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Some results on uniform spaces with linearly ordered bases

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Abstract. The paper is concerned with uniform spaces having a base linearly ordered by inclusion of entourages (or by refinement of uniform coverings, respectively). By a well-known fact, these spaces coincide with the socalled ω_μ-metric spaces, for which several well-known metrization theorems can be extended. — Amongst several applications, two characterizations of compact metric spaces and of separable metric spaces are derived.

§ 1. **Introduction.** The uniform structure $\mathfrak{U}_d$ of any metric space (X, d) obviously has a linearly ordered base $\mathfrak{B}$ consisting of entourages

$$U_n = \{(x, y)|\ d(x, y) < 1/n\}, \quad n = 1, 2, \ldots$$

More generally, it is interesting to study uniform spaces $(X, \mathfrak{U})$ with linearly ordered bases $\mathfrak{B}$ ($U_i < U_j$ iff $U_i \supset U_j$ for $U_i, U_j \in \mathfrak{B}$). Such spaces have been investigated by many authors and under several aspects: R. Sikorski [37], F. Hausdorff [12, p. 285 ff], L. W. Cohen and C. Goffman [5], F. W. Stevenson and W. J. Thron [39], Shu-Tang Wang [42], P. Nyikos and H. C. Reichel [27], [28], A. Hayes [15], P. Nyikos [25], R. Paintandre [29], E. M. Alfsen and O. Njastad [2], M. Fréchet [9].

If $(X, \mathfrak{U})$ is a uniform space with a linearly ordered base $\mathfrak{B}$ and $\aleph_\mu$ is the least power of such a base, then there exists an equivalent well ordered base of power $\aleph_\mu$ ([39]). (Obviously, such a space is metrizable iff $\mu = 0$). Moreover, F. W. Stevenson and W. J. Thron [39] showed that any such space $(X, \mathfrak{U})$ is ω_μ-metrizable in the sense of R. Sikorski. That means: there is a linearly ordered abelian group G which has a decreasing ω_μ-sequence converging to 0 in the order topology, and a "distance function" ϱ: $X \times X \to G$ satisfying the usual axioms for a metric on X, which generates the topology of X. (Here ω_μ denotes the μth infinite cardinal). — Conversely, any ω_μ-metric ϱ on X induces a uniformity $\mathfrak{U}_\varrho$ on X, a base of which consists of all sets $U_a = \{(x, y)|\ \varrho(x, y) < a\}$, $a \in G$, $a > 0$. Many properties of metric spaces have their analogues in the theory of ω_μ-metric spaces, however the uncountable case usually regards different proofs and methods (see for example: [27], [28], [42]).

Shu-Tang Wang [42], for example, proved an analogue of the Nagata–Smirnov metrization theorem: a regular space X is ω_μ-metrizable iff it has an open base $\mathfrak{B}$ for its topology consisting of an ω_μ-sequence of locally finite collections

$$\mathfrak{B}_\alpha = \{B_{\alpha i} \mid i \in I\}, \quad \alpha < \omega_\mu,$$

and the intersection of fewer than ω_μ distinct open sets is open in X again. Several other ω_μ-metrization theorems have been presented by P. Nyikos and H. C. Reichel in [28]. Moreover, special theorems for $\mu>0$ can be obtained which do not have analogues for $\mu = 0$ ([27], [28], [39], [47] for example).

It seems interesting to study to what extent "countability" inherent in metrization theorems can be generalized using the well ordering of natural numbers instead of their cardinality. (See e.g. Theorems 5.1 and 5.3). — For other investigations with similar intention compare also a paper of J. E. Vaughn [41] on linearly stratifiable spaces.

EXAMPLE 1.1. Let A be any set and B be a well ordered set of order type ω_μ. Consider the set A^B of all ω_μ-sequences (x_α), $x_\alpha \in A$, $\alpha < \omega_\mu$. The "natural topology" $\mathscr{V}$ on A^B is defined by the base $\mathfrak{N}$ consisting of the sets

$$x(\beta) = \{y \in A^B \mid y_\alpha = x_\alpha \text{ for } \alpha < \beta\}, \quad \beta < \omega_\mu \text{ and } x \in A^B.$$

(This topology has been defined by A. K. Steiner and E. F. Steiner in [38].) If $\omega_\mu = \omega_0$, the spaces $(A^B, \mathscr{V})$ exactly coincide with Baire's zerodimensional sequence spaces, as defined by F. Hausdorff [14]. A detailed study of the importance of Baire's sequence-spaces can be found in J. Nagata's book [21] or in [22] and [27].

The natural topology on A^B can be induced by a uniformity $\mathfrak{U}$ with a linearly ordered base $\mathfrak{B}$ consisting of all

$$B_\beta = \{((x_\alpha), (y_\alpha)) \mid x_\alpha = y_\alpha \text{ for } \alpha < \beta\}, \quad \beta < \omega_\mu.$$

The class of all non-metrizable ω_μ-metric spaces (i.e. $\mu>0$) can be characterized by the following proposition (P. Nyikos and H. C. Reichel [27] and I. Juhász [45]).

PROPOSITION 1.2. *Let $\mu>0$; a topological space is ω_μ-metrizable iff X can be embedded as a subspace of a suitable space $(A^B, \mathscr{V})$, $B = \omega_\mu$.*

Remark. An ω_μ-metric space X is ω_μ-additive in the sense of R. Sikorski: for any collection $\{O_\alpha \mid \alpha \leqslant \beta < \omega_\mu\}$ of open sets, $\bigcap O_\alpha$ $(\alpha \leqslant \beta)$ is open. In this respect, ω_μ-metric spaces have been used also in the theory of Boolean algebras (Sikorski [37]). Moreover, every closed set $F \subset X$ is the intersection of a system $\mathfrak{S}$ of open sets $O_i \subset X$, where card $\mathfrak{S} \leqslant \omega_\mu$.

The usual characterization of separability in metric spaces and the metrization theorem of P. Urysohn can be generalized, too:

PROPOSITION 1.3. *Equivalent are*

(i) *X is an ω_μ-additive space with a basis $\mathfrak{A}$ of cardinality $\leqslant \omega_\mu$.*

(ii) *X is an ω_μ-metric space with a dense subset Y of cardinality $\leqslant \omega_\mu$.*

Proof. (i)⇒(ii): In [37] R. Sikorski proved an ω_μ-analogue of Urysohn's metrization theorem by which follows ω_μ-metrizability of X. Moreover, for any $A_\tau \in \mathfrak{A}$, $\tau < \omega_\mu$, take a point $y_\tau \in X$, then $Y = \{y_\tau \mid \tau < \omega_\mu\}$ is a dense subset of X.

(ii)⇒(i): Use the theorem of Stevenson and Thron cited above and note that, if (ε_τ), $\tau < \omega_\mu$, is a decreasing ω_μ-sequence in G converging to $0 \in G$ in the order-topology, the system of all

$$B_{y\tau} = \{x \in X \mid \varrho(x, y) < \varepsilon_\tau\}, \quad \tau < \omega_\mu,\ y \in Y,$$

is a basis $\mathfrak{A}$ of X with cardinality $\leqslant \aleph_\mu \cdot \aleph_\mu = \aleph_\mu$.

Using Proposition 1.2, § 2 characterizes *complete* uniformities with linearly ordered bases.

Since every uniform space with a linearly ordered base is paracompact, the question arises to characterize those paracompact spaces which have compatible uniform structures with linearly ordered bases in a purely topological manner. More generally, § 3 presents topological criteria for completely regular spaces to have compatible uniform structures with linearly ordered bases (Theorems 3.2 and 3.3).

§ 4 deals with *k-bounded* uniformities with linearly ordered bases; generalizing inverse limit methods used in this paragraph, we obtain a characterization of the class of N-compact spaces, too.

Compact Hausdorff spaces X have unique uniform structures. However, spaces with a unique uniform structure $\mathfrak{U}$ need not be compact. In § 5 we show that if the unique uniformity $\mathfrak{U}$ on X has a linearly ordered base, X must be a compact metric space (Theorem 5.1). Thus we obtain a new characterization of compact metric spaces. With similar methods we characterize the class of separable metric spaces and we study completely regular spaces with several compatible uniform structures but which have exactly one uniformity with a linearly ordered base.

All topological spaces are Hausdorff, all uniformities are separated.

§ 2. Complete uniform spaces with linearly ordered bases. An ω_μ-metric space, (X, ϱ) is ω_μ-complete if and only if every Cauchy-ω_μ-sequence converges [39]. F. W. Stevenson and W. J. Thron showed the following

LEMMA 2.1. *An ω_μ-metric space (X, ϱ) is ω_μ-complete if and only if $(X, \mathfrak{U}_\varrho)$ is complete in the uniform sense.*

Obviously, for $\mu = 0$, $\mathfrak{U}_\varrho$ is complete iff X is completely metrizable. We shall now characterize all complete uniform spaces with linearly ordered bases, which are *not* metrizable.

THEOREM 2.2. *The following assertions are equivalent*:

(i) *$(X, \mathfrak{U})$ is a complete uniform space with a linearly ordered base of least power $\aleph_\mu$, $\mu>0$.*

(ii) *$(X, \mathfrak{U})$ can be embedded as a closed subspace of a suitable space $(A^B, \mathscr{V})$, $B = \omega_\mu$, $\mu>0$.*

Remark. For $\mu = 0$, the topology $\mathscr{V}$ on A^B coincides with the product topology on the countable product of the discrete spaces $A_i = A$, $i = 1, 2, \ldots$, [38].

Proof. Since the embedding mapping in Theorem 1.1 is a unimorphic embedding we only have to show that the uniform structure $\mathfrak{U}_{\mathscr{V}}$ of $(A^B, \mathscr{V})$ is complete. And this will be assured by

THEOREM 2.3. *The uniformity $\mathfrak{U}_{\mathscr{V}}$ of any space $(A^B, \mathscr{V})$ is complete.*

Proof. Let $B = \omega_\mu$ $(\mu \geqslant 0)$, then, by the theorem of Stevenson and Thron (§ 1), $(A^B, \mathscr{V})$ is an ω_μ-metrizable space. Moreover, there is a compatible ω_μ-metric d: $A^B \times A^B \to G$ with the following property: there is an ω_μ-sequence (ε_τ), $\tau < \omega_\mu$, $\varepsilon_\tau \in G$, converging to $0 \in G$ in the order topology of G such that for all $\tau < \omega_\mu$ and $y \in A^B$:

$$K_{\varepsilon_\tau}(y) = \{x \in A^B |\ d(x, y) < \varepsilon_\tau\} = y(\tau) = \{x \in A^B |\ x_\alpha = y_\alpha \forall \alpha < \tau\}\,.$$

We shall show that every Cauchy-ω_μ-sequence converges, i.e.: (A^B, d) is ω_μ-complete.

Let (x^σ), $\sigma < \omega_\mu$, be a Cauchy-ω_μ-sequence, in other words: for every $\tau < \omega_\mu$ there is a $\delta(\tau) < \omega_\mu$ such that $d(x^{\sigma_1}, x^{\sigma_2}) < \varepsilon_\tau$ whenever $\sigma_1, \sigma_2 > \delta$. Thus we have

$$x^{\sigma_1} \in K_{\varepsilon_\tau}(x^{\sigma_2}) = x^{\sigma_2}(\tau) \quad \text{and} \quad x^{\sigma_2} \in K_{\varepsilon_\tau}(x^{\sigma_1}) = x^{\sigma_1}(\tau)\,.$$

Therefore $x^{\sigma_1}(\tau) = x^{\sigma_2}(\tau)$. Moreover, since ω_μ is a limit ordinal, and $y(\tau') \supset y(\tau'')$ whenever $\tau' < \tau''$, we can chose $\delta = \delta(\tau)$ so that $\delta \geqslant \tau$ and $\delta(\tau) > \delta(\tau')$ for all $\tau' < \tau$. That means: for any $\sigma > \delta(\tau)$, we have $x^\sigma \in x^{\delta(\tau)+1}(\tau)$.

Now construct $x \in A^B$ by defining its γ-coordinates by $x_\gamma = (x^{\delta(\gamma)+1})_\gamma$, the γ-coordinate of $x^{\delta(\gamma)+1}$, for all $\gamma < \omega_\mu$. Now we claim that x is the limit of the transfinite sequence (x^σ), $\sigma < \omega_\mu$.

Since the sets $x(\varrho)$, $\varrho < \omega_\mu$, form a local base of x, we only have to show that, for all $\varrho < \omega_\mu$, there is a $\xi(\varrho)$ such that $x^\sigma \in x(\varrho)$ if $\sigma > \xi$. But this is obvious if we take $\xi(\varrho) = \delta(\varrho)$. Remember that $x^\sigma \in x^{\delta(\varrho)+1}(\varrho)$, because $\sigma > \delta(\varrho)$, and therefore, for all γ-coordinates, $\gamma < \varrho$, we have $(x^\sigma)_\gamma = (x^{\delta(\varrho)+1})_\gamma = (x^{\delta(\gamma)+1})_\gamma = x_\gamma$, which yields $x^\sigma \in x(\varrho)$.

Thus (A^B, d) is ω_μ-complete and $(A^B, \mathfrak{U}_{\mathscr{V}})$ is complete by Lemma 2.1.

§ 3. Criteria for topological spaces to have compatible uniformities with linearly ordered bases. In [15] A. Hayes has shown that any uniform space $(X, \mathfrak{U})$ with a linearly ordered base $\mathfrak{B}$ is paracompact. Conversely, the question arises to give a simple *topological* criterion for a paracompact space $(X, \mathfrak{T})$ that its topology $\mathfrak{T}$ can be induced by a uniformity with a linearly ordered base of least power ω_μ. (Clearly, every such condition, specialized to $\mu = 0$, must yield metrizability of X).

DEFINITION 3.1. Let X be a topological space. An open base $\mathfrak{B}$ of X is called an *ω_μ-uniform base* if and only if

(i) $\bigcap B_\alpha$ $(B_\alpha \in \mathfrak{B}, \alpha \leqslant \tau < \omega_\mu)$ is open, and

(ii) for any $p \in X$ and any neighbourhood $V(p)$, we have:

$$\operatorname{card}\{B |\ B \in \mathfrak{B}, p \in B, B \not\subset V(p)\} < \omega_\mu\,.$$

THEOREM 3.2. *The topology of any topological space $(X, \mathfrak{T})$ can be induced by a uniform structure $\mathfrak{U}$ with a linearly ordered base $\mathfrak{B}$ if and only if $(X, \mathfrak{T})$ is para-*

compact and has an ω_μ-uniform base for some cardinal ω_μ. (Remark that the existence of an ω_μ-uniform base is a purely topological criterion.)

Remark. Obviously, for $\mu = 0$, the theorem yields a metrization theorem of P. Alexandroff [1]. Compare also the remark at the end of the paper.

Proof. Necessity: Let $(X, \mathfrak{U})$ be a uniform space with a linearly ordered base of least power $\aleph_\mu$. If $\mu = 0$, the space is metrizable. In this case define $\mathfrak{U}_n$ to be a locally finite refinement of the open covering consisting of the balls

$$B_n(x) = \{y |\ d(x, y) < 1/n\}, \quad x \in X,\ n = 1, 2, \ldots$$

Then $\mathfrak{U} = \bigcup \mathfrak{U}_n$ $(n = 1, 2, \ldots)$ is an ω_0-uniform base of X.

If $\mu \neq 0$, we use our Theorem 1.2, by which X can be embedded homeomorphically into a suitable space $(A^B, \mathscr{V})$. The base $\mathfrak{N}$ of $(A^B, \mathscr{V})$ as described in Example 1.1 has then the following properties:

(i) $\{x(\beta) |\ \beta < \omega_\mu\}$ is a well ordered local base of $x \in X$: $x(\beta) \subset x(\gamma)$ iff $\beta > \gamma$;

(ii) two basis sets $x(\beta)$, $y(\gamma) \in \mathfrak{N}$ either have empty intersection, or one is contained in the other; and

(iii) if $y \in x(\beta)$ then $x(\beta) = y(\beta)$ for all $\beta < \omega_\mu$.

Therefore $\mathfrak{N}$ is an ω_μ-uniform base for $(A^B, \mathscr{V})$, and $\{N \cap X |\ N \in \mathfrak{N}\}$ is an ω_μ-uniform base of X.

Sufficiency: Let $\mathfrak{B}$ be an ω_μ-uniform base for X, then for any $B \in \mathfrak{B}$, $\operatorname{card}\{C |\ C \in \mathfrak{B}, C \supset B\} = : c(B) < \omega_\mu$, as follows from the definition using any point $p \in B$ and letting $V(p) = B$. For $\tau < \omega_\mu$, let $\mathfrak{B}_\tau = \{B \in \mathfrak{B} |\ c(B) \geqslant \tau\} \cup \{\{p\} |\ p$ is an isolated point of $X\}$. Then $\mathfrak{B}_\tau$ is an open covering of X. Chose a locally finite open refinement $\mathfrak{W}_\tau$ of $\mathfrak{B}_\tau$, then $\mathfrak{W} = \bigcup \mathfrak{W}\,(\tau < \omega_\mu)$ is a topological base of X. To show this, let $p \in X$ and $V(p)$ be any neighbourhood of p, then

$$\operatorname{card}\{B \in \mathfrak{B} |\ p \in B, B \cap (X \setminus V) \neq \emptyset\} = \sigma < \omega_\mu\,.$$

So, for any B, $p \in B$, $B \cap (X \setminus V) \neq \emptyset$ we have $c(B) \leqslant \sigma$, and, for every $\tau > \sigma$ and $C \in \mathfrak{B}_\tau$, $p \in C$, we have $c(C) \geqslant \tau > \sigma$. Therefore $C \subset V$. Consequently, there is an $U \in \mathfrak{W}_\tau$ with $p \in U \subset C \subset V$.

So $\mathfrak{W}$ consists of an ω_μ-sequence of locally finite open coverings of X, which assures ω_μ-metrizability of X by Shu-Tang Wang's generalization of the Nagata–Smirnov theorem (§ 1). Obviously, we can use this theorem, since, for $\mu > 0$, the intersection of fewer than ω_μ distinct open sets is open. To see this, remember that $\mathfrak{B}$ is an ω_μ-uniform open base of X sharing this property. ■

There is another topological characterization of uniform spaces with linearly ordered bases:

Let X be a strongly zerodimensional metric space $(\operatorname{Ind} X = \dim X = 0)$, then X is metrizable in a non-archimedean way, i.e.: there is a compatible metric d for X which satisfies $d(x, y) \leqslant \max(d(x, z), d(z, y))$ for all $x, y, z \in X$ (F. Hausdorff [13], J. de Groot [10], J. Nagata [22]). Conversely, every n.-a. metric space has $\dim X = 0$. (Note that by J. de Groots theorem n.-a. metrics can be compatible even with the Euclidean topology of the rationals or the Cantor discontinuum, for example). By the strong triangle inequality, two balls B_ε, B_δ with radii ε and δ respectively, either have empty intersection or one containes the other. More generally, A. F. Monna [19] defined a topological space X to be "non-archimedean" if X has a topological base $\mathfrak{B}$ sharing this property ($\mathfrak{B}$ is called a "non-archimedean" base if any two members of $\mathfrak{B}$ either have empty intersection or one contains the other). Non-archimedean spaces have been studied in several papers ([3], [19]), [47], [26], [30], [32], [44] and others). If a n.-a. topological space is metrizable it is non-archimedeanly metrizable. Sometimes, non-archimedean bases $\mathfrak{B}$ are called "bases of rank 1".

Now let $(X, \mathfrak{U})$ be a uniform space with a linearly ordered base $\mathfrak{B}$ of least power $\aleph_\mu$. If $\mu>0$, Theorem 1.2 asserts that X is homeomorphic with a subspace of a certain space $(A^B, \mathscr{V})$ as described in Example 1.1. It is easy to see that the "natural" topological base $\mathfrak{C} = \{x(\gamma)|\ x \in A^B, \gamma \in B\}$ of $(A^B, \mathscr{V})$ has the property that $x(\alpha) \cap y(\beta)$ is either empty or equals $x(\alpha)$ or $y(\beta)$. So we get the following proposition:

Every *uniform space* $(X, \mathfrak{U})$ with a *linearly ordered base* $\mathfrak{B}$ of least cardinality $\aleph_\mu$ is a *n.-a. topological space* in the sense of A. F. Monna if $\mu>0$ and — as we just saw — for $\mu = 0$, X is n.-a. iff $\dim X = 0$.

Surprisingly, we can prove a *converse* for ω_μ-compact spaces:

One can show that any compact n.-a. topological space is metrizable (Nyikos–Reichel [26]); now, as a corollary of Theorem 3.2, we can prove the generalization of this theorem to higher cardinals, but we have to be careful: it is easy to see that every non-archimedean base $\mathfrak{B}$ of a compact space X is a tree with respect to inverse inclusion as order relation: $B<C \Leftrightarrow B\supset C$. (As usual, a tree is a partially ordered set such that the set of predecessors of any element is well ordered.) As the proof of the following theorem will show, it is this property we have to assume explicitely in case $\omega_\mu>\omega_0$. Moreover, recall that a space X is "ω_μ-compact" ([37], [17]) iff every open cover of X has a subcover consisting of fewer than ω_μ sets [1].

THEOREM 3.3. *Let X be an ω_μ-compact T_2-space, $\mu>0$, then the following is equivalent*:

(i) *X has a compatible uniformity $\mathfrak{U}$ with a linearly ordered base $\mathfrak{B}$ of least cardinality $\aleph_\mu$*,

(ii) *X is an ω_μ-additive space* ([37]) *admitting a non-archimedean base $\mathfrak{C}$ for its topology such that $\mathfrak{C}$ is a tree* (*w.r.t. inverse inclusion as its order relation*).

If $\mu = 0$, (i) *is equivalent with metrizability of X, and the implication* (i)$\Rightarrow$(ii) *is true only for spaces X with* $\dim X = 0$.

[1] In several papers the term "ω_μ-compact" is used in the sense that every open cover of cardinality $\leqslant\omega_\mu$ has a finite subcover. In the terminology of Aleksandroff–Urysohn (1929) the term "ω_μ-compact" in the just mentionedsense is called "initially ω_μ-compact" and in the sense of this paper "finally ω_μ-compact".

Proof. (i)$\Rightarrow$(ii) follows from the remarks cited above, note that the base $\mathfrak{C}$ constructed above is a tree. The proof of the converse uses Theorem 3.2: let $\mathfrak{C} = \{C_i|\ i \in I\}$ be a non-archimedean base of X which, moreover, is a tree. Then every C_i is open and closed in X, since there are no basissets intersecting both, C_i and its complement. Moreover, by the non-archimedean property described above, any collection of basissets $\{C_j|\ j \in J\subset I\}$ is a chain (with respect to the inclusion) whenever there is a point $x \in X$ with $x \in C_j$ for all $j \in J$. We shall show that $\mathfrak{C}$ is an ω_μ-uniform base for X. Suppose there is a point $x \in X$ and a neighbourhood $U(x)$ of x such that $\operatorname{card}\{C_j|\ x \in C_j \text{ and } C_j \cap (X\setminus U)\neq\emptyset\}\geqslant\omega_\mu$. Without loss of generality we can assume that $U \in \mathfrak{C}$. Since all C_j are clopen sets and $\mathfrak{C}$ is a tree, it would be possible then to construct a partition of $X\setminus U$ into ω_μ or more clopen sets $C_k\setminus C_l$, $k, l \in J$, and a suitable clopen set $X\setminus C_i$ $(i \in J)$, respectively. But this would yield a contradiction, because the closed subspace $X\setminus U$ is ω_μ-compact.

Therefore $\operatorname{card}\{C_j|\ x \in C_j \text{ and } C_j \cap (X\setminus U) \neq \emptyset\}<\omega_\mu$ for every point $x \in X$ and every neighbourhood U of x; thus the base $\mathfrak{C}$ is an ω_μ-uniform base.

By a theorem proved in [26] and [44], any non-archimedean topological space is (hereditarely) paracompact, and we can make use of Theorem 3.2. Thus X has a compatible uniform structure with a linearly ordered base.

For the second part of our proof, we assumed that ω_μ was a regular cardinal; if ω_μ is singular then X would be a discrete space of cardinality $<\omega_\mu$, and there is nothing to prove.

Remark. If in (ii) of Theorem 3.3 we do not assume that that $\mathfrak{C}$ is a tree then our methods used above would only guarantee that

$$\operatorname{card}\{C_j|\ x \in C_j \text{ and } C_j \cap (X\setminus U) \neq \emptyset\}\leqslant\omega_\mu ,$$

since this is then a totally ordered set in which every subset has a coinitial and cofinal subset of cardinal $<\omega_\mu$ (by ω_μ-compactness of X). Thus CGH implies the result if ω_μ is a regular cardinal; if ω_μ is singular, there is nothing to prove again.

§ 4. k-bounded uniform spaces with linearly ordered bases.

DEFINITION ([31], [40]). An uniform space $(X, \mathfrak{U})$ is *k-bounded* iff for every $U \in \mathfrak{U}$ there exists a set A of cardinality $<k$ such that $U(A) = \bigcup\{U(x)|\ x \in A\} = X$. $(X, \mathfrak{U})$ is *strictly k-bounded* iff k is the least cardinal number for which $(X, \mathfrak{U})$ is k-bounded. Equivalently, $(X, \mathfrak{U})$ is k-bounded iff $\mathfrak{U}$ has an associated family of k-bounded pseudometrics, where a pseudometric p is k-bounded if for each $\varepsilon>0$, X can be written as a union of fewer than k sets, each of p-diameter not exceeding ε. We shall characterize the class of k-bounded uniform spaces with linearly ordered bases:

EXAMPLE 4.1. *Let I be a linearly ordered countable index set and $\{X_i, \pi_{ij}|\ i \in I\}$, π_{ij}: $X_i\to X_j$ for $i\geqslant j$, an inverse limiting system consisting of discrete spaces X_i with* $\operatorname{card} X_i<k$. *If $\varprojlim X_i = : Y \neq \emptyset$, the topology of Y can be induced by a k-bounded uniformity $\mathfrak{U}$ on X with a linearly ordered base.*

Proof. Let $p_i\colon Y\to X_i$, $i\in I$, be the canonical projections. For each $i\in I$, we obtain partitions

$$\mathfrak{A}_i = \{O^i_\alpha = p_i^{-1}(x)|\ x\in X_i\}$$

of Y consisting of fewer than k open and closed sets O^i_α. Thus $\mathfrak{A}_i$, $i\in I$, are open coverings of Y and $\mathfrak{A}_i$ is a refinement of $\mathfrak{A}_j$ if $i\geqslant j$, since for every $O^i_\alpha\in\mathfrak{A}_i$ there is a point $x\in X_i$ such that $O^i_\alpha = p_i^{-1}(x)$, and $O^i_\alpha\subset p_j^{-1}(\pi_{ij}(x))\in\mathfrak{A}_j$. Therefore the partitions $\mathfrak{A}_i$, $i\in I$, induce a uniform structure $\mathfrak{U}$ on Y with a linearly ordered base $\mathfrak{B} = \{O^i_\alpha\times O^i_{\alpha}|\ O^i_\alpha\in\mathfrak{A}_i;\ i\in I\}$.

Moreover, since $\operatorname{card}\mathfrak{A}_i<k$ for every $i\in I$, $\mathfrak{U}$ is k-bounded, and so is every subspace. Clearly, the topology on Y coincides with the topology induced by $\mathfrak{U}$.

Now let $(X, \mathfrak{U})$ be an arbitrary k-bounded uniform space with a linearly ordered base $\mathfrak{B} = \{B_\tau|\ \tau<\omega_\mu\}$, $\mu>0$.

Without loss of generality, we can assume that $\mathfrak{B}_\tau = \{B_\tau(x)|\ x\in X\}$ is a *partition* of X for every $\tau<\omega_\mu$. To prove this, let $\mathfrak{C} = \{C_\tau|\ \tau<\omega_\mu\}$ be any linearly ordered base of $\mathfrak{U}$, and for every C_τ take a sequence C_{τ_n}, $n = 1, 2, \ldots$, such that $\{C_{\tau_{n+1}}(x)|\ x\in X\}$ is a star-refinement of $\{C_n(x)|\ x\in X\}$. Then $B_\tau = \bigcap C_{\tau_n}$ ($n = 1, 2, \ldots$) is an entourage of $\mathfrak{U}$, because $\mu>0$, and $\{B_\tau(x)|\ x\in X\}$ is a partition of X (a star-refinement of itself). In fact, $\bigcap B_\tau$ ($\tau\leqslant\sigma<\omega_\mu$) is always an entourage of $\mathfrak{U}$ again.

$(X, \mathfrak{U})$ is k-bounded, so $\mathfrak{B}_\tau = \{B_\tau(x)|\ x\in X\}$ consists of fewer than k clopen sets. Moreover, $\mathfrak{B}_\sigma$ is a refinement of $\mathfrak{B}_\tau$ iff $\sigma\geqslant\tau$. Consider now the collections $\mathfrak{B}_\tau$ as topological spaces X_τ with discrete topology and define $\pi_{\sigma\tau}\colon \mathfrak{B}_\sigma\to\mathfrak{B}_\tau$ ($\sigma\geqslant\tau$) in such a way that $\pi_{\sigma\tau}(B_\sigma(x)) = B_\tau(y)$ for any $x, y\in X$, if and only if $B_\sigma(x)\subset B_\tau(y)$. Thus $\{X_\tau = \mathfrak{B}_\tau;\ \pi_{\sigma\tau}|\ \tau<\omega_\mu\}$ is a linearly ordered inverse limiting system and $(X, \mathfrak{T}_{\mathfrak{U}})$ (¹) is a subspace of $Y = \varprojlim X_\tau$. This follows immediately from the fact that for each $x\in X$ and $\tau<\omega_\mu$ there exists exactly one $B_\tau(x)\in\mathfrak{B}_\tau$ containing x so that the ω_μ-sequence $(B_\tau(x))$, $\tau<\omega_\mu$, is a point of $\varprojlim X_\tau$. Moreover, since

$$\bigcup\{B_\tau(x)\in\mathfrak{B}_\tau|\ \tau<\omega_\mu,\ x\in X\}$$

is a topological base of X, the mapping f, defined by $f(x) = (B_\tau(x))_{\tau<\omega_\mu}$ is a continuous injection, $f\colon X\to Y$. But f^{-1} *need not be continuous if* $\omega_\mu>\omega_0$: For $x\in X$ and any fixed $B_\alpha(x)\subset X$, $\alpha<\omega_\mu$, $f(B_\alpha(x))$ consists of all transfinite sequence $(b_\tau)_{\tau<\omega_\mu}$ in Y with α-coordinate $b_\alpha = B_\alpha(x)$. As mentioned above, $B_{x\sigma} = \bigcap B_\tau(x)$, $\tau<\sigma<\omega_\mu$, is an open set and $f(B_{x\sigma}) = \{(b_\tau)|\ b_\tau = B_\tau(x)$ for all $\tau<\sigma\}$. But for $\sigma>\omega_0$, this set need not be open in the topology on Y which in fact is the topology inherited from the product topology on the set ΠX_τ ($\tau<\omega_\mu$).

On the other side, if we provide the set ΠX_τ ($\tau<\omega_\mu$) with the "natural" uniformity $\mathfrak{U}_{\mathscr{V}}$ described in Example 1.1, f and f^{-1} are uniformly continuous injections. So, if A denotes a set with $\operatorname{card}A = \sup\operatorname{card}X_\tau$, $\tau<\omega_\mu$, we obtain the following strengthening of Theorem 1.2:

(¹) $\mathfrak{T}_{\mathfrak{U}}$ denotes the topology induced by $\mathfrak{U}$.

THEOREM 4.2. *If $(X, \mathfrak{U})$ is a k-bounded uniform space with a linearly ordered base of least cardinality $\aleph_\mu$, $\mu>0$, then X is unimorphic to a subspace of a suitable space $(A^B, \mathfrak{U}_{\mathscr{V}})$, $B = \omega_\mu$, $\operatorname{card}A\leqslant k$.*

Remark 4.3. Conversely, a space $(A^B, \mathfrak{U}_{\mathscr{V}})$, $B = \omega_\mu$ and $\operatorname{card}A = k>\aleph_0$, *need not be k-bounded*: For $\omega_0<\alpha\leqslant\omega_\mu$, let $B_\alpha = \{((x_\tau), (y_\tau))|\ x_\tau = y_\tau$ for $\tau<\alpha\}$, then $B_\alpha\in\mathfrak{U}_{\mathscr{V}}$, but $\mathfrak{B}_\alpha = \{B_\alpha(x)|\ x\in X\}$ is a partition of X which may consist of more than k clopen sets (in fact, for $k\leqslant\alpha\leqslant\omega_\mu$, $\mathfrak{B}_\alpha$ can consist of $k^\alpha>k$ clopen sets). Thus $\mathfrak{U}_{\mathscr{V}}$ need *not* be k-bounded if $\mu>0$.

If $k\leqslant\omega_\mu$, every $\mathfrak{B}_\alpha$ consists of at most $k^\alpha\leqslant k^{\omega_\mu}$ clopen sets, so $\mathfrak{U}_{\mathscr{V}}$ is $2^{2^{\omega_\mu}}$-*bounded*. If $k = 2^{\omega_\mu}$, then $k^{\omega_\mu} = k = 2^{\omega_\mu}$ and $\mathfrak{U}_{\mathscr{V}}$ is strictly $2^{2^{\omega_\mu}}$-*bounded*. Finally, if $k>\omega_\mu$ we have $k^\alpha\leqslant k^{\omega_\mu}\leqslant 2^k$, and $\mathfrak{U}_{\mathscr{V}}$ is 2^{2^k}-*bounded*.

Remark 4.4. Let $(X, \mathfrak{U})$ be any uniform space with a linearly ordered base, then there is always a cardinal k such that $\mathfrak{U}$ is k-bounded (embed X into a space $(A^B, \mathfrak{U}_{\mathscr{V}})$ and let $k = \sup\operatorname{card}\mathfrak{B}_\alpha$ ($\alpha<\omega_\mu$)).

Concerning totally bounded uniformities with linearly ordered bases see Theorem 5.3.

Remark 4.5. In the light of Example 4.1 and Theorem 4.2 it might be interesting to study inverse systems not only with linearly ordered index sets and to describe $Y = \varprojlim X_i$ where the inverse limit is obtained from an *arbitrary* spectrum consisting of discrete spaces X_i of cardinality $<k$. In the following we use several methods also used by K. Nagami [20], P. Nyikos [23] and J. Flachsmeyer [8].

Remember that a space is N-*compact* iff X is homeomorphic with a closed subset of a product of the discrete space $N = \{1, 2, 3, \ldots\}$. (As a reference see e.g. [17]).

The category of N-compact spaces is reflexive in the category of T_2-spaces, i.e. closed hereditary and productive [16]. By a theorem of H. Herrlich, X is N-compact iff every clopen ultrafilter with the countable intersection property is fixed in X.

Every discrete space X of non-measurable cardinal is N-compact, since X is realcompact and strongly zerodimensional [17], [24]; and so is every closed subspace of any product of such discrete spaces.

Conversely, let X be an N-compact space, and let k be an infinite non-measurable cardinal. Consider all partitions $\mathfrak{B}$ of X into fewer than k clopen sets. These partitions obviously generate a uniform structure $\mathfrak{U}_k$ compatible with the topology of X (compare the proof of Theorem 4.1). Define $\mathfrak{B}>\mathfrak{B}'$ iff $\mathfrak{B}$ refines $\mathfrak{B}'$ and note that for any pair $\mathfrak{B}, \mathfrak{B}'$, $\mathfrak{B}\wedge\mathfrak{B}' = \{B\cap B'|\ B\in\mathfrak{B},\ B'\in\mathfrak{B}'\}$ is again a partition of X into fewer than k clopen sets. Now for every such partition $\mathfrak{B}$ let $X_{\mathfrak{B}}$ be the discrete space whose points are the elements of $\mathfrak{B}$, and for $\mathfrak{B}>\mathfrak{B}'$ define $\pi_{\mathfrak{B}\mathfrak{B}'}\colon X_{\mathfrak{B}}\to X_{\mathfrak{B}'}$ by associating to every member $B\in\mathfrak{B}$ the member $B'\in\mathfrak{B}'$ which contains B. All spaces $X_{\mathfrak{B}}$, considered as discrete uniform spaces are complete, therefore $Y_k = \varprojlim X_{\mathfrak{B}}$ with the uniformity $\mathfrak{W}_k$ inherited by the product uniformity, is again complete. By the methods developed by J. Flachsmeier in [8], we easily learn that X can be embedded as a dense subspace of Y_k and $\mathfrak{U}_k$ is the subspace uniformity inherited by the uniformity $\mathfrak{W}_k$ of Y_k:

In fact, since all projections $\varphi_{\mathfrak{V}}: X \to X_{\mathfrak{V}}$ are uniformly continuous, so is the embedding i of X into Y_k and since all $\varphi_{\mathfrak{V}}$ are quotient maps, i is an uniform embedding. So $(Y_k, \mathfrak{W}_k)$ is the completion of $(X, \mathfrak{U}_k)$. Moreover, Y_k can be considered as a subspace of $Y_{\aleph_0}$, the completion of $(X, \mathfrak{U}_{\aleph_0})$ constructed in the same way as Y_k. But as it is well known (A. C. M. van Rooij [35]), [23] the completion of $(X, \mathfrak{U}_{\aleph_0})$ is, homeomorphic with the N-compactification $\hat{X}$ of X, i.e. the reflection of X in the category of N-compact spaces. Therefore, since X is N-compact, we have $\hat{X} \simeq X \simeq Y_k \simeq Y_{\aleph_0}$, and we have gotten:

THEOREM 4.3. *A T_2-space X is N-compact iff X is homeomorphic with the inverse limit* $\varprojlim X_i$ *of an inverse spectrum consisting of discrete spaces X_i where* card $X_i \leqslant k$ *and k is a non-measurable cardinal.*

§ 5. Spaces having unique uniform structures with linearly ordered bases. As it is well known, compact spaces X have unique uniform structures $\mathfrak{U}$ compatible with their topology $\mathfrak{T}$: a base of $\mathfrak{U}$ consisting of all neighbourhoods of the diagonal $\{(x, x) | \ x \in X\}$ which are open in the space $X \times X$. Conversely, there are noncompact spaces having unique uniform structures. For uniformities with linearly ordered bases we obtain

THEOREM 5.1. *A topological space X is a compact metric space if and only if the topology of X can be induced by a unique uniform structure which has a linearly ordered base.*

Proof. Since the "only-if-part" is obvious, let us prove sufficiency. Let X have a unique compatible uniform structure $\mathfrak{U}$ and let $\mathfrak{B} = \{V_\tau | \ \tau < \omega_\mu\}$ be a linearly ordered base of $\mathfrak{U}$, then the topology induced by $\mathfrak{U}$ on X is paracompact (see § 3). So there is a complete uniformity $\mathfrak{W}$ compatible with the topology. Because of uniqueness, $\mathfrak{W} = \mathfrak{U}$.

On the other side, the completion of X with respect to $\mathfrak{U}$ must be the Stone–Čech compactification βX of X. Hence $X = \beta X$, and X is compact. But any compact uniform space with a linearly ordered base is metrizable. This follows from a more general theorem of P. Nyikos and H. C. Reichel in [28]. Nevertheless, let us prove this special case just by a few words: let $\{B_\tau | \ \tau < \omega_\mu\}$ be a base for $\mathfrak{U}$. We can assume then that $B_\tau \subset B_\sigma$ iff $\tau > \sigma$, that the intersection of fewer than ω_μ many B_τ's in an entourage again and that every B_τ is clopen in $X \times X$ (compare § 1 and § 4). So either $\omega_\mu = \omega_0$, or there is an open covering $\{X \setminus B_i | \ i < \omega_0\}$ of the clopen, and hence compact subspace $X \setminus \bigcap B_i \ (i < \omega_0)$ which has no finite subcover. Thus X is metrizable.

Theorem 5.1 characterizes those spaces X having a unique uniformity $\mathfrak{U}$ with the additional property that $\mathfrak{U}$ has a linearly ordered base. It might now be interesting to characterize those topological spaces X which may have several compatible uniform structures but exactly one of them with a linearly ordered base. In this respect we can prove the following

PROPOSITION 5.2. *Let $\omega_\mu > \omega_0$. An ω_μ-compact Hausdorff space* (§ 3) *has at most one uniform structure with a linearly ordered base of power $\aleph_\mu$.*

Proof. Let $\mathfrak{U}$ be a compatible uniformity with a linearly ordered base $\mathfrak{B}$ of power $\aleph_\mu$, then, as we have shown in § 4, we can assume that all systems $\mathfrak{U}_\tau = \{U_\tau(x) | \ U_\tau \in \mathfrak{B}\}$ are partitiones of X into open and closed sets. Moreover, card $\mathfrak{U}_\tau < \omega_\mu$ for all τ. Now let $\mathfrak{A} = \{A_\tau | \ \tau \leqslant \alpha < \omega_\mu\}$ be an arbitrary partition of X into fewer than ω_μ open and closed sets, then $\mathfrak{A}$ is a uniform cover with respect to $\mathfrak{U}$, in other words: there is a $V \in \mathfrak{U}$ such that $\{V(x) | \ x \in X\}$ refines $\mathfrak{A}$. Of course, since $\mathfrak{U}$ generates the topology and all A_τ are open sets, for every $A_\tau \in \mathfrak{A}$ there is an $U_\tau \in \mathfrak{B}$ such that $\mathfrak{U}_\tau$ refines the partition $\{A_\tau, X \setminus A_\tau\}$. And since $V = \bigcap U_\tau \ (\tau \leqslant \alpha < \omega_\mu)$ is an entourage of $\mathfrak{U}$ again as we have seen in § 3, V is the entourage we were looking for.

So $\mathfrak{U}$ is the unique uniform structure of X generated by the system of all partitions of X into closed and open sets.

By similar methods we obtain another metrization theorem:

THEOREM 5.3. *A topological space X is a separable metric space if and only if its topology can be induced by a totally bounded uniformity with a linearly ordered base.*

Proof. Let $(A^B, \mathfrak{U}_\mathscr{V})$ be as in Example 1.1; then, by Theorem 2.3, $\mathfrak{U}$ is complete. So, if $\mathfrak{U}_\mathscr{V}$ is totally bounded, $(A^B, \mathfrak{U}_\mathscr{V})$ is compact and hence metrizable (and therefore separable) by the argument used in the proof above. Now let $(X, \mathfrak{U})$ be any totally bounded uniform space with a linearly ordered base, then $(X, \mathfrak{U})$ is unimorphic with a subspace Y of a certain space $(A^B, \mathfrak{U}_\mathscr{V})$ as mentioned in § 1, and the arguments above applies to the closure $\bar{Y}$ of Y in $(A^B, \mathfrak{U}_\mathscr{V})$. Thus $(X, \mathfrak{U})$ is metrizable, and a metric space is separable iff it is metrizable in a totally bounded manner.

§ 6. Pseudometrics compatible with uniformities with linearly ordered bases. As it is well known, uniform spaces $(X, \mathfrak{U})$ can be described by families of pseudometrics $d: X \times X \to \boldsymbol{R}$. As usual, a pseudometric d on X is called *uniform relative* to $\mathfrak{U}$, if for every $\varepsilon > 0$ there exists an $U \in \mathfrak{U}$ such that $(x, y) \in U$ implies $d(x, y) < \varepsilon$. For a given uniform space $(X, \mathfrak{U})$ the family $\mathfrak{P}$ of all pseudometrics d on X uniform relative to $\mathfrak{U}$ is

(i) separating, i.e.: for every pair $x, y \in X$ there exists a $d \in \mathfrak{P}$ with $d(x, y) > 0$;
(ii) full, i.e. if $d_1, d_2 \in \mathfrak{P}$, then $\max(d_1, d_2) \in \mathfrak{P}$.

Conversely, any full separating family $\mathfrak{P}$ of pseudometrics d on X determines a uniformity $\mathfrak{U}$ on X such that all $d \in \mathfrak{P}$ are uniform relative to $\mathfrak{U}$. In fact, the system of all

$$U_{dn} = \{(x, y) | \ d(x, y) < 2^{-n}\}, \quad n = 1, 2, \ldots, \ d \in \mathfrak{P},$$

is a base of $\mathfrak{U}$.

A pseudometric d is called *non-archimedean* [19] iff

$$d(x, y) \leqslant \max(d(x, z), d(z, y)) \quad \text{for all } x, y, z \in X.$$

(F. Hausdorff [13] and some other authors call them ultra(pseudo)metrics). J. de Groot [10] has proved the following

THEOREM. *A metric space X is strongly zerodimensional* ($\dim X = \operatorname{Ind} X = 0$) *if and only if X is non-archimedeanly metrizable* (1).

There is a corresponding characterization of weakly zerodimensional spaces X ($\operatorname{ind} X = 0$):

PROPOSITION 6.1. *A completely regular space X is weakly zerodimensional* ($\operatorname{ind} X = 0$) *iff X has a compatible uniformity determined by a family $\mathfrak{P}$ of non-archimedean pseudometrics.*

Proof. Note that for any pair of n.-a. pseudometrics d_1, d_2 on X, $d = \max(d_1, d_2)$ is a n.-a. pseudometric again. Let $\mathfrak{P} = \{d_i | i \in I\}$ be a full family of n.-a. pseudometrics, then for every $x \in X$, the "balls" $U_{dn}(x) = \{y | d(x, y) < 1/n\}$, $n = 1, 2, \ldots, d \in \mathfrak{P}$, are clopen sets and form a neighbourhood base of x. Conversely, for any clopen set $A \subset X$, define $d_A(x, y) = 0$ iff x and y are elements of A or $X \setminus A$, otherwise let $d_A(x, y) = 1$. Then the system $\mathfrak{D} = \{d_A | A$ is clopen in $X\}$ is a separating full system of n.-a. pseudometrics, and the uniformity $\mathfrak{U}$ determined by $\mathfrak{D}$ is compatible with the topology of X. (Compare also [19] and [4].)

Using n.-a. pseudometrics we can formulate a nice characterization of uniform spaces $(X, \mathfrak{U})$ with linearly ordered base: let $\mathfrak{B} = \{B_\tau | \tau < \omega_\mu\}$, $\mu > 0$, be such a base, then, as mentioned in § 4, we can assume that for every $\tau < \omega_\mu$, $\mathfrak{B}_\tau = \{B_\tau(x) | x \in X\}$ is a clopen partition of X, $\mathfrak{B}_\tau$ refining $\mathfrak{B}_\sigma$ iff $\tau > \sigma$. If we let $d_\tau(y, z) = 0$ iff y, z belong to the same set $B_\tau(x)$ and $d_\tau(y, z) = 1$ otherwise, we obtain a linearly ordered system of n.-a. pseudometrics compatible with $\mathfrak{U}$. Hereby we have $d_i \geqslant d_j$ iff $d_i(x, y) < \varepsilon$ implies $d_j(x, y) < \varepsilon$ for every $\varepsilon > 0$. Conversely, if a ω_1-*uniformity* $\mathfrak{U}$ on X (i.e. every countable intersection of entourages is an entourage) is induced by a linearly ordered, full and separating system of pseudometrics $\{d_i | i \in I\}$, then $\mathfrak{U}$ has a linearly ordered base. Hence X is either a discrete space (if I is countable) or X is a strongly zerodimensional space. Moreover, $\{d_i | i \in I\}$ can be replaced by a linearly ordered, full and separating system of non-archimedean pseudometrics for $\mathfrak{U}$.

Thus, summarizing results of this paper (combined with well-known results cited above) we obtain a corollary which supplements the theorem of de Groot and the remarks above. (Obviously, a countable system of n.-a. pseudometrics for $\mathfrak{U}$ can always be replaced by a n.-a. metric for $\mathfrak{U}$.)

COROLLARY 6.2. *For any uniform space $(X, \mathfrak{U})$ are equivalent:*

(i) *$\mathfrak{U}$ has a linearly ordered base and is not metrizable.*

(ii) *$\mathfrak{U}$ is a non-discrete ω_1-uniformity which can be generated by a linearly ordered, full and separating system of pseudometrics.*

(iii) *$\mathfrak{U}$ is a non-discrete ω_1-uniformity which can be generated by a linearly ordered, full and separating system of non-archimedean pseudometrics.*

(iv) *$\mathfrak{U}$ is non-metrizable but can be generated by a linearly ordered, full and separating system of discrete pseudometrics.*

(1) The theorem partly was known to F. Hausdorff [13]. See also J. Nagata [22].

Proof. The proof follows from the remarks above: (iv)⇒(iii)⇒(ii)⇒(i)⇒(iv).

Remark. Having concluded this paper I was informed that I. Juhász in *Untersuchungen über ω_μ-metrisierbare Räume* (Annale Univ. Sci., Section Mathematicai, Budapest, Tom. VIII (1965), pp. 129–145) proved a theorem which — in a great part — is analogue to Proposition 1.2 in the introduction of this paper; however, Juhász used different definitions and methods. (Amongst others, Juhász used the generalized continuum hypothesis for his proof; later, in 1973, it was shown by M. M. Čoban that the assumption of CGH can be deleted (Akad. Nauk. Moldavskat SSR. Izvest. Bul. Akad. RSS Moldovenest 3 (1973), pp. 12–19 & 91, in Russian)). Moreover, Theorem 3.2 could also be deduced from a theorem in Juhász' paper, by applying the theorem of Stevenson and Thron (§ 1). On the other hand, the proof of our Theorem 3.2 uses different methods and seems to be shorter and more direct.

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Added in proof (further and new references since 1975).

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