$ and $\bigcap_{n<\omega} \mathrm{St}(x, \mathfrak{G}_n) \subset X$ for every $x \in X$ (see Arhangel'skiĭ [2]). If $G \in \bigcup_{n<\omega} \mathfrak{G}_n$, define $\tilde{G} = \mathrm{Int}_{\beta X}(r^{-1}(G \cap X)) \cap G$. The sets $\tilde{G}$ are open in βX and $\tilde{G} \cap X = G \cap X$. Therefore the sets $G_n = \bigcup_{G \in \mathfrak{G}_n} \tilde{G}$ are open in βX and contain X. Clearly, in order to prove our lemma it is enough to show that $X = \bigcap_{n<\omega} G_n$.

Assume that $y \in \bigcap_{n<\omega} G_n \setminus X$. There exists an $n<\omega$ such that $y \notin \mathrm{St}(r(y), \mathfrak{G}_n)$ and $G \in \mathfrak{G}_n$ such that $y \in \tilde{G} \subset r^{-1}(G \cap X) \cap G$. Consequently, $r(y) \in G$ and $y \in \mathrm{St}(r(y), \mathfrak{G}_n)$, which is a contradiction. ∎

Our last lemma is a generalization of the well-known Bing's example [3] of a normal space which is not $\aleph_1$-collectionwise normal (see also [18]).

Lemma 3 (Generalized Bing's example). *For a paracompact space X and a cardinal number $\tau<l(X)$ there exists a space $Z=Z(X,\tau)$ with the following properties:*

(a) *X is a closed G_δ-subspace of Z.*

(b) *$Z\setminus X$ is discrete.*

(c) *Z is τ-collectionwise normal.*

(d) *Z is not τ^+-collectionwise normal.*

(e) *Z is perfect iff X is perfect.*

(f) *X is not a retract of Z.*

Proof. Let $\mathfrak{F}$ be the family of all continuous mappings $f\colon X\to J_f(\tau)$ of X into $J_f(\tau)=J(\tau)$. The diagonal mapping $\psi=\underset{f\in F}{\Delta} f\colon X\to\prod_{f\in F}J_f(\tau)$ is a homeomorphic embedding of X into $T=\prod_{f\in F}J_f(\tau)$ (cf. [10]; The Diagonal Lemma, p. 78 and Fact 5), so that we can identify X with $\psi(X)\subset T$. Let Y be the set T with the topology obtained from the topology of T by means of making the points of $T\setminus X$ isolated (see [10]; Example 5.1.2).

The space Y containing X as a closed subspace is τ-collectionwise normal. Indeed, let $\{F_\alpha\}_{\alpha<\tau}$ be a discrete family of closed subsets of Y. Since $Y\setminus X$ is discrete, we may assume without the loss of generality that F_α's are contained in X. The mapping $\varphi\colon\bigcup_{\alpha<\tau}F_\alpha\to J(\tau)$ defined by $\varphi(F_\alpha)\subset\{(1,\alpha)\}$, where the set $J(\tau)$ is represented in the form $\bigcup_{\alpha<\tau}(I\times\{\alpha\})$, with the points $\{(0,\alpha)\}_{\alpha<\tau}$ identified, has — by Fact 5 — a continuous extension $f\colon X\to J(\tau)$ onto X.

Since $f\in\mathfrak{F}$, there exists a projection $\pi_f\colon Y\to J_f(\tau)$. If $y=\psi(x)$ belongs to Y, then clearly $\pi_f(y)=f(x)$ and hence — is virtue of our identification — π_f is a continuous extension of f. Therefore, the family $\{V_\alpha\}_{\alpha<\tau}$, where $V_\alpha=\pi_f^{-1}((0,1]\times\{\alpha\})$, consists of disjoint open subsets of Y and $F_\alpha\subset V_\alpha$.

The space Y is not τ^+-collectionwise normal. First of all let us note that in X there exists a discrete family of cardinality τ^+ of non-empty closed subsets. Indeed, since $l(X)>\tau$ one can find an open covering $\mathfrak{U}$ of X such that its every subcovering has cardinality $\geqslant\tau^+$. As X is paracompact, $\mathfrak{U}$ admits a refinement $\mathfrak{F}=\bigcup_{n<\omega}\mathfrak{F}_n$, where each family $\mathfrak{F}_n$ is discrete in X and consists of non-empty closed sets. Clearly, one of the families $\mathfrak{F}_n$ must have cardinality $\geqslant\tau^+$.

If the space Y was τ^+-collectionwise normal, then this family could be separated by a disjoint family of cardinality τ^+ consisting of open subsets of Y. It follows from the definition of topology on Y, that this would imply the existence of a disjoint family of cardinality τ^+ of non-empty open subsets of T, which is impossible (see [6]; Corollary 14).

Let $Z=Z(X,\tau)$ be the set $(X\times\{0\})\cup\bigcup_{n=1}^{\infty}((Y\setminus X)\times\{1/n\})$ with the topology of the subspace of the product space $Y\times I$. Obviously, Z satisfies (a) and (b). Properties (c) and (d) easily follow from analogous properties of Y and property (e) is a simple consequence of (a) and (b). The condition (f) follows from collectionwise normality of X and properties (b) and (d). ∎

Remark 2. For X denoting the discrete space of cardinality τ^+, the space $Z=Z(X,\tau)$ is a perfectly normal, τ-collectionwise normal space, which is not τ^+-collectionwise normal (cf. [18]; Theorem 1).

For an arbitrary paracompact space X and $\tau<l(X)$ the normal space $Z=Z(X,\tau)$ contains X as a closed P^τ-embedded, but not P^{τ^+}-embedded subset (see [19]; Theorem 2). ∎

Proof of Theorem 1. (i)⇒(iv). It follows from Proposition 1 that X is a closed subspace of $M\times I^\varkappa$, where M is a complete metrizable space of weight $\leqslant\tau$. By Lemma 1 M can be embedded as closed subspace into $J(\tau)^{\aleph_0}$ and therefore X is a closed subspace of $J(\tau)^{\aleph_0}\times I^\varkappa$ (cf. Šostak [22]).

(iv)⇒(iii). Since X is an AR(P)-space there exists a retraction $r\colon J(\tau)^{\aleph_0}\times I^\varkappa\to X$ of $J(\tau)^{\aleph_0}\times I^\varkappa$ onto X. Let $f\colon F\to X$ be a continuous mapping of a closed subspace F of a τ-collectionwise normal space Z into X. By Facts 5 and 7 there exists a continuous extension $\varphi\colon Z\to J(\tau)^{\aleph_0}\times I^\varkappa$ of f into $J(\tau)^{\aleph_0}\times I^\varkappa$. The composition $\bar f=r\circ f\colon Z\to X$ is a required extension of f.

(iii)⇒(ii). This implication is obvious.

(ii)⇒(i). The Čech-completeness of X follows from Lemma 2 and the inequality $l(X)\leqslant\tau$ is a consequence of Lemma 3. ∎

Corollary 1. *Let X be an absolute retract for paracompact p-spaces. Then:*

(a) *X is an absolute retract (extensor) iff X is Čech-complete and Lindelöf.*

(b) *X is an absolute retract (extensor) for collectionwise normal spaces iff X is Čech-complete.* ∎

Corollary 2. *For a paracompact p-space X and a cardinal number τ the following conditions are equivalent:*

(i) *X is an absolute retract for paracompact p-spaces, $l(X)\leqslant\tau$ and X is Čech-complete.*

(ii) *X is an absolute retract for τ-collectionwise normal spaces.*

(iii) *X is an absolute extensor for τ-collectionwise normal spaces.*

(iv) *X is a retract of $J(\tau)^{\aleph_0}\times I^\varkappa$, for some $\varkappa$.*

Proof. The implication (i)⇒(iv) follows from Theorem 1. If (iv) holds, then by Proposition 1 X is an AR(P)-space and hence in virtue of Theorem 1 (iii) is true.

The implication (iii)⇒(ii) is obvious and if X satisfies (ii) then *a fortiori* X is an AR(P)-space and again by Theorem 1 (i) holds. ∎

The next corollary, which is an easy consequence of Theorem 1, Fact 6 and Lemma 1, is a generalization of the result obtained independently (for $\tau=\aleph_0$ or ∞) by Dowker [7], Hanner [12] and Michael [15] (see also Arens [1] and Bogatyj [4]).

Corollary 3. *For an absolute retract for metrizable spaces M and a cardinal number τ the following conditions are equivalent:*

(i) *M is complete of weight* $\leqslant\tau$.

(ii) *M is an absolute retract for* τ*-collectionwise normal spaces.*

(iii) *M is an absolute extensor for* τ*-collectionwise normal spaces.*

(iv) *M is a closed subspace of* $J(\tau)^{\aleph_0}$. ■

COROLLARY 4. *For a metrizable space M and a cardinal number* τ *the following conditions are equivalent*:

(i) *M is a complete absolute retract for metrizable spaces, and* $w(M)\leqslant\tau$.

(ii) *M is an absolute retract for* τ*-collectionwise normal spaces.*

(iii) *M is an absolute extensor for* τ*-collectionwise normal spaces.*

(iv) *M is retract of* $J(\tau)^{\aleph_0}$. ■

For a compact space X we shall denote by $C(X)$ the space of continuous real-valued functions defined on X with the topology of uniform convergence. It is well-known that $C(X)$ is a Banach space and $w(C(X)) = \aleph_0\cdot w(X)$. Consequently, Fact 3 and Corollary 3 imply:

COROLLARY 5. *A Banach space M is an absolute retract (extensor) for* τ*-collectionwise normal spaces if and only if* $w(M)\leqslant\tau$. ■

COROLLARY 6 (cf. Starbird [21]). *Let X be compact. The space* $C(X)$ *is an absolute retract (extensor) for* τ*-collectionwise normal spaces if and only if* $w(X)\leqslant\tau$. ■

THEOREM 2. *For an absolute retract for paracompact p-spaces X and a cardinal number* τ *the following conditions are equivalent*:

(i) $l(X)\leqslant\tau$.

(ii) *X is an absolute extensor for perfectly* τ*-collectionwise normal spaces.*

(iii) *Every continuous mapping* $f\colon F\to X$ *from a closed* G_δ*-subspace F of a* τ*-collectionwise normal space Z into X is continuously extendable onto Z.*

Proof. (i)⇒(iii). It follows from Proposition 1 that there exists a metrizable space M of weight $\leqslant\tau$ such that X is a closed subspace of $M\times I^\varkappa$ for some $\varkappa$ and we can clearly assume that M is contained in $J(\tau)^{\aleph_0}$. Let $f\colon F\to X$ be a continuous mapping of a closed G_δ-subspace F of a τ-collectionwise normal space Z. Since Z is normal, there exists a continuous function $\psi\colon Z\to I$ such that $\psi^{-1}(0) = F$. By Facts 5 and 7 there is a continuous extension $\varphi\colon Z\to J(\tau)^{\aleph_0}\times I^\varkappa$ of f into $J(\tau)^{\aleph_0}\times I^\varkappa$.

If $g = \varphi\Delta\psi\colon Z\to J(\tau)^{\aleph_0}\times I^\varkappa\times I$ is a diagonal mapping, then

$$g(Z)\subset T = (J(\tau)^{\aleph_0}\times I^\varkappa\times(0,1])\cup(M\times I^\varkappa\times\{0\})$$

and clearly X is a closed subspace of T. Consider the projection $\pi\colon J(\tau)^{\aleph_0}\times I^\varkappa\times I\to J(\tau)^{\aleph_0}\times I$ parallel to the compact space $I^\varkappa$. Since π is perfect and

$$T = \pi^{-1}((J(\tau)^{\aleph_0}\times(0,1])\cup(M\times\{0\}))$$

is an inverse image under π of a metric space, we conclude that T is a paracompact p-space. Let $r\colon T\to X$ be a retraction of T onto X. The composition $\tilde{f} = r\circ g\colon Z\to X$ is a required continuous extension of f.

(iii)⇒(ii). This implication is obvious.

(ii)⇒(i). Assume that $l(X)>\tau$. As in the proof of Lemma 3 we note that X contains a discrete family $\{F_\alpha\}_{\alpha<\tau^+}$ of cardinality τ^+ of non-empty closed subsets. Let $\{U_\alpha\}_{\alpha<\tau^+}$ be a disjoint family of open subsets of X such that $F_\alpha\subset U_\alpha$ and choose an $x_\alpha\in F_\alpha$ for $\alpha<\tau^+$. For the discrete space $D = \{d_\alpha\}_{\alpha<\tau^+}$ of cardinality τ^+ the space $Z = Z(D,\tau)$ from Lemma 3 is perfectly τ-collectionwise normal. Define the mapping $f\colon D\to X$ by putting $f\{d_\alpha\} = x_\alpha$. By our assumption there exists a continuous extension $\tilde{f}\colon Z\to X$ of f onto Z. It follows that the family $\{\tilde{f}^{-1}(U_\alpha)\}$ consists of disjoint open subsets of Z and $d_\alpha\in U_\alpha$, which clearly implies that Z is collectionwise normal. This contradicts Lemma 3. ■

Remark 3. In general, the space X satisfying conditions of Theorem 2 need not be an absolute *retract* for perfectly τ-collectionwise normal spaces, since it need not be perfect. ■

COROLLARY 7. *Let X be an absolute retract for paracompact p-spaces. Then*:

(a) *X is an absolute extensor for perfectly normal spaces iff X is Lindelöf.*

(b) *X is an absolute extensor for perfectly collectionwise normal spaces.* ■

The next corollary is a generalization of the result obtained independently (for $\tau = \aleph_0$ or ∞) by Hanner [12] and Michael [15] (see also Dowker [7] and Bogatyj [4]).

COROLLARY 8. *For an absolute retract for metrizable spaces M and a cardinal number* τ *the following conditions are equivalent*:

(i) $w(M)\leqslant\tau$.

(ii) *M is an absolute retract for perfectly* τ*-collectionwise normal spaces.*

(iii) *M is an absolute extensor for perfectly* τ*-collectionwise normal spaces.*

(iv) *Every continuous mapping* $f\colon F\to M$ *from a closed* G_δ*-subspace of a* τ*-collectionwise normal space Z into M is continuously extendable onto Z.*

Proof. By Theorem 2 and Fact 6 it suffices to prove the implication (ii)⇒(i) and this is a consequence of Lemma 3. ■

COROLLARY 9. *A normed linear space M is an absolute retract (extensor) for perfectly* τ*-collectionwise normal spaces if and only if* $w(M)\leqslant\tau$. ■

§ 3. Collectionwise normality and absolute neighbourhood retracts. In this section we shall formulate for absolute *neighbourhood* retracts (extensors) the counterparts of the results obtained in § 2 for absolute retracts (extensors). Proofs are very similar and therefore are omitted.

PROPOSITION 1*. *A paracompact p-space X in an* ANR(P)*-space if and only if X is a retract of an open subspace of the product* $M\times C$, *where M is an* ANR(M)*-space and C is an* ANR(C)*-space.* ■

PROPOSITION 2*. *A paracompact p-space is an* ANR(P)*-space if and only if it is an* ANE(P)*-space.* ■

THEOREM 1*. *For an absolute neighbourhood retract for paracompact p-spaces X and a cardinal number τ the following conditions are equivalent:*

(i) *X is Čech-complete and $l(X) \leqslant \tau$.*

(ii) *X is an absolute neighbourhood retract for τ-collectionwise normal spaces.*

(iii) *X is an absolute neighbourhood extensor for τ-collectionwise normal spaces.*

(iv) *X is a closed subspace of $J(\tau)^{\aleph_0} \times I^{\varkappa}$, for some $\varkappa$.* ■

COROLLARY 1*. *Let X be an absolute neighbourhood retract for paracompact p-spaces. Then:*

(a) *X is an absolute neighbourhood retract (extensor) iff X is Čech-complete and Lindelöf.*

(b) *X is an absolute neighbourhood retract (extensor) for collectionwise normal spaces iff X is Čech-complete.* ■

COROLLARY 2*. *For a paracompact p-space X and a cardinal number τ the following conditions are equivalent:*

(i) *X is an absolute neighbourhood retract for paracompact p-spaces, $l(X) \leqslant \tau$ and X is Čech-complete.*

(ii) *X is an absolute neighbourhood retract for τ-collectionwise normal spaces.*

(iii) *X is an absolute neighbourhood extensor for τ-collectionwise normal spaces.*

(iv) *X is a retract of an open subspace of $J(\tau)^{\aleph_0} \times I^{\varkappa}$.* ■

COROLLARY 3*. *For an absolute neighbourhood retract for metrizable spaces M and a cardinal number τ the following conditions are equivalent:*

(i) *M is complete of weight $\leqslant \tau$.*

(ii) *M is an absolute neighbourhood retract for τ-collectionwise normal spaces.*

(iii) *M is an absolute neighbourhood extensor for τ-collectionwise normal spaces.*

(iv) *M is a closed subspace of $J(\tau)^{\aleph_0}$.* ■

COROLLARY 4*. *For a metrizable space M and a cardinal number τ the following conditions are equivalent:*

(i) *M is a complete absolute neighbourhood retract for metrizable spaces, and weight $M \leqslant \tau$.*

(ii) *M is an absolute neighbourhood retract for τ-collectionwise normal spaces.*

(iii) *M is an absolute neighbourhood extensor for τ-collectionwise normal spaces.*

(iv) *M is a retract of an open subspace of $J(\tau)^{\aleph_0}$.* ■

THEOREM 2*. *For an absolute neighbourhood retract for paracompact p-spaces and a cardinal number τ the following conditions are equivalent:*

(i) *$l(X) \leqslant \tau$.*

(ii) *X is an absolute neighbourhood extensor for perfectly τ-collectionwise normal spaces.*

(iii) *Every continuous mapping $f\colon F \to X$ of a closed G_δ-subspace F of a τ-collectionwise normal space Z into X is continuously extendable onto a neighbourhood U of F in Z.* ■

COROLLARY 7*. *Let X be an absolute neighbourhood retract for paracompact p-spaces. Then:*

(a) *X is an absolute neighbourhood extensor for perfectly normal spaces iff X is Lindelöf.*

(b) *X is an absolute neighbourhood extensor for perfectly collectionwise normal spaces.* ■

COROLLARY 8*. *For an absolute neighbourhood retract for metrizable spaces M and a cardinal number τ the following conditions are equivalent:*

(i) *$w(M) \leqslant \tau$.*

(ii) *M is an absolute neighbourhood retract for perfectly τ-collectionwise normal spaces.*

(iii) *M is an absolute neighbourhood extensor for perfectly τ-collectionwise normal spaces.*

(iv) *Every continuous mapping $f\colon F \to M$ of a closed G_δ-subspace F of a τ-collectionwise normal space Z into M is continuously extendable onto a neighbourhood U of F in Z.* ■

§ 4. Homotopy extension theorem. To illustrate possible applications of the results obtained in the preceding sections we shall derive from them a generalization of the Brosuk homotopy extension theorem, which is a slight strengthening and refolmulation of the recent result of Morita [17] and Starbird [20]. Throughout this section P is assumed to be a paracompact p-space.

THEOREM 3. *Let F be a closed subspace of a τ-collectionwise normal space X and let P be an absolute neighbourhood retract for τ-collectionwise normal spaces.*

Every continuous mapping $f\colon (X \times \{0\}) \cup (F \times I) \to P$ is continuously extendable onto $X \times I$.

COROLLARY 10. *Let P be a closed subspace of a normal space X and let P be an absolute neighbourhood retract.*

Every continuous mapping $f\colon (X \times \{0\}) \cup (F \times I) \to P$ is continuously extendable onto $X \times I$.

For a metrizable space M we shall denote by $C(I, M)$ the metric space of all continuous mappings $f\colon I \to M$ with the topology of uniform convergence. Let us recall the following facts:

(1) $w(C(I, M)) \leqslant \aleph_0 \cdot w(M)$.

(2) $C(I, M)$ is complete if so is M.

(3) For every space M the exponential mapping $\Lambda\colon M^{Z \times I} \to C(I, M)^Z$, defined for $z \in Z$ and $t \in I$ by $[\Lambda(f)(z)](t) = f(z, t)$ establishes a one-to-one correspondence between continuous mappings $f\colon Z \times I \to M$ and continuous mappings $\Phi\colon Z \to C(I, M)$ (see [9]; Chapter XII, Theorems 3.1, 5.3 and 8.2).

Lemma 4. *If a metrizable space M is an absolute retract for τ-collectionwise normal spaces, then so is $C(I, M)$.*

Proof. Assume that M is an absolute retract for τ-collectionwise normal spaces. It follows from Corollary 4, that M is a complete AR$(\mathcal{M})$-space of weight $\leqslant\tau$. By (1) and (2) $C(I, M)$ is also complete and has weight $\leqslant\tau$. Therefore, again by Corollary 4, to prove our lemma it is enough to show that $C(I, M)$ is an AE$(\mathcal{M})$-space (see [13]; § 53, II, Theorem 5).

Let $g: F\to C(I, M)$ be a continuous mapping of a closed subspace F of a metrizable space Z. By (3) the mapping $\Lambda(g)$: $F\times I\to M$ is continuous and hence there exists an extension $\psi: X\times I\to M$ of $\Lambda(g)$ onto $X\times I$. Clearly $\tilde{g} = \Lambda^{-1}(\psi): Z\to C(I, M)$ is a required continuous extension of g. ■

Proof of Theorem 3. In virtue of Corollary 2* P is a retract of an open subspace U of $J(\tau)^{\aleph_0}\times I^{\varkappa}$. Let r: $U\to P$ be the retraction.

We shall first show, that there exists a continuous extension $\varphi: X\times I\to J(\tau)^{\aleph_0}\times I^{\varkappa}$ of f into $J(\tau)^{\aleph_0}\times I^{\varkappa}$. To this end it suffices to prove that if M is a factor of the product space $J(\tau)^{\aleph_0}\times I^{\varkappa}$ (i.e. if $M = J(\tau)$ or I) and g: $(X\times\{0\})\cup(F\times I)\to M$ is a continuous mapping, then there exists a continuous extension $\tilde{g}$: $X\times I\to M$ of g.

According to (3), the mapping $G = g|(F\times I)$: $F\times I\to M$ induces a continuous mapping $\Lambda(G)$: $F\to C(I, M)$. Lemma 4 and Corollary 4 imply that $C(I, M)$ is an absolute extensor for τ-collectionwise normal spaces. Let Φ: $X\to C(I, M)$ be a continuous extension of $\Lambda(G)$. Clearly, the mapping $\tilde{G} = \Lambda^{-1}(\Phi)$: $X\times I\to M$ is a continuous extension of G.

The argument used below is a generalization of an idea due to Starbird [21]. Since the set $K = \{x\in X|\ g(x, 0) = \tilde{G}(x, 0)\}$ containing F is functionally closed, there exists a continuous function ψ: $X\to I$ such that $K = \psi^{-1}(0)$. The mapping h: $X\to M$ defined by putting $h(x) = \tilde{G}(x, \psi(x))$, for $x\in X$, is continuous and since M is contractible there exists a homotopy between $g|(X\times\{0\})$ and h (see [5]; Corollary IV. 2.3.), i.e. a continuous mapping H: $X\times I\to M$ such that $H(x, 0) = g(x, 0)$ and $H(x, 1) = h(x)$, for $x\in X$.

One easily checks that the mapping $\tilde{g}$: $X\times I\to M$ defined by

$$\tilde{g}(x, t) = \begin{cases} \tilde{G}(x, t), & \text{if } t\geqslant\psi(x),\\ H(x, t/\psi(x)), & \text{otherwise} \end{cases}$$

is a required continuous extension of g.

Let φ: $X\times I\to J(\tau)^{\aleph_0}\times I^{\varkappa}$ be a continuous extension of f and $V = \varphi^{-1}(U)$. Since I is compact and $V\supset F\times I$ there exists an open subset G of X such that $F\subset G$ and $G\times I$ is contained in V. Take a continuous mapping θ: $X\to I$ such that $\theta(F)\subset\{1\}$ and $\theta(X\setminus G)\subset\{0\}$. The mapping $\tilde{f}$: $X\times I\to P$ defined by $\tilde{f}(x, t) = r(\varphi(x, \theta(x)\cdot t))$ is a required continuous extension of f. ■

Corollary 10 is an immediate consequence of Theorem 3. ■

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